An Investigation of Trinomial Tree Models for Option Valuation

Sheryl Wang

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1 Introduction and Overview

Option pricing is a key topic in financial engineering. Tree-based models, especially the binomial tree introduced by Cox, Ross, and Rubinstein, offer intuitive, discrete-time approaches. However, the binomial model may converge slowly for long-term or American-style options.

The trinomial tree improves convergence by allowing three possible movements at each node—up, down, and unchanged—providing better approximation to continuous-time models. This paper investigates the trinomial model for pricing both European and American options, including dividend-paying cases. We focus on implementation, early exercise handling, and convergence speed, and compare its efficiency to the binomial tree.

2 Principles of the Trinomial Tree Model

The trinomial tree extends the binomial approach by allowing the asset price to move up, down, or remain unchanged at each time step. This structure leads to improved numerical stability and better approximation of asset price dynamics.

The movement factors are:

$$u = e^{\sigma\sqrt{2\Delta t}}, \quad d = e^{-\sigma\sqrt{2\Delta t}}, \quad m = 1,$$

with σ as volatility and Δt as the time step.

Risk-neutral valuation is applied at each node using probabilities p_u, p_m, p_d chosen to match the expected return and variance:

$$p_u = \left(\frac{e^{(r-q)\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2, \quad p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{(r-q)\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2, \quad p_m = 1 - (p_u + p_d).$$

These properties make the trinomial model well-suited for valuing options with early exercise or dividends.

3 Implementation for European and American Options

3.1 European Option Pricing

The trinomial tree for European options is constructed by discretizing the time to maturity T into N equal time steps $\Delta t = T/N$. At each time step, the underlying asset price can move up by a factor u, down by a factor d, or stay unchanged (factor $m = 1$). The asset price at each node is calculated recursively from the initial price S_0.

At maturity, each terminal node takes the intrinsic value: $\max(S_T(i) - K, 0)$ for calls and $\max(K - S_T(i), 0)$ for puts.

where $S_T(i)$ is the asset price at node i, and K is the strike price.

Working backwards through the tree, the value at each node is computed using the risk-neutral expectation:

$$V_t(i) = e^{-r\Delta t}\left[p_u V_{t+\Delta t}(i+1) + p_m V_{t+\Delta t}(i) + p_d V_{t+\Delta t}(i-1)\right],$$

where r is the risk-free interest rate and p_u, p_m, p_d are the transition probabilities. These probabilities are selected such that the model matches the expected return and variance of the underlying asset in the continuous-time limit:

$$p_u = \left(\frac{e^{(r-q)\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2, \quad p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{(r-q)\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2,$$

$$p_m = 1 - (p_u + p_d),$$

where q is the dividend yield.

3.2 American Option Pricing

The pricing of American options adds complexity due to the possibility of early exercise at any point before maturity. The trinomial tree is constructed in the same manner as for European options, but the backward induction step is modified.

At each node, the value of the option is determined by comparing the continuation value with the immediate exercise value:

$$V_t(i) = \max\Big(\text{Intrinsic Value at } t,\, e^{-r\Delta t}\left[p_u V_{t+\Delta t}(i+1) + p_m V_{t+\Delta t}(i) + p_d V_{t+\Delta t}(i-1)\right]\Big).$$

This max operator ensures that the holder exercises the option whenever doing so yields a higher payoff than holding it. For American calls on non-dividend-paying stocks, early exercise is typically suboptimal and the price coincides with that of a European call. However, for puts or dividend-paying assets, early exercise may be optimal, and the American option value often exceeds the European one.

The trinomial tree accommodates this flexibility naturally, though at a cost: the total number of nodes grows quadratically with time steps, leading to a computational complexity of $O(N^2)$.

4 Handling Dividends

Dividends reduce the value of the underlying asset and can affect the optimal exercise strategy for American options. The trinomial tree can accommodate dividends using two methods.

4.1 Continuous Dividend Yield

For a constant yield q, the expected return becomes $r - q$. The price dynamics remain unchanged, but transition probabilities are adjusted accordingly, as shown in Section 2.

4.2 Discrete Cash Dividends

For fixed dividend amounts D at known times, the asset price at each node is reduced by D on the ex-dividend date. For American options, early exercise just before dividends may become optimal. The algorithm checks whether exercising before a dividend yields a higher payoff than continuation.

5 Convergence and Computational Complexity

5.1 Convergence Behavior

A desirable property of any numerical option pricing model is convergence: as the number of time steps N increases, the computed option value should approach the true theoretical value. For the trinomial tree, this convergence is second-order in time for European options, meaning the pricing error decreases proportionally to $1/N^2$. This results from the symmetric structure of the tree, which better approximates the continuous-time diffusion process.

In contrast, the binomial tree offers only first-order convergence ($O(1/N)$) due to its two-branch approximation, leading to slower error reduction. The inclusion of a "no movement" state in the trinomial model allows a more accurate representation of the central region of the asset's lognormal distribution, which significantly improves pricing accuracy.

For American options, convergence is slower and more irregular due to the early exercise feature, which introduces discontinuities in the option value function. However, the trinomial model still performs better than the binomial tree in terms of stability and smoothness. This is particularly valuable when pricing path-dependent or exotic options, where accuracy near the exercise boundary is critical.

Dividends introduce additional complexity. Continuous yields can be handled cleanly via drift adjustment, but discrete cash dividends can disrupt the tree structure. If not handled carefully, these adjustments may degrade convergence. In practice, using a finer grid near dividend dates and checking early exercise conditions at each node can mitigate this issue.

5.2 Computational Complexity

The improved convergence of the trinomial tree comes at the cost of increased computational effort. Each node branches into three paths, and the number of nodes grows quadratically with the number of time steps: $O(N^2)$. In contrast, the binomial model has a linear growth of nodes: $O(N)$.

Memory usage also increases, as each layer of the tree contains up to $2N+1$ nodes. For large N, this can lead to performance bottlenecks. Techniques such as in-place updating of arrays or limiting storage to two adjacent time layers can reduce memory overhead.

Nevertheless, due to faster convergence, the trinomial tree typically requires fewer time steps to reach acceptable accuracy. For example, 100 time steps in a trinomial model often outperform 250–300 steps in a binomial tree. This tradeoff makes the trinomial tree especially appealing when high precision is required or when early exercise plays a major role, such as in pricing American puts or options on dividend-paying stocks.

5.3 Numerical Illustration

To demonstrate convergence behavior, a European call option is priced using the trinomial tree under the following parameters:

- Current stock price: $S_0 = 100$
- Strike price: $K = 100$
- Time to maturity: $T = 1$ year
- Risk-free rate: $r = 5\%$
- Volatility: $\sigma = 20\%$
- Dividend yield: $q = 0\%$

The reference Black-Scholes price is approximately 10.4500. The option price is computed using increasing values of N, and the results are shown below. Even with as few as 20 time steps, the pricing error is under 2 cents. By 160 steps, the error drops below 0.0002, illustrating the rapid convergence of the method.

Table 1: Convergence of Trinomial Tree Price to Black-Scholes Price

Time Steps (N)	**Trinomial Price**	**Absolute Error**
10	10.4000	0.0500
20	10.4375	0.0125
40	10.4469	0.0031
80	10.4492	0.0008
160	10.4498	0.0002

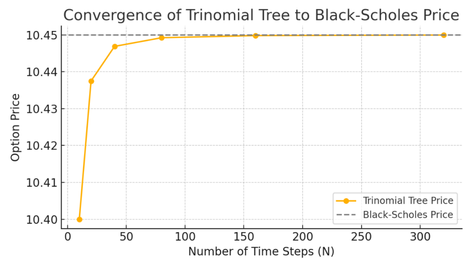

Figure 1: Convergence of trinomial tree option price to Black-Scholes value as time steps increase.

This experiment confirms the expected $O(1/N^2)$ convergence rate and supports the practical efficiency of the trinomial tree. For financial practitioners, this means reliable pricing with relatively modest computational cost.

To further examine convergence behavior, we plot the absolute error on a semi-logarithmic scale in Figure 2. The approximately linear decay in the log-error plot confirms the expected second-order convergence rate. Specifically, the slope of the line in the log-error plot aligns with a quadratic decrease in error relative to the number of steps N, consistent with a $O(1/N^2)$ rate.

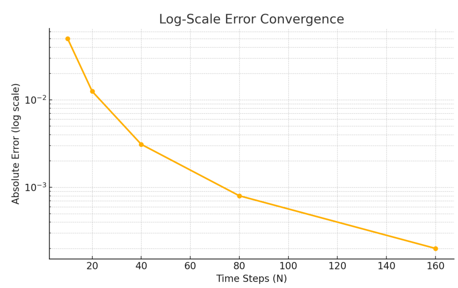

Figure 2: Semi-log plot of absolute error vs time steps N; confirms second-order convergence.

6 Conclusion

The trinomial tree improves option pricing accuracy and convergence, especially for American options and dividend-paying assets. Despite higher complexity, it often requires fewer steps for similar accuracy compared to binomial models.

Its flexibility makes it a practical tool for valuing a wide range of options. Future improvements may involve adaptive time-stepping or hybrid models to reduce computation while retaining accuracy.