Math 545 - Assignment 3

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```r
library(itsmr)
library(forecast)
```

3.1

Causality will be implied by $\phi(z) = 0$ only being solved by roots of absolute value greater than 1, and invertibility will be implied by $\theta(z) = 0$ only being solved by roots of absoltue value greater than 1.

a)

$\phi(z) = 1 + 0.2z - 0.48z^2 = 0 \implies z = \frac{-0.2\pm\sqrt{0.2^2-(4*0.48*1)}}{-2*0.48} = \frac{-0.2\pm\sqrt{1.96}}{-0.96}$ which is equal to -1.25 or 1.66, both of which are greater in absolute value than 1, thus it is causal. $\theta(z) = 1$ and this is never 0, so it is invertible.

b)

$\phi(z) = 1 + 1.9z + 0.88z^2 = 0 \implies z = \frac{-1.9\pm\sqrt{1.9^2-(4*0.88*1)}}{2*0.88} = \frac{-1.9\pm\sqrt{0.3}}{1.76}$ which is equal to -0.909 or 01.25, the former being smaller than 1 in absolute value implies the process is not causal. $\theta(z) = 1 + 0.2z + 0.7z^2 = 0 \implies z = \frac{-0.2\pm\sqrt{0.2^2-(4*0.7*1)}}{2*0.7}$ which is -0.1428+1.186i or -0.1428-1.186i, where both absolute values are approximately $1.32 > 1$, thus invertible.

c)

$\phi(z) = 1+0.6z = 0 \implies |z| = 1/0.6 = 1.66 > 1$, thus it is causal. $\theta(z) = 1+1.2z \implies |z| = 1/1.2 = 0.83 < 1$, thus it is not invertible.

d)

$\phi(z) = 1 + 1.8z + 0.81z^2 = 0 \implies |z| = |\frac{-1.8\pm\sqrt{1.8^2-(4*0.81)}}{2*0.81}| = |-1.11| = 1.11 > 1$, thus causal. $\theta(z) = 1$ and this is never 0, so it is invertible.

e)

$\phi(z) = 1 + 1.6z = 0 \implies |z| = 1/1.6 = 0.625 < 1$, thus it is not causal. $\theta(z) = 1 - 0.4z + 0.04z^2 = 0 \implies z = \frac{0.4\pm\sqrt{0.4^2-(4*0.04)}}{2*0.04} \implies |z| = 5 > 1$ thus invertible.

3.5

a)

Since $Y_t = X_t + W_t$,we have that $E[Y_t] = E[X_t] + E[W_t] = E[X_t] + 0$ since W_t is white noise. Since X_t was said to be ARMA, we know it is stationary thus this has no t dependence. Let $E[X_t] = \mu \implies E[Y_t] = \mu$. $\implies \gamma_y(h) = E(Y_{t+h}.Y_t) - \mu^2$
$= E[(X_{t+h} + W_{t+h})(X_t + W_t)] - \mu^2$

$= E(X_{t+h} * X_t) + E(X_{t+h} * W_t) + E(W_{t+h} * X_t) + E(W_{t+h} * W_t) - \mu^2$
. We know that X_t is ARMA, which is a linear process and a function of Z_t, which is uncorrelated with W_t, thus we can say that X_t is uncorrelated with W_t, and thus:
$= E(X_{t+h} * X_t) + E(X_{t+h})E(W_t) + E(W_{t+h})E(X_t) + E(W_{t+h})E(W_t) - \mu^2$
$= E(X_{t+h} * X_t) - \mu^2 = \gamma_x(h)$. Since X_t is ARMA and stationary, this has no t dependence. In addition;
$\gamma_y(0) = E(Y_t^2) - \mu^2$
$= E[(X_t + W_t)(X_t + W_t)] - \mu^2$
$= E(X_tX_t) + E(X_t)E(W_t) + E(X_t)E(W_t) + E(W_t^2) - \mu^2$
$= E(X_tX_t) + \sigma_w^2 - \mu^2$
$= \gamma_x(0) + \sigma_w^2$, which once again has no t dependence due to the stationarity of X_t. Thus we conclude that Y_t is stationary.

The autocorrelation function of Y is thus given by $\gamma_y(h) = \gamma_x(h)$ if h>0, and $\gamma_y(h) = \gamma_x(h) + \sigma_w^2$ if h = 0.

b)

$U_t = \phi(B)Y_t$
$= \phi(B)(X_t + W_t)$
$= \phi(B)X_t + \phi(B)W_t)$
, but $\phi(B)X_t = \theta(B)Z_t$, so
$= \theta(B)Z_t + \phi(B)W_t)$. So
$U_t = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q} + W_t - \phi_1 W_{t-1} - ... - \phi_p W_{t-p}$. If $h \leq r$, then this will have variance terms (the likes of $E(Z_t^2)$) which will give a non-zero autocovariance. However, if $|h| > r$, then there will be no variance term possible and due to the uncorrelation of W_t and Z_t, everything will become zero, so we have that $\gamma_u(h) = 0$ if $|h| > r$, thus it is r correlated. Hence we conclude that it is an MA(r) process. What we've effectively showed here, is that $\phi(B)Y_t = U_t =$ some polynomial of coefficients and white noise terms; an MA(r) process. This is the defintion an ARMA(p,r) process (since LHS is AR and RHS is MA by the respective polynomials). So we conclude that Y_t is an ARMA(p,r) process.

3.9)

a)

$E[Y_t] = E[Y_{t+h}] = E[\mu + Z_t + \theta_1 Z_{t-1} + \theta_{12} Z_{t-12}] = \mu$ as Z_t is WN and has mean 0. $\gamma_y(h) = E[(\mu + Z_t + \theta_1 Z_{t-1} + \theta_{12} Z_{t-12} - \mu)(\mu + Z_{t+h} + \theta_1 Z_{t+h-1} + \theta_{12} Z_{t+h-12} - \mu)]$

$= E(Z_t Z_{t+h}) + \theta_1 E[Z_t Z_{t+h-1}] + \theta_{12} E[Z_t Z_{t+h-12}] + \mu E[Z_t] + \theta_1 E[Z_{t-1} Z_{t+h}] + \theta_1^2 E[Z_{t-1} Z_{t+h-1}] + \theta_1 \theta_{12} E[Z_{t-1} Z_{t+h-12}] + \theta_{12} E[Z_{t-12} Z_{t+h}] + \theta_{12}\theta_1 E[Z_{t-12} Z_{t+h-1}] + \theta_{12}^2 E[Z_{t-12} Z_{t+h-12}]$

If h = 0, $= E[Z_t^2] + 0 + 0 + 0 + E[\theta_1^2 Z_{t-1}^2] + 0 + 0 + 0 + E[\theta_{12}^2 Z_{t-12}^2] = \sigma^2 + \theta_1^2\sigma^2 + \theta_{12}^2\sigma^2 = \sigma^2(1 + \theta_1^2 + \theta_{12}^2)$
If h = 1, $= 0 + E[\theta_1 Z_t^2] + 0 + 0 + 0 + 0 + 0 + 0 = \theta_1\sigma^2$
if h = 11, $= 0 + 0 + 0 + 0 + 0 + \theta_1\theta_{12} E[Z_{t-1}^2] = \theta_1\theta_{12}\sigma^2$
if h = 12, $= 0 + 0 + \theta_{12} E[Z_t^2] + 0 + 0 + 0 + 0 + 0 + 0 = \theta_{12}\sigma^2$
.

Thus the autovariance funcction: $\gamma_y =$
$\sigma^2(1 + \theta_1^2 + \theta_{12}^2)$ if h = 0;
$\theta_1\sigma^2$ if |h| = 1,
$\theta_1\theta_{12}\sigma^2$ if |h|=11,
$\theta_{12}\sigma^2$ if |h|=12.
0 else.

b)

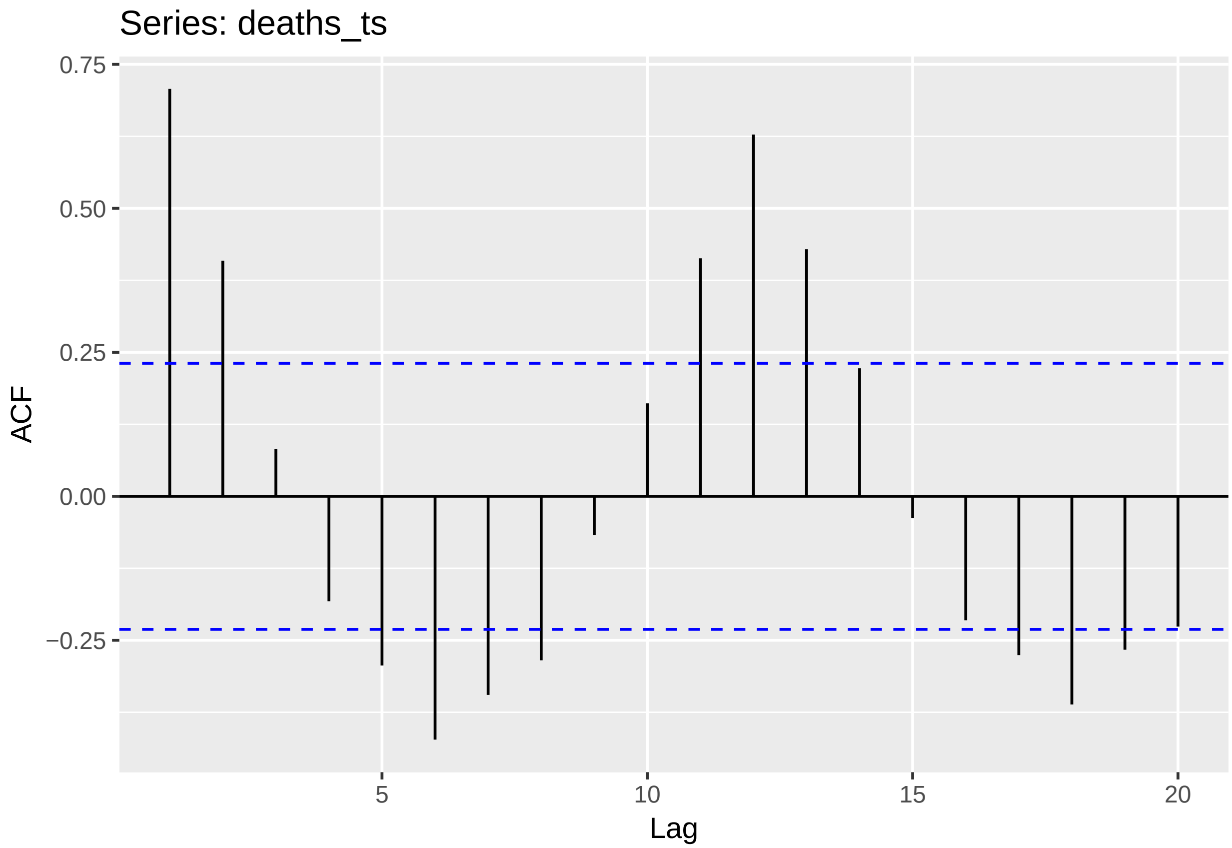
First we obtain our data.

```
deaths_ts = ts(deaths, frequency=12, start=c(1973,1))
deaths_ts
```

```
##         Jan   Feb   Mar   Apr   May   Jun   Jul   Aug   Sep   Oct   Nov
## 1973  9007  8106  8928  9137 10017 10826 11317 10744  9713  9938  9161
## 1974  7750  6981  8038  8422  8714  9512 10120  9823  8743  9129  8710
## 1975  8162  7306  8124  7870  9387  9556 10093  9620  8285  8433  8160
## 1976  7717  7461  7776  7925  8634  8945 10078  9179  8037  8488  7874
## 1977  7792  6957  7726  8106  8890  9299 10625  9302  8314  8850  8265
## 1978  7836  6892  7791  8129  9115  9434 10484  9827  9110  9070  8633
##         Dec
## 1973  8927
## 1974  8680
## 1975  8034
## 1976  8647
## 1977  8796
## 1978  9240
```

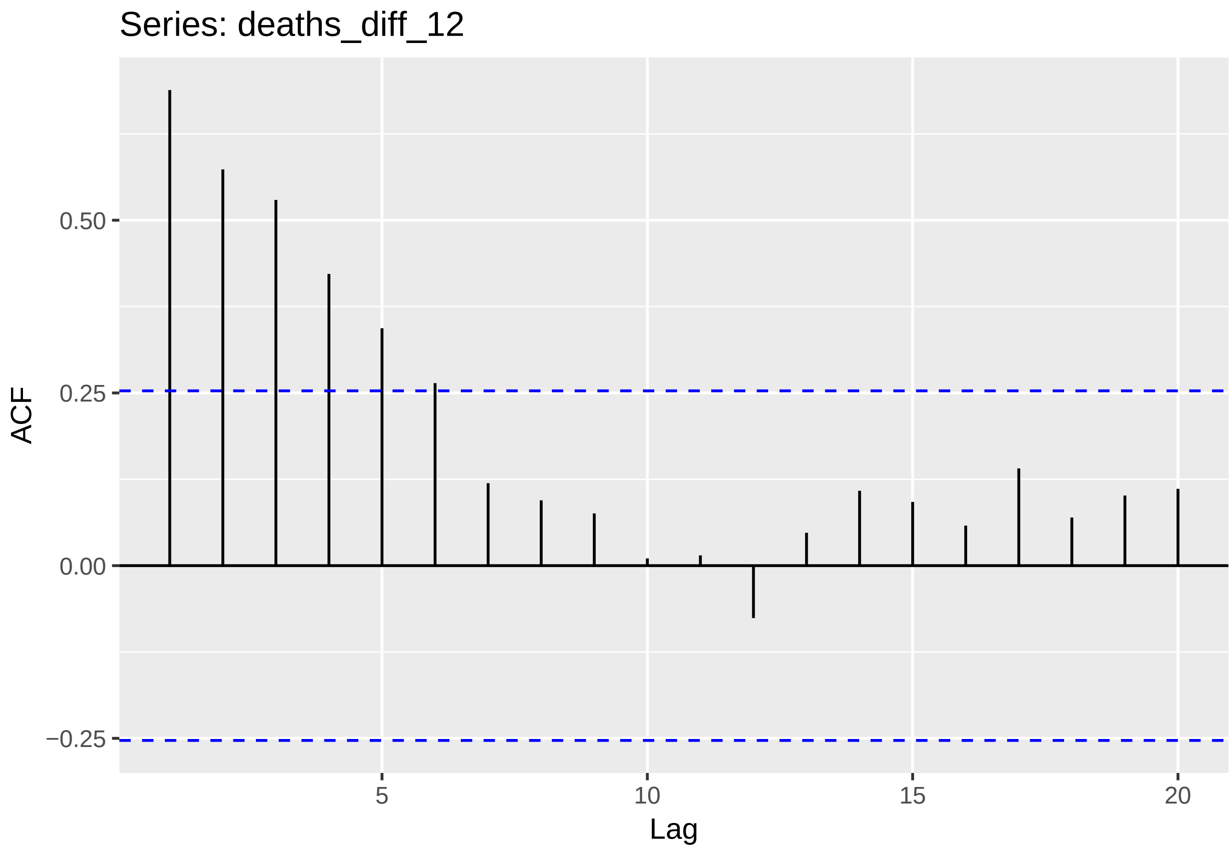
Below is the autocorrelation function upto lag 20 of the original series.

```
ggAcf(deaths_ts,lag=20)
```



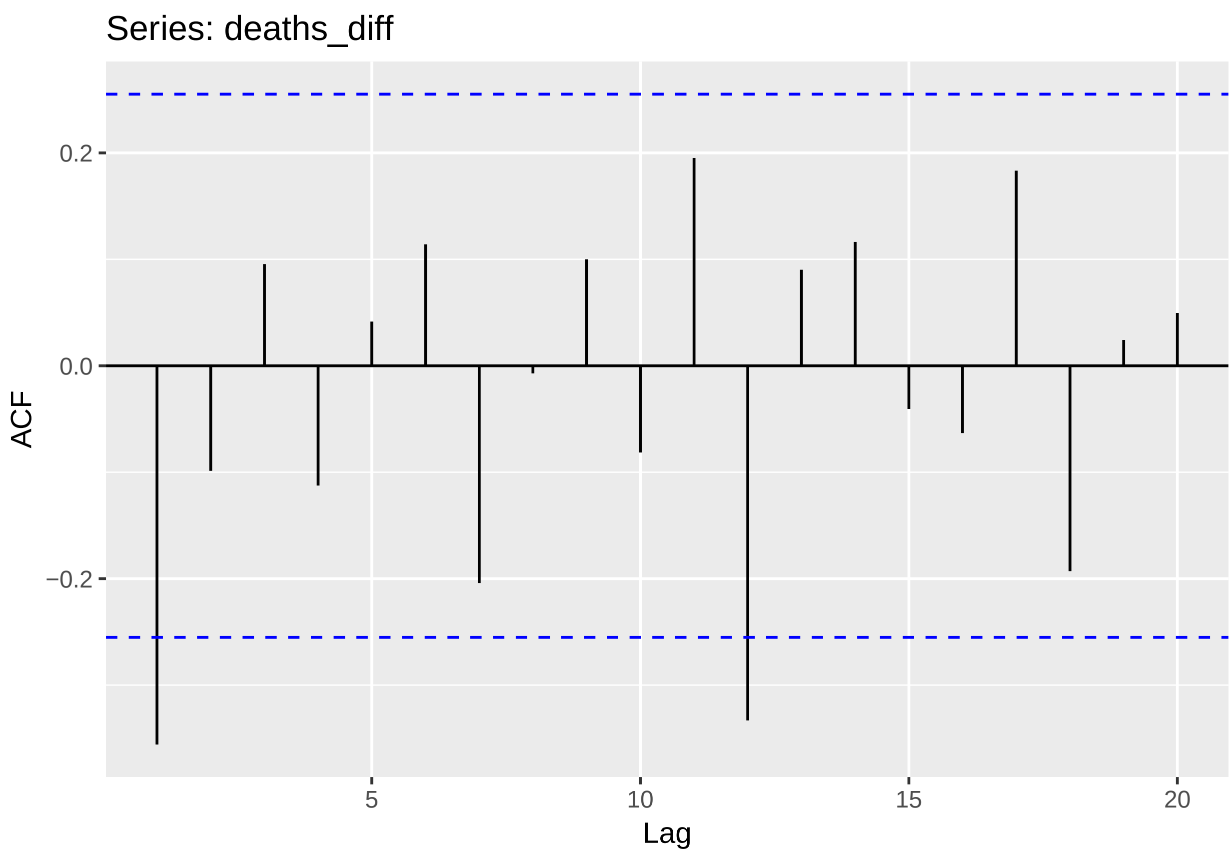
Below is the ACF as we apply ∇_{12}

```r
deaths_diff_12 = deaths_ts %>% diff(.,lag=12)
ggAcf(deaths_diff_12,lag=20)
```



And we apply ∇ to this:

```r
deaths_diff = deaths_diff_12 %>% diff(.,lag=1)
ggAcf(deaths_diff,lag=20)
```



The correlations seems to have calmed down a bit. Below is the autocorrelation $\rho(h)$ at different lags upto 20 , in addition to the sample mean of the series.

```
ggacf_saved = ggAcf(deaths_diff,lag=20)
acf_values = ggacf_saved[["data"]]
acf_values
```

```
##    Lag          ACF
## 2    1 -0.355843707
## 3    2 -0.098720893
## 4    3  0.095530249
## 5    4 -0.112515462
## 6    5  0.041529223
## 7    6  0.114108535
## 8    7 -0.204130053
## 9    8 -0.007123311
## 10   9  0.100066891
## 11  10 -0.081448224
## 12  11  0.195205609
## 13  12 -0.333182813
## 14  13  0.090181278
## 15  14  0.116314298
## 16  15 -0.040607730
## 17  16 -0.063250388
## 18  17  0.183280640
## 19  18 -0.192938261
## 20  19  0.024188047
```

```
## 21  20  0.049580453
```

```r
cat("Sample Mean:", mean(deaths_diff), "
")
```

```
## Sample Mean: 28.83051
```

We now take a look at the sample autocovariances upto lag 20:

```r
lag<-0:20
acvf=acvf(deaths_diff,20)
cbind(lag,acvf)
```

```
##       lag       acvf
##  [1,]   0 152669.632
##  [2,]   1 -54326.528
##  [3,]   2 -15071.682
##  [4,]   3  14584.568
##  [5,]   4 -17177.694
##  [6,]   5   6340.251
##  [7,]   6  17420.908
##  [8,]   7 -31164.460
##  [9,]   8  -1087.513
## [10,]   9  15277.175
## [11,]  10 -12434.670
## [12,]  11  29801.969
## [13,]  12 -50866.898
## [14,]  13  13767.943
## [15,]  14  17757.661
## [16,]  15  -6199.567
## [17,]  16  -9656.413
## [18,]  17  27981.388
## [19,]  18 -29455.813
## [20,]  19   3692.780
## [21,]  20   7569.430
```

c)

We have $\hat{\gamma}_x(1) = -54326.528$
$\hat{\gamma}_x(11) = 29801.969$
$\hat{\gamma}_x(12) = -50866.898$
For μ we can simply use the mean of the differenced series, thus $\mu = 28.83051$

From part a) we have $\gamma_x(1) = \hat{\gamma}_x(1) \implies \theta_1\sigma^2 = -54326.528$

$$\gamma_x(11) = \hat{\gamma}_x(11) \implies \theta_1\theta_{12}\sigma^2 = 29801.969$$

$$\gamma_x(12) = \hat{\gamma}_x(12) \implies \theta_{12}\sigma^2 = -50866.898$$

We have 3 equations with 3 unknowns.

Substituting equation 1 into equation 2 gives $-54326.528 * \theta_{12} = 29801.969 \implies \theta_{12} = -0.548571$. Plugging this into equation 3 gives $-0.548571 * \sigma^2 = -50866.898 \implies \sigma^2 = 92726.2$. Finally, plugging this into equaiton 1 gives $\theta_1 * 92726.2 = -54326.528 \implies \theta_1 = -0.585881$.

Thus in the form of part a), the model would be $X_t = 28.83051 + Z_t - 0.585881 Z_{t-1} - 0.548571 Z_{t-12}$ where Z_t white noise with mean 0 and variance $\sigma^2 = 92726.2$.

5.3

a)

The model is given as $X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t$. Causality means $\phi(z) = 0$ only if $|z| > 1$. In this case we have that $\phi(z) = 1 - \phi z - \phi^2 z^2 = 0 \implies z^2 + \frac{z}{\phi} - \frac{1}{\phi^2} = 0 \implies \frac{1}{2}[\frac{-1}{\phi} \pm \sqrt{\frac{1+4}{\phi^2}}] = \frac{1}{2}[\frac{-1}{\phi} \pm \frac{\sqrt{5}}{\phi}]$ thus the solutions are $\frac{-1+\sqrt{5}}{2\phi} = z_1$, $\frac{-1-\sqrt{5}}{2\phi} = z_2$
$|z_1| > 1 \implies |\frac{-1+\sqrt{5}}{2\phi}| > 1 \implies \frac{-1+\sqrt{5}}{2|\phi|} > 1$ as $-1+\sqrt{5}$ is positive, $\implies |\phi| < \frac{-1+\sqrt{5}}{2} = 0.618$
Also $|z_2| > 1 \implies |\frac{-1-\sqrt{5}}{2\phi}| > 1 \implies \frac{|-(1+\sqrt{5})|}{2|\phi|} > 1 \implies \frac{(1+\sqrt{5})}{2|\phi|} > 1$ as $1+\sqrt{5}$ is positive, $\implies |\phi| < \frac{1+\sqrt{5}}{2} = 1.618$. This is already implied by $|\phi| < 0.618$ anyway, so we impose $|\phi| < 0.618$ as our causality condiiton.

b)

We are given $\hat{\gamma}(0) = 6.06$ and $\hat{\rho}(1) = 0.687 \implies \hat{\gamma}(1) = \hat{\rho}(1) * \hat{\gamma}(0) = 0.687 * 6.06 = 4.16322$
. These will be used later. Note that this model is simply an AR model; so $\theta(z) = 1$, $\theta_0 = 1$ and $\theta_j = 0$ for any $j \neq 0$. In addition, we also had $\phi(z) = 1 - \phi z - \phi^2 z^2$. Note also that since this is an AR model it is linear and can be expressed as $X_t = \sum(\psi_j Z_{t-j})$ where $\psi_j = \phi^j$ (we had shown this in class). These will come in handy later.

The Yule-Walker equation provides $\hat{\gamma}(k) - \phi\hat{\gamma}(k-1) - \phi^2\hat{\gamma}(k-2) = \sigma^2 \sum_{j=k}^{2} \theta_j \phi^j$
For $k = 0$ we have using the fact that $\gamma(h) = \gamma(-h)$:
$\hat{\gamma}(0) - \phi\hat{\gamma}(-1) - \phi^2\hat{\gamma}(-2) = \hat{\gamma}(0) - \phi\hat{\gamma}(1) - \phi^2\hat{\gamma}(2) = \sigma^2[\theta_0\phi^0 + \theta_1\phi^1\theta_2\phi^2] = \sigma^2[1+0+0] = \sigma^2$
$\implies \hat{\gamma}(0) - \phi\hat{\gamma}(1) - \phi^2\hat{\gamma}(2) = \sigma^2 => eq1$
For $k = 1$, we have $\hat{\gamma}(1) - \phi\hat{\gamma}(0) - \phi^2\hat{\gamma}(1) = \sigma^2[\theta_1\phi^1\theta_2\phi^2] = 0$
$\implies \hat{\gamma}(1) - \phi\hat{\gamma}(0) - \phi^2\hat{\gamma}(1) = 0 => eq2$
For $k = 2$ we have $\hat{\gamma}(2) - \phi\hat{\gamma}(1) - \phi^2\hat{\gamma}(0) = \sigma^2[\theta_2\phi^2] = 0$
$\implies \hat{\gamma}(2) - \phi\hat{\gamma}(1) - \phi^2\hat{\gamma}(0) = 0 => eq3$

So now we have 3 equations, and since we have information about $\hat{\gamma}(0)$ and $\hat{\rho}(1)$ (and therefore $\hat{\gamma}(1)$), if we can get rid of $\hat{\gamma}(2)$ terms, we would be able to solve for wat we need.

From eq2 we have $\implies \hat{\gamma}(1) - \phi\hat{\gamma}(0) - \phi^2\hat{\gamma} = 0 \implies \hat{\gamma}(1)(1-\phi^2) - \phi\hat{\gamma}(0) = 0 \implies \hat{\gamma}(1)(1-\phi^2) = \phi\hat{\gamma}(0) \implies \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)}(1-\phi^2) = \phi \implies \hat{\rho}(1)(1-\phi^2) = 0$
Expanding and solving this; $\hat{\rho}(1)(1-\phi^2) - \phi = 0$
$\hat{\rho}(1) - \phi^2\hat{\rho}(1) - \phi = 0$
$\phi^2 + \frac{\phi}{\hat{\rho}(1)} - 1 = 0$
$\phi = \frac{1}{2}[\frac{-1}{\hat{\rho}(1)} \pm \sqrt{\frac{1}{\hat{\rho}(1)^2} + 4}]$

Thus plugging in the known value of $\hat{\rho}(1)$ we get $\phi = \frac{1}{2}[\frac{-1}{0.687} \pm \sqrt{\frac{1}{0.687^2} + 4}] \implies \phi = 0.509$ or $\phi = -1.96461$
but $|-1.96461| > |0.618|$
so this would make it non-causal (by part a), thus we take $\phi = 0.509$

Now we multiply eq3 by ϕ^2 and add it to eq1 to get $\hat{\gamma}(0) - \phi\hat{\gamma}(1) - \phi^2\hat{\gamma}(2) + \phi^2\hat{\gamma}(2) - \phi^3\hat{\gamma}(1) - \phi^4\hat{\gamma}(0) = \sigma^2 \implies -\phi^3\hat{\gamma}(1) - \phi\hat{\gamma}(1) - \phi^4\hat{\gamma}(0) + \hat{\gamma}(0) = \sigma^2$
Plugging in our now know value of ϕ and $\hat{\gamma}(1)$ & $\hat{\gamma}(0)$ we get $-0.509^3 * 4.16322 - 0.509 * 4.16322 - 0.509^4 * 6.06 + 6.06 = \sigma^2$
$\implies \sigma^2 = 2.98514$.

5.4)

a)

For this we can construct a confidence bound for the autocorrelation $\rho(k)$ and see if it is included. By our usual confidence bounds for autocorrelations, we have $\rho(k) \sim N(0, \frac{1}{n})$ so a 95% confidence interval would be constructed as $\rho(k) = \hat{\rho}(k) \pm z_{\alpha/2} * \frac{1}{\sqrt{200}}$
$= \hat{\rho}(k) \pm \frac{1.95}{\sqrt{200}}$
$= \hat{\rho}(k) \pm 0.137886$. Thus using our estimates we get:
0.427 ± 0.137886
0.475 ± 0.137886
0.169 ± 0.137886
0 is not included in any of these; so it's not reasonable to assume that $X_t - \mu$ is white noise.

b)

We can use our sample mean for the estimate of μ, so we estimate it as 3.82 For estimates of ϕ we need to solve the following system: $\hat{R}\hat{\phi} = \hat{\rho}(k)$ where:

$$\hat{R} = \begin{matrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{matrix}$$

$\hat{\phi} = [\hat{\phi}_1, \hat{\phi}_2]$, and $\hat{\rho}(k) = [\hat{\rho}(1), \hat{\rho}(2)]^T$ Thus we will have $\hat{\phi} = \hat{R}^{-1}\hat{\rho}(k)$. In our case: $\hat{\rho}(k) = [0.427, 0.475]^T$ and

$$\hat{R} = \begin{matrix} 1 & 0.427 \\ 0.427 & 1 \end{matrix}$$

and

$$\hat{R}^{-1} = \begin{matrix} 1.22299 & -0.522215 \\ -0.522215 & 1.22299 \end{matrix}$$

Thus $\hat{\phi} = \hat{R}^{-1}\hat{\rho}(k) = [0.274163, 0.357932]^T$
So our estimates are $\hat{\phi}(1) = 0.274163$, and $\hat{\phi}(2) = 0.357932$
The estimate for σ^2 will be given by $\hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\rho}_2^T \hat{R}_2^{-1} \hat{\rho}_2)$. We know our $\hat{R}_2^{-1}$ and $\hat{\rho}(2) = [0.427, 0.475]^T$ and we know that $\hat{\gamma}(0) = 1.15$. $\hat{\rho}_2^T \hat{R}_2^{-1} \hat{\rho}_2$ gives 0.287085, so $\hat{\sigma}^2 = 1.15(1 - 0.287085) = 0.81985$

c)

The sample mean itself already hints this isn't true, but let's take a closer look. We know that a 95% confidence interval for μ would be given by $\bar{X}_n \frac{\pm 1.96 v^{1/2}}{\sqrt{n}}$ where $v = \sum \gamma(h)$ for all finite valued h.

We can estimate this v by summing all the covariances we can get that is $v \approx \hat{\gamma}(-3) + \hat{\gamma}(-2) + \hat{\gamma}(-1) + \hat{\gamma}(0) + \hat{\gamma}(1) + \hat{\gamma}(2) + \hat{\gamma}(3)$. We can acquire those values from given values of $\hat{\rho}(k)$ using the fact that $\hat{\gamma}(k) = \hat{\rho}(k) * \hat{\gamma}(0)$ and $\hat{\gamma}(k) = \hat{\gamma}(-k)$
Thus we get $\hat{\gamma}(3) = \hat{\gamma}(-3) = 0.169 * 1.15 = 0.19435$
$\hat{\gamma}(2) = \hat{\gamma}(-2) = 0.475 * 1.15 = 0.54625$
$\hat{\gamma}(1) = \hat{\gamma}(-1) = 0.427 * 1.15 = 0.49105$
Thus $v = 2 * 0.19435 + 2 * 0.54625 + 2 * 0.49105 + 1.15 = 3.6133$
So our confidence interval for μ is $3.82 \pm \frac{1.96 * 3.6133^{1/2}}{\sqrt{200}}$
$= [3.556553, 4.083447]$. Since 0 is not in this interval, we reject the hypothesis that $\mu = 0$

d)

We know that $\phi \sim (\phi, n^{-1}\sigma^2\Gamma_p^-1)$ where Γ_p=

$$\Gamma = \begin{matrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{matrix} = \begin{matrix} 1.15 & 0.49105 \\ 0.49105 & 1.15 \end{matrix}$$

So we have

$$\Gamma^{-1} = \begin{matrix} 1.06347 & -0.4541 \\ -0.4541 & 1.06347 \end{matrix}$$

And $n^{-1}\sigma^2\Gamma_p^-1$ using our estimated σ^2 and n=200 is

$$\begin{matrix} 0.00435941 & -0.00186147 \\ -0.00186147 & 0.00435941 \end{matrix}$$

Thus the main diagonal of this will give our variances. That is; $\hat{\phi}_1 \sim N(\phi_1, 0.00435941)$
$\hat{\phi}_2 \sim N(\phi_2, 0.00435941)$
Thus the 95% confidence intervals using our estimates $\hat{\phi}(1)$ and $\hat{\phi}(2)$ are given as $\phi_1 = \hat{\phi}(1) \pm z_{\alpha/2} * \sqrt{var} = 0.274163 \pm 1.96 * \sqrt{0.00435941} = [0.144752, 0.403574]$
$\phi_2 = \hat{\phi}(2) \pm z_{\alpha/2} * \sqrt{var} = 0.357932 \pm 1.96 * \sqrt{0.00435941} = [0.228521, 0.487343]$

e)

By the definition of PACF, $\alpha(0) = 1$ and $\alpha(h) = \phi_{hh}$ where ϕ_{nn} is the last element of $\tilde{\phi}_h = \Gamma_h^{-1}\tilde{\gamma}_h$. Here we have $\Gamma_1^{-1} = \frac{1}{\hat{\gamma}(0)}$ and $\gamma_1 = \hat{\gamma}(1)$, thus $\alpha(1) = \frac{1}{\hat{\gamma}(0)} * \hat{\gamma}(1) = \hat{\rho}(1) = 0.427$
and $\alpha(2)$ would be given by the last element of $\Gamma_2^{-1}\hat{\gamma}_2$.

$$\Gamma_2^{-1} = \begin{matrix} 1.06347 & -0.4541 \\ -0.4541 & 1.06347 \end{matrix}$$

as we found earlier, and $\hat{\gamma}_2 = [\hat{\gamma}(1), \hat{\gamma}(2)]^T = [0.49105, 0.54625]$
$\implies \Gamma_2^{-1}\tilde{\gamma}_2 = [0.274165, 0.357935]$, thus $\alpha(2) = 0.357935$.