Recursion and Lists

Concepts of Programming Languages
Lecture 2

CAS CS 320

Practice Problem

None today...

Outline

» Cover the **basic expressions** we need to start programming in OCaml, look at some examples

» Discuss **lists** so we can write actually interesting programs

» Talk about **tail recursion** and how that affects performance of OCaml programs

Recall: Functional vs. Imperative

OCaml is a **functional language.** This means a couple things:

- No state (which means no loops!)
- We don't think of a program as describing a procedure, but as defining a value using an **expression**

Recall: The Three Components

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Syntax: What a *well-formed* program in your PL?

```
def f():
    return 3
```

✓

```
define f():
    3 return
```

✗

Recall: The Three Components

Syntax: What a *well-formed* program in your PL?

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Type System (Static Semantics): What is a *valid* program in your PL?

```
x = 2 + 2
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```
x = 2 + "two"
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```

Semantics (Dynamic Semantics): What is the *output* of a (valid) program?

```
>>> 2 + 2
4
```

```
>>> 2 + 2
False
```

For every possible expression, we'll define the syntax rules, the typing rules, and the semantic rules

One Last Point: Building Interpreters

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PL is math but we still like to *use* PLs. The three components of a PL correspond to the three things we need to *implement* in an **interpreter** of a PL.

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```
parse : string -> expr
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» **Syntax** is implemented by a **parser**

```
parse : string -> expr
```

» **Type system** is implemented by a **type checker**

```
type_check : expr -> bool (* valid or not *)
```

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» **Syntax** is implemented by a **parser**

```
parse : string -> expr
```

» **Type system** is implemented by a **type checker**

```
type_check : expr -> bool (* valid or not *)
```

» **(Dynamic) semantics** is implemented by an **evaluator**

```
eval : expr -> value
```

Basic Expressions

Basic Expressions

» Literals

» Let-expressions (local variables)

» If-expressions

» Functions

» Applications

Basic Expressions

- » **Literals**
- » Let-expressions (local variables)
- » If-expressions
- » Functions
- » Applications

Primitive Types and Literals

OCaml has a collection of standard literals and types

Type	Literals	Operators
int	0, -2, 13, -023	+, -, *, /, mod
float	3., -1.01	+., -., *., /.
bool	true, false	&&, ||, not
char	'b', 'c'	
string	"word", "@*&#"	^

Basic Expressions

- Literals
- **Let-expressions (local variables)**
- If-expressions
- Functions
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Let-Expressions (Informal)

body

```
let x = 2 in x + x
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let x = 2 in x + x
```

syntax: `let VARIABLE = EXPRESSION in BODY`

typing: the type is the same as that of BODY *given BODY is well-typed after substituting the VARIABLE in BODY*

semantics: the is the same as the value of BODY *after substituting the VARIABLE in BODY*

Example: Ill-Typed Let-Expression

```
let x = 2 in "two" <> x
```

An ill-typed expression will throw a type error when you type it into utop

Note that the body of a let-expression may be ill-typed *depending on the value assigned to its variable*

A Note on Substitution

$\boxed{\texttt{let x = 2 in x + x}} \longrightarrow \boxed{\texttt{2 + 2}}$

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```
let x = 2 in x + x
```

$\longrightarrow$

```
2 + 2
```

Formally, we write $[v/x]e$ to mean "substitute v for x in e", e.g., $[3/x](x+x)$ is the same as $3+3$

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Intuitively, substitution is simple: **replace the variable**

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Intuitively, substitution is simple: **replace the variable**

Turns out, this is **very hard** to do correctly, *it's subtle* and a source of a lot of mistakes in PL implementations

Recall: Anatomy of an OCaml Program

```
let x = 3

let y = "string"

(* function definition *)
let square x = x * x

(* recursive function definition *)
let rec f x = if x = 0 then 0 else x + f (x - 1)

(* We can't just print , we assign to wildcard *)
let _ = print_endline("Hello world")
```

An OCaml Program consists of <u>top-level let-expressions</u>, i.e., it is a **collection of named expressions**

OCaml Programs are Expressions

```
let x = 3 in

let y = "string" in

(* function definition *)
let square x = x * x in

(* recursive function definition *)
let rec f x = if x = 0 then 0 else x + f (x - 1) in

(* We can't just print , we assign to wildcard *)
let _ = print_endline("Hello world") in

()
```

This sequence of top–level let expressions is really shorthand for a **collection of nested local variables**

(This is a lie, but its a useful one for now)

Basic Expressions

- » Literals
- » Let-expressions (local variables)
- » **If-expressions**
- » Functions
- » Applications

If-Expressions

```
let abs x = if x > 0 then x else -x
```

Note: OCaml is whitespace agnostic!

If-Expressions

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OCaml has expressions for conditional reasoning

If-Expressions

```
let abs x = if x > 0 then x else -x
```

Note: OCaml is whitespace agnostic!

OCaml has expressions for conditional reasoning

Note: The **else** case is *required* and the **then** and **else** cases must be the *same type* (why?)

If-Expressions

```
let foo x =
  if x < 0 then
    "negative"
  else if x = 0 then
    "zero"
  else
    "positive"
```

Answer: Remember, all we have is expressions. So every if-expression <u>must have a value and a type</u> (and therefore, an **else** case of the same type)

We can do **else if** just by nesting if-expressions! (neat)

Aside: If-Expressions in Python

```
if x < 0:
  return -1
else:
  return 1
```

if-stmt (Python)

```
return (-1 if x < 0 else 1)
```

if-expr (Python)

If-*statements* in Python are different from if-expressions, but **both are available**

Statements don't have a value, expressions do

If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

If-Expressions (Informal)

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Syntax: `if CONDITION then TRUE–CASE else FALSE–CASE`

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Typing: CONDITION must be a Boolean and TRUE–CASE and FALSE–CASE must be the same type. The type is then the same as that of TRUE–CASE and FALSE–CASE

If-Expressions (Informal)

```
let abs x = if x > 0 then x else -x
```

Syntax: if CONDITION then TRUE–CASE else FALSE–CASE

Typing: CONDITION must be a Boolean and TRUE–CASE and FALSE–CASE must be the same type. The type is then the same as that of TRUE–CASE and FALSE–CASE

Semantics: If CONDITION holds, then we get the TRUE–CASE, otherwise we get the FALSE–CASE

Basic Expressions

» Literals

» Let-expressions (local variables)

» If-expressions

» **Functions**

» Applications

Functions

```
let f x y z = x + y + z
let f (x : int) (y : int) (z : int) : int = x + y + z
```

There are a couple ways of defining functions in OCaml

Note that let-expression can take arguments. ***How should we interpret this? If everything is an expression?***

Anonymous Functions

```
let f = fun x -> fun y -> fun z -> x + y + z
```

Answer: It must be that **functions are expressions as well**!

In OCaml, we can define *anonymous* functions, which are just **functions without names**

You should think of:

```
let f x y z = x + y + z
```

as shorthand for the above

Aside: Type Annotations

```
let rec fact (n : int) : int =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

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OCaml has <u>type inference</u> which means we rarely have to *specify* the types of expression in our program

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let rec fact (n : int) : int =
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  then 1
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```

OCaml has type inference which means we rarely have to *specify* the types of expression in our program

That said, you **should** include type annotations, especially at the beginning, because they're useful for *documentation* and for *code clarity*

Aside: Anonymous Functions in Python

```
lambda x: x + 1
```

Python

```
fun x -> x + 1
```

OCaml

$$\lambda x\,.\,x + 1$$

Math

There are also anonymous function in Python!

They're called *lambdas*, based on the **lambda calculus**, a mathematical formulation of a functional PL that dates back to the 1930s, invented by **Alonzo Church**

(You'll find a lot of functional ideas hidden in languages like Python)

Functions (Informal)

body

```
fun x -> x + 1
```

Syntax: fun VAR–NAME –> EXPR

Typing: the type of a function is **T1 –> T2** where T1 is the type of the input and T2 is the type of the output

Semantics: A function will evaluate to special ***function value*** (printed as <fun> by utop)

Important: Curried Functions

```
let f = fun x -> fun y -> fun z -> x + y + z
```

The only kind of function we have is *single argument*

This seems restrictive, but ultimately it doesn't affect us <u>at all</u>

We can *simulate* multi-argument functions with nested functions. This is called **Currying** after Haskell Curry

Important: Curried Functions

```
let f = fun x -> fun y -> fun z -> x + y + z
```

We should think of the above function as something which takes an input and returns **another function**

In other words, we *partially apply* the function

Basic Expressions

- » Literals
- » Let-expressions (local variables)
- » If-expressions
- » Functions
- » **Applications**

Application

```
(fun x -> fun y -> x + y + 1) 3 2
```

Application is done by *juxtaposition* which means we put the arguments next to the function

Application is *left-associative*, which means we pass arguments from left to right

Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
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Application (Informally)

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(fun x -> fun y -> x + y + 1) 3 2
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`Syntax:` `FUNCTION–EXPR ARG–EXPR`

Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
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Syntax: FUNCTION–EXPR ARG–EXPR

Typing: If FUNCTION–EXPR is of type T1 –> T2, and ARG–EXPR is of type T1, then the type is T2

Application (Informally)

```
(fun x -> fun y -> x + y + 1) 3 2
```

Syntax: FUNCTION–EXPR ARG–EXPR

Typing: If FUNCTION–EXPR is of type T1 –> T2, and ARG–EXPR is of type T1, then the type is T2

Semantics: Substitute the value of ARG–EXPR into the body of FUNCTION–EXPR and evaluate that

Application (Example)

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```
(fun x ->  fun y -> x + y  + 1)  3  2
```

Application (Example)

```
(fun x ->  fun y -> x + y  + 1)  3  2
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is the same as

```
(fun x -> (fun y -> x + y  + 1)) 3  2
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Application (Example)

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(fun x ->  fun y -> x + y  + 1)  3  2        is the same as

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```

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           (fun y -> 3 + y  + 1)     2      evaluates to

                     3 + 2  + 1
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           (fun y -> 3 + y  + 1)      2      evaluates to

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                    (3 + 2) + 1
```

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          (fun y -> 3 + y  + 1)     2       evaluates to

                    3 + 2  + 1              is the same as

                   (3 + 2) + 1              evaluates to

                         5 + 1
```

Application (Example)

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(fun x -> (fun y -> x + y  + 1)) 3  2        is the same as
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          (fun y -> 3 + y  + 1)     2        evaluates to
                    3 + 2  + 1               is the same as
                   (3 + 2) + 1               evaluates to
                        5  + 1               evaluates to
                        6
```

demo

(extended example)

Lists

What is a list?

```
let _ = 1 :: 2 :: 3 :: []
let _ = 1 :: (2 :: (3 :: []))
let _ = [1; 2; 3]
```

A list is an ordered ***variable-length*** ***homogeneous*** collection of data

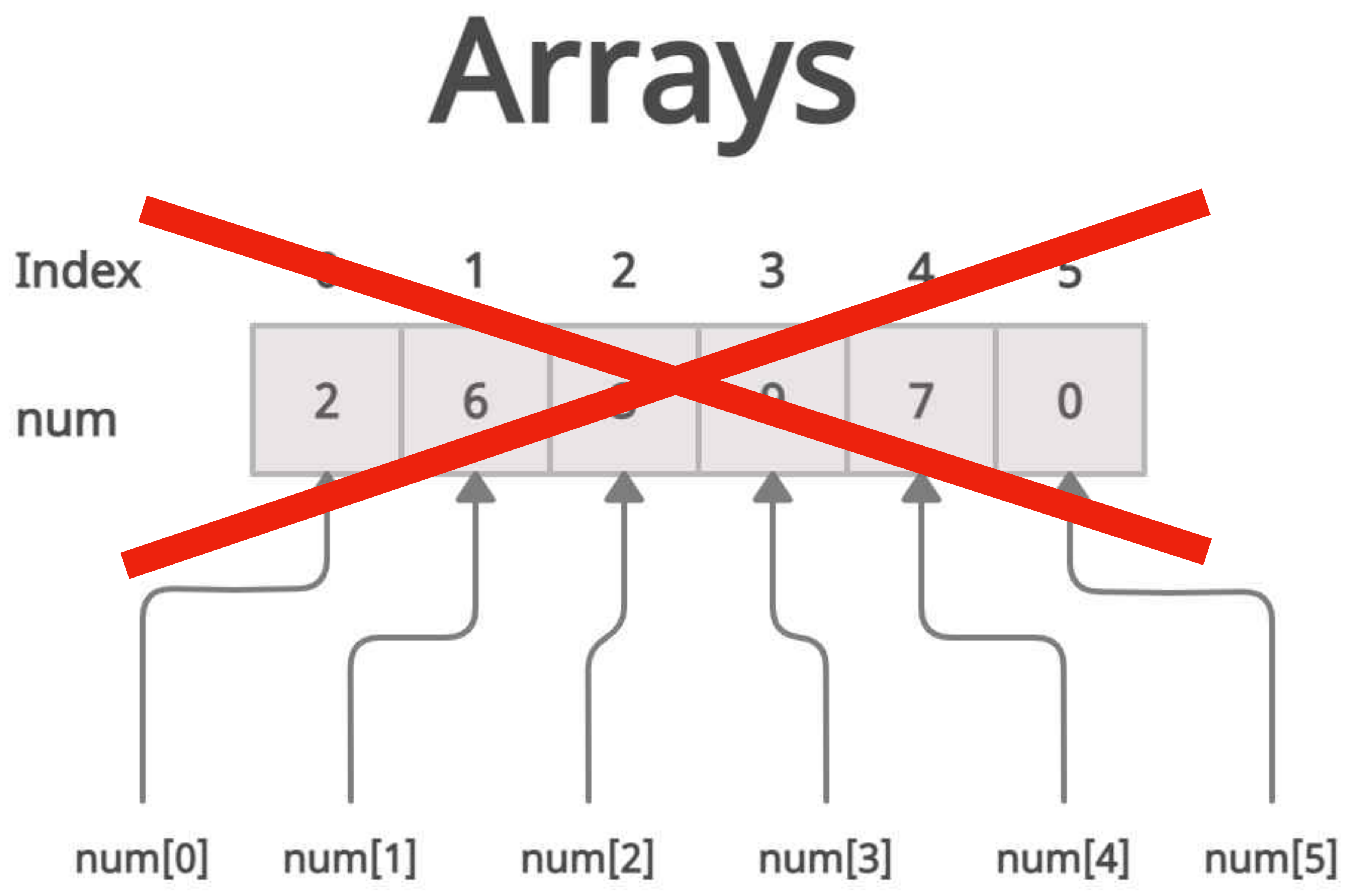
Many important operations on data can be represented as operations on lists (e.g., updating all users in a database)

What is a list not?

A list is *not* an array. We don't have constant–time indexing

A list is *not* mutable. **No data structures in FP are mutable**

(You should think of a list structurally as more like a linked list, sort of)



https://usemynotes.com/what-are-arrays-in-java/

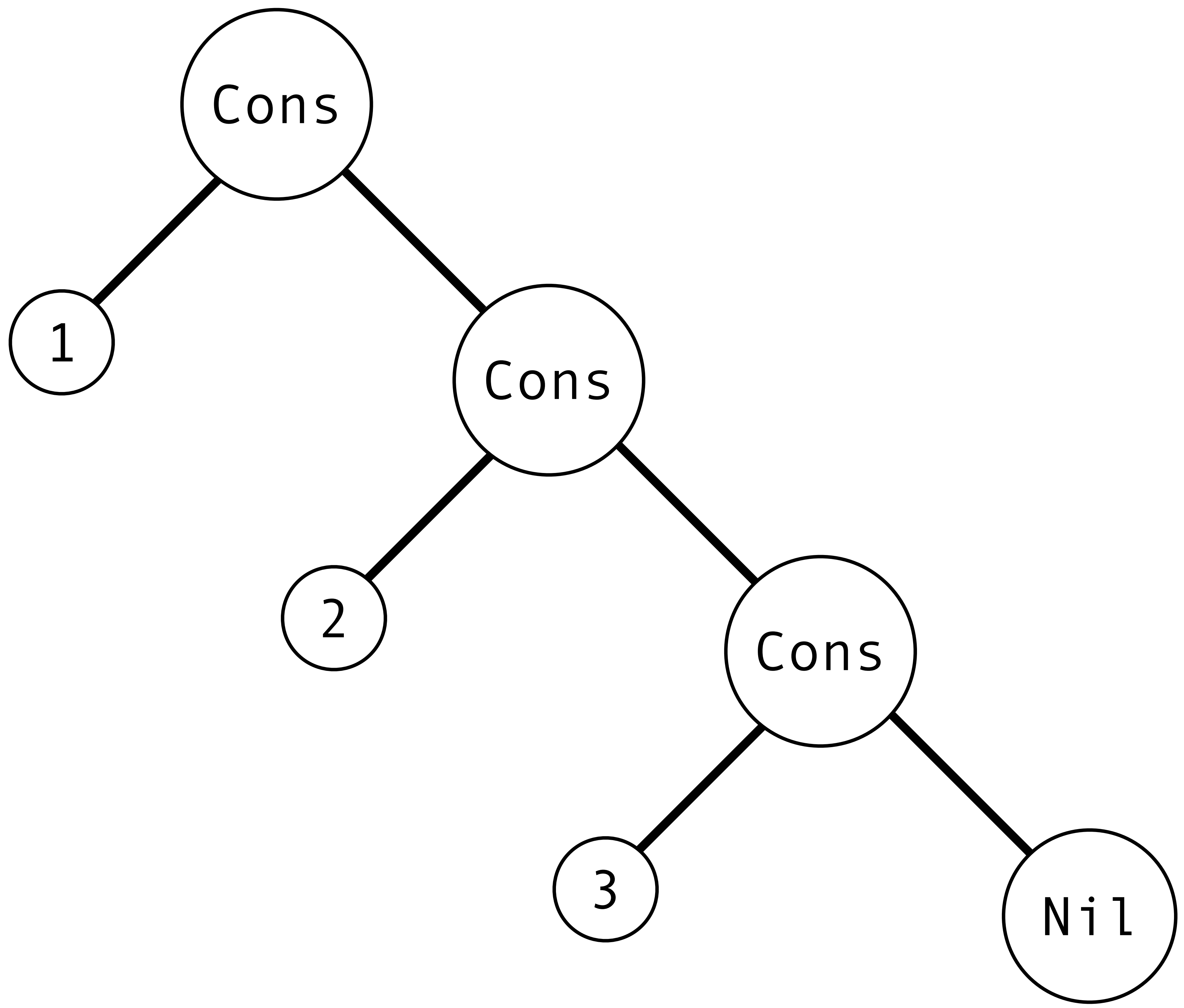
The Picture

We can think of the list

`1 :: 2 :: 3 :: []`

as a leaning tree with data a leaves

(this will generalize to other *algebraic* data types)



List Syntax (Informally)

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`x :: xs` stands for the list `xs` with `x` prepended to it. The symbol `::` is pronounced "cons" and is a *right associative* operator

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`x :: xs` stands for the list xs with x prepended to it. The symbol `::` is pronounced "cons" and is a *right associative* operator

`[x1; x2;...; xn]` is a list literal. It's shorthand for a list of a known length

Example

Construct a function ***generate*** *which, given integers* ***n,*** *returns a list consisting of the first* ***n*** *positive integers*

Lists Semantics (Informally)

$$\texttt{[2 + 3; 4 * 12; 2 - 1]} \Downarrow \texttt{[5; 48; 1]}$$

We evaluate the list [e_1; e_2; ... ; e_k] by evaluating each element of the list (from right to left)

Destructing Lists

```
match l with
| [] -> (* something *)
| x :: xs -> (* something else *)
| ... (* other patterns??? *)
```

As with any type in OCaml, we can use **pattern matching** to destruct lists

With pattern matching, we describe the value we want based on the *shape* of the list we're matching on

Example

*Implement the function **double** where **double l** is the same as the list **l** but with every element doubled*

Lists are Immutable

```
val remove_all_negatives : int list -> int list
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Lists are Immutable

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```

All data structures in FP are <u>immutable</u>

If we want to "update" a list, we have to produce an *entirely new* list

In reality the data is not literally duplicated, there are optimizations which allow for shared data

Practice Problem

Implement the function

remove_all_negatives : int list -> int list

*where **remove_all_negative l** is the same as the list **l** but with all negative numbers removed*

Deep Pattern Matching

```
match <expr> with
| [] -> <expr>
| [h1; h2] -> <expr>
| h1::h2::t -> <expr>
| h::t -> <expr>
| ......
```

Pattern matching is very general. We can match on more complex patterns than just empty and nonempty

Example

Implement the function

delete_every_other : int list -> int list

*such that **delete_every_other l** is the first, third, fifth,..., and so on elements of **l***

A Note on Polymorphism

```
let rec length l =
  match l with
  | [] -> 0
  | x :: xs -> 1 + length xs
```

What is the type of the length function?

Does this function depend on the values in the list?

The List Type

`[1;2;3]`	`["1";"2";"3"]`	`[[1;1];[2;2];[3;3]]`
`int list`	`string list`	`int list list`

The List Type

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The list type is an example of a **parametrized** type

The List Type

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[1;2;3]          ["1";"2";"3"]          [[1;1];[2;2];[3;3]]
```

int list   **string list**   **int list list**

The list type is an example of a **parametrized** type

A function on lists is *polymorphic* (with respect to the list parameter) if it can be apply to a list parametrized by *any* type

The List Type

`[1;2;3]` — **int list**

`["1";"2";"3"]` — **string list**

`[[1;1];[2;2];[3;3]]` — **int list list**

The list type is an example of a **parametrized** type

A function on lists is *polymorphic* (with respect to the list parameter) if it can be apply to a list parametrized by *any* type

For this, we need *type parameters* to stand for *any* type:

The List Type

`[1;2;3]`	`["1";"2";"3"]`	`[[1;1];[2;2];[3;3]]`
int list	**string list**	**int list list**

The list type is an example of a **parametrized** type

A function on lists is *polymorphic* (with respect to the list parameter) if it can be apply to a list parametrized by *any* type

For this, we need *type parameters* to stand for *any* type:

'a, 'b, 'c,...

Not all functions can be polymorphic

```
let rec sum l =
  match l with
  | [] -> 0
  | x :: xs -> x + sum xs
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Can this function be applied to a list parametrized by any type?

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Can this function be applied to a list parametrized by any type?

Answer: No, it can only be applied to **int lists**

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Can this function be applied to a list parametrized by any type?

Answer: No, it can only be applied to **int lists**

OCaml's type inference is good at "guessing" when functions are polymorphic

Practice Problem

Implement the function

reverse : 'a list -> 'a list

such that ***reverse l*** *is the same as* ***l*** *but in reverse order*

Tail Recursion

demo

(even the wrong way)

Tail Recursion

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

not tail recursive

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

tail recursive

A recursive function is **tail recursive** if it does not perform any computations on the result of a recursive call

Why do we care?

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Recursive functions are *expensive* with respect to the call-stack

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Tail-call elimination is an optimization implemented by OCaml's compiler which *reuses* stack frames

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Tail-recursive functions "behave iteratively"

The Picture

```
fact 5
```

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

The Picture

fact 5

fact 4

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

The Picture

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let rec fact n =
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  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

The Picture

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fact 5

fact 4

fact 3

fact 2

fact 1

The Picture

```
let rec fact n =
  if n <= 0
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  else n * fact (n - 1)
```

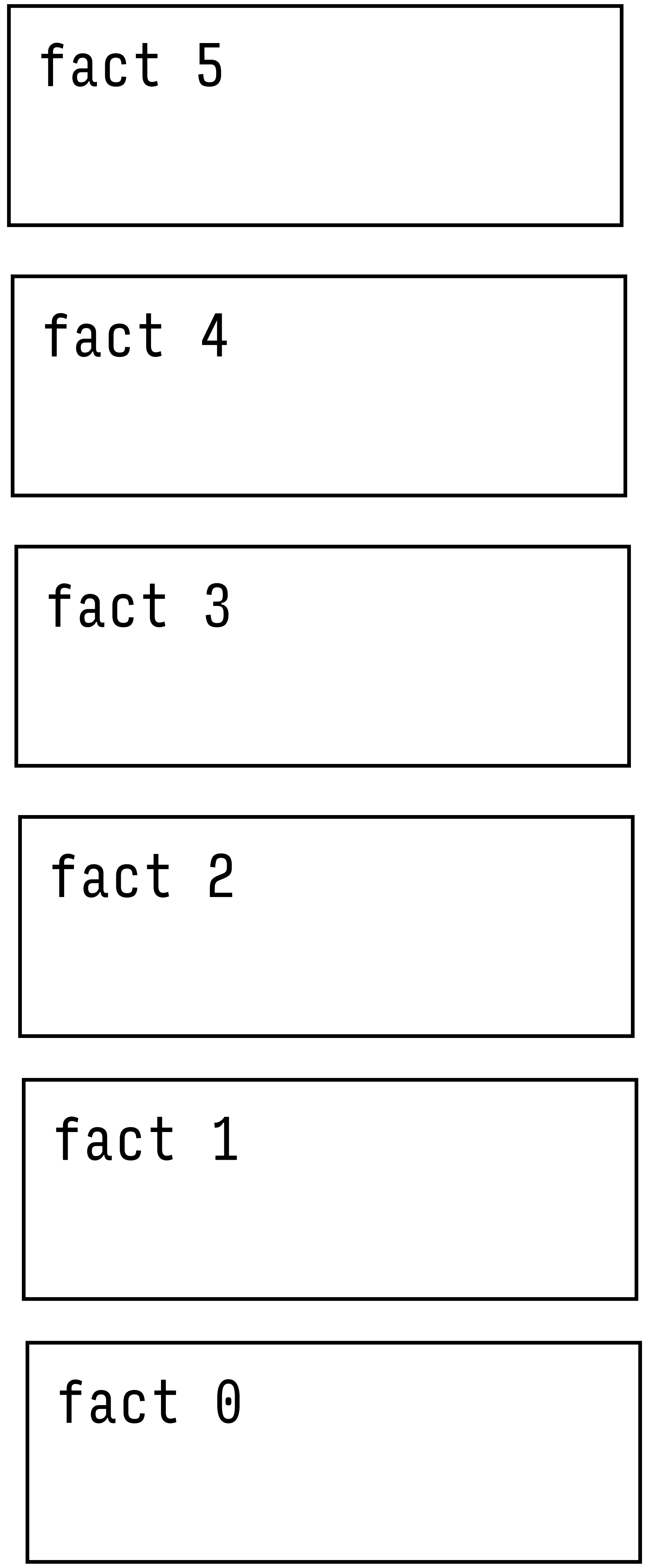

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

fact 5
fact 4
fact 3
fact 2
fact 1
fact 0 $\Longrightarrow$ **1**

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

fact 5
fact 4
fact 3
fact 2
fact 1 $\Longrightarrow$ **1 * 1 = 1**

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

fact 2

$\Longrightarrow$ **2 * 1 = 2**

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

fact 5

fact 4

fact 3

$\Longrightarrow$ **3 * 2 = 6**

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

```
fact 5
```

```
fact 4
⟹ 4 * 6 = 24
```

The Picture

```
fact 5
⟹ 5 * 24 = 120
```

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

The Picture

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

```
fact 5
⟹ 5 * 24 = 120

fact 4
⟹ 4 * 6 = 24

fact 3
⟹ 3 * 2 = 6

fact 2
⟹ 2 * 1 = 2

fact 1
⟹ 1 * 1 = 1

fact 0
⟹ 1
```

1 frame per recursive call

The Picture

```
loop 1 5
```

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

The Picture

```
loop 1 5
```

```
loop 5 4
```

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

```
fact 60 2
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

```
fact 60 2
```

```
fact 120 1
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

```
fact 60 2
```

```
fact 120 1
```

```
fact 120 0
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

```
fact 60 2
```

```
fact 120 1
```

```
fact 120 0

⟹ 120
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

```
fact 60 2
```

```
fact 120 1

⟹ 120
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
```

```
fact 60 2
```

$\Longrightarrow$ **120**

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

```
loop 5 4
```

```
loop 20 3
⟹ 120
```

The Picture

```
loop 1 5
```

```
loop 5 4
⟹ 120
```

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

The Picture

```
loop 1 5
⟹ 120
```

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

The Picture

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
⟹ 120

loop 5 4
⟹ 120

loop 20 3
⟹ 120

fact 60 2
⟹ 120

fact 120 1
⟹ 120

fact 120 0
⟹ 120
```

1 frame per recursive call

BUT THE VALUE DOESN'T CHANGE ON IT'S WAY UP THE CALL STACK

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 1 5
```

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 5 4
```

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 20 3
```

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 120 1
```

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 120 0
```

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 120 0
```

$\Longrightarrow$ **120**

The Picture (Optimized)

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

```
loop 120 0

⟹ 120
```

1 frame *for every* recursive call

Tail Position

```
let rec fact n =
  if n <= 0
  then 1
  else n * fact (n - 1)
```

computation after the recursive call

not tail recursive

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

tail recursive

Tail-call optimizations apply to functions whose recursive calls are in **tail position**

Intuition: A call is in tail position if there is no computation *after* the recursive call

Aside: Tail Position More Formally

```
let rec f x1 x2 ... xk = e
```

A recursive call **f e1 e2 ... ek** is in tail position in **e** if:

- it does not appear in **e**, or **e** is the recursive call itself
- e = **if e1 then e2 else e3** and the call does not appear in **e1** and it is in tail position in **e2** and **e3**
- e is a **match-expression** and the call is in tail position in every branch, and does not appear in the matched expression
- e = **let x = e1 in e2** and the call does not appear in the **e1** and it is in tail position in **e2**

Aside: Tail Position More Formally

```
let rec f x1 x2 ... xk = e
```

A recursive call **f e1 e2 ... ek*** is in tail position in **e** if:

- it does not appear in **e**, or **e** is the recursive call itself
- e = **if e1 then e2 else e3** and the call does not appear in **e1** and it is in tail position in **e2** and **e3**
- e is a **match-expression** and the call is in tail position in every branch, and does not appear in the matched expression
- e = **let x = e1 in e2** and the call does not appear in the **e1** and it is in tail position in **e2**

* f cannot appear in e1 ... ek

Tail Recursion and Lists

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

Tail Recursion and Lists

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

We need to take care with tail-recursion and lists

Tail Recursion and Lists

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

We need to take care with tail-recursion and lists

Does the above program concatenate two lists?

```ocaml
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```ocaml
append [1;2;3] [4;5;6]
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [1;2;3] [4;5;6]
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
match [1;2;3] with
| [] -> [4;5;6]
| x :: xs -> loop xs (x :: [4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
match 1 :: [2;3] with
| [] -> [4;5;6]
| x :: xs -> loop xs (x :: [4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [2;3] (1 :: [4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [2;3] [1;4;5;6]
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
match [2;3] with
| [] -> [1;4;5;6]
| x :: xs -> loop xs (x :: [1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
match 2 :: [3] with
| [] -> [1;4;5;6]
| x :: xs -> loop xs (x :: [1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [3] (2 :: [1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [3] [2;1;4;5;6]
```

```ocaml
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```ocaml
match [3] with
| [] -> [2;1;4;5;6]
| x :: xs -> loop xs (x :: [2;1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
match 3 :: [] with
| [] -> [2;1;4;5;6]
| x :: xs -> loop xs (x :: [2;1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [] (3 :: [2;1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
loop [] [3;2;1;4;5;6]
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

```
match [] with
| [] -> [3;2;1;4;5;6]
| x :: xs -> loop xs (x :: [3;2;1;4;5;6])
```

```
let append l r =
  let rec loop acc l =
    match l with
    | [] -> acc
    | x :: xs -> loop xs (x :: acc)
  in loop l r
```

[3;2;1;4;5;6]

whoops!

Tail Recursion and Lists

```
let append l r =
  let rec loop l acc =
    match l with
    | [] -> acc
    | x :: xs -> loop (x :: acc) xs
  in loop l r
```

We need to take care with tail-recursion and lists

Does the above program concatenate two lists?

Tail Recursion and Lists

```
let append l r =
  let rec loop l acc =
    match l with
    | [] -> acc
    | x :: xs -> loop (x :: acc) xs
  in loop l r
```

should be (List.rev l)

We need to take care with tail-recursion and lists

Does the above program concatenate two lists?

Accumulators

```
let fact n =
  let rec loop acc n =
    if n <= 0
    then acc
    else loop (n * acc) (n - 1)
  in loop 1 n
```

Our accumulator pattern is almost always tail recursive

Code Example

Implement the function

reverse : 'a list -> 'a list

The implementation must be tail recursive

Summary

OCaml is a language of **expressions**, everything is an expression

Lists are used to process collections of homogeneous data

We can use **tail-recursion** to make our implementations more memory efficient, but we have to be careful when working with lists