## APPENDIX

**Theorem 6** (Theorem 7 in [14])**.** *If $\boldsymbol{x}$ and $\boldsymbol{y}$ are confusable, then they have to be in one of the following forms.*
*(A) $\boldsymbol{x} = \boldsymbol{u\alpha\omega\beta v}, \boldsymbol{y} = \boldsymbol{u\bar{\alpha}\omega\bar{\beta}v}$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are alternating sequences of length at least two.*
*(B) $\boldsymbol{x} = \boldsymbol{u\alpha}\bar{a}\gamma\bar{b}\boldsymbol{\beta v}, \boldsymbol{y} = \boldsymbol{u}\overline{\boldsymbol{\alpha}a\gamma b\boldsymbol{\beta}}\boldsymbol{v}$, where $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$ and $\gamma$ are alternating sequences. Here, $\boldsymbol{\alpha}$ is of length at least two and ends with $a$, $\boldsymbol{\beta}$ is of length at least two and starts with $b$, and $\gamma$ starts with $\overline{a}$ and ends with $\overline{b}$.*

**Theorem 7** (Theorem 4.1 in [13])**.** *For two sequences $\boldsymbol{x} = \boldsymbol{u\tilde{x}v}$ and $\boldsymbol{y} = \boldsymbol{u\tilde{y}v}$ in $\Sigma^n$ with $d_L(\boldsymbol{x},\boldsymbol{y}) \geq 2$ , if $|\mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y})| \geq 5$, then one of the following holds:*

*(A) $\boldsymbol{\tilde{x}} = \boldsymbol{a\omega b}$ and $\boldsymbol{\tilde{y}} = \boldsymbol{\bar{a}\omega\bar{b}}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are alternating sequences of length at least two and $\boldsymbol{\omega}$ is a combination of periodic sequences. More precisely, suppose $\boldsymbol{a}$ ends with $\alpha$ and $\boldsymbol{b}$ starts with $\beta'$,then $\boldsymbol{\omega} = \boldsymbol{u'\omega'v'}$ and one of the following holds:*

- *$\boldsymbol{u'}$ is a run sequence starting with $\alpha$, $\boldsymbol{\omega'}$ is an alternating sequence of length at least one starting with $\bar{\alpha}$ and ending with $\beta'$, $\boldsymbol{v'}$ is a run sequence starting with $\bar{\beta}'$*
- *$\boldsymbol{u'}$ is a run sequence starting with $\bar{\alpha}$, $\boldsymbol{\omega'}$ is an alternating sequence of length at least one starting with $\alpha$ and ending with $\bar{\beta}'$, $\boldsymbol{v'}$ is a run sequence starting with $\beta'$.*

*(B) $\boldsymbol{\tilde{x}} = \boldsymbol{a}\bar{\alpha}\boldsymbol{\omega}\bar{\beta}\boldsymbol{b}, \boldsymbol{\tilde{y}} = \boldsymbol{a}\alpha\bar{\boldsymbol{\omega}}\beta\boldsymbol{b}$, where $a$ is an alternating seqence of length at least two ending with $\alpha$, $\boldsymbol{\omega}$ is an alternating sequence starting with $\bar{\alpha}$ and ending with $\bar{\beta}'$ , and $b$ is an alternating sequence of length at least two starting with $\beta'$.*

*Particularly, $|\mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y})| = 6$ if and only if $\boldsymbol{\tilde{x}} = \boldsymbol{a\omega b}$ and $\boldsymbol{\tilde{y}} = \boldsymbol{\bar{a}\omega\bar{b}}$, where $\boldsymbol{a}$ is an alternating sequence of length at least two ending with $\alpha$, $\boldsymbol{b}$ is an alternating sequence of length at least two starting with $\beta'$, and one of the following holds:*

- *if $\boldsymbol{\omega}$ starts with $\alpha'$ and ends with $\beta'$, $\boldsymbol{\omega}$ is a alternating sequence of length at least two;*
- *if $\boldsymbol{\omega}$ starts with $\alpha'$ and ends with $\bar{\beta}'$, $\boldsymbol{\omega}$ is a alternating sequence of length at least three;*
- *if $\boldsymbol{\omega}$ starts with $\bar{\alpha}'$ and ends with $\beta'$, $\boldsymbol{\omega}$ is a alternating sequence of length at least three;*
- *if $\boldsymbol{\omega}$ starts with $\bar{\alpha}'$ and ends with $\bar{\beta}'$, $\boldsymbol{\omega}$ is a alternating sequence of length at least two;*

**Lemma 10** (Lemma 12(ii) in [14])**.** *Suppose that $\boldsymbol{x} = \alpha\boldsymbol{r}\beta$ and $\boldsymbol{y} = \bar{\alpha}\boldsymbol{s}\bar{\beta}$ and $d_L(\boldsymbol{x},\boldsymbol{y}) \geq 2$. Set $\mathcal{S} = \mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y})$. If the first two bits of $\boldsymbol{x}$ are equal, i.e. $\boldsymbol{r} = \alpha\boldsymbol{r'}$, then $|\boldsymbol{\mathcal{S}}| \leq 4$. Hence by symmetry, if $\boldsymbol{r} = \boldsymbol{r'}\beta$ or $\boldsymbol{s} = \bar{\alpha}\boldsymbol{s'}$ or $\boldsymbol{s} = \boldsymbol{s'}\bar{\beta}$, i.e., the first or last two bits of $\boldsymbol{x}$ or $\boldsymbol{y}$ are equal, then $|\boldsymbol{\mathcal{S}}| \leq 4$.*

**Lemma 7** Let $\boldsymbol{x},\boldsymbol{y}$ be two binary sequences of length $n$ such that $d_L(\boldsymbol{x},\boldsymbol{y}) \geq 3$, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, then $\boldsymbol{x} = \boldsymbol{uas\omega^1 tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\omega^2 t\bar{b}v}$, where

- $\boldsymbol{u},\boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,
- $\boldsymbol{a},\boldsymbol{b},\boldsymbol{s},\boldsymbol{t}$ are maximal alternating sequence of length at least 2.
- $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$.

*Proof:* According Lemma 6, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, then they are of the form $\boldsymbol{x} = \boldsymbol{ua\omega^1 bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}\omega^2\bar{b}v}$, such that

- $\boldsymbol{u},\boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$;
- $\boldsymbol{a},\boldsymbol{b}$ are maximal alternating sequences with $|\boldsymbol{a}|,|\boldsymbol{b}| \geq 2$;
- $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ are two distinct sequences of length $\ell$, where $\omega_1^1 = \omega_1^2, \omega_\ell^1 = \omega_\ell^2$.

Suppose $\boldsymbol{a}$ starts with $i$, ends with $\alpha$, $\boldsymbol{b}$ starts with $\beta$ ends with $j$. For the sequence $\boldsymbol{x},\boldsymbol{y}$ satisfying the above requirements, there are four possibilities as follows:
1) $\boldsymbol{x} = \boldsymbol{ua}\alpha\omega_2^1 \cdots \omega_{\ell-1}^1\beta\boldsymbol{bv}$ , $\boldsymbol{y} = \boldsymbol{u\bar{a}}\alpha\omega_2^2 \cdots \omega_{\ell-1}^2\beta\boldsymbol{\bar{b}v}$ ;
2) $\boldsymbol{x} = \boldsymbol{ua}\alpha\omega_2^1 \cdots \omega_{\ell-1}^1\bar{\beta}\boldsymbol{bv}$ , $\boldsymbol{y} = \boldsymbol{u\bar{a}}\alpha\omega_2^2 \cdots \omega_{\ell-1}^2\bar{\beta}\boldsymbol{\bar{b}v}$ ;
3) $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\omega_2^1 \cdots \omega_{\ell-1}^1\beta\boldsymbol{bv}$ , $\boldsymbol{y} = \boldsymbol{u\bar{a}}\bar{\alpha}\omega_2^2 \cdots \omega_{\ell-1}^2\beta\boldsymbol{\bar{b}v}$ ;
4) $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\omega_2^1 \cdots \omega_{\ell-1}^1\bar{\beta}\boldsymbol{bv}$ , $\boldsymbol{y} = \boldsymbol{u\bar{a}}\bar{\alpha}\omega_2^2 \cdots \omega_{\ell-1}^2\bar{\beta}\boldsymbol{\bar{b}v}$ .
where (3)(4) are equivalent to (1)(2), so it is sufficient to discuss only (1)(2).

Denote $\omega_3^1 \cdots \omega_\ell^1$ as $\boldsymbol{c}$, $\omega_3^2 \cdots \omega_\ell^2$ as $\boldsymbol{d}$, $\mathcal{D}_3(\boldsymbol{a\omega^1 b}) \cap \mathcal{D}_3(\boldsymbol{\bar{a}\omega^2\bar{b}})$ as $\boldsymbol{\mathcal{S}}$. Without prejudice to generality, let $\omega_1^1 = \omega_1^2 = \alpha$, and We proceed with the following three cases.

*Case (A):* $\omega_2^1 = \alpha, \omega_2^2 = \alpha$

$$\boldsymbol{a\omega^1 b} = \boldsymbol{a}\ \alpha\alpha\boldsymbol{c}\ \boldsymbol{b} = i\cdots\alpha\alpha\alpha\boldsymbol{c}\beta\cdots j$$
$$\boldsymbol{\bar{a}\omega^2\bar{b}} = \boldsymbol{\bar{a}}\ \alpha\alpha\boldsymbol{d}\ \boldsymbol{\bar{b}} = \bar{i}\cdots\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta}\cdots\bar{j}$$

Since $\boldsymbol{a},\boldsymbol{b}$ are alternating sequences, $\boldsymbol{\mathcal{S}}_j^i = (\boldsymbol{a}\alpha) \circ (\mathcal{D}_3(\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_1(\boldsymbol{d})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$. According to Lemma 4, $|\boldsymbol{\mathcal{S}}_j^i| \leq 4$.

$\boldsymbol{\mathcal{S}}_{\bar{j}}^i = (\boldsymbol{a}\alpha) \circ (\mathcal{D}_2(\alpha\boldsymbol{c}) \cap \mathcal{D}_2(\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$. According to Lemma 1, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^i| \leq 6$.

$\boldsymbol{\mathcal{S}}_j^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_2(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_2(\bar{\alpha}\alpha\alpha\boldsymbol{d})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$. We know that the first bit of the sequence of centers of the two error balls is different, then according to Lemma 10, $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| \leq 4$.

$\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$.The sequence in $\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})$ only can begin with $\alpha$, $|\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})| = |\mathcal{D}_1(\boldsymbol{c}) \cap \mathcal{D}_2(\boldsymbol{d}\bar{\beta})|$, then according Lemma 5, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| \leq 3$.

In this case, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = |\boldsymbol{\mathcal{S}}| \leq 17$.

*Case (B):* $\omega_2^1 = \alpha, \omega_2^2 = \bar{\alpha}$

$$\boldsymbol{a\omega^1 b} = \boldsymbol{a}\ \alpha\alpha\boldsymbol{c}\ \boldsymbol{b} = i\cdots\alpha\alpha\alpha\boldsymbol{c}\beta\cdots j$$
$$\boldsymbol{\bar{a}\omega^2\bar{b}} = \boldsymbol{\bar{a}}\ \alpha\bar{\alpha}\boldsymbol{d}\ \boldsymbol{\bar{b}} = \bar{i}\cdots\bar{\alpha}\alpha\bar{\alpha}\boldsymbol{d}\bar{\beta}\cdots\bar{j}$$

Since $\boldsymbol{a},\boldsymbol{b}$ are alternating sequences, $\boldsymbol{\mathcal{S}}_j^i = \boldsymbol{a} \circ (\mathcal{D}_3(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_1(\bar{\alpha}\boldsymbol{d})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$. There is at most one sequence $\boldsymbol{d}$ that begins with $\alpha$ and two sequences $\mathcal{D}_1(\boldsymbol{c}\beta) \cap \mathcal{D}_1(\bar{\alpha}\boldsymbol{d})$ that begins with $\bar{\alpha}$ in the set $\mathcal{D}_3(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_1(\bar{\alpha}\boldsymbol{d})$ (It request $\mathcal{D}_3(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_1(\bar{\alpha}\boldsymbol{d})$ begins with $\bar{\alpha}$ and Except for the common prefix and common suffix, the remainder of $\boldsymbol{c}\beta$ and $\bar{\alpha}\boldsymbol{d}$ are alternating sequences with different start symbols). Thus, $|\boldsymbol{\mathcal{S}}_j^i| \leq 3$.

$\boldsymbol{\mathcal{S}}_{\bar{j}}^i = \boldsymbol{a} \circ (\mathcal{D}_2(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_2(\bar{\alpha}\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$. According to Lemma 10, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^i| \leq 4$.

$\boldsymbol{\mathcal{S}}_j^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_2(\alpha\alpha\boldsymbol{c}\beta) \cap \mathcal{D}_2(\bar{\alpha}\alpha\bar{\alpha}\boldsymbol{d})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$. According to Lemma 10, $|\boldsymbol{\mathcal{S}}_j^{\bar{i}}| \leq 4$.

$\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$.The sequence in $\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})$ only can begin with $\alpha$, $|\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})| = |\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_2(\alpha\alpha\boldsymbol{d}\bar{\beta})|$, then according Lemma 5, $|\boldsymbol{\mathcal{S}}_{j}^{\bar{i}}| \leq 3$.

In sum, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \leq 14$.

*Case (C):* $\omega_2^1 = \bar{\alpha}, \omega_2^2 = \alpha$

$$\boldsymbol{a\omega^1 b} = \boldsymbol{a}\alpha\bar{\alpha}\omega_3^1 \cdots \omega_\ell^1 \boldsymbol{b} = i \cdots \alpha\alpha\bar{\alpha}\boldsymbol{c}\beta \cdots j$$
$$\boldsymbol{\bar{a}\omega^2 \bar{b}} = \boldsymbol{\bar{a}}\alpha\alpha\omega_3^2 \cdots \omega_\ell^2 \boldsymbol{\bar{b}} = \bar{i} \cdots \bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta} \cdots \bar{j}$$

Similarly to *Case (A)*, $|\boldsymbol{\mathcal{S}}_j^i| \leq 4, |\boldsymbol{\mathcal{S}}_{\bar{j}}^{\bar{i}}| \leq 4$. $\boldsymbol{\mathcal{S}}_{\bar{j}}^{i} = (a_1 \cdots \alpha_{|\boldsymbol{a}|-1})\alpha \circ (\mathcal{D}_2(\bar{\alpha}\boldsymbol{c}) \cap \mathcal{D}_2(\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$. According to Lemma 1 and Theorem 6, $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| \leq 6$, and if $|\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| \geq 5$ , then $\bar{\alpha}\boldsymbol{c}, \boldsymbol{d}\bar{\beta}$ satisfies one of the following two structures

(i) $\bar{\alpha}\boldsymbol{c} = \boldsymbol{s_1 t_1 t_2 t_3 s_2}$, $\boldsymbol{d}\bar{\beta} = \boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3} s_2}$, or
(ii) $\bar{\alpha}\boldsymbol{c} = \boldsymbol{s_1 t_1}\bar{\gamma}\boldsymbol{t_2}\bar{\lambda}\boldsymbol{t_3 s_2}$, $\boldsymbol{d}\bar{\beta} = \boldsymbol{s_1 \bar{t_1}}\bar{\gamma}\boldsymbol{\bar{t_2}}\bar{\lambda}\boldsymbol{\bar{t_3} s_2}$,

where $\boldsymbol{t_1}, \boldsymbol{t_2}, \boldsymbol{t_3}$ are alternating sequences of length at least two, and their lengths are denoted $\ell_1, \ell_2, \ell_3$ respectively. In particular, in case (ii), $\boldsymbol{t_1}$ ends with $\gamma$, $\boldsymbol{t_2}$ starts with $\bar{\gamma}$ and ends with $\bar{\lambda}$, $\boldsymbol{t_3}$ starts with $\lambda$.

In case (i), $\boldsymbol{\mathcal{S}}_{j}^{\bar{i}} = (a_2 \cdots \alpha_{|\boldsymbol{a}|}) \circ (\mathcal{D}_1(\alpha\bar{\alpha}\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})) \circ (b_1 \cdots b_{|\boldsymbol{b}|-1})$, where $\mathcal{D}_1(\alpha\bar{\alpha}\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta}) = \mathcal{D}_1(\alpha\boldsymbol{s_1 t_1 t_2 t_3 s_2}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3} s_2}) = \mathcal{D}_1(\alpha\boldsymbol{s_1 t_1 t_2 t_3}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}})$. For convenience, we suppose that $\boldsymbol{t_3}$ ends with $\theta$, and in this and subsequent proofs, we use $s_1[1]$,$s_1[1,t]$ to denote the first bit of the sequence $\boldsymbol{s_1}$ and subsequence formed by the first bit to the t-th bit respectively, then $\mathcal{D}_1(\alpha\alpha\boldsymbol{c}) \cap \mathcal{D}_3(\bar{\alpha}\alpha\alpha\boldsymbol{d}\bar{\beta})$ consists of the following parts:

- $\mathcal{D}_0(\boldsymbol{s_1 t_1 t_2 t_3}) \cap (\bar{\alpha} \circ \mathcal{D}_2(\alpha\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3-2]) \circ \theta)$, Note that we have known $\boldsymbol{s}$ begins with $\bar{\alpha}$, $\boldsymbol{t_3}$ ends with $\theta$.
- $\alpha \circ (\mathcal{D}_1(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3-1]) \cap \mathcal{D}_1(\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3-2])) \circ \theta$
- $\alpha \circ (\mathcal{D}_0(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3-2]) \cap \mathcal{D}_2(\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3-1])) \circ \bar{\theta}$

Firstly, denote $\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3 - 1]$ as $\tilde{\boldsymbol{x}}$, $\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3 - 2]$ as $\tilde{\boldsymbol{y}}$ and their length is $\ell$. Since $\boldsymbol{s_1}$ begins with $\bar{\alpha}$, $\boldsymbol{t_3}$ is alternating sequence, then $\tilde{x}_1 \neq \tilde{y}_1$ and the last $\ell_3 - 2$ bits of $x, y$ is equal. According to 2, $D_1(\tilde{x}) \cap D_1(\tilde{y}) = 2$ if and only if the first $\ell - \ell_3 + 2$ bits of $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ are alternating sequences begining with different symbols. And since $\boldsymbol{t_1}$ is alternating of length at least 2, then exsits at least one bit $\tilde{x}_{|\boldsymbol{s_1}|} + 2$ equals to $\tilde{y}_{|\boldsymbol{s_1}|} + 2$, then $|\mathcal{D}_1(\tilde{x}) \cap \mathcal{D}_1(\tilde{y})| \leq 1$.

Next, we asume that there exists $|\mathcal{D}_0(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3-2]) \cap \mathcal{D}_2(\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3 - 1])| = 1$, then in order to get the same prefix $\boldsymbol{s_1 t_1 [1, \ell_1 - 1]}$, it is necessary to delete $\alpha$ in the beganning and $\bar{t_1}[1]$, since $\boldsymbol{t_2}$ is alternating sequence, it is obviously that $t_2[1] = t_2[2]$ and then $t_1[\ell]t_2[1] \neq t_2[1]t_2[2]$, a contradiction. Thus, $(\alpha \circ (\mathcal{D}_0(\boldsymbol{s_1 t_1 t_2 t_3}[1, \ell_3 - 2]) \cap \mathcal{D}_2(\alpha\boldsymbol{s_1 \bar{t_1} t_2 \bar{t_3}}[1, \ell_3 - 1])) \circ \bar{\theta})| = 0$.

In sum, $|\boldsymbol{\mathcal{S}}_{j}^{\bar{i}}| \leq 2$ . In case (ii), it can be derived similarly. So with all that,we have $|\boldsymbol{\mathcal{S}}_{j}^{\bar{i}}| + |\boldsymbol{\mathcal{S}}_{\bar{j}}^{i}| \leq 8$.

$|\boldsymbol{\mathcal{S}}_{j}^{\bar{i}}| = |\mathcal{D}_2(\alpha\bar{\alpha}\boldsymbol{c}\beta) \cap \mathcal{D}_2(\bar{\alpha}\alpha\alpha\boldsymbol{d})|$, according Theorem 3, if $|\boldsymbol{\mathcal{S}}_{j}^{\bar{i}}| = 6$ , then $\alpha\bar{\alpha}\boldsymbol{c}\beta = \boldsymbol{t_1 t_2 t_3 s_2}, \bar{\alpha}\alpha\alpha\boldsymbol{d} = \boldsymbol{\bar{t_1} t_2 \bar{t_3} s_2}$, where $\boldsymbol{t_1} = \alpha\bar{\alpha}, \boldsymbol{t_2}, \boldsymbol{t_3}$ are alternating sequences and $\boldsymbol{t_2}$ begins with $\alpha$. We have:

$$\boldsymbol{x} = \boldsymbol{u}i \cdots a_{|\boldsymbol{a}|-1}\alpha \ \alpha\bar{\alpha}\alpha t_2[2] \cdots t_2[\ell_2]\boldsymbol{t_3 s_2} b_2 \cdots j\boldsymbol{v}$$
$$\boldsymbol{y} = \boldsymbol{u}\bar{i} \cdots \bar{a}_{|\boldsymbol{a}|-1}\bar{\alpha} \ \alpha\alpha t_2[2] \cdots t_2[\ell_2]\boldsymbol{\bar{t_3} s_2}\bar{b}_1\bar{b}_2 \cdots \bar{j}\boldsymbol{v}$$

The subsequence obtained by deleting $\bar{i}, \bar{j}$ from $\boldsymbol{y}$ is also obtained by deleting the one of the two last bits of $\boldsymbol{t_2}$ and last bit of $\boldsymbol{t_3}$ from $\boldsymbol{x}$. It implies that $d_L(\boldsymbol{x}, \boldsymbol{y}) = 2$, a contradiction. Then $|\boldsymbol{\mathcal{S}}_{j}^{\bar{i}}| \leq 5$.

Thus, if $\omega_2^1 = \bar{\alpha}, \omega_2^2 = \alpha$ , then $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = |\boldsymbol{\mathcal{S}}| \leq 4 + 8 + 5 = 17$.

*Case (D):* $\omega_2^1 = \bar{\alpha}, \omega_2^2 = \bar{\alpha}$

$$\boldsymbol{a\omega^1 b} = \boldsymbol{a}\alpha\bar{\alpha}\omega_3^1 \cdots \omega_\ell^1 \boldsymbol{b} = i \cdots \alpha\alpha\bar{\alpha}\boldsymbol{c}\beta \cdots j$$
$$\boldsymbol{\bar{a}\omega^2 \bar{b}} = \boldsymbol{\bar{a}}\alpha\bar{\alpha}\omega_3^2 \cdots \omega_\ell^2 \boldsymbol{\bar{b}} = \bar{i} \cdots \bar{\alpha}\alpha\bar{\alpha}\boldsymbol{d}\bar{\beta} \cdots \bar{j}$$

In this case, if $\omega_{\ell-1}^1 \neq \omega_{\ell-1}^2$ is equivalent to Case (B) or Case (C); if $\omega_\ell^1 = \omega_\ell^2 = \omega_{\ell-1}^1 = \omega_{\ell-1}^2$is equivalent to (B). Thus, when $\omega_1^1 = \omega_1^2 = \alpha$, $|\boldsymbol{\mathcal{S}}| > 17$ holds only if $\omega_2^1 = \omega_2^2 = \bar{\alpha}$ and $\omega_\ell^1 = \omega_\ell^2, \omega_{\ell-1}^1 = \omega_{\ell-1}^2, \omega_{\ell-1}^1 = \overline{\omega_\ell^1}$. There are two possible structures according as follows:

- $\boldsymbol{x} = \boldsymbol{ua}\alpha\bar{\alpha}\omega_3^1 \cdots \omega_{\ell-2}^1\bar{\beta}\beta\boldsymbol{bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}}\alpha\bar{\alpha}\omega_3^2 \cdots \omega_{\ell-2}^2\bar{\beta}\beta\boldsymbol{\bar{b}v}$
- $\boldsymbol{x} = \boldsymbol{ua}\alpha\bar{\alpha}\omega_3^1 \cdots \omega_{\ell-2}^1\beta\bar{\beta}\boldsymbol{bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}}\alpha\bar{\alpha}\omega_3^2 \cdots \omega_{\ell-2}^2\beta\bar{\beta}\boldsymbol{\bar{b}v}$

Similarly, if $\omega_2^1 = \omega_2^2 = \bar{\alpha}$, we can gets symmetrically

- $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\alpha\omega_3^1 \cdots \omega_{\ell-2}^1\bar{\beta}\beta\boldsymbol{bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}}\bar{\alpha}\alpha\omega_3^2 \cdots \omega_{\ell-2}^2\bar{\beta}\beta\boldsymbol{\bar{b}v}$
- $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\alpha\omega_3^1 \cdots \omega_{\ell-2}^1\beta\bar{\beta}\boldsymbol{bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}}\bar{\alpha}\alpha\omega_3^2 \cdots \omega_{\ell-2}^2\beta\bar{\beta}\boldsymbol{\bar{b}v}$

The above is equivalent to $\boldsymbol{x} = \boldsymbol{uas\omega^1 tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\omega^2 t\bar{b}v}$, where

- $\boldsymbol{u}, \boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,
- $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t}$ are maximal alternating sequence of length at least 2.
- $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$.

∎

**Lemma 11.** *Let $\boldsymbol{x}, \boldsymbol{y}$ be two binary sequences of length $n$ such that $\mathcal{D}_L(\boldsymbol{x}, \boldsymbol{y}) \geq 3$, if $\mathcal{D}_2(\boldsymbol{x}) \cap \mathcal{D}_2(\boldsymbol{y}) = 5$, then $\boldsymbol{x}, \boldsymbol{y}$ must holds one of the following four structures:*

*(i)* $\boldsymbol{x} = \boldsymbol{ua\omega bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}\omega\bar{b}v}$, *where* $\boldsymbol{\omega} = \boldsymbol{u'\omega'v'}$, *and satisfies*
  - *(i) $\boldsymbol{u'}$ is a run consisting of $\alpha$ , $\boldsymbol{\omega'}$ is alternating sequence starts with $\bar{\alpha}$ and ends with $\beta$ ,$\boldsymbol{v'}$ is a run consisting of $\bar{\beta}$, or (ii) $\boldsymbol{u'}$ is a run consisting of $\bar{\alpha}$ , $\omega$ is alternating sequence starts with $\alpha$ and ends with $\bar{\beta}$ ,$\boldsymbol{v'}$ is a run consisting of $\beta$ .*
  - *at least one of $|\boldsymbol{u'}| \geq 2, |\boldsymbol{v'}| \geq 2$ holds*
  - *$|\boldsymbol{\omega'}| \geq 1$, and the equality is allowed to hold only if $|\bar{\alpha}| = \beta$.*

*(ii)* $\boldsymbol{x} = \boldsymbol{ua\omega bv}, \boldsymbol{y} = \boldsymbol{u\bar{a}\bar{\omega}\bar{b}v}$, *$\boldsymbol{\omega}$ is alternating sequence and satisfies $\boldsymbol{\omega} = \alpha\bar{\beta}$ or $\boldsymbol{\omega} = \bar{\alpha}\beta$.*

*(iii)* $\boldsymbol{x} = \boldsymbol{ua\omega bv}, \boldsymbol{y} = \boldsymbol{u}\bar{a}\boldsymbol{\bar{\omega}}\bar{b}\boldsymbol{v}$, *$\boldsymbol{\omega} = \alpha = \bar{\beta}$ or $\boldsymbol{\omega} = \bar{\alpha} = \beta$.*

*(iv)* $\boldsymbol{x} = \boldsymbol{ua}\bar{\alpha}\boldsymbol{\omega}\bar{\beta}\boldsymbol{bv}, \boldsymbol{y} = \boldsymbol{u}\overline{\boldsymbol{a\alpha\omega\beta b}}\boldsymbol{v}$ *,where $\boldsymbol{\omega}$ is an alternating sequence starts with $\bar{\alpha}$ and ends with $\bar{\beta}$.*

*In (i)-(iv), $\boldsymbol{u}, \boldsymbol{v}$ is the longest common prefix and suffix of $\boldsymbol{x}, \boldsymbol{y}$; $\boldsymbol{a}, \boldsymbol{b}$ is the longest alternating sequence that holds the above structure and $\boldsymbol{a}$ ends with $\alpha$,$\boldsymbol{b}$ starts with $\beta$.*

*Proof:* This lemma can be deduced from Theorem 3. ∎

**Lemma 8.** Let $\boldsymbol{x}, \boldsymbol{y}$ be two binary sequences of length $n$ such that $\mathcal{D}_L(\boldsymbol{x}, \boldsymbol{y}) > 2$ and $\boldsymbol{x} = \boldsymbol{u\tilde{x}v}, \boldsymbol{y} = \boldsymbol{u\tilde{y}v}$, where $\boldsymbol{u}, \boldsymbol{v}$ is the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$. Denote $\boldsymbol{x}$ starts with $i$, ends with $j$, $\mathcal{S} = \mathcal{D}_3(\tilde{\boldsymbol{x}}) \cap \mathcal{D}_3(\tilde{\boldsymbol{y}})$. If $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, then following holds:

1) If $|\boldsymbol{\mathcal{S}}^{i}_{\bar{j}}| = 6$, then $|\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = 4$.
2) If $|\boldsymbol{\mathcal{S}}^{i}_{\bar{j}}| = 6$, then $|\boldsymbol{\mathcal{S}}^{\bar{i}}_{j}| = 4$.

*Proof:* According to Lemma 7, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) \geq 18$, $\boldsymbol{x} = \boldsymbol{uas\omega^1tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\omega^2t\bar{b}v}$, where

- $\boldsymbol{u}, \boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$,
- $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t}$ are maximal alternating sequence of length at least 2.
- $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$.

Suppose $\boldsymbol{a}$ ends with $\alpha$, $\boldsymbol{b}$ starts with $\beta$. Since $\boldsymbol{a}, \boldsymbol{b}$ are alternating sequences, then $\mathcal{S}^{i}_{\bar{j}} = D_2(\boldsymbol{as\omega^1t}b_1b_2...b_{|\boldsymbol{b}|-1}) \cap D_2(\bar{a}_2 \cdots \bar{a}_{|\boldsymbol{b}|}\boldsymbol{s\omega^1t\bar{b}}) = (a_1a_2\cdots a_{|\boldsymbol{a}|-1}) \circ (D_2(\alpha\boldsymbol{s\omega^1t}) \cap D_2(\boldsymbol{s\omega^2t}\bar{\beta})) \circ (b_1b_2\cdots b_{|\boldsymbol{b}|-1}), |\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = D_1(a_2\cdots a_{|\boldsymbol{a}|}\boldsymbol{s\omega^1t}b_1\cdots b_{|\boldsymbol{b}|} - 1) \cap D_3(\boldsymbol{as\omega^2tb}) = (a_2\cdots a_{|\boldsymbol{a}|}) \circ (D_1(\boldsymbol{s\omega^1t}) \cap D_3(\bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta})) \circ (b_1b_2\cdots b_{|\boldsymbol{b}|-1})|$.

Let $\tilde{\boldsymbol{x}} = \alpha\boldsymbol{s\omega^1t}\beta$ , $\tilde{\boldsymbol{y}} = \bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta}, |\tilde{\boldsymbol{x}}| = |\tilde{\boldsymbol{y}}| = \ell$, then $|\boldsymbol{\mathcal{S}}^{i}_{\bar{j}}| = |D_2(\tilde{x}_1\cdots\tilde{x}_{\ell-1}) \cap D_2(\tilde{y}_2\cdots\tilde{y}_\ell)|$, $|\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = |D_1(\tilde{x}_2\cdots\tilde{x}_{\ell-1}) \cap D_3(\tilde{y}_1\cdots\tilde{y}_\ell)|$. Due to $\tilde{x}_2\tilde{x}_3$ is the first two bits of $\boldsymbol{s}$, $\tilde{x}_{\ell-1}\tilde{x}_\ell$ is the last two bits of $\boldsymbol{t}$, thus $\tilde{x}_2 = \tilde{y}_2 = \overline{\tilde{x}_3} = \overline{\tilde{y}_3}$, $\tilde{x}_{\ell-1} = \tilde{y}_{\ell-1} = \overline{\tilde{x}_\ell} = \overline{\tilde{y}_\ell}$.

According to Lemma 3, $|\boldsymbol{\mathcal{S}}^{i}_{\bar{j}}| = 6$ if and only if $\tilde{x}_1\cdots\tilde{x}_{\ell-1}, \tilde{y}_2\cdots\tilde{y}_\ell$ satisfies the following structure in Fig.1 (1), where $\psi_2, \psi_3, \psi_4$ are alternating sequence of length at least 2, and only when $\psi_3$ is completely reversed or zero-reversed in $\tilde{\boldsymbol{x}}, |\psi_3| \geq 2$, otherwise $|\psi_3| \geq 3$. Denote $|\psi_1|, |\psi_2|, |\psi_3|, |\psi_4|, |\psi_5|$ as $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$. A categorical discussion of the different values of $\gamma, \theta$ follows.

*Case (A):* $\gamma = \alpha, \theta = \beta$, refer to Fig.1(2) and $\psi_1 = \alpha, \psi_5$ is empty.(The parts marked in yellow and green are $\boldsymbol{s\omega^1t}$ and $\bar{\alpha}\boldsymbol{s\omega^2t}\bar{\beta}$, respectively.)

$$\begin{aligned}
&|\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = |D_1(\alpha\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}\beta) \cap D_3(\bar{\alpha}\alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta\bar{\beta})| \\
&= |\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| \\
&+ |\alpha \circ ((\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta)) \circ \bar{\beta}| \\
&+ |\bar{\alpha} \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \bar{\beta}|
\end{aligned}$$

*(A1)* Since $\boldsymbol{\psi_2}, \boldsymbol{\psi_3}$ are alternating sequences, $|\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| = |D_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap D_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})|$, according Theorem 3, we have:

(i) If $\tilde{x}_{\ell_1+\ell_2} = \tilde{x}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{x}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 2$
(ii) If $\tilde{x}_{\ell_1+\ell_2} = \tilde{x}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \bar{\tilde{x}}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 3$,
(iii) If $\tilde{x}_{\ell_1+\ell_2} = \bar{\tilde{x}}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{x}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 3$
(iv) If $\tilde{x}_{\ell_1+\ell_2} = \bar{\tilde{x}}_{\ell_1+\ell_2+1}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3} = \bar{\tilde{x}}_{\ell_1+\ell_2+\ell_3+1}$, then $|\boldsymbol{\psi_3}| \geq 2$

In case (i), $\tilde{x}_{\ell_1+\ell_2+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} = \tilde{y}_{1+\ell_1+\ell_2+\ell_3}, (\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3}), (\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1})$ are alternating sequences of length $|\boldsymbol{\psi_3}| \geq 2$ and begin with different symbol, then $|\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = |\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1})| = 2$ .

In case (ii), $\tilde{x}_{\ell_1+\ell_2+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2}$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} = \bar{\tilde{y}}_{1+\ell_1+\ell_2+\ell_3}$,$(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}), (\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})$ are alternating sequences of length $|\boldsymbol{\psi_3}| - 1 \geq 2$and begin with different symbol, then $|\mathcal{D}_1(\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3+1}) \cap \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3})| = 2$ .

Due to the symmetry, the same result can be obtained in in case (iii),case (iv). Thus, $|\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| = 2$.

*(A2)* Since $\boldsymbol{\psi_4}$ is alternating suquence, it implies that $\tilde{x}_{\ell_1+\ell_2+\ell_3+1}\cdots\tilde{x}_{\ell-3}\bar{\beta} = \tilde{y}_{1+\ell_1+\ell_2+\ell_3+2}\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta\bar{\beta}$, and $\boldsymbol{\psi_3} = \tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3}$, then $|\alpha \circ ((\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_2(\bar{\alpha}\tilde{y'}_4\cdots\tilde{t'}_{\ell-3}\bar{\beta}\beta)) \circ \bar{\beta}| = |\tilde{x}_{\ell_1+\ell_2+1}\cdots\tilde{x}_{\ell-3}) \cap D_2(\tilde{y'}_{1+\ell_1+\ell_2}\cdots\tilde{t'}_{\ell-3}\bar{\beta}\beta)| = 1$.

*(A3)* Since $\boldsymbol{\psi_2}, \boldsymbol{\psi_3}$ is alternating suquence, it implies that $\tilde{x}_4\cdots\tilde{x}_{\ell_1+\ell_2} = \alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{1+\ell_1+\ell_2-3}$, $\psi_3[3,\ell_3] = \tilde{x}_{\ell_1+\ell_2+3}\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3} = \tilde{y}_{1+\ell_1+\ell_2+1}\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3-2}$, and $\tilde{x}_{\ell_1+\ell_2+1}\tilde{x}_{\ell_1+\ell_2+2} \in \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2-2}\tilde{y}_{1+\ell_1+\ell_2-1}\tilde{y}_{1+\ell_1+\ell_2})$, $\tilde{x}_{\ell_1+\ell_2+\ell_3+1} \in \mathcal{D}_1(\tilde{y}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{y}_{1+\ell_1+\ell_2+\ell_3})$, thus $|\bar{\alpha} \circ (\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \bar{\beta}| = |(\tilde{x}_4\cdots\tilde{x}_{\ell_1+\ell_2+\ell_3}) \cap D_2(\alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{1+\ell_1+\ell_2+\ell_3}))| = 1$.

The analysis of the rest three cases is similar, and here we only give the expression.

*Case (B):* $\gamma = \alpha, \theta = \bar{\beta}$, there exists a common suffix $\bar{\beta}$.

$$\begin{aligned}
&|\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = |D_1(\alpha\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta\bar{\beta}) \cap D_3(\bar{\alpha}\alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta\bar{\beta}\bar{\beta})| \\
&= |D_1(\alpha\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_3(\bar{\alpha}\alpha\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta\bar{\beta}| \\
&= |\alpha \circ (D_1(\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_1(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3})) \circ \beta| \\
&+ |\alpha \circ ((\bar{\alpha}\tilde{x}_4\cdots\tilde{x}_{\ell-4}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta)) \circ \bar{\beta}| \\
&+ |\bar{\alpha} \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\bar{\alpha}\tilde{y'}_4\cdots\tilde{y}_{\ell-3})) \circ \beta| \\
&= 2 + 1 + 1 = 4.
\end{aligned}$$

*Case (C):* $\gamma = \bar{\alpha}, \theta = \beta$, there exists a common prefix $\bar{\alpha}$.

$$\begin{aligned}
&|\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = |D_1(\bar{\alpha}\alpha\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}\beta) \cap D_3(\bar{\alpha}\bar{\alpha}\alpha\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta\bar{\beta})| \\
&= |D_1(\alpha\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}\beta) \cap D_3(\bar{\alpha}\alpha\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta\bar{\beta}| \\
&= |\alpha \circ (D_1(\tilde{x}_4\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_1(\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| \\
&+ |\alpha \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\bar{\beta}\beta)) \circ \bar{\beta}| \\
&+ |\bar{\alpha} \circ ((\tilde{x}_5\cdots\tilde{x}_{\ell-3}\bar{\beta}) \cap D_2(\alpha\tilde{y'}_4\cdots\tilde{y}_{\ell-3}\bar{\beta})) \circ \beta| \\
&= 2 + 1 + 1 = 4
\end{aligned}$$

*Case (D):* $\gamma = \bar{\alpha}, \theta = \bar{\beta}$, there exists a common prefix $\bar{\alpha}$ and a common suffix $\bar{\beta}$.

$$\begin{aligned}
&|\boldsymbol{\mathcal{S}}^{\bar{i}}_{\bar{j}}| = |D_1(\bar{\alpha}\alpha\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta\bar{\beta}) \cap D_3(\bar{\alpha}\bar{\alpha}\alpha\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta\bar{\beta}\bar{\beta})| \\
&= |D_1(\alpha\tilde{x}_4\cdots\tilde{x}_{\ell-3}\beta) \cap D_3(\bar{\alpha}\alpha\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta\bar{\beta}| \\
&= |\alpha \circ (D_1(\tilde{x}_4\cdots\tilde{x}_{\ell-3}) \cap D_1(\tilde{y}_4\cdots\tilde{y}_{\ell-3})) \circ \beta| \\
&+ |\alpha \circ ((\tilde{x}_4\cdots\tilde{x}_{\ell-4}) \cap D_2(\bar{\alpha}\tilde{y}_4\cdots\tilde{y}_{\ell-3}\beta)) \circ \bar{\beta}| \\
&+ |\bar{\alpha} \circ ((\tilde{x}_5\cdots\tilde{x}_{\ell-3}) \cap D_2(\alpha\tilde{y'}_4\cdots\tilde{y}_{\ell-3})) \circ \beta| \\
&= 2 + 1 + 1 = 4
\end{aligned}$$

■

**Theorem 3.** Let $\boldsymbol{x}, \boldsymbol{y}$ be two binary sequences of length n such that $d_L(\boldsymbol{x}, \boldsymbol{y}) \geq 3$, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = 19$ if and only they are of the form $\boldsymbol{x} = \boldsymbol{uas\omega tbv}, \boldsymbol{y} = \boldsymbol{u\bar{a}s\bar{\omega}t\bar{b}v}$, such that

- $\boldsymbol{u}, \boldsymbol{v}$ are the longest common prefix and suffix of $\boldsymbol{x}$,$\boldsymbol{y}$.
- $\boldsymbol{a}, \boldsymbol{b}$ are maximal alternating sequences of length $\geq 2$.

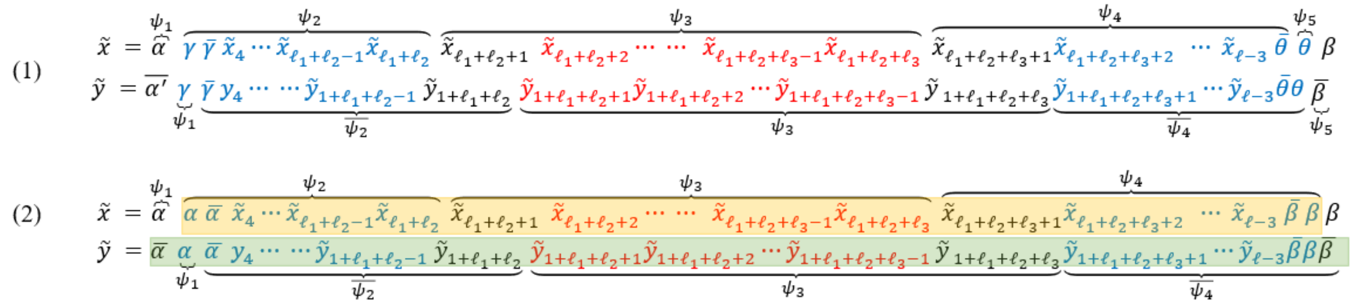

Fig. 1. Illustrations of $\tilde{x}$ and $\tilde{y}$ when $|D_2(\alpha \boldsymbol{s\omega^1 t}) \cap D_2(\boldsymbol{s\omega^2 t}\beta)| = 6$. (1) General case. (2) Case (A) ($\gamma = \alpha, \theta = \bar{\beta}$)

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- $\boldsymbol{s}, \boldsymbol{t}$ are maximal alternating sequences. If $\boldsymbol{s}$ is completely reversed or zero-reversed between $\boldsymbol{a}$ and $\boldsymbol{w}$, then $|\boldsymbol{s}| \geq 2$, otherwise $|\boldsymbol{s}| \geq 3$. The same applies to $\boldsymbol{t}$.
- $|\boldsymbol{\omega}| = 3$ and $\boldsymbol{\omega}$ is an alternating sequence which is neither zero-reversed nor completely reversed.

*Proof:* Let $\boldsymbol{x} = \boldsymbol{u\tilde{x}v}, \boldsymbol{y} = \boldsymbol{u\tilde{y}v}, |\boldsymbol{\tilde{x}}| = |\boldsymbol{\tilde{y}}| = n$, then $\tilde{x}_1 \neq \tilde{y}_1, \tilde{x}_n \neq \tilde{y}_n$. Denote $\mathcal{D}_3(\boldsymbol{\tilde{x}}) \cap \mathcal{D}_3(\boldsymbol{\tilde{y}}) = \mathcal{S}$ ,then $|\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y})| = \mathcal{D}_3(\boldsymbol{\tilde{x}}) \cap \mathcal{D}_3(\boldsymbol{\tilde{y}}) = \mathcal{S}$.

According to Theorem 7, if $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = 19$, then $\boldsymbol{\tilde{x}} = \boldsymbol{as\omega^1 tb}, \boldsymbol{\tilde{y}} = \boldsymbol{\bar{a}s\omega^2 t\bar{b}}$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{t}$ are alternating sequences of length at least 2 and $\boldsymbol{\omega^1} \neq \boldsymbol{\omega^2}$. Suppose $\boldsymbol{a}$ starts with $i$, ends with $\alpha$, $\boldsymbol{b}$ starts with $\beta$, ends with $j$, denote $\boldsymbol{\tilde{x}} = \alpha \boldsymbol{s\omega^1 t}\beta, \boldsymbol{\tilde{y}} = \bar{\alpha}\boldsymbol{s\omega^2 t}\bar{\beta}, |\boldsymbol{\tilde{x}}| = |\boldsymbol{\tilde{y}}| = \ell$.

According to Lemma 8, $\mathcal{D}_3(\boldsymbol{x}) \cap \mathcal{D}_3(\boldsymbol{y}) = 19$ holds if and only if:

*Type (A):* $|\boldsymbol{S}_j^{\bar{i}}| = 6, |\boldsymbol{S}_{\bar{j}}^{i}| = 5, |\boldsymbol{S}_j^{i}| = 4, |\boldsymbol{S}_{\bar{j}}^{\bar{i}}| = 4$, or

*Type (B):* $|\boldsymbol{S}_j^{\bar{i}}| = 5, |\boldsymbol{S}_{\bar{j}}^{i}| = 6, |\boldsymbol{S}_j^{i}| = 4, |\boldsymbol{S}_{\bar{j}}^{\bar{i}}| = 4$.

We begin by discussing the Type (A). $|\boldsymbol{S}_j^{\bar{i}}| = 6$ holds if and only if the structures in Fig. 2 (1) is satisfied. In Fig.2 (1), $\boldsymbol{\phi_2}, \boldsymbol{\phi_3}, \boldsymbol{\phi_4}$ are alternating sequence of length at least 2, and only when $\boldsymbol{\phi_3}$ is completely reversed or zero-reversed in $\boldsymbol{\tilde{x}}, |\boldsymbol{\phi_3}| \geq 2$ ,otherwise $|\boldsymbol{\phi_3}| \geq 3$. Denote $|\boldsymbol{\phi_1}|, |\boldsymbol{\phi_2}|, |\boldsymbol{\phi_3}|, |\boldsymbol{\phi_4}|, |\boldsymbol{\phi_5}|$ as $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ .

According to Lemma 11, if $|\boldsymbol{S}_{\bar{j}}^{i}| = 5$, holds, there are four possible structures. Next we discuss the likelihood that these four structures hold in the case where $|\boldsymbol{S}_j^{\bar{i}}| = 6$ holds. As before, the discussion is based on the example of $\gamma = \alpha$

**(A)** $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \boldsymbol{\psi_1 \psi_2 \psi_3 \psi_4 \psi_5}, \tilde{y}_2 \cdots \tilde{y}_\ell = \boldsymbol{\psi_1 \overline{\psi_2} \psi_3 \overline{\psi_4} \psi_5}$, where $\boldsymbol{\psi_3} = \boldsymbol{u' \psi_3' v'}$ and at least one of $\boldsymbol{u'}, \boldsymbol{v'}$ is a run of length at least 2. Since $\boldsymbol{\phi_2}, \boldsymbol{\phi_3}, \boldsymbol{\phi_4}$ are alternating sequences, it is clear that we cannot find a $k$ such that $\tilde{x}_k = \tilde{x}_{k+1} = \tilde{x}_{k+1} = \tilde{y}_{k+2}$.

**(B)** $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \boldsymbol{\psi_1 \psi_2 \psi_3 \psi_4 \psi_5}, \tilde{y}_2 \cdots \tilde{y}_\ell = \boldsymbol{\psi_1 \overline{\psi_2} \psi_3 \overline{\psi_4} \psi_5}$, where $|\boldsymbol{\psi_3}| = 2$, and $\boldsymbol{\psi_3}$ is neither zero-reversed nor completely reversed in $\boldsymbol{\tilde{x}'}$.

*(B1)* If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$ , then in order to satisfy the condition of $\boldsymbol{\psi_3}$ , then there must be $\ell_3 = 2$ , and $\tilde{x}_{1+\ell_1+\ell_2} = \tilde{x}_{1+\ell_1+\ell_2+1}$, Obviously, when $\boldsymbol{\phi_2}, \boldsymbol{\phi_3}$ are alternating sequence , $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$ and $\tilde{x}_{1+\ell_1+\ell_2} = \tilde{x}_{1+\ell_1+\ell_2+1}$ cannot hold at the same time.

*(B2)* If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+1}\overline{\tilde{y}}_{\ell_1+\ell_2+2}$, There are two possible structures that make the condition of $\psi_3$ satisfied:

(i) $\tilde{x}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \tilde{y}_{\ell_1+\ell_2+\ell_3+1}\tilde{y}_{\ell_1+\ell_2+\ell_3+2}$ and $\ell_3 = 2$,

(ii) $\tilde{x}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+1}\overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+2}$ and $\ell_3 = 4$.

In case (i), $\boldsymbol{\psi_3}$ corresponds to $\tilde{x}_{1+\ell_1+\ell_2+1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3}$ and we have $\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+2} = \tilde{x}_{1+\ell_1+\ell_2+1}, \tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \tilde{y}_{\ell_1+\ell_2+\ell_3+2} = \tilde{x}_{1+\ell_1+\ell_2+\ell_3+1}$, i.e. $\omega$ is completely reversed in $\tilde{x}$, a contradiction.

In case (ii), $\psi_3$ corresponds to $|\tilde{x}_{1+\ell_1+\ell_2+1}\tilde{x}_{1+\ell_1+\ell_2+2}|$. We have $\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+2} = \tilde{x}_{1+\ell_1+\ell_2+1}$ and $\tilde{x}_{1+\ell_1+\ell_2+2} = \overline{\tilde{x}}_{1+\ell_1+\ell_2+3}$. Thus, under the condition of $|\boldsymbol{S}_j^{\bar{i}}| = 6$, (B) can hold simultaneously in case (ii).

**(C)** $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \boldsymbol{\psi_1 \psi_2 \psi_3 \psi_4 \psi_5}, \tilde{y}_2 \cdots \tilde{y}_\ell = \boldsymbol{\psi_1 \overline{\psi_2} \psi_3 \overline{\psi_4} \psi_5}$ where $|\boldsymbol{\psi_3}| = 1$ and $\boldsymbol{\psi_3}$ is neither zero-reversed nor completely reversed in $\tilde{x}$.

*(C1)* If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$, it is clear that $|\boldsymbol{\psi_3}| > 1$.

*(C2)* If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+1}\overline{\tilde{y}}_{\ell_1+\ell_2+2}$, there are only one possible structures that make the condition of $\boldsymbol{\psi_3}$ satisfied: $\ell_3 = 3$ and $\tilde{x}_{1+\ell_1+\ell_2+\ell_3-1}\tilde{x}_{1+\ell_1+\ell_2+\ell_3} = \overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+1}\overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+2}$ then we have $\overline{\alpha}\alpha\tilde{x}_4 \cdots \tilde{x}_{1+\ell_1+\ell_2} = y_4 \cdots \tilde{y}_{\ell_1+\ell_2+3}$, $\tilde{x}_{1+\ell_1+\ell_2+1} \cdots \tilde{x}_{\ell-3} = \tilde{y}_{\ell_1+\ell_2+\ell_3+1} \cdots \tilde{y}_{\ell-3}\bar{\theta}\theta$, and either $\bar{\theta} = \bar{\beta}$ or $\theta = \bar{\beta}$ holds. Thus $d_L(\tilde{x}, \tilde{y}) = 1$, then $d_L(x, y) = 2$, a contradiction.

**(D)** $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \psi_1\psi_2\overline{\mu}\psi_3\xi\psi_4\psi_5, \tilde{y}_2 \cdots \tilde{y}_\ell = \psi_1\overline{\psi}_2\overline{\mu}\overline{\psi}_3\overline{\xi}\overline{\psi}_4\psi_5$

*(D1)* If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \tilde{y}_{\ell_1+\ell_2+1}\tilde{y}_{\ell_1+\ell_2+2}$, then $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2}$ corresponds to $\overline{\mu}\psi_3[1]$ it is clear that $\overline{\mu}\psi_3[1] \neq \overline{\mu}\overline{\psi_3}[1]$, a contradiction.

*(D2)* If $\tilde{x}_{1+\ell_1+\ell_2-1}\tilde{x}_{1+\ell_1+\ell_2} = \overline{\tilde{y}}_{\ell_1+\ell_2+1}\overline{\tilde{y}}_{\ell_1+\ell_2+2}$. Since the same segment following the complementary segment corresponds to $\overline{\mu}$, only $\ell_3 = 3$ can satisfies that the length of the segment equals 1, then neither $\tilde{x}_{1+\ell_1+\ell_2+2} = \tilde{y}_{\ell_1+\ell_2+\ell_3+1}$ nor $\tilde{x}_{1+\ell_1+\ell_2+2} = \overline{\tilde{y}}_{\ell_1+\ell_2+\ell_3+1}$ can satisfies (D).

In sum, if $|\boldsymbol{S}_j^{\bar{i}}| = 6, |S_{\bar{j}}^{i}|$ hold simultaneously, then $\tilde{x}_1 \cdots \tilde{x}_{\ell-1} = \psi_1\psi_2\psi_3\psi_4\psi_5, \tilde{y}_2 \cdots \tilde{y}_\ell = \psi_1\overline{\psi_2}\psi_3\overline{\psi_4}\psi_5$, where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$ are alternating sequences, $|\boldsymbol{\psi_3}| = 2$ and $\boldsymbol{\psi_3}$

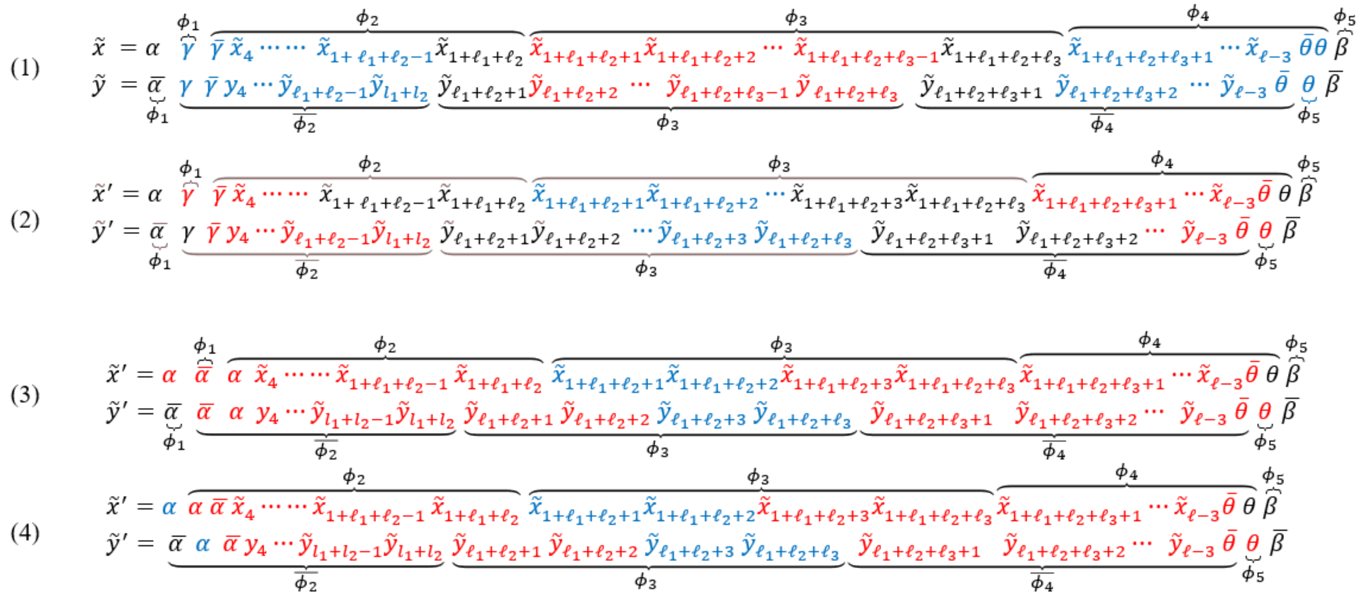

Fig. 2. Illustrations of $\tilde{x}$ and $\tilde{y}$ when $|\boldsymbol{S}_{j}^{\bar{i}}| = 6, |\boldsymbol{S}_{\bar{j}}^{i}| = 5$. In the figure, complementary alternating segments and identical alternating segments are represented in red and blue, respectively. Since $\boldsymbol{a}, \boldsymbol{b}$ are alternating sequences, $\boldsymbol{S}_{j}^{\bar{i}} = a_2 \cdots a_{|\boldsymbol{a}|} \circ (\mathcal{D}_2(\tilde{x}_2 \cdots \tilde{x}_\ell) \cap \mathcal{D}_2(\tilde{y}_1 \cdots \tilde{y}_{\ell-1})) \circ (b_2 \cdots b_{|\boldsymbol{b}|})$, then $|\mathcal{S}_{j}^{\bar{i}}| = 6$ if and only if $\tilde{x}_2 \cdots \tilde{x}_\ell = \boldsymbol{\phi_1 \phi_2 \phi_3 \phi_4 \phi_5}, \tilde{y}_1 \cdots \tilde{1}_{\ell-1} = \boldsymbol{\phi_1 \bar{\phi}_2 \phi_3 \bar{\phi}_4 \phi_5}$ as (1). Based on the known sequence features of $\boldsymbol{\phi_i}, \boldsymbol{s}, \boldsymbol{t}$, we can obtain (2) and if $\gamma = \alpha, |\boldsymbol{\psi_1}| = 0$, else $|\boldsymbol{\psi_1}| = 1$. (3)(4) are the structures for the cases where $|\boldsymbol{\phi}| = 4$ and $\gamma$ takes different values, respectively.

is neither zero reversed nor completely reversed in $\tilde{x}$, i.e. $\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{y}}'$ satisfies the structures in Fig.2 (3).

Denote $|\boldsymbol{\psi_1}|, |\boldsymbol{\psi_2}|, |\boldsymbol{\psi_3}|, |\boldsymbol{\psi_4}|, |\boldsymbol{\psi_5}|$ as $\ell_1', \ell_2', \ell_3', \ell_4', \ell_5'$. It is clear that $\ell_2' = \ell_2 + 2, \ell_3' = 2, \ell_4' = \ell_4 + 2$. Next, we analyze the structure of $\boldsymbol{s}, \boldsymbol{\omega^1}, \boldsymbol{\omega^2}, \boldsymbol{t}$ when all the conditions are satisfied simultaneously.

(A)Since we have established that $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{s\omega^1 t}\beta, \tilde{\boldsymbol{y}} = \alpha \boldsymbol{s\omega^2 t}\bar{\beta}$ and $\boldsymbol{s}, \boldsymbol{t}$ are alternating sequence. Combined with the structure in Fig.2 (3), it is easy to see that the $\boldsymbol{\omega^1} = \tilde{x}_{1+\ell_1+\ell_2+1}\tilde{x}_{1+\ell_1+\ell_2+2}\tilde{x}_{\underline{1+}\ell_1+\ell_2+3}, \boldsymbol{\omega^2} = \tilde{y}_{\ell_1+\ell_2+2}\tilde{y}_{\ell_1+\ell_2+3}\tilde{y}_{\ell_1+\ell_2+\ell_3}$, and $\boldsymbol{\omega^1} = \overline{\boldsymbol{\omega}^2}$, where $\omega_1^1 = s_{|s|}, \omega_3^1 = \bar{t_1}$.

(B)In Fig.2 (3), we assume that $\gamma = \bar{\alpha}$, and $|\boldsymbol{s}| = \ell_1 + \ell_2 = \ell_2' - 1$. In order to satisfy $\ell_2 \geq 2, \ell_2' \geq 2$, then $|\boldsymbol{s}| \geq 3$. In the other case $\gamma = \alpha$, we show in Fig2 (4),$|\boldsymbol{s}| = \ell_2 = 1 + \ell_2' - 1$, and in order to satisfy $\ell_2 \geq 2, \ell_2' \geq 2$, $|\boldsymbol{s}| \geq 2$. In sum, we get $|\boldsymbol{s}| \geq 2$ if completely reversed in $\tilde{x}$ and $|\boldsymbol{s}| \geq 3$ if $s_1 = \bar{\alpha}, s_{|\boldsymbol{s}|} = \omega_1^1$.

(C)Due to symmetry, we obtain $|\boldsymbol{t}| \geq 2$ if zero-reversed in $\tilde{x}$ and $|\boldsymbol{t}| \geq 3$ if $t_1 = \omega_{|\boldsymbol{\omega}|}^1, t_{|\boldsymbol{t}|} = \beta$.

In summary, we have obtained a necessary condition for $|\boldsymbol{S}_{\bar{j}}^{i}| = 5, |\boldsymbol{S}_{j}^{\bar{i}}| = 6$, i.e. $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{s\omega t}\beta, \tilde{\boldsymbol{y}} = \alpha \boldsymbol{s\bar{\omega} t}\bar{\beta}$, such that

- $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{\omega}$ are alternating sequences,
- $|\boldsymbol{\omega}| = 3, \omega_1 = s_{|s|}, \omega_3 = \overline{t_1}$ ,
- if $|\boldsymbol{s}|$ is completely reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{s}| \geq 2$, else $|\boldsymbol{s}| \geq 3$
- if $|\boldsymbol{t}|$ is zero-reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{t}| \geq 2$, else $|\boldsymbol{t}| \geq 3$

Next we prove that the above condition is also sufficient.

Firstly, it is clear that $|\boldsymbol{S}_{j}^{\bar{i}}| = 6$, then $|\boldsymbol{S}_{\bar{j}}^{i}| = 4$ according Lemma 8. Thus, we only need to prove that $|\boldsymbol{S}_{\bar{j}}^{i}| = 5$, and $|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| = 4$.

Since $\boldsymbol{a}, \boldsymbol{b}$ are alternating sequences, then $\boldsymbol{S}_{\bar{j}}^{i} = a_1 \cdots a_{|\boldsymbol{a}|-1} \circ (\mathcal{D}_2(\alpha \boldsymbol{s\omega t}) \cap \mathcal{D}_2(\boldsymbol{s\bar{\omega}t}\bar{\beta})) \circ b_1 \cdots b_{|\boldsymbol{b}|-1}$. Since $\boldsymbol{s}, \boldsymbol{\omega}, \boldsymbol{t}$ are alternating sequences, and $\omega_1 = s_{|s|}, \omega_3 = \bar{t_1}$, then $\alpha\boldsymbol{s\omega t}, \boldsymbol{s\bar{\omega}t}\bar{\beta}$ can be written as $\alpha \boldsymbol{s\omega t} = \boldsymbol{\psi_1\psi_2\psi_3\psi_4\psi_5}$ , $\boldsymbol{s\bar{\omega}t}\bar{\beta} = \boldsymbol{\psi_1\bar{\psi}_2\psi_3\bar{\psi}_4\psi_5}$, where $\boldsymbol{\psi_i}$ is alternating, $|\boldsymbol{\psi_1}|, |\boldsymbol{\psi_5}| \leq 2$ ,and $\boldsymbol{\psi_3}$ corresponds to $\omega_1\omega_2$. Denote $|\psi_1|, |\psi_2|, |\psi_3|, |\psi_4|, |\psi_5|$ as $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$

Firstly, $|(\mathcal{D}_2(\alpha \boldsymbol{s\omega t}) \cap \mathcal{D}_2(\boldsymbol{s\bar{\omega}t}\bar{\beta}))| < 6$ according to Theorem3 and we can get the set

$$
\{
\begin{aligned}
&\boldsymbol{\psi_1}\psi_2[1, \ell_2 - 1]\boldsymbol{\psi_3}\psi_4[1, \ell_4 - 1]\boldsymbol{\psi_5},\\
&\boldsymbol{\psi_1}\psi_2[1, \ell_2 - 1]\boldsymbol{\psi_3}\psi_4[2, \ell_4]\boldsymbol{\psi_5},\\
&\boldsymbol{\psi_1}\psi_2[2, \ell_2]\boldsymbol{\psi_3}\psi_4[1, \ell_4 - 1]\boldsymbol{\psi_5},\\
&\boldsymbol{\psi_1}\psi_2[2, \ell_2]\boldsymbol{\psi_3}\psi_4[2, \ell_4]\boldsymbol{\psi_5},\\
&\boldsymbol{\psi_1\bar{\psi}_2\psi_4\psi_5}
\end{aligned}
\} \subseteq (\mathcal{D}_2(\alpha \boldsymbol{s\omega t}) \cap \mathcal{D}_2(\boldsymbol{s\bar{\omega}t}\bar{\beta})).
$$

Thus $|\boldsymbol{S}_{\bar{j}}^{i}| = |(\mathcal{D}_2(\alpha \boldsymbol{s\omega t}) \cap \mathcal{D}_2(\boldsymbol{s\bar{\omega}t}\bar{\beta}))| = 5$.

$\boldsymbol{S}_{\bar{j}}^{\bar{i}} = a_2 \cdots a_{|\boldsymbol{a}|} \circ (\mathcal{D}_1(\boldsymbol{s\omega t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s\bar{\omega}t}\bar{\beta})) \circ b_1 \cdots b_{|\boldsymbol{b}|-1}$. Firstly $|(\mathcal{D}_1(\boldsymbol{s\omega t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s\bar{\omega}t}\bar{\beta}))| < 4$ according to Lemma 4 and we can get the set

$$
\{
\begin{aligned}
&\boldsymbol{s}\omega_1\omega_2\boldsymbol{t},\\
&\boldsymbol{s}\omega_2\omega_3\boldsymbol{t},\\
&\bar{\alpha}s_3 \cdots s_{|\boldsymbol{s}|}\boldsymbol{\omega t},\\
&\boldsymbol{s\omega}t_1 \cdots t_{|\boldsymbol{t}|-3}\bar{\beta},
\end{aligned}
\} \subseteq (\mathcal{D}_1(\boldsymbol{s\omega t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s\bar{\omega}t}\bar{\beta})).
$$

Thus $|\boldsymbol{S}_{\bar{j}}^{\bar{i}}| = |(\mathcal{D}_1(\boldsymbol{s\omega t}) \cap \mathcal{D}_3(\bar{\alpha}'\boldsymbol{s\bar{\omega}t}\bar{\beta}))| = 5$ So far, we have proved the sufficient and necessary condition condition for Type (A).

Symmetrically, we can get the sufficient and necessary condition condition for Type (B) is that $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{s} \boldsymbol{\omega} \boldsymbol{t} \beta, \tilde{\boldsymbol{y}} = \alpha \boldsymbol{s} \bar{\boldsymbol{\omega}} \boldsymbol{t} \bar{\beta}$ such that

- $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{\omega}$ are alternating sequences
- $|\boldsymbol{\omega}| = 3, \omega_1 = \overline{s_{|\boldsymbol{s}|}}, \omega_3 = t_1,$
- if $|\boldsymbol{s}|$ is zero-reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{s}| \geq 2$, else $|\boldsymbol{s}| \geq 3$.
- if $|\boldsymbol{t}|$ is completely reversed in $\tilde{\boldsymbol{x}}$,$|\boldsymbol{t}| \geq 2$, else $|\boldsymbol{t}| \geq 3$.

Combining the above two types, we obtain Theorem 3. ■