APPENDIX

Theorem 6 (Theorem 7 in [14]). *If $\boldsymbol{x}$ and $\boldsymbol{y}$ are confusable, then they have to be in one of the following forms.*
(A) $\boldsymbol{x} = \boldsymbol{u\alpha\omega\beta v}, \boldsymbol{y} = \boldsymbol{u\bar{\alpha}\omega\bar{\beta}v}$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are alternating sequences of length at least two.
(B) $\boldsymbol{x} = \boldsymbol{u\alpha}\bar{a}\gamma\bar{b}\boldsymbol{\beta v}, \boldsymbol{y} = \boldsymbol{u}\overline{\boldsymbol{\alpha}a\gamma b\boldsymbol{\beta}}\boldsymbol{v}$, where $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are alternating sequences. Here, $\boldsymbol{\alpha}$ is of length at least two and ends with a, $\boldsymbol{\beta}$ is of length at least two and starts with b, and $\boldsymbol{\gamma}$ starts with $\bar{a}$ and ends with $\bar{b}$.

Theorem 7 (Theorem 4.1 in [13]). *For two sequences $\boldsymbol{x} = \boldsymbol{u\tilde{x}v}$ and $\boldsymbol{y} = \boldsymbol{u\tilde{y}v}$ in Σ^n with $d_L(\boldsymbol{x},\boldsymbol{y}) \geq 2$, if $|\mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y})| \geq 5$, then one of the following holds:*

(A) $\boldsymbol{\tilde{x}} = \boldsymbol{a\omega b}$ and $\boldsymbol{\tilde{y}} = \boldsymbol{\bar{a}\omega\bar{b}}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are alternating sequences of length at least two and $\boldsymbol{\omega}$ is a combination of periodic sequences. More precisely, suppose $\boldsymbol{a}$ ends with α and $\boldsymbol{b}$ starts with β',then $\boldsymbol{\omega} = \boldsymbol{u'\omega'v'}$ and one of the following holds:

- *$\boldsymbol{u'}$ is a run sequence starting with α, $\boldsymbol{\omega'}$ is an alternating sequence of length at least one starting with $\bar{\alpha}$ and ending with β', $\boldsymbol{v'}$ is a run sequence starting with $\bar{\beta}'$*
- *$\boldsymbol{u'}$ is a run sequence starting with $\bar{\alpha}$, $\boldsymbol{\omega'}$ is an alternating sequence of length at least one starting with α and ending with $\bar{\beta}'$, $\boldsymbol{v'}$ is a run sequence starting with β'.*

(B) $\boldsymbol{\tilde{x}} = \boldsymbol{a}\bar{\alpha}\boldsymbol{\omega}\bar{\beta}\boldsymbol{b}, \boldsymbol{\tilde{y}} = \boldsymbol{a}\alpha\bar{\boldsymbol{\omega}}\beta\boldsymbol{b}$, where a is an alternating seqence of length at least two ending with α, $\boldsymbol{\omega}$ is an alternating sequence starting with $\bar{\alpha}$ and ending with $\bar{\beta}'$, and b is an alternating sequence of length at least two starting with β'.

Particularly, $|\mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y})| = 6$ if and only if $\boldsymbol{\tilde{x}} = \boldsymbol{a\omega b}$ and $\boldsymbol{\tilde{y}} = \boldsymbol{\bar{a}\omega\bar{b}}$, where $\boldsymbol{a}$ is an alternating sequence of length at least two ending with α, $\boldsymbol{b}$ is an alternating sequence of length at least two starting with β', and one of the following holds:

- *if $\boldsymbol{\omega}$ starts with α' and ends with β', $\boldsymbol{\omega}$ is a alternating sequence of length at least two;*
- *if $\boldsymbol{\omega}$ starts with α' and ends with $\bar{\beta}'$, $\boldsymbol{\omega}$ is a alternating sequence of length at least three;*
- *if $\boldsymbol{\omega}$ starts with $\bar{\alpha}'$ and ends with β', $\boldsymbol{\omega}$ is a alternating sequence of length at least three;*
- *if $\boldsymbol{\omega}$ starts with $\bar{\alpha}'$ and ends with $\bar{\beta}'$, $\boldsymbol{\omega}$ is a alternating sequence of length at least two;*

Lemma 10 (Lemma 12(ii) in [14]). *Suppose that $\boldsymbol{x} = \alpha\boldsymbol{r}\beta$ and $\boldsymbol{y} = \bar{\alpha}\boldsymbol{s}\bar{\beta}$ and $d_L(\boldsymbol{x},\boldsymbol{y}) \geq 2$. Set $\mathcal{S} = \mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y})$. If the first two bits of $\boldsymbol{x}$ are equal, i.e. $\boldsymbol{r} = \alpha\boldsymbol{r'}$, then $|\boldsymbol{\mathcal{S}}| \leq 4$. Hence by symmetry, if $\boldsymbol{r} = \boldsymbol{r'}\beta$ or $\boldsymbol{s} = \bar{\alpha}\boldsymbol{s'}$ or $\boldsymbol{s} = \boldsymbol{s'}\bar{\beta}$, i.e., the first or last two bits of $\boldsymbol{x}$ or $\boldsymbol{y}$ are equal, then $|\boldsymbol{\mathcal{S}}| \leq 4$.*

Lemma 7 Let $\boldsymbol{x},\boldsymbol{y}$ be two binary sequences of length n such that $d_L(\boldsymbol{x},\boldsymbol{y}) \geq 3$, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, then $\boldsymbol{x} = \boldsymbol{uas\omega^1tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\omega^2t\bar{b}v}$, where

- $\boldsymbol{u},\boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,
- $\boldsymbol{a},\boldsymbol{b},\boldsymbol{s},\boldsymbol{t}$ are maximal alternating sequence of length at least 2.
- $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$.

Proof: According Lemma 6, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, then they are of the form $\boldsymbol{x} = \boldsymbol{ua\omega^1bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}\omega^2\bar{b}v}$, such that

- $\boldsymbol{u},\boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$;
- $\boldsymbol{a},\boldsymbol{b}$ are maximal alternating sequences with $|\boldsymbol{a}|, |\boldsymbol{b}| \geq 2$;
- $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ are two distinct sequences of length ℓ, where $\omega_1^1 = \omega_1^2, \omega_\ell^1 = \omega_\ell^2$.

Suppose $\boldsymbol{a}$ starts with i, ends with α, $\boldsymbol{b}$ starts with β ends with j. For the sequence $\boldsymbol{x},\boldsymbol{y}$ satisfying the above requirements, there are four possibilities as follows:

1) $\boldsymbol{x} = \boldsymbol{ua}\alpha\omega_2^1 \cdots \omega_{\ell-1}^1\beta\boldsymbol{bv}$, $\boldsymbol{y} = \boldsymbol{u\bar{a}}\alpha\omega_2^2 \cdots \omega_{\ell-1}^2\beta\boldsymbol{\bar{b}v}$;
2) $\boldsymbol{x} = \boldsymbol{ua}\alpha\omega_2^1 \cdots \omega_{\ell-1}^1\bar{\beta}\boldsymbol{bv}$, $\boldsymbol{y} = \boldsymbol{u\bar{a}}\alpha\omega_2^2 \cdots \omega_{\ell-1}^2\bar{\beta}\boldsymbol{\bar{b}v}$;
3) $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\omega_2^1 \cdots \omega_{\ell-1}^1\beta\boldsymbol{bv}$, $\boldsymbol{y} = \boldsymbol{u\bar{a}}\bar{\alpha}\omega_2^2 \cdots \omega_{\ell-1}^2\beta\boldsymbol{\bar{b}v}$;
4) $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\omega_2^1 \cdots \omega_{\ell-1}^1\bar{\beta}\boldsymbol{bv}$, $\boldsymbol{y} = \boldsymbol{u\bar{a}}\bar{\alpha}\omega_2^2 \cdots \omega_{\ell-1}^2\bar{\beta}\boldsymbol{\bar{b}v}$.

where (3)(4) are equivalent to (1)(2), so it is sufficient to discuss only (1)(2).

Denote $\omega_3^1 \cdots \omega_l^1$ as $\boldsymbol{c}$, $\omega_3^2 \cdots \omega_l^2$ as $\boldsymbol{d}$, $\mathcal{D}_3(\boldsymbol{a\omega^1b}) \cap \mathcal{D}_3(\boldsymbol{\bar{a}\omega^2\bar{b}})$ as $\boldsymbol{\mathcal{S}}$. Without prejudice to generality, let $\omega_1^1 = \omega_1^2 = \alpha$, and We proceed with the following three cases.

Case (A): $\omega_1^1 = \alpha, \omega_2^1 = \alpha$

$$\boldsymbol{a\omega^1b} = \boldsymbol{a}\ \alpha\alpha\boldsymbol{c}\ \boldsymbol{b}\ =\ i \cdots \alpha\alpha\alpha\boldsymbol{c}\beta \cdots j$$
$$\boldsymbol{\bar{a}\omega^2\bar{b}} = \boldsymbol{\bar{a}}\ \alpha\omega_2^2\boldsymbol{d}\ \boldsymbol{\bar{b}} = \bar{i} \cdots \bar{\alpha}\alpha\omega_2^2\boldsymbol{d}\bar{\beta} \cdots \bar{j}$$

Since $\boldsymbol{a},\boldsymbol{b}$ are alternating sequences, $\boldsymbol{\mathcal{S}}_j^i = (a_1 \cdots \alpha_{|\boldsymbol{a}|-1}) \circ (\mathcal{D}_3(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_1(\omega_2^2\boldsymbol{d})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$. According Lemma 4, $|\boldsymbol{\mathcal{S}}_j^i| \leq 4$.

$\boldsymbol{\mathcal{S}}_{\bar{j}}^i = (a_1 \cdots \alpha_{|\boldsymbol{a}|-1}) \circ (\mathcal{D}_2(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_2(\omega_2^2\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$. According to Lemma 1, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^i| \leq 6$.

$\boldsymbol{\mathcal{S}}_j^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_2(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_2(\bar{\alpha}\alpha\omega_2^2\boldsymbol{d})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$. We know that the first bit of the sequence of centers of the two error balls is different, then according to Lemma 10, $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| \leq 4$.

$\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\omega_2^2\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$. According Lemma 4, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| \leq 4$.

In this case, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = |\boldsymbol{\mathcal{S}}| \leq 17$.

Case (B): $\omega_2^1 = \bar{\alpha}, \omega_2^2 = \alpha$

$$\boldsymbol{a\omega^1b} = \boldsymbol{a}\alpha\bar{\alpha}\omega_3^1 \cdots \omega_l^1\boldsymbol{b} = i \cdots \alpha\alpha\bar{\alpha}\boldsymbol{c}\beta \cdots j$$
$$\boldsymbol{\bar{a}\omega^2\bar{b}} = \boldsymbol{\bar{a}}\alpha\alpha\omega_3^2 \cdots \omega_l^2\boldsymbol{\bar{b}} = \bar{i} \cdots \bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta} \cdots \bar{j}$$

Similarly to *Case (A)*, $|\boldsymbol{\mathcal{S}}_j^i| \leq 4, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| \leq 4$. $\boldsymbol{\mathcal{S}}_{\bar{j}}^i = (a_1 \cdots \alpha_{|\boldsymbol{a}|-1})\alpha \circ (\mathcal{D}_2(\alpha\boldsymbol{c}) \cap \mathcal{D}_2(\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$. According to Lemma 1, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^i| \leq 6$ and according Theorem 6, if $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{l}| \geq 5$, then $\bar{\alpha}\boldsymbol{c}, \boldsymbol{d}\bar{\beta}$ satisfies one of the following two structures

(i) $\bar{\alpha}\boldsymbol{c} = \boldsymbol{s_1t_1t_2t_3s_2},\ \boldsymbol{d}\bar{\beta} = \boldsymbol{s_1\bar{t_1}t_2\bar{t_3}s_2}$, or
(ii) $\bar{\alpha}\boldsymbol{c} = \boldsymbol{s_1t_1}\bar{\gamma}\boldsymbol{t_2}\bar{\lambda}\boldsymbol{t_3s_2},\ \boldsymbol{d}\bar{\beta} = \boldsymbol{s_1\bar{t_1}}\bar{\gamma}\boldsymbol{\bar{t_2}}\bar{\lambda}\boldsymbol{\bar{t_3}s_2}$

, where $\boldsymbol{t_1},\boldsymbol{t_2},\boldsymbol{t_3}$ are alternating sequences, and their lengths are denoted ℓ_1, ℓ_2, ℓ_3 respectively. In particular, in case (ii), $\boldsymbol{t_1}$ ends with γ, $\boldsymbol{t_3}$ starts with λ. For convenience, we suppose that $\boldsymbol{t_3}$ ends with θ , in case (i),

$$\begin{aligned}
|\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| &= |\mathcal{D}_1(\alpha\boldsymbol{s_1t_1t_2t_3s_2}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{s_1\bar{t_1}t_2\bar{t_3}s_2})| \\
&= |\mathcal{D}_1(\alpha\boldsymbol{s_1t_1t_2t_3}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{s_1\bar{t_1}t_2\bar{t_3}})| \\
&= |(\boldsymbol{s_1t_1t_2t_3}) \cap (\bar{\alpha} \circ \mathcal{D}_2(\alpha\alpha\boldsymbol{s_1\bar{t_1}t_2\bar{t_3}}[1, \ell_3 - 2]) \circ \theta)| \\
&+ |\alpha \circ (\mathcal{D}_1(\boldsymbol{s_1t_1t_2t_3}[1, \ell_3 - 1]) \cap \mathcal{D}_1(\alpha\boldsymbol{s_1\bar{t_1}t_2\bar{t_3}}[1, \ell_3 - 2])) \circ \theta| \\
&+ |\alpha \circ (\mathcal{D}_0(\boldsymbol{s_1t_1t_2t_3}[1, \ell_3 - 2]) \cap \mathcal{D}_2(\alpha\boldsymbol{s_1\bar{t_1}t_2\bar{t_3}}[1, \ell_3 - 1])) \circ \bar{\theta}|
\end{aligned}$$

Note that $(\boldsymbol{s_1 t_1 t_2 t_3})$ starts with $\bar{\alpha}'$, and in particular, we use $s_1[1]$ to denote the first bit of the sequence $\boldsymbol{s_1}$ in this section.

Firstly, according Lemma 2, $|\mathcal{D}_1(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3 - 1]) \cap \mathcal{D}_1(\alpha \boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3 - 2]) \leq 1$. Next, we asume that there exists $|\mathcal{D}_0(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3 - 2]) \cap \mathcal{D}_2(\alpha \boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3 - 1])| = 1$, then in order to get the same prefix $\boldsymbol{s_1 t_1 [1, \ell_1 - 1]}$, it is necessary to delete α in the begining and $\bar{t_1}$, since $\boldsymbol{t_2}$ is alternating sequence, it is obviously that at this point $t_1[\ell] t_2[1] \neq t_2[1] t_2[2]$, then $(\alpha \circ (\mathcal{D}_0(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3 - 2]) \cap \mathcal{D}_2(\alpha \boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3 - 1])) \circ \bar{\theta})| = 0$.

Thus, $|\boldsymbol{S}_j^{\bar{i}}| \leq 2$. In case (ii), it can be derived similarly. So with all that,we have $|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| + |\boldsymbol{S}_{\bar{j}}^{i}| \leq 8$.

$|\boldsymbol{S}_j^{\bar{i}}| = |\mathcal{D}_2(\alpha \bar{\alpha} \boldsymbol{c} \beta) \cap \mathcal{D}_2(\bar{\alpha} \alpha \alpha \boldsymbol{d})|$, according Theorem 3, if $|\boldsymbol{S}_j^{\bar{i}}| = 6$, then $\alpha \bar{\alpha} \boldsymbol{c} \beta = \boldsymbol{t_1 t_2 t_3 s_2}, \bar{\alpha} \alpha \alpha \boldsymbol{d} = \boldsymbol{\bar{t_1} t_2 \bar{t_3} s_2}$, where $\boldsymbol{t_1}, \boldsymbol{t_2}, \boldsymbol{t_3}$ are alternating sequences. It implies that $d_L(\boldsymbol{x}, \boldsymbol{y}) = 2$, a contradiction. Then $|\boldsymbol{S}_j^{\bar{i}}| \leq 5$.

Thus, if $\omega_2^1 = \bar{\alpha}, \omega_2^2 = \alpha$, then $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = |\boldsymbol{\mathcal{S}}| \leq 4 + 8 + 5 = 17$.

Through (A),(B), we get only when $\omega_2^1 = \bar{\alpha}, \omega_2^2 = \bar{\alpha}$, is possiable, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$. In this case, if $\omega_l^1 = \omega_l^2 = \beta, \omega_{l-1}^1 = \omega_l^1$ or $\omega_l^1 = \omega_l^2 = \bar{\beta}, \omega_{l-1}^2 = \omega_l^2$, is equivalent to (A), and if $\omega_l^1 = \omega_l^2 = \beta, \omega_{l-1}^1 = \omega_l^1, \omega_{l-1}^2 = \omega_l^1$ or $\omega_l^1 = \omega_l^2 \bar{\beta}, \omega_{l-1}^2 = \omega_l^2, \omega_{l-1}^1 = \omega_l^2$, is equivalent to (B). Thus, when $\omega_1^1 = \omega_1^2 = \alpha$, $|\boldsymbol{\mathcal{S}}| > 17$, only if $\omega_2^1 = \omega_2^2 = \bar{\alpha}$ and $\omega_l^1 = \omega_l^2, \omega_{l-1}^1 = \omega_{l-1}^2, \omega_l^1 = \omega_{l-1}^1$. There are two possible structures according as follows: then $\boldsymbol{x}, \boldsymbol{y}$ must holds one of the following two structures:

(i) $\boldsymbol{x} = \boldsymbol{ua} \alpha \bar{\alpha} \omega_3^1 \cdots \omega_{\ell-2}^1 \bar{\beta} \beta \boldsymbol{bv}$,
$\boldsymbol{y} = \boldsymbol{u \bar{a}} \alpha \bar{\alpha} \omega_3^2 \cdots \omega_{\ell-2}^2 \bar{\beta} \beta \boldsymbol{\bar{b} v}$

(ii) $\boldsymbol{x} = \boldsymbol{ua} \alpha \bar{\alpha} \omega_3^1 \cdots \omega_{\ell-2}^1 \beta \bar{\beta} \boldsymbol{bv}$,
$\boldsymbol{y} = \boldsymbol{u \bar{a}} \alpha \bar{\alpha} \omega_3^2 \cdots \omega_{\ell-2}^2 \beta \bar{\beta} \boldsymbol{\bar{b} v}$

The above is equivalent to $\boldsymbol{x} = \boldsymbol{uas\omega^1 tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\omega^2 t\bar{b}v}$, where

- $\boldsymbol{u}, \boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,
- $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t}$ are maximal alternating sequence of length at least 2.
- $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$.

■

Lemma 11. *Let $\boldsymbol{x}, \boldsymbol{y}$ be two binary sequences of length n such that $\mathcal{D}_L(\boldsymbol{x}, \boldsymbol{y}) \geq 3$, if $\mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y}) = 5$, then $\boldsymbol{x}, \boldsymbol{y}$ must holds one of the following four structures:*

(i) $\boldsymbol{x} = \boldsymbol{u}a\omega b\boldsymbol{v}, \boldsymbol{y} = \boldsymbol{u}\bar{a}\omega\bar{b}\boldsymbol{v}$, where $\omega = \boldsymbol{u}'\omega'\boldsymbol{v}'$, and satisfies

- *$\boldsymbol{u}'$ is the run consisting of α , ω is alternating sequence starts with $\bar{\alpha}$ and ends with β ,$\boldsymbol{v}'$ is the run consisting of $\bar{\beta}$,or $\boldsymbol{u}'$ is the run consisting of $\bar{\alpha}$, ω is alternating sequence starts with α and ends with $\bar{\beta}$,$\boldsymbol{v}'$ is the run consisting of β .*
- *at least one of $|\boldsymbol{u}'| \geq 2, |\boldsymbol{v}'| \geq 2$ holds*
- *$|\boldsymbol{\omega}'| \geq 1$, and the equality is allowed to hold only if $|\bar{\alpha}| = \beta$.*

(ii) $\boldsymbol{x} = \boldsymbol{ua}\omega b\boldsymbol{v}, \boldsymbol{y} = \boldsymbol{u}\bar{a}\omega\bar{b}\boldsymbol{v}$, $\boldsymbol{\omega}$ is alternating sequence and satisfies $\boldsymbol{\omega} = \alpha\bar{\beta}$ or $\boldsymbol{\omega} = \bar{\alpha}\beta$.

(iii) $\boldsymbol{x} = \boldsymbol{ua\omega bv}, \boldsymbol{y} = \boldsymbol{u}\bar{a}\boldsymbol{\omega}\bar{b}\boldsymbol{v}$, $\boldsymbol{\omega} = \alpha = \bar{\beta}$ or $\boldsymbol{\omega} = \bar{\alpha} = \beta$.

(iv) $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\boldsymbol{\omega}\bar{\beta}\boldsymbol{bv}, \boldsymbol{y} = \boldsymbol{u}\overline{\boldsymbol{a}\alpha\boldsymbol{\omega}\beta\bar{\boldsymbol{b}}}\boldsymbol{v}$,where $\boldsymbol{\omega}$ is an alternating sequence starts with $\bar{\alpha}$ and ends with $\bar{\beta}$.

In (i)-(iv),$\boldsymbol{u}, \boldsymbol{v}$ is the longest common prefix and suffix of $\boldsymbol{x}, \boldsymbol{y}$;$\boldsymbol{a}, \boldsymbol{b}$ is the longest alternating sequence that holds the above structure and $\boldsymbol{a}$ ends with α,$\boldsymbol{b}$ starts with β.

Proof: This lemma can be deduced from Theorem 3. ■

Lemma 8. Let $\boldsymbol{x}, \boldsymbol{y}$ be two binary sequences of length n such that $\mathcal{D}_L(\boldsymbol{x}, \boldsymbol{y}) > 2$ and $\boldsymbol{x} = \boldsymbol{u\tilde{x}v}, \boldsymbol{y} = \boldsymbol{u\tilde{y}v}$, where $\boldsymbol{u}, \boldsymbol{v}$ is the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$. Denote $\boldsymbol{x}$ starts with i, ends with j, $\mathcal{S} = \mathcal{D}_3(\boldsymbol{\tilde{x}}) \cap \mathcal{D}_3(\boldsymbol{\tilde{y}})$. If $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, then following holds:

1) If $|\boldsymbol{S}_{\bar{j}}^{i}| = 6$, then $|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| = 4$.
2) If $|\boldsymbol{S}_{j}^{\bar{i}}| = 6$, then $|\boldsymbol{S}_{\bar{j}}^{i}| = 4$.

Proof: According to Lemma 7, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, $\boldsymbol{x} = \boldsymbol{uas\omega^1 tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\omega^2 t\bar{b}v}$, where

- $\boldsymbol{u}, \boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,
- $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t}$ are maximal alternating sequence of length at least 2.
- $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$.

Suppose $\boldsymbol{a}$ starts with i, ends with $\alpha, \boldsymbol{b}$ starts with β, ends with j.

Let $\boldsymbol{\tilde{x}} = \alpha \boldsymbol{s\omega^1 t}\beta$, $\boldsymbol{\tilde{y}} = \bar{\alpha}\boldsymbol{s\omega^2 t}\bar{\beta}$. Since $\boldsymbol{a}, \boldsymbol{b}$ are alternating sequences, then $\mathcal{S}_{\bar{j}}^{i} = D_2(\boldsymbol{as\omega^1 t} b_1 b_2 ... b_{|\boldsymbol{b}|-1}) \cap D_2(\bar{a}_2 ... \bar{a}_{|\boldsymbol{b}|} \boldsymbol{s\omega^1 t\bar{b}}) = (a_1 a_2 \cdots a_{|\boldsymbol{a}|-1}) \circ (D_2(a_{|\boldsymbol{a}|}\boldsymbol{s\omega^1 t}) \cap D_2(\boldsymbol{s\omega^2 t}\bar{b}_1)) \circ (b_1 b_2 \cdots b_{|\boldsymbol{b}|-1}) = (a_1 a_2 \cdots a_{|\boldsymbol{a}|-1}) \circ (D_2(\alpha \boldsymbol{s\omega^1 t}) \cap D_2(\boldsymbol{s\omega^2 t}\beta)) \circ (b_1 b_2 \cdots b_{|\boldsymbol{b}|-1})$.

$|\boldsymbol{S}_{\bar{j}}^{i}| = 6$ if and only if $|D_2(\alpha \boldsymbol{s\omega^1 t}) \cap D_2(\boldsymbol{s\omega^2 t}\beta)| = 6$, according to Theorem 3, $\alpha \boldsymbol{s\omega^1 t}, \boldsymbol{s\omega^2 t}\beta$ satisfies the following structure in Fig.1 (1), where ψ_2, ψ_3, ψ_4 are alternating sequence of length at least 2, and only when ψ_3 is completely reversed or zero-reversed in $\boldsymbol{\tilde{x}}, |\psi_3| \geq 2$, otherwise $|\psi_3| \geq 3$. Denote $|\psi_1|, |\psi_2|, |\psi_3|, |\psi_4|, |\psi_5|$ as $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$.

A categorical discussion of the different values of γ, θ follows.

(A) $\gamma = \alpha, \theta = \bar{\beta}$, refer to Fig.1(2) (The parts marked in yellow and green are $\boldsymbol{s\omega^1 t}$ and $\bar{\alpha}\boldsymbol{s\omega^2 t}\bar{\beta}$, respectively).

$$
\begin{aligned}
|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| &= D_1(\boldsymbol{s\omega^1 t}) \cap D_3(\bar{\alpha}\boldsymbol{s\omega^2 t}\bar{\beta}) \\
&= |\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4 \cdots \tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4 \cdots \tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| \\
&+ |\alpha \circ ((\bar{\alpha}\tilde{x}_4 \cdots \tilde{x}_{\ell-3}) \cap D_2(\bar{\alpha}\tilde{y}_4 \cdots \tilde{y}_{\ell-3}\bar{\beta}\beta)) \circ \bar{\beta}| \\
&+ |\bar{\alpha} \circ ((\tilde{x}_4 \cdots \tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\bar{\alpha}\tilde{y}_4 \cdots \tilde{y}_{\ell-3}\bar{\beta})) \circ \bar{\beta}|
\end{aligned}
$$

(A1) Since $\bar{\alpha}\tilde{x}_4 \cdots \tilde{x}_{\ell_1+\ell_2} = \bar{\alpha}\tilde{y}_4 \cdots \tilde{y}_{1+\ell_1+\ell_2-1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+2} \cdots \tilde{x}_{\ell-3}\bar{\theta} = \tilde{y}_{1+\ell_1+\ell_2+\ell_3+1} \cdots \tilde{y}_{\ell-3}\bar{\theta}$, then $|\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4 \cdots \tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4 \cdots \tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| = |D_1(\tilde{x}_{\ell_1+\ell_2+1} \cdots \tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap D_1(\tilde{y}_{1+\ell_1+\ell_2} \cdots \tilde{y}_{1+\ell_1+\ell_2+\ell_3})|$, according Theorem 3, we have:

(i) If $\tilde{x}_{\ell_1+\ell_2} = \tilde{x}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{x}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 2$
(ii) If $\tilde{x}_{\ell_1+\ell_2} = \tilde{x}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \bar{\tilde{x}}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 3$,
(iii) If $\tilde{x}_{\ell_1+\ell_2} = \bar{\tilde{x}}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{x}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 3$

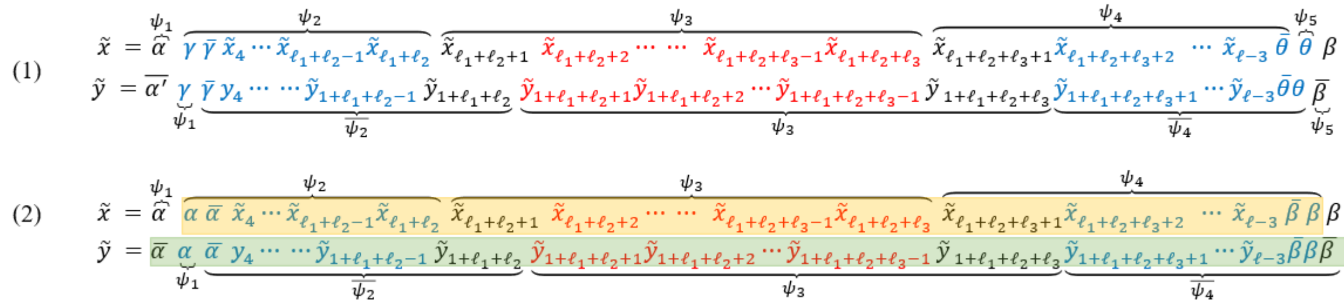

Fig. 1. Illustrations of $\tilde{x}$ and $\tilde{y}$ when $|D_2(\alpha\boldsymbol{s\omega^1t}) \cap D_2(\boldsymbol{s\omega^2t}\beta)| = 6$. (1) General case. (2) Case (A) ($\gamma = \alpha, \theta = \bar{\beta}$)

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(iv) If $\tilde{x}_{\ell_1+\ell_2} = \bar{\tilde{x}}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \bar{\tilde{x}}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 2$

In case (i), $\tilde{x}_{\ell_1+\ell_2+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} = \tilde{y}_{1+\ell_1+\ell_2+\ell_3}$, $(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3})$, $(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1})$ are alternating sequences of length at least 2 and begin with different symbol, then $|\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = |\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1})| = 2$.

In case (ii), $\tilde{x}_{\ell_1+\ell_2+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2+\ell_3}$,$(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1})$, $(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})$ are alternating sequences of length at least 2 and begin with different symbol, then $|\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = 2$.

In case (iii), $\tilde{x}_{\ell_1+\ell_2+1} = \tilde{y}_{1+\ell_1+\ell_2}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} = \tilde{y}_{1+\ell_1+\ell_2+\ell_3}$, $(\tilde{x}_{\ell_1+\ell_2+2}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3})$, $(\tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1})$ are alternating sequences of length at least 2 and begin with different symbol, then $|\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = |\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+2}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1})| = 2$.

In case (iv), $\tilde{x}_{\ell_1+\ell_2+1} = \tilde{y}_{1+\ell_1+\ell_2}, \tilde{x}_{\ell_1+\ell_2+\ell_3+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2+\ell_3}$, $(\tilde{x}_{\ell_1+\ell_2+2}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1})$, $(\tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})$ are alternating sequences of length at least 2 and begin with different symbol, then $|\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = |\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+2}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = 2$.

(A2) Since $\boldsymbol{\psi_4}$ is alternating suquence, it implies that $\tilde{x}_{\ell_1+\ell_2+\ell_3+1}\cdots\tilde{x}_{\ell-3}\bar{\beta} = \tilde{y}_{1+\ell_1+\ell_2+\ell_3+2}\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta\bar{\beta}$, and $\boldsymbol{\psi_3} = \tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3}$, then $|\alpha \circ ((\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_2(\bar{\alpha}y'_4\cdots t'_{\ell-3}\bar{\beta}\beta)) \circ \bar{\beta}| = |\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell-3}) \cap D_2(\tilde{y'}_{1+\ell_1+\ell_2}\cdots\tilde{t'}_{\ell-3}\bar{\beta}\beta)| = 1$.

(A3) Since $\boldsymbol{\psi_2}, \boldsymbol{\psi_3}$ is alternating suquence, it implies that $\tilde{x}_4\cdots\tilde{x}_{\ell_1+\ell_2} = \alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{1+\ell_1+\ell_2-3}$, $\psi_3[3,\ell_3] = \tilde{x}_{\ell_1+\ell_2+3}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-2}$, and $\tilde{x}_{\ell_1+\ell_2+1}\tilde{x}_{\ell_1+\ell_2+2} \in \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2-2}\tilde{y}_{1+\ell_1+\ell_2-1}\tilde{y}_{1+\ell_1+\ell_2})$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} \in \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{y}_{1+\ell_1+\ell_2+\ell_3})$, thus $|\bar{\alpha} \circ (\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\bar{\alpha}\tilde{y'}_4\cdots\tilde{t'}_{\ell-3}\bar{\beta})) \circ \bar{\beta}| = |(\tilde{x}_4\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3}) \cap D_2(\alpha\bar{\alpha}\tilde{y'}_4\cdots\tilde{t'}_{1+\ell_1+\ell_2+\ell_3}))| = 1$.

The analysis of the rest three cases is similar, and here we only give the expression.

(B) $\gamma = \alpha, \theta = \beta$.

$$
\begin{aligned}
|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| &= D_1(\boldsymbol{s\omega^1t}) \cap D_3(\bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta}) \\
&= |\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3})) \circ \beta\bar{\beta}| \\
&+ |\alpha \circ ((\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-4}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta)) \circ \bar{\beta}\bar{\beta}| \\
&+ |\bar{\alpha} \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\bar{\alpha}\tilde{y'}_4\cdots\tilde{y}_{\ell-3})) \circ \beta\bar{\beta}| \\
&= 2+1+1 = 4.
\end{aligned}
$$

(C) $\gamma = \bar{\alpha}', \theta = \beta$.

$$
\begin{aligned}
|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| &= D_1(\boldsymbol{s\omega^1t}) \cap D_3(\bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta}) \\
&= |\bar{\alpha}\alpha \circ (D_1(\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_1(\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| \\
&+ |\bar{\alpha}\alpha \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta)) \circ \bar{\beta}| \\
&+ |\bar{\alpha}\bar{\alpha} \circ ((\tilde{x}_5\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\tilde{y'}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| \\
&= 2+1+1 = 4
\end{aligned}
$$

(D) $\gamma = \bar{\alpha}', \theta = \bar{\beta}$.

$$
\begin{aligned}
|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| &= D_1(\boldsymbol{s\omega^1t}) \cap D_3(\bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta}) \\
&= |\bar{\alpha}\alpha \circ (D_1(\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_1(\tilde{y}_4\cdots\tilde{y}_{\ell-3})) \circ \beta\bar{\beta}| \\
&+ |\bar{\alpha}\alpha \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-4}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta)) \circ \bar{\beta}\bar{\beta}| \\
&+ |\bar{\alpha}\bar{\alpha} \circ ((\tilde{x}_5\cdots\tilde{x}_{\ell-3}) \cap D_2(\alpha\tilde{y'}_4\cdots\tilde{y}_{\ell-3})) \circ \beta\bar{\beta}| \\
&= 2+1+1 = 4
\end{aligned}
$$

■

Theorem 3. Let $\boldsymbol{x}, \boldsymbol{y}$ be two binary sequences of length n such that $\mathcal{D}_L(\boldsymbol{x}, \boldsymbol{y}) > 2$, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = 19$ if and only $\boldsymbol{x} = \boldsymbol{uabcdev}, \boldsymbol{y} = \boldsymbol{u\bar{a}b\bar{c}d\bar{e}v}$**)**,where $\boldsymbol{u}, \boldsymbol{v}$ is the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,and satisfies (1)$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ are alternating sequence. (2)$|\boldsymbol{a}|, |\boldsymbol{e}| \geq 2$. (3)For $\boldsymbol{b}, \boldsymbol{d}$, if $\boldsymbol{b}$ is completely reversed or zero-reversed in $\boldsymbol{x}$, then $|\boldsymbol{b}| \geq 2$,otherwise $|\boldsymbol{b}| \geq 3$.The same applies to $\boldsymbol{d}$. (4) Denote $\boldsymbol{b}$ ends with α,$\boldsymbol{d}$ starts with β,then $\alpha = \bar{\beta}$. (5) $|\boldsymbol{c}| = 3$.

Proof:

Let $\boldsymbol{x} = \boldsymbol{u\tilde{x}v}, \boldsymbol{y} = \boldsymbol{u\tilde{y}v}, |\tilde{\boldsymbol{x}}| = |\tilde{\boldsymbol{y}}| = n$, then $\tilde{x}_1 \neq \tilde{y}_1, \tilde{x}_n \neq \tilde{y}_n$. Denote $\mathcal{D}_3(\tilde{\boldsymbol{x}}) \cap \mathcal{D}_3(\tilde{\boldsymbol{y}}) = \mathcal{S}$,then $|\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y})| = \mathcal{D}_3(\tilde{\boldsymbol{x}}) \cap \mathcal{D}_3(\tilde{\boldsymbol{y}}) = \mathcal{S}$.

According to Theorem 7, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = 19$, then $\tilde{\boldsymbol{x}} = \boldsymbol{as\omega^1tb}, \tilde{\boldsymbol{y}} = \boldsymbol{\bar{a}s\omega^2t\bar{b}}$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t}$ are alternating sequences of length at least 2 and $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$. Suppose $\boldsymbol{a}$ starts with i, ends with α, $\boldsymbol{b}$ starts with β, ends with j, denote $\tilde{\boldsymbol{x}} = \alpha\boldsymbol{s\omega^1t}\beta, \tilde{\boldsymbol{y}} = \bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta}, |\tilde{\boldsymbol{x}}| = |\tilde{\boldsymbol{y}}| = \ell$.

According to Lemma 8, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = 19$ holds if and only if:

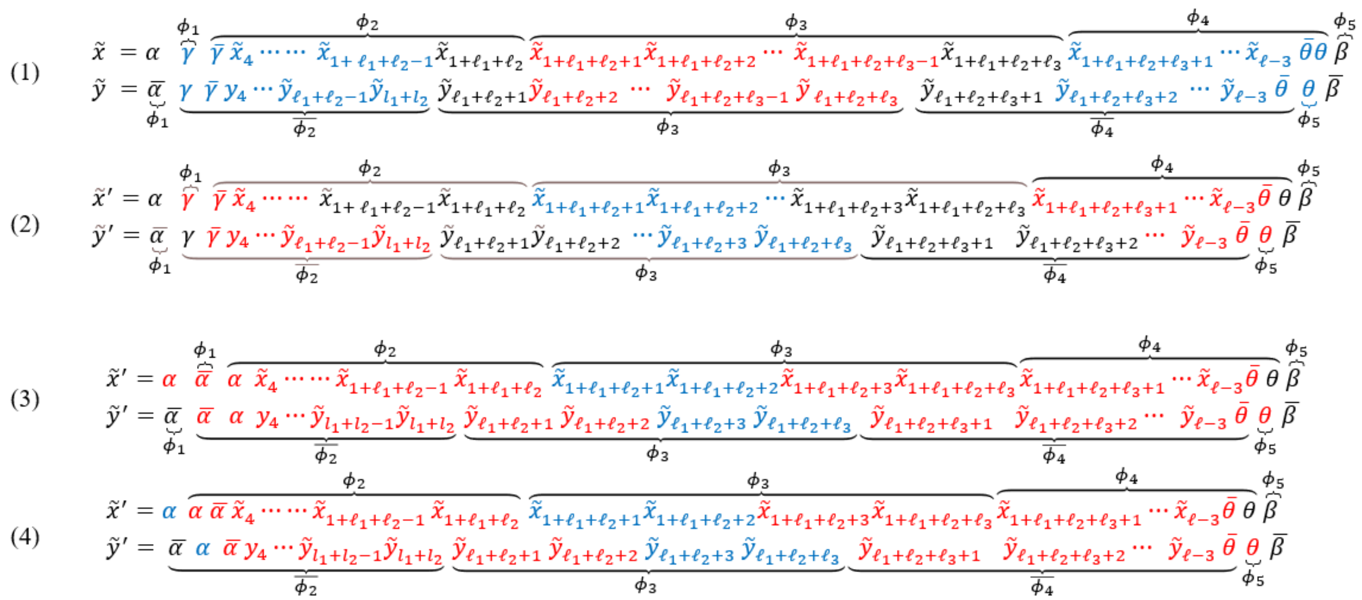

Fig. 2. Illustrations of $\tilde{x}$ and $\tilde{y}$ when $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| = 6, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| = 5$. In the figure, complementary alternating segments and identical alternating segments are represented in red and blue, respectively. Since $\boldsymbol{a}, \boldsymbol{b}$ are alternating sequences, $\boldsymbol{\mathcal{S}}_j^{\bar{i}} = a_2 \cdots a_{|\boldsymbol{a}|} \circ (\mathcal{D}_2(\tilde{x}_2 \cdots \tilde{x}_\ell) \cap \mathcal{D}_2(\tilde{y}_1 \cdots \tilde{y}_{\ell-1})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$, then $|\mathcal{S}_j^{\bar{i}}| = 6$ if and only if $\tilde{x}_2 \cdots \tilde{x}_\ell = \boldsymbol{\phi_1 \phi_2 \phi_3 \phi_4 \phi_5}, \tilde{y}_1 \cdots \tilde{1}_{\ell-1} = \boldsymbol{\phi_1 \bar{\phi}_2 \phi_3 \bar{\phi}_4 \phi_5}$ as (1). Based on the known sequence features of $\boldsymbol{\phi_i}, \boldsymbol{s}, \boldsymbol{t}$, we can obtain (2) and if $\gamma = \alpha, |\boldsymbol{\psi_1}| = 0$, else $|\boldsymbol{\psi_1}| = 1$. (3)(4) are the structures for the cases where $|\boldsymbol{\phi}| = 4$ and γ takes different values, respectively.

Type (A): $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| = 6, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| = 5, |\boldsymbol{\mathcal{S}}_j^{i}| = 4, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| = 4$, or

Type (B): $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| = 5, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| = 6, |\boldsymbol{\mathcal{S}}_j^{i}| = 4, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| = 4$.

We begin by discussing the Type (A). $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| = 6$ holds if and only if the structures in Fig. 2 (1) is satisfied. In Fig.2 (1), $\boldsymbol{\phi_2}, \boldsymbol{\phi_3}, \boldsymbol{\phi_4}$ are alternating sequence of length at least 2, and only when $\boldsymbol{\phi_3}$ is completely reversed or zero-reversed in $\tilde{\boldsymbol{x}}, |\boldsymbol{\phi_3}| \geq 2$,otherwise $|\boldsymbol{\phi_3}| \geq 3$. Denote $|\boldsymbol{\phi_1}|, |\boldsymbol{\phi_2}|, |\boldsymbol{\phi_3}|, |\boldsymbol{\phi_4}|, |\boldsymbol{\phi_5}|$ as $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$.

According to Lemma 11, if $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| = 5$, holds, there are four possible structures. Next we discuss the likelihood that these four structures hold in the case where $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| = 6$ holds. As before, the discussion is based on the example of $\gamma = \alpha$

(A) $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \boldsymbol{\psi_1 \psi_2 \psi_3 \psi_4 \psi_5}, \tilde{y}_2 \cdots \tilde{y}_\ell = \boldsymbol{\psi_1 \overline{\psi_2} \psi_3 \overline{\psi_4} \psi_5}$, where $\boldsymbol{\psi_3} = \boldsymbol{u' \psi_3' v'}$ and at least one of $\boldsymbol{u'}, \boldsymbol{v'}$ is a run of length at least 2. Since $\boldsymbol{\phi_2}, \boldsymbol{\phi_3}, \boldsymbol{\phi_4}$ are alternating sequences, it is clear that we cannot find a k such that $\tilde{x}_k = \tilde{x}_{k+1} = \tilde{x}_{k+1} = \tilde{y}_{k+2}$.

(B) $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \boldsymbol{\psi_1 \psi_2 \psi_3 \psi_4 \psi_5}, \tilde{y}_2 \cdots \tilde{y}_\ell = \boldsymbol{\psi_1 \overline{\psi_2} \psi_3 \overline{\psi_4} \psi_5}$, where $|\boldsymbol{\psi_3}| = 2$, and $\boldsymbol{\psi_3}$ is neither zero-reversed nor completely reversed in $\tilde{\boldsymbol{x}}'$.

(B1) If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$, then in order to satisfy the condition of $\boldsymbol{\psi_3}$, then there must be $\ell_3 = 2$, and $\tilde{x}_{1+\ell_1+\ell_2} = \tilde{x}_{1+\ell_1+\ell_2+1}$, Obviously, when $\boldsymbol{\phi_2}, \boldsymbol{\phi_3}$ are alternating sequence , $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$ and $\tilde{x}_{1+\ell_1+\ell_2} = \tilde{x}_{1+\ell_1+\ell_2+1}$ cannot hold at the same time.

(B2) If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+1}\overline{\tilde{y}}_{\ell_1+\ell_2+2}$, There are two possible structures that make the condition of ψ_3 satisfied:

(i) $\tilde{x}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \tilde{y}_{\ell_1+\ell_2+\ell_3+1}\tilde{y}_{\ell_1+\ell_2+\ell_3+2}$ and $\ell_3 = 2$,

(ii) $\tilde{x}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+1}\overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+2}$ and $\ell_3 = 4$.

In case (i), $\boldsymbol{\psi_3}$ corresponds to $\tilde{x}_{1+\ell_1+\ell_2+1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3}$ and we have $\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+2} = \tilde{x}_{1+\ell_1+\ell_2+1}, \tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \tilde{y}_{\ell_1+\ell_2+\ell_3+2} = \tilde{x}_{1+\ell_1+\ell_2+\ell_3+1}$, i.e. ω is completely reversed in $\tilde{x}$, a contradiction.

In case (ii), ψ_3 corresponds to $|\tilde{x}_{1+\ell_1+\ell_2+1}\tilde{x}_{1+\ell_1+\ell_2+2}|$. We have $\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+2} = \tilde{x}_{1+\ell_1+\ell_2+1}$ and $\tilde{x}_{1+\ell_1+\ell_2+2} = \tilde{x}_{1+\ell_1+\ell_2+3}$. Thus, under the condition of $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| = 6$, (B) can hold simultaneously in case (ii).

(C) $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \boldsymbol{\psi_1 \psi_2 \psi_3 \psi_4 \psi_5}, \tilde{y}_2 \cdots \tilde{y}_\ell = \boldsymbol{\psi_1 \overline{\psi_2} \psi_3 \overline{\psi_4} \psi_5}$ where $|\boldsymbol{\psi_3}| = 1$ and $\boldsymbol{\psi_3}$ is neither zero-reversed nor completely reversed in $\tilde{x}$.

(C1) If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$, it is clear that $|\boldsymbol{\psi_3}| > 1$.

(C2) If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+1}\overline{\tilde{y}}_{\ell_1+\ell_2+2}$, there are only one possible structures that make the condition of $\boldsymbol{\psi_3}$ satisfied: $\ell_3 = 3$ and $\tilde{x}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+1}\overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+2}$ then we have $\overline{\alpha}\alpha\tilde{x}_4 \cdots \tilde{x}_{1+\ell_1+\ell_2} = y_4 \cdots \tilde{y}_{\ell_1+\ell_2+3}$, $\tilde{x}_{1+\ell_1+\ell_2+1} \cdots \tilde{x}_{\ell-3} = \tilde{y}_{\ell_1+\ell_2+\ell_3+1} \cdots \tilde{y}_{\ell-3}\overline{\theta}\theta$, and either $\overline{\theta} = \overline{\beta}$ or $\theta = \overline{\beta}$ holds. Thus $d_L(\tilde{x}, \tilde{y}) = 1$, then $d_L(x, y) = 2$, a contradiction.

(D) $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \psi_1 \psi_2 \overline{\mu} \psi_3 \overline{\xi} \psi_4 \psi_5, \tilde{y}_2 \cdots \tilde{y}_\ell = \psi_1 \bar{\psi}_2 \overline{\mu} \bar{\psi}_3 \overline{\xi} \bar{\psi}_4 \psi_5$

(D1) If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$, then $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2}$ corresponds to $\overline{\mu}\psi_3[1]$ it is clear that $\overline{\mu}\psi_3[1] \neq \overline{\mu}\overline{\psi_3}[1]$, a contradiction.

(D2) If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+1}\overline{\tilde{y}}_{\ell_1+\ell_2+2}$. Since the same segment following the complementary segment cor-

responds to $\overline{\mu}$, only $\ell_3 = 3$ can satisfies that the length of the segment equals 1, then neither $\tilde{x}_{1+\ell_1+\ell_2+2} = \tilde{y}_{\ell_1+\ell_2+\ell_3+1}$ nor $\tilde{x}_{1+\ell_1+\ell_2+2} = \overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+1}$ can satisfies (D).

In sum, if $|\boldsymbol{S}^{\bar{i}}_{j}| = 6, |S^{i}_{\bar{j}}|$ hold simultaneously, then $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \psi_1\psi_2\psi_3\psi_4\psi_5, \tilde{y}_2 \cdots \tilde{y}_\ell = \psi_1\overline{\psi_2}\psi_3\overline{\psi_4}\psi_5$, where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$ are alternating sequences, $|\boldsymbol{\psi_3}| = 2$ and $\boldsymbol{\psi_3}$ is neither zero reversed nor completely reversed in $\tilde{x}$, i.e. $\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{y}}'$ satisfies the structures in Fig.2 (3).

Denote $|\boldsymbol{\psi_1}|, |\boldsymbol{\psi_2}|, |\boldsymbol{\psi_3}|, |\boldsymbol{\psi_4}|, |\boldsymbol{\psi_5}|$ as $\ell'_1, \ell'_2, \ell'_3, \ell'_4, \ell'_5$. It is clear that $\ell'_2 = \ell_2+2, \ell'_3 = 2, \ell'_4 = \ell_4+2$. Next, we analyze the structure of $\boldsymbol{s}, \boldsymbol{\omega^1}, \boldsymbol{\omega^2}, \boldsymbol{t}$ when all the conditions are satisfied simultaneously.

(A)Since we have established that $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{s}\boldsymbol{\omega^1}\boldsymbol{t}\beta, \tilde{\boldsymbol{y}} = \alpha \boldsymbol{s}\boldsymbol{\omega^2}\boldsymbol{t}\bar{\beta}$ and $\boldsymbol{s}, \boldsymbol{t}$ are alternating sequence. Combined with the structure in Fig.2 (3), it is easy to see that the $\boldsymbol{\omega^1} = \tilde{x}_{1+\ell_1+\ell_2+1}\tilde{x}_{1+\ell_1+\ell_2+2}\overline{\tilde{x}_{1+\ell_1+\ell_2+3}}, \boldsymbol{\omega^2} = \tilde{y}_{\ell_1+\ell_2+2}\tilde{y}_{\ell_1+\ell_2+3}\tilde{y}_{\ell_1+\ell_2+\ell_3}$, and $\boldsymbol{\omega^1} = \overline{\boldsymbol{\omega}^2}$, where $\omega^1_1 = s_{|s|}, \omega^1_3 = \bar{t_1}$.

(B)In Fig.2 (3), we assume that $\gamma = \bar{\alpha}$, and $|\boldsymbol{s}| = \ell_1 + \ell_2 = \ell'_2 - 1$. In order to satisfy $\ell_2 \geq 2, \ell'_2 \geq 2$, then $|\boldsymbol{s}| \geq 3$. In the other case $\gamma = \alpha$, we show in Fig2 (4),$|\boldsymbol{s}| = \ell_2 = 1 + \ell'_2 - 1$, and in order to satisfy $\ell_2 \geq 2, \ell'_2 \geq 2$, $|\boldsymbol{s}| \geq 2$. In sum, we get $|\boldsymbol{s}| \geq 2$ if completely reversed in $\tilde{x}$ and $|\boldsymbol{s}| \geq 3$ if $s_1 = \bar{\alpha}, s_{|\boldsymbol{s}|} = \omega^1_1$.

(C)Due to symmetry, we obtain $|\boldsymbol{t}| \geq 2$ if zero-reversed in $\tilde{x}$ and $|\boldsymbol{t}| \geq 3$ if $t_1 = \omega^1_{|\boldsymbol{\omega}|}, t_{|\boldsymbol{t}|} = \beta$.

In summary, we have obtained a necessary condition for $|\boldsymbol{S}^{i}_{\bar{j}}| = 5, |\boldsymbol{S}^{\bar{i}}_{j}| = 6$, i.e. $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}\beta, \tilde{\boldsymbol{y}} = \alpha \boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}$, such that

- $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{\omega}$ are alternating sequences,
- $|\boldsymbol{\omega}| = 3, \omega_1 = s_{|s|}, \omega_3 = \overline{t_1}$,
- if $|\boldsymbol{s}|$ is completely reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{s}| \geq 2$, else $|\boldsymbol{s}| \geq 3$
- if $|\boldsymbol{t}|$ is zero-reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{t}| \geq 2$, else $|\boldsymbol{t}| \geq 3$

Next we prove that the above condition is also sufficient.

Firstly, it is clear that $|\boldsymbol{S}^{\bar{i}}_{j}| = 6$, then $|\boldsymbol{S}^{i}_{j}| = 4$ according Lemma 8. Thus, we only need to prove that $|\boldsymbol{S}^{i}_{\bar{j}}| = 5$, and $|\boldsymbol{S}^{\bar{i}}_{\bar{j}}| = 4$.

Since $\boldsymbol{a}, \boldsymbol{b}$ are alternating sequences, then $\boldsymbol{S}^{i}_{\bar{j}} = a_1 \cdots a_{|\boldsymbol{a}|-1} \circ (\mathcal{D}_2(\alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_2(\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta})) \circ b_1 \cdots b_{|\boldsymbol{b}|-1}$. Since $\boldsymbol{s}, \boldsymbol{\omega}, \boldsymbol{t}$ are alternating sequences, and $\omega_1 = s_{|s|}, \omega_3 = \bar{t_1}$, then $\alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}, \boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}$ can be written as $\alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t} = \boldsymbol{\psi_1\psi_2\psi_3\psi_4\psi_5}$, $\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta} = \boldsymbol{\psi_1\bar{\psi}_2\psi_3\bar{\psi}_4\psi_5}$, where $\boldsymbol{\psi_i}$ is alternating, $|\boldsymbol{\psi_1}|, |\boldsymbol{\psi_5}| \leq 2$,and $\boldsymbol{\psi_3}$ corresponds to $\omega_1\omega_2$. Denote $|\boldsymbol{\psi}_1|, |\boldsymbol{\psi}_2|, |\boldsymbol{\psi}_3|, |\boldsymbol{\psi}_4|, |\boldsymbol{\psi}_5|$ as $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$

Firstly, $|(\mathcal{D}_2(\alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_2(\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}))| < 6$ according to Theorem3 and we can get the set

$\{$

$\boldsymbol{\psi_1}\psi_2[1, \ell_2 - 1]\boldsymbol{\psi_3}\psi_4[1, \ell_4 - 1]\boldsymbol{\psi_5}$,

$\boldsymbol{\psi_1}\psi_2[1, \ell_2 - 1]\boldsymbol{\psi_3}\psi_4[2, \ell_4]\boldsymbol{\psi_5}$,

$\boldsymbol{\psi_1}\psi_2[2, \ell_2]\boldsymbol{\psi_3}\psi_4[1, \ell_4 - 1]\boldsymbol{\psi_5}$,

$\boldsymbol{\psi_1}\psi_2[2, \ell_2]\boldsymbol{\psi_3}\psi_4[2, \ell_4]\boldsymbol{\psi_5}$,

$\boldsymbol{\psi_1\bar{\psi}_2\psi_4\psi_5}$

$\} \subseteq (\mathcal{D}_2(\alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_2(\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}))$.

Thus $|\boldsymbol{S}^{i}_{\bar{j}}| = |(\mathcal{D}_2(\alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_2(\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}))| = 5$.

$\boldsymbol{S}^{\bar{i}}_{\bar{j}} = a_2 \cdots a_{|\boldsymbol{a}|} \circ (\mathcal{D}_1(\boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta})) \circ b_1 \cdots b_{|\boldsymbol{b}|-1}$. Firstly $|(\mathcal{D}_1(\boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}))| < 4$ according to Lemma 4 and we can get the set

$\{$

$\boldsymbol{s}\omega_1\omega_2\boldsymbol{t}$,

$\boldsymbol{s}\omega_2\omega_3\boldsymbol{t}$,

$\bar{\alpha}s_3 \cdots s_{|\boldsymbol{s}|}\boldsymbol{\omega}\boldsymbol{t}$,

$\boldsymbol{s}\boldsymbol{\omega}t_1 \cdots t_{|\boldsymbol{t}|-3}\bar{\beta}$,

$\} \subseteq (\mathcal{D}_1(\boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}))$.

Thus $|\boldsymbol{S}^{\bar{i}}_{\bar{j}}| = |(\mathcal{D}_1(\boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}))| = 5$ So far, we have proved the sufficient and necessary condition condition for Type (A).

Symmetrically, we can get the sufficient and necessary condition condition for Type (B) is that $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{s}\boldsymbol{\omega}\boldsymbol{t}\beta, \tilde{\boldsymbol{y}} = \alpha \boldsymbol{s}\bar{\boldsymbol{\omega}}\boldsymbol{t}\bar{\beta}$ such that

- $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{\omega}$ are alternating sequences
- $|\boldsymbol{\omega}| = 3, \omega_1 = \overline{s_{|\boldsymbol{s}|}}, \omega_3 = t_1$,
- if $|\boldsymbol{s}|$ is zero-reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{s}| \geq 2$, else $|\boldsymbol{s}| \geq 3$.
- if $|\boldsymbol{t}|$ is completely reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{t}| \geq 2$, else $|\boldsymbol{t}| \geq 3$.

Combining the above two types, we obtain Theorem 3. ■