Computer Problem Set 3.2

Local volatility and the Dupire formula

The present problem set is attached to Chapter 10 of the lectures notes. We denote by $C(T,K)$ the time 0−price of a European call option with maturity T and strike K. We shall denote C_T, C_K and C_{KK} the corresponding partial derivative with respect to T, with respect to K, and second partial derivative with respect to K, respectively. We recall the Dupire formula

$$\sigma^2(T,K) \quad = \quad 2\frac{C_T(T,K)+rKC_K(T,K)}{K^2C_{KK}(T,K)}. \qquad (1)$$

In terms of the implied volatilities $I(T,K)$, obtained by inversion of the Black-Scholes formula C^{BS}, i.e. $C(T,K)=C^{BS}(T,K,I(T,K))$, this formula reduces to

$$\sigma^2(T,K) \quad = \quad \frac{\frac{I}{T}+2I_T+2rKI_K}{K^2\left(\frac{1}{K^2IT}+2\frac{\mathbf{d}_+}{KI\sqrt{T}}I_K+\frac{\mathbf{d}_+\mathbf{d}_-}{I}I_K^2+I_{KK}\right)} \qquad (2)$$

where $\mathrm{d}_\pm$ is the standard function involved in the Black-Scholes formula, and subscripts indicate again partial derivatives.

The file optionprices.txt contains call options prices $C(T_i,K_j)$ for a spot price $S_0=100$, spot interest rate $r=0$, maturities $T_i := i\frac{T}{n}$, $n=8$, $T=0.9$, $i=0,\ldots,n$, and strikes $K_j := 80+j10^{-1}$, $j=0,\ldots,400$.

1. Provide an approximation $\bar\sigma^2(T_i,K_j)$ of the Dupire local volatility function by using the Dupire formula (1). Comment on the encountered numerical difficulties, if any.

2. We next turn to an alternative approximation method of the Dupire local volatility function.

   (a) Deduce from the provided data the corresponding implied volatilities $I(T_i,K_j)$, $i=0,\ldots,n$ and $j=0,\ldots,400$.

   (b) Provide an alternative approximation $\hat\sigma^2(T_i,K_j)$ of the Dupire local volatility function by using the Dupire formula (2).

   (c) Build a program which produces a linear interpolation in the variables (T,K) of the points $T_i\hat\sigma^2(T_i,K_j)$, $i=0,\ldots,n$, $j=0,\ldots,400$.

3. We finally verify numerically the validity of the Dupire formula. Consider the local volatility model $d\hat S_t = \hat S_t\hat\sigma(t,S_t)dB_t$, where B is a Brownian motion under the risk-neutral measure $\mathbb{Q}$.

   (a) By using an Euler discretization scheme for the process $\hat S$, provide Monte-Carlo approximations of $\hat C(T_i,K_j)$, $i=0,\ldots,n$, $j=0,\ldots,400$.

   (b) Compare the data $\hat C(T_i,K_j) := \mathbb{E}[(\hat S_{T_i}-K_j)^+]$ to the initial data $C(T_i,K_j)$.