DR. G. B. DAVIES

SIMPLE HARMONIC MOTION

NAME:

CLASS:

Contents

1 Simple Harmonic Motion

Learning Objectives and Prior Knowledge

In this section you'll learn:

- What simple harmonic motion is.
- The conditions necessary for simple harmonic motion to occur.

Simple harmonic motion is a type of *periodic motion* or *vibration* around a *fixed* or *equilibrium* point. It's important because it turns out that there are many examples of simple harmonic motion in physics - e.g. a pendulum bob swinging, a vibrating mass on a spring - and also because many other types of more complicated vibration or period motion can be approximated as simple harmonic motion.

In order for simple harmonic motion to occur a system needs to have some *equilibrium* or *fixed* point that the system would like to be in if it wasn't somehow stopped from doing so. For example, a pendulum bob would like to come to rest with the string pointing straight down if it isn't lifted to the side, and a mass on a spring would like to rest at the spring's *natural length* if the spring isn't extended or compressed.

Another way to think of the fixed or equilibrium point is the position the system would come to rest if it were to be in its lowest energy state. In the pendulum bob for example, any position of the bob that isn't directly down will have a higher gravitational potential energy, and therefore the position with the bob at the bottom of the swing is its lowest energy state.

The other condition is that the force acting on a body is directly proportional to the displacement from this fixed point and is directed opposite to the displacement.

We can state this mathematically as follows:

$$\boxed{\begin{aligned} F &\propto \Delta x = x - x_0 \\ F &= k\Delta x = k\left(x - x_0\right) \end{aligned}}$$

You may notice that this looks like Hooke's law: that's no coincidence, because a mass on a spring is an example of a system that moves simply harmonically.

where x_0 is the position of the fixed point.

In words, we can say:

> If an object moves in such a way that its acceleration is directly proportional to its distance from a fixed point and the acceleration is directed towards that fixed point, the object is undergoing simple harmonic motion.

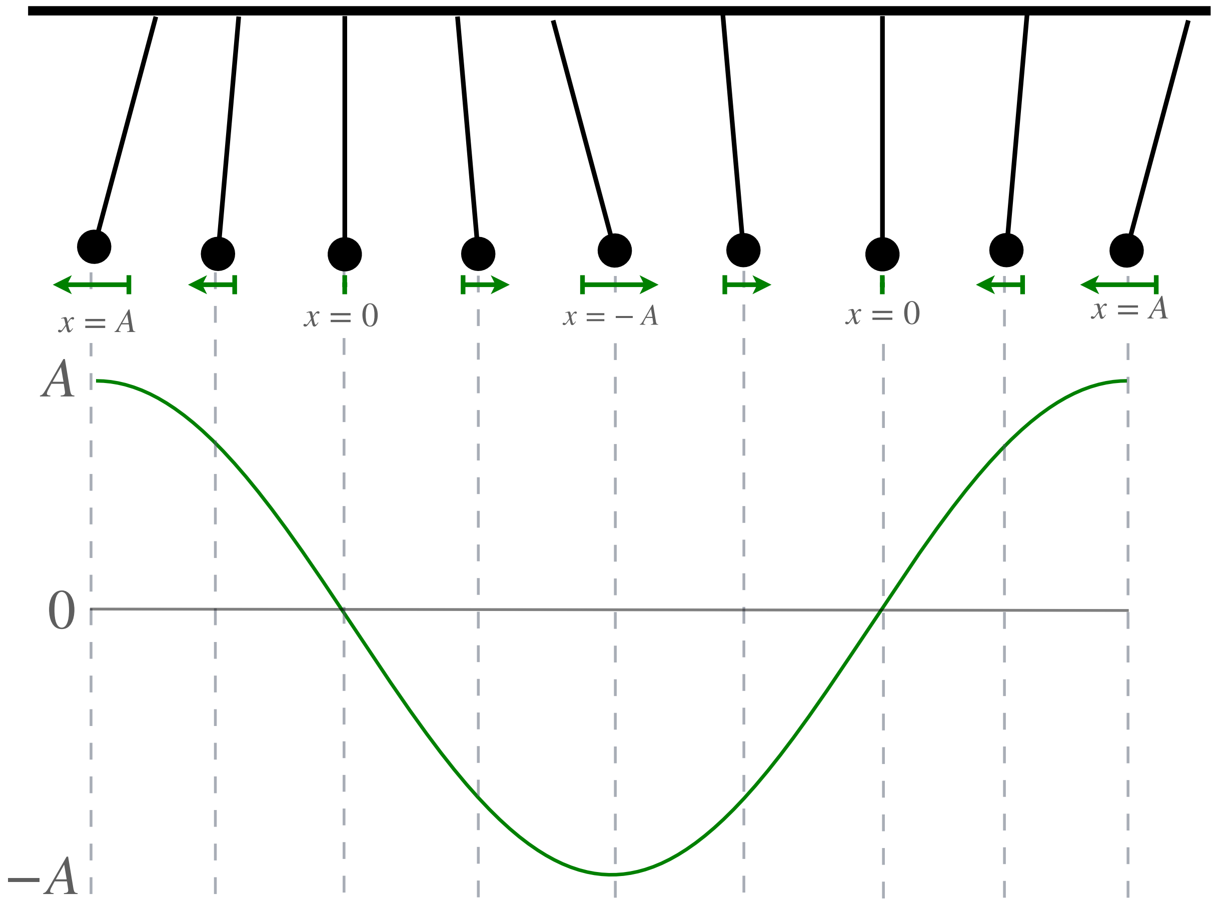
2 The Mathematics of Simple Harmonic Motion

Learning Objectives and Prior Knowledge

In this section you'll learn:

- How to describe the position of an object undergoing simple harmonic motion using $x = A\cos\omega t$.
- How to derive the velocity $v = -A\omega\sin\omega t$ and the acceleration $a = -\omega^2 A\cos\omega t = -\omega^2 x$ of simple harmonic motion.
- How to draw and interpret graphs of position x, velocity v, and time-period t of objects moving simple harmonically.

If we watch an object moving simple harmonically, for example a pendulum bob, and plot its displacement x around the equilibrium point x_0 we will see a graph that looks like the following:



This graph may look familiar to you: it's a cos graph. The object oscillates from a maximum displacement $x = A$ to the left to another maximum displacement $x = -A$ to the right where A is called the *amplitude* of the oscillations.

Whether you choose displacement to the left to be positive or negative is up to you or the question you're given.

The reasons behind why an object moving simple harmonically trace a cos graph are slightly complicated and for now you just have to take it as a fact.

It may not be obvious why the angular frequency ω that we encountered in circular motion comes into play here. It turns out that if the pendulum bob were to move in a complete circle with a constant centripetal force, the time it would take to do one revolution is the same as the time it takes to carry out one full simple harmonic oscillation.

We can describe this motion using some general cosine function that is true for any objecting moving simple harmonically:

$$x = A\cos\omega t$$

In Appendix A we show a proof of this result but for now you can take it as a given. One way to think of it is as follows: in uniform circular motion the pendulum bob would have a constant centripetal force acting on it and move with constant speed.

In simple harmonic motion, however, the speed varies because the force acting on it changes depending on how far away it is from the equilibrium point ($F \propto \Delta x$).

it turns out that the time it would take the pendulum bob moving at constant speed to complete a full circle is the same as the time taken for the pendulum bob to oscillate back and forth with varying speed as in simple harmonic motion.

In a way you can think of simple harmonic motion as a 1D projection of 2D motion.

where A is the amplitude of the oscillations and ω is the angular frequency. Remember that angular frequency ω is related to the time-period T via $\omega = \dfrac{2\pi}{T}$.

Before we look at this intuitively from a physics standpoint, let's note that we can mathematically derive the velocity $v = \dfrac{dx}{dt}$ and acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ by differentiating this function:

$$\begin{aligned} x &= A\cos\omega t \\ v &= -\omega A\sin\omega t \\ a &= -\omega^2 A\cos\omega t \end{aligned}$$

If you haven't covered it in A-level Maths yet, the rule for differentiating sine and cosine is as follows:

$$\begin{aligned} y =& \sin(kx) \\ \implies \frac{dy}{dx} =& k\cos(kx) \end{aligned}$$

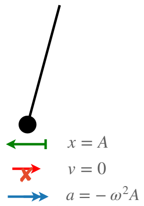
and

$$\begin{aligned} y =& \cos(kx) \\ \implies \frac{dy}{dx} =& -k\sin(kx) \end{aligned}$$

Since the factor $x = A\cos\omega t$ we can write $a = -\omega^2 A\cos\omega t$ in a simple alternative form:

$$\begin{aligned} a =& -\omega^2 A\cos\omega t \\ a =& -\omega^2 x \end{aligned}$$

1



POSITIVE DISPLACEMENTS ARE TO THE LEFT, SO ACCELERATIONS AND VELOCITIES TO THE RIGHT WILL BE NEGATIVE.

AT THE MAXIMUM DISPLACEMENT, THE VELOCITY IS 0 AND THE ACCELERATION IS ALSO A MAXIMUM BUT POINTING TO THE RIGHT, OPPOSITE THE DISPLACEMENT.

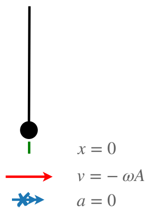
KINETIC ENERGY IS ZERO AND POTENTIAL ENERGY IS AT A MAXIMUM.

2

SINCE THE BOB ACCELERATES TO THE RIGHT, ITS VELOCITY INCREASES TO THE RIGHT AS IT HEADS BACK TOWARDS THE EQUILIBRIUM POSITION.

AS IT GETS CLOSER TO EQUILIBRIUM, ITS ACCELERATION DECREASES

3



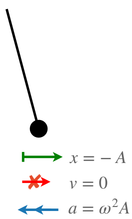
AT THE EQUILIBRIUM POSITION, THE DISPLACEMENT AND ACCELERATION ARE ZERO, AND VELOCITY IS AT A MAXIMUM DIRECTED TO THE RIGHT, SO IT IS STILL NEGATIVE.

KINETIC ENERGY IS AT ITS MAXIMUM AND POTENTIAL ENERGY ZERO.

4

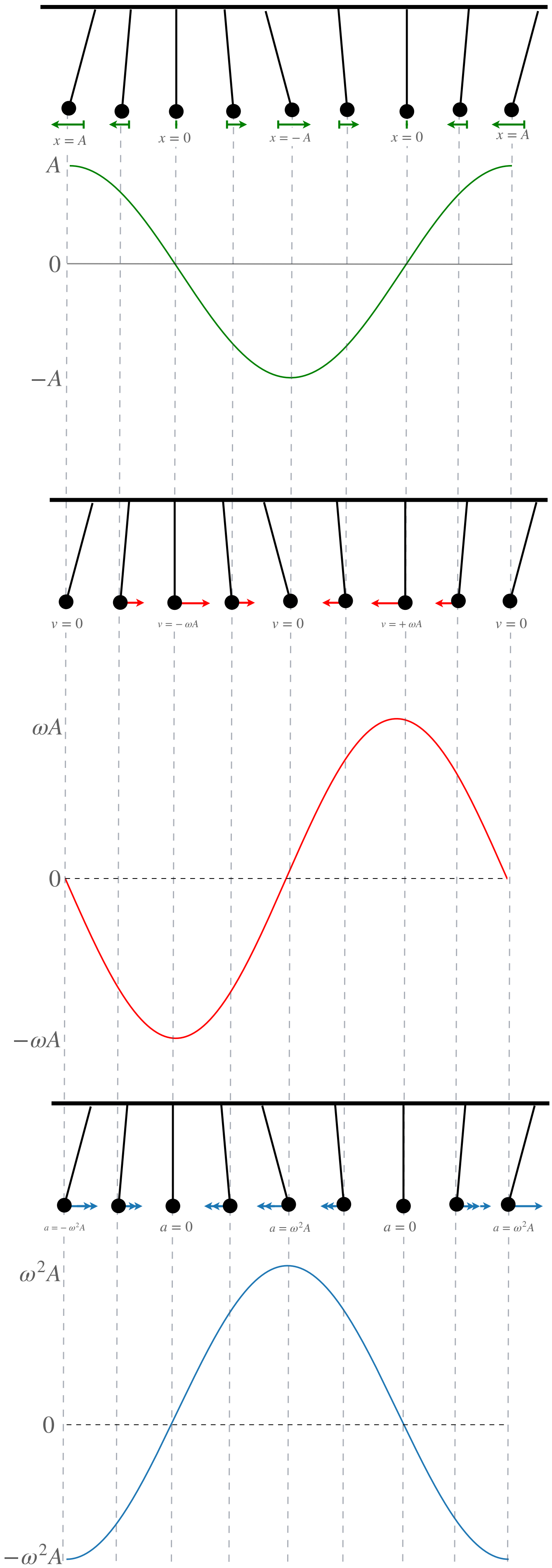
PAST THE EQUILIBRIUM POSITION, THE VELOCITY IS STILL DIRECTED TO THE RIGHT BUT IT DECREASES BECAUSE THE ACCELERATION POINTS TO THE LEFT BACK TOWARDS THE EQUILIBRIUM POSITION

5



AT THE END OF THE SWING, THE DISPLACEMENT IS A MAXIMUM TO THE RIGHT, HENCE NEGATIVE. THERE IS NO VELOCITY AND ACCELERATION IS A MAXIMUM DIRECTED TOWARDS THE EQUILIBRIUM POSITION TO THE LEFT, AND SO IS POSITIVE.

KINETIC ENERGY IS ZERO AND POTENTIAL ENERGY IS AT ITS MAXIMUM



2.1 Maximum displacement, velocity, and acceleration

Worked Example 2-1 - Calculating the maximum acceleration of a simple harmonic oscillator

Q: A speaker vibrates with frequency 50 Hz with an amplitude of 2 cm. Calculate the maximum acceleration of the cone.

A: A speaker vibrates with frequency 50 Hz with an amplitude of 2 cm. Calculate the maximum acceleration of the cone.

Practice Questions 2-1 - Calculating the frictional force on a mass on a turntable

3 Energy in Simple Harmonic Motion

Learning Objectives and Prior Knowledge

In this section you'll learn:

- How to describe energy transfers between kinetic and potential energy in simple harmonic oscillators.
- How to derive the expression for the total energy of a simple harmonic system $E = \frac{1}{2}m\omega^2 A^2$.

We saw that in the example of a pendulum bob we convert gravitational potential energy to kinetic energy which gets converted back to gravitational potential energy during the swing.

In the case of the mass-spring system, the elastic potential energy stored in the compressed spring is converted into kinetic energy before being converted into elastic potential energy again as the spring extends.

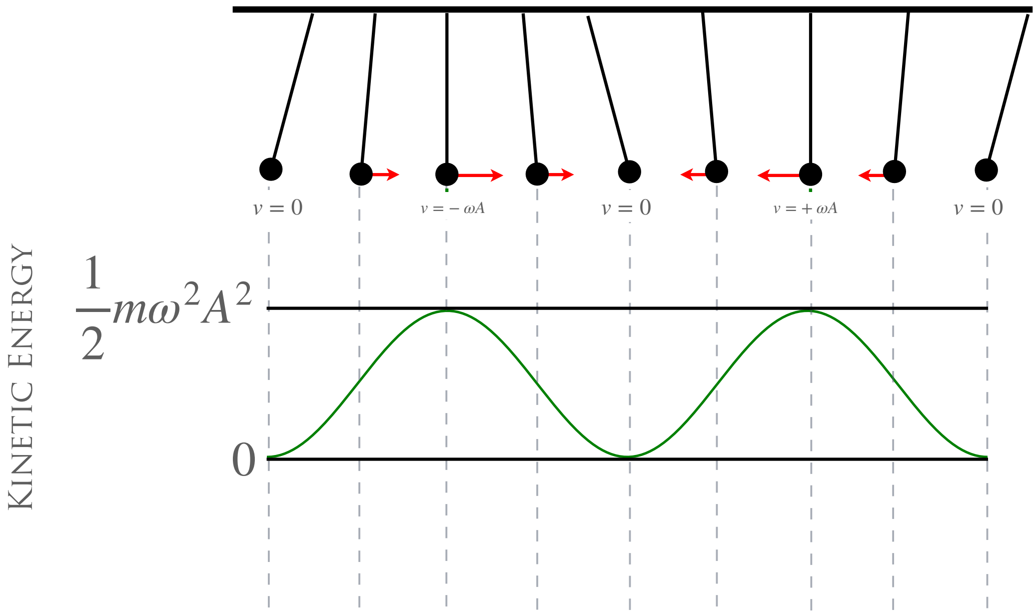
In both cases, potential energy is converted to kinetic energy and back again over and over. This is a general feature of simple harmonic motion: the conversion of potential energy to kinetic energy.

3.1 Kinetic Energy of a Simple Harmonic Oscillator

The kinetic energy of an object of mass m moving with velocity v is given by $KE = \frac{1}{2}mv^2$. We also know the velocity of an object as it undergoes simple harmonic motion: $v = -\omega A \sin \omega t$ and so we can obtain the following expression for the kinetic energy of a body undergoing simple harmonic motion:

$$\begin{aligned} KE &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(-\omega A \sin \omega t\right)^2 \\ &= \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t \end{aligned}$$

The graph below shows how the kinetic energy varies during a complete oscillation of a pendulum bob.



We can see that the maximum kinetic energy occurs when $\sin^2 \omega t = 1$, which is when the pendulum bob is in the equilibrium position.

3.2 *Total Energy*

The easiest way to find the total energy of a simple harmonic oscillator is to look at what happens when all of the energy is of a particular type because we know during simple harmonic motion that the energy continually transfers between kinetic energy and potential energy.

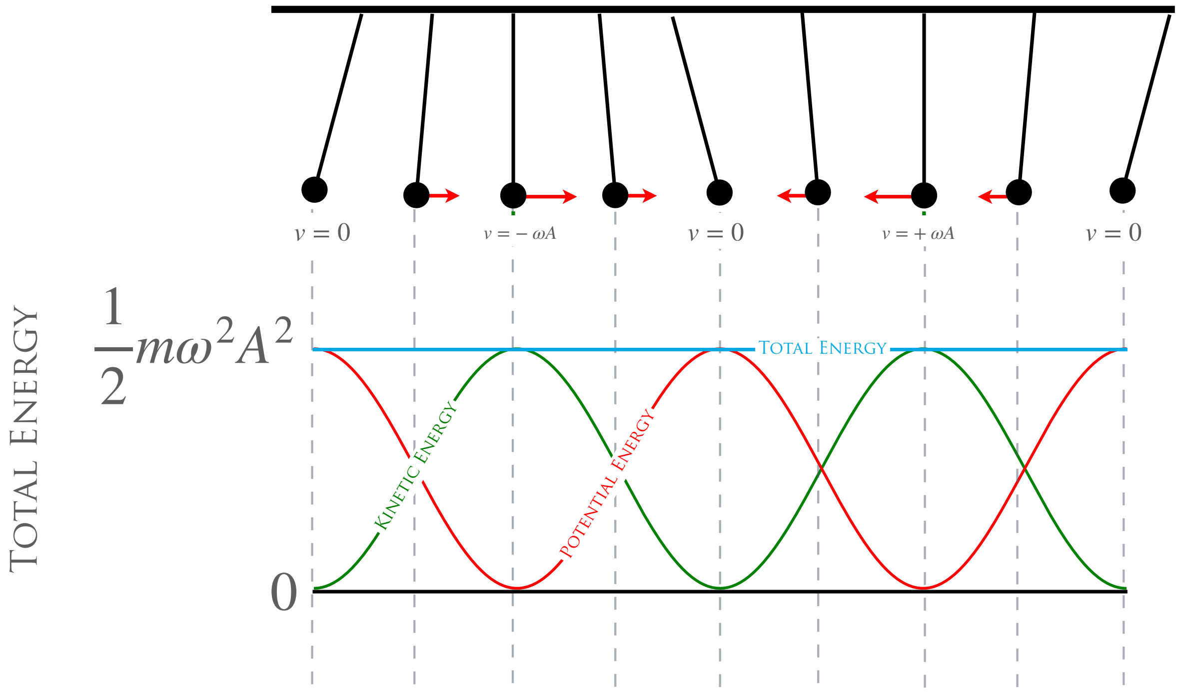
Therefore, when the kinetic energy is a maximum $KE = \frac{1}{2}m\omega^2 A^2$ the potential energy is 0.

Similarly, when the kinetic energy is 0 that means the oscillator has converted all of its kinetic energy to potential energy and the potential energy must therefore be $PE = \frac{1}{2}m\omega^2 A^2$ due to conservation of energy.

Therefore, we can conclude that the *total energy* is:

$$\boxed{\text{Total Energy} = \frac{1}{2}m\omega^2 A^2}$$

and the only thing that changes is how the total energy is distributed between kinetic energy and potential energy as shown in the image below.



4 *Simple Harmonic Systems*

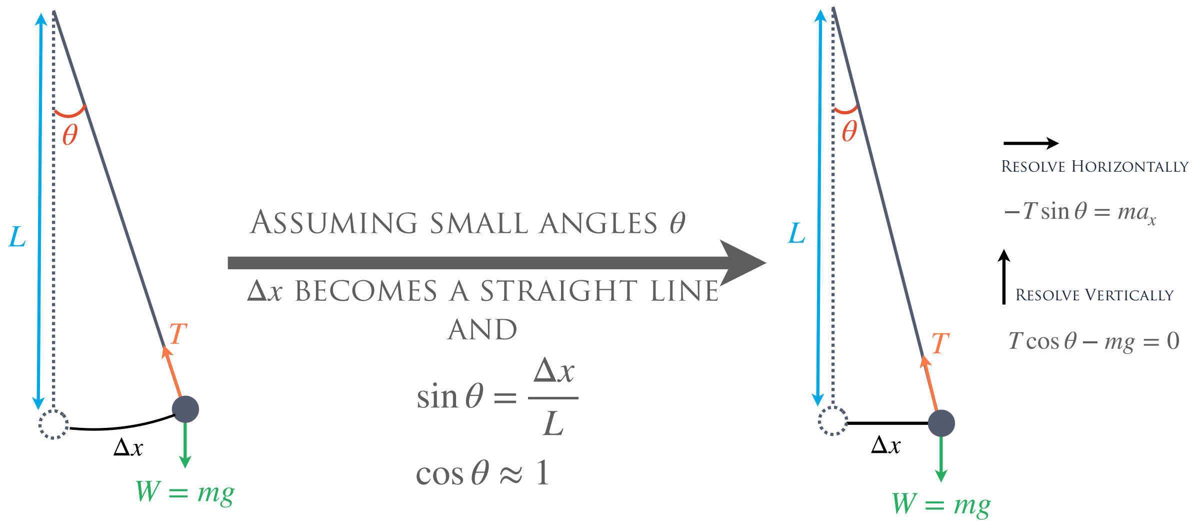
Learning Objectives and Prior Knowledge

In this section you'll learn:

- About the two most important simple harmonic systems: the pendulum bob and the mass-spring system. These systems can be used as "model systems" for many other, more complicated, oscillating systems.
- How to derive the expression $T = 2\pi\sqrt{\frac{l}{g}}$ for the time-period of a pendulum system.
- How to derive the expression $T = 2\pi\sqrt{\frac{m}{k}}$ for the time-period of a mass-spring system.

4.1 *The Pendulum Bob*

Let's analyse the forces on a simple pendulum to see whether $F = k\Delta x$: if it is, then the simple pendulum will oscillate simple harmonically.



Having resolved the forces, we can now combine our force analysis with the small angle approximations. In the vertical direction $T\cos\theta = mg$ which becomes simply $T = mg$ since $\cos\theta \approx 1$ for small angles.

In the horizontal direction:

$$
\begin{aligned}
-T\sin\theta &= ma_x \\
-T\frac{\Delta x}{L} &= ma_x \\
\implies a_x &= -\frac{T\Delta x}{mL}
\end{aligned}
$$

and now using the fact that $T = mg$ we obtain:

$$
\begin{aligned}
a_x &= -\frac{T\Delta x}{mL} \\
a_x &= -\frac{mg\Delta x}{mL} \\
\implies a_x &= -\frac{g}{L}\Delta x
\end{aligned}
$$

Therefore, $a_x = -k\Delta x$ with $k = \frac{g}{L}$ which fulfils the criteria for simple harmonic motion.

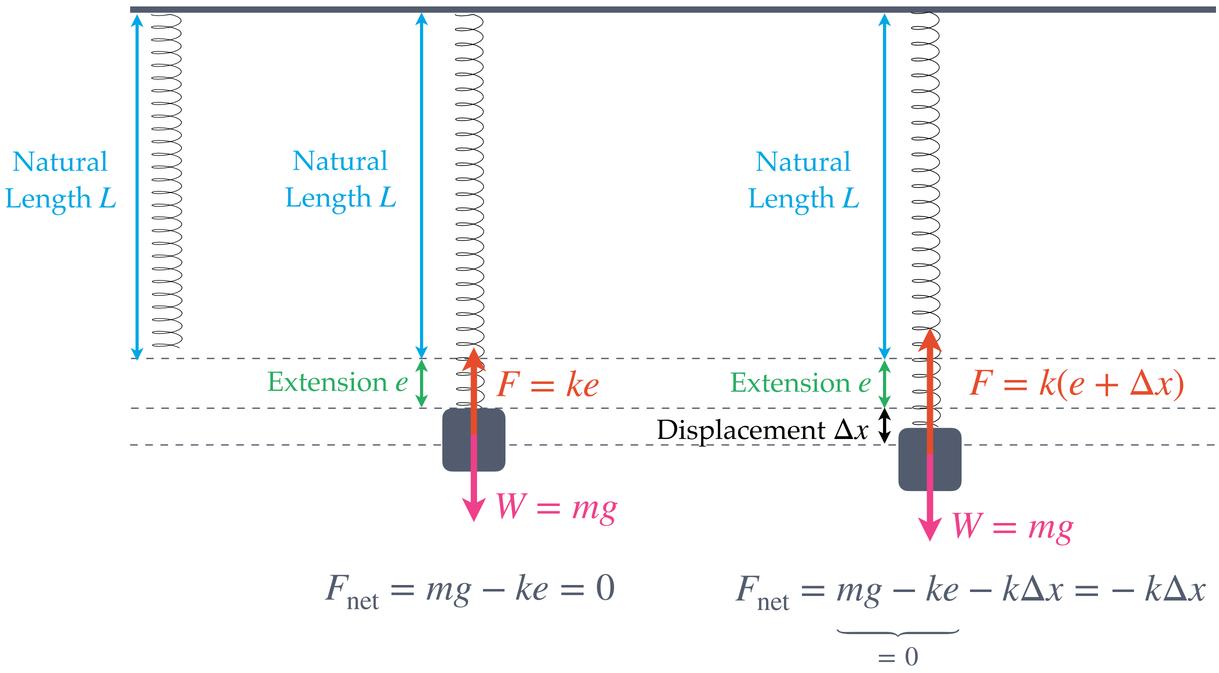
Since we also know that for any simple harmonic oscillator $a = -\omega^2\Delta x$ then we can find the angular frequency and time-period of the pendulum bob:

$$
\boxed{
\begin{aligned}
a_x = \frac{g}{L}\Delta x &= -\omega^2\Delta x \\
\implies \frac{g}{L} &= \omega^2 \\
\implies \omega &= \sqrt{\frac{g}{L}} \\
\implies T &= 2\pi\sqrt{\frac{L}{g}}
\end{aligned}
}
$$

Notice that the time-period T does not depend on the amplitude of the swing - no matter how high you release the pendulum, it still takes the same amount of time to complete an oscillation - which might seem a surprising result. The reason for this is because as we increase the amplitude, the restoring force F gets larger because the displacement from equilibrium Δx is larger and so the bob goes faster.

4.2 *A Mass-Spring System*

Imagine a spring with natural length L that has a mass m attached such that the spring extends by an amount e until it is in equilibrium. In this case, $F = ke = mg$ as shown in the image below.



If we then pull down the mass such that it is displaced a small amount Δx from its equilibrium position, it will experience an additional force $F = -\Delta x$ and so we immediately achieve the required condition for circular motion.

Comparing with $F = ma$, we obtain:

$$\boxed{\begin{aligned} ma &= -k\Delta x \\ \implies a &= -\frac{k}{m}\Delta x \end{aligned}}$$

Comparing with the acceleration of simple harmonic oscillators $a = -\omega^2 \Delta x$ we find:

$$
\begin{aligned}
a = -\frac{k}{m}\Delta x = & -\omega^2 \Delta x \\
\implies \frac{k}{m} = & \ \omega^2 \\
\implies \omega = & \sqrt{\frac{k}{m}} \\
\implies T = & \ 2\pi\sqrt{\frac{m}{k}}
\end{aligned}
$$

Challenge Questions 4-1 - Simple Harmonic Systems - A Floating Ball

1. A ball of density ρ_s floats on a fluid of density ρ_f such that exactly half of the ball is submerged.
   (a) Draw a free-body force diagram of the system.
   (b) Show that the upthrust force is given by $U = \frac{2}{3}\rho_f \pi r^3 g$
   (c) Show that the weight W can be expressed in terms of the density of the ball as $\frac{4}{3}\rho_s \pi r^3 g$
   (d) Hence show that the density of the ball is related to the density of the fluid via the following relation $\rho_s = \frac{1}{2}\rho_f$.
2. The ball is now displaced a tiny amount $\Delta x << r$ below the surface.
   (a) By approximating the additional volume displaced as a disc, show that the additional volume displaced by the ball is $V = \pi r^2 \Delta x$
   (b) Show that the net restoring force acting on the ball is $F = -k\Delta x$ where $k = \rho_f \pi r^2 g$ and therefore that the motion is simple harmonic.
   (c) Hence show that the time-period T of the ball's oscillations is given by

$$T = 2\pi\sqrt{\frac{2r}{3g}}$$

Challenge Questions 4-2 - Simple Harmonic Systems - Water in a U-Tube

A U-tube is filled with liquid of density ρ such that the liquid in both arms reaches a height h as shown in the diagram.

1. Suction is applied to one arm of the tube such that one column rises by an amount y and the other column sinks by an amount y.
   (a) Write down the pressure P acting on the point $h - y$ on the raised column.
   (b) Hence show that the force acting on the raised column $F = \frac{2mgy}{h}$
   (c) Hence show that the time-period of oscillations of the liquid T is given by

$$T = 2\pi\sqrt{\frac{h}{g}}$$

.

Challenge Questions 4-3 - Simple Harmonic Systems - Gravity Tubes

A hole is drilled from one side of the Earth to the other, passing through the centre of the Earth.

In this question, we will prove that the motion of a ball dropped into the hole is simple harmonic and find its time-period T.

1. Write down the gravitational field strength on the surface of the earth g_s in terms of its radius R_E and mass M_E.
2. Show that the gravitational field strength on the surface of the earth g_s can be written in terms of its density ρ as $g_s = \frac{4}{3}G\pi\rho R_E$.
3. The gravitational field strength at some point r inside the surface of the earth, g, is caused only by the mass of inside a sphere of radius r.
   (a) Show that the mass inside the sphere of radius r, M_r, can be expressed as
   $$M_r = M\frac{r^3}{R_E^3}$$
   .
   (b) Hence show that the gravitational field strength g at radius r can be expressed as
   $$g = GM\frac{r}{R_E^3}$$
   (c) Show that the gravitational field strength g at radius r can be expressed alternatively as
   $$g = g_s\frac{r}{R_E}$$
   (d) Hence show that the force F acting on the ball is of the form $F = -k\Delta x$ with
   $$k = \frac{mg_s}{R_E}$$
   .
   (e) Hence show that the time-period T of the oscillations is given by
   $$T = 2\pi\sqrt{\frac{R_E}{g_s}}$$
4. The radius of the Earth is $R_E = 6400$ km.
   (a) A person drops a ball through the hole. Calculate how long it takes for the ball to return to the person.

Challenge Questions 4-4 - Simple Harmonic Systems - Charged particles

For this question, you need to know the binomial expansion from further maths $(1+x)^n = 1 + nx + n\frac{(n-1)}{2!}(x^2)$ where $x << 1$ and Coulomb's law $F = \frac{kQq}{r^2}$ from the electric fields chapter.

1. A small positive charge $+q$ is placed in the middle of a line of length r joining two equal positive charges $+Q$.
   (a) If the positive charged is displaced a small amount $\Delta x << r$, prove that the motion of the charge is simple harmonic.
   (b) Find the time-period T of the motion.
   (c) If the small charge were instead negative, would the charge still oscillate simple harmonically? Explain.

Challenge Questions 4-5 - Simple Harmonic Systems - Atomic Lattices

To model the interaction of atoms, chemists and physicists often use the *Lennard-Jones* potential given by the following formula:

$$V_{\mathrm{LJ}}(r) = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

This potential tells you the potential energy between two particles as the distance r between them varies.

1. Show that distance at which the potential energy is a minimum (the equilibrium distance) is $r_m = 2^{\frac{1}{6}}\sigma$ and that the potential energy at this distance is $V = -\epsilon$.
2. Sketch the graph of this function $V(r)$.
3. Using ideas about increases and decreases in potential energy, write down the range of distances for which the atoms attract, and the range for which they repel.
4. There is a general relationship in physics between force and energy: $F = -\dfrac{dV}{dr}$.
   (a) Find a formula that expresses the force $F(r)$ as a function of distance r between the atoms.
   (b) Sketch a graph of the force F against distance r.
5. Using the binomial expansion again, show that an atom displaced a small amount $\Delta x << r$ from the equilibrium distance $r_m = 2^{\frac{1}{6}}\sigma$ will oscillate simple harmonically.
6. Find the time-period T of the motion.

5 *Damping*

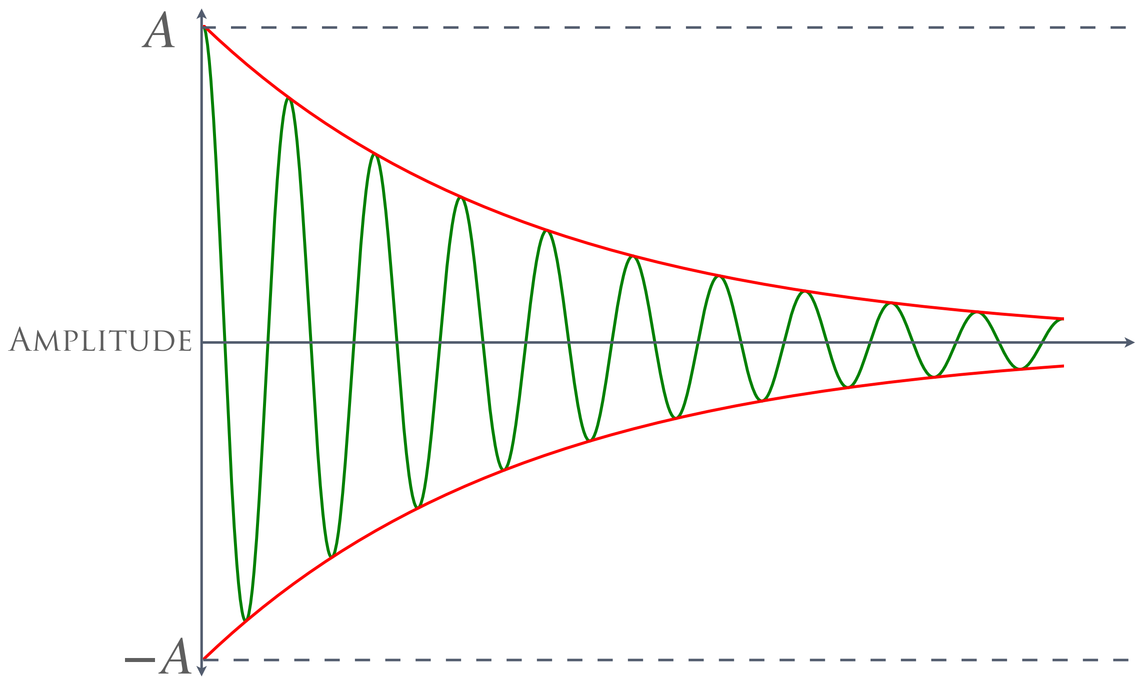
Learning Objectives and Prior Knowledge

In this section you'll learn:

- How to define damping.
- The difference between underdamped, overdamped, and critically damped systems.
- How to plot the graphs for underdamped, overdamped, and critically damped systems.

Most systems oscillating simple harmonically will experience *damping*. Damping is where the oscillating system loses energy because of resistive forces such as friction and drag.

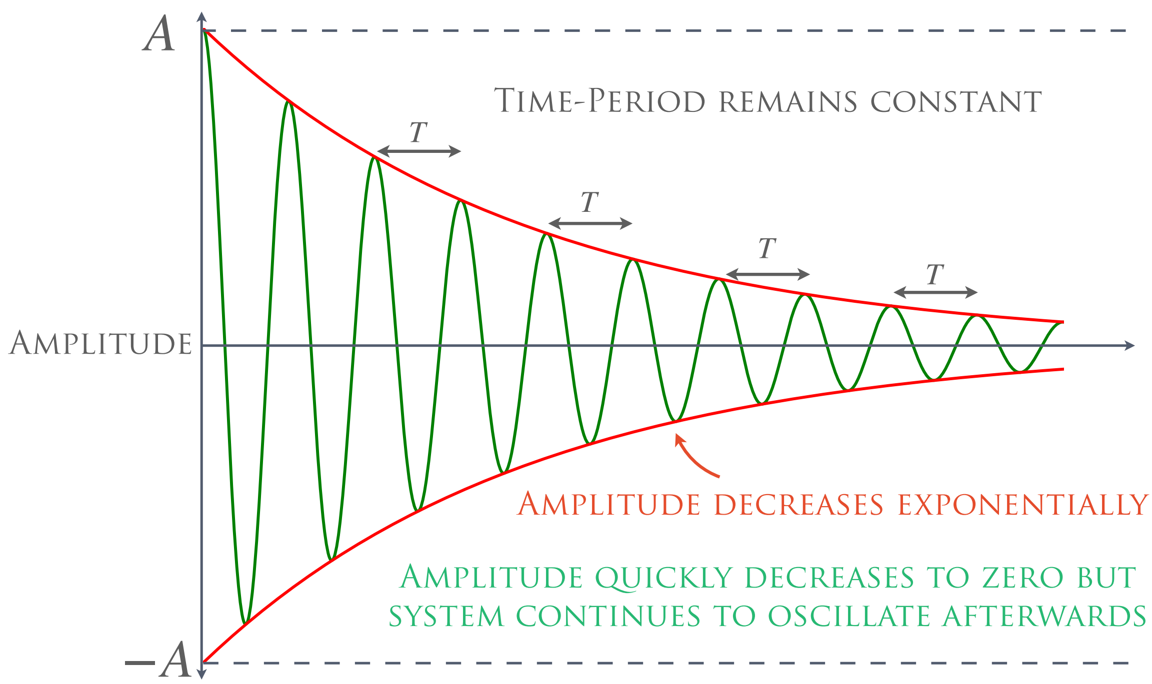
As the system loses energy, its **amplitude decreases**. The oscillations of a system might look similar to the image below:



The amplitude decreases because the total energy of a harmonic oscillator is $E = \frac{1}{2}m\omega^2 A^2$ and so decreasing energy implies decreasing amplitude provided the mass m and angular frequency ω remain the same (which they do - remember frequency is independent of amplitude for a simple harmonic oscillator).

However, the image above is only one example of a damped system known as an *underdamped* system

5.1 *Underdamping*

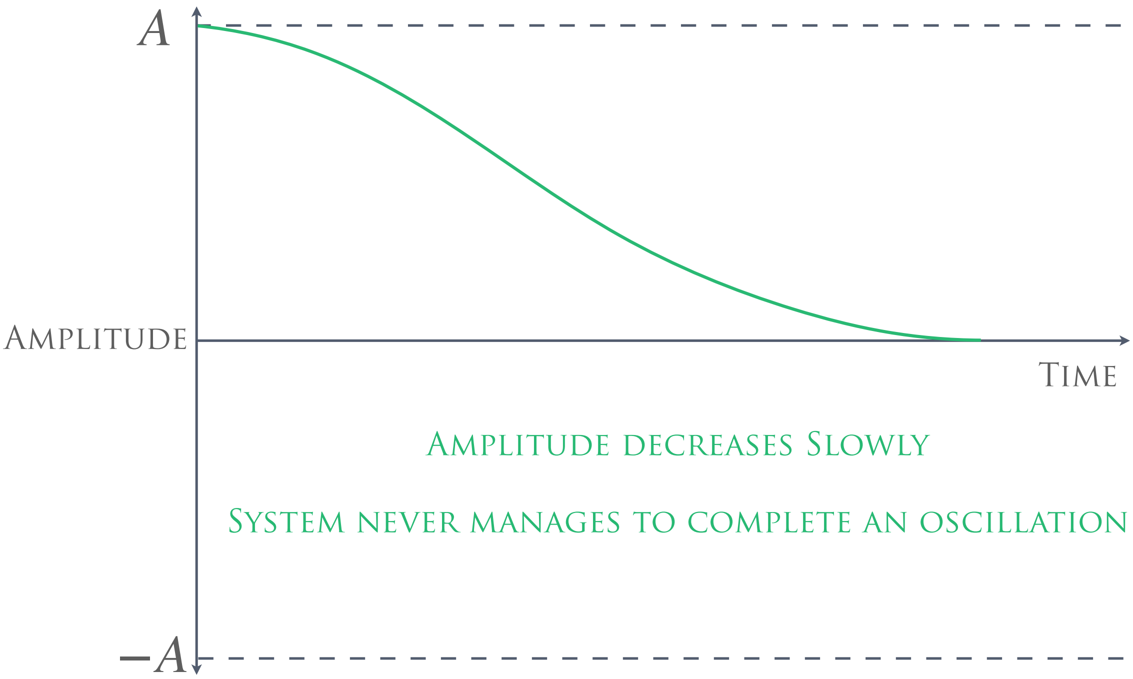


In an underdamped system, the oscillations quickly reach 0 amplitude but the oscillations continue afterwards with slowly reducing amplitude.

It turns out that the amplitude decays exponentially.

To see this in action, take any kind of pendulum on a string. The oscillations are usually underdamped because the only resistive force is a small drag force on the pendulum bob and so it takes many oscillations to decay to zero amplitude.

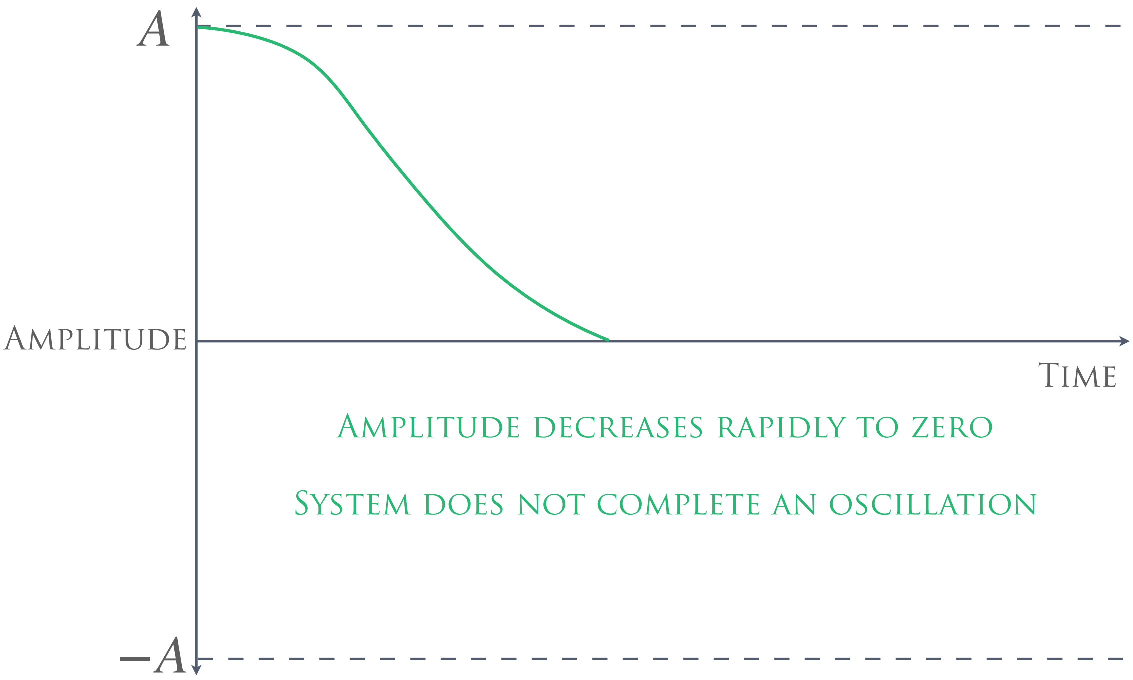
5.2 *Overdamping*



In overdamping, the system has no chance to even complete a single oscillation since the resistive forces are so large, however, reaching 0 amplitude takes longer than if the system is critically damped.

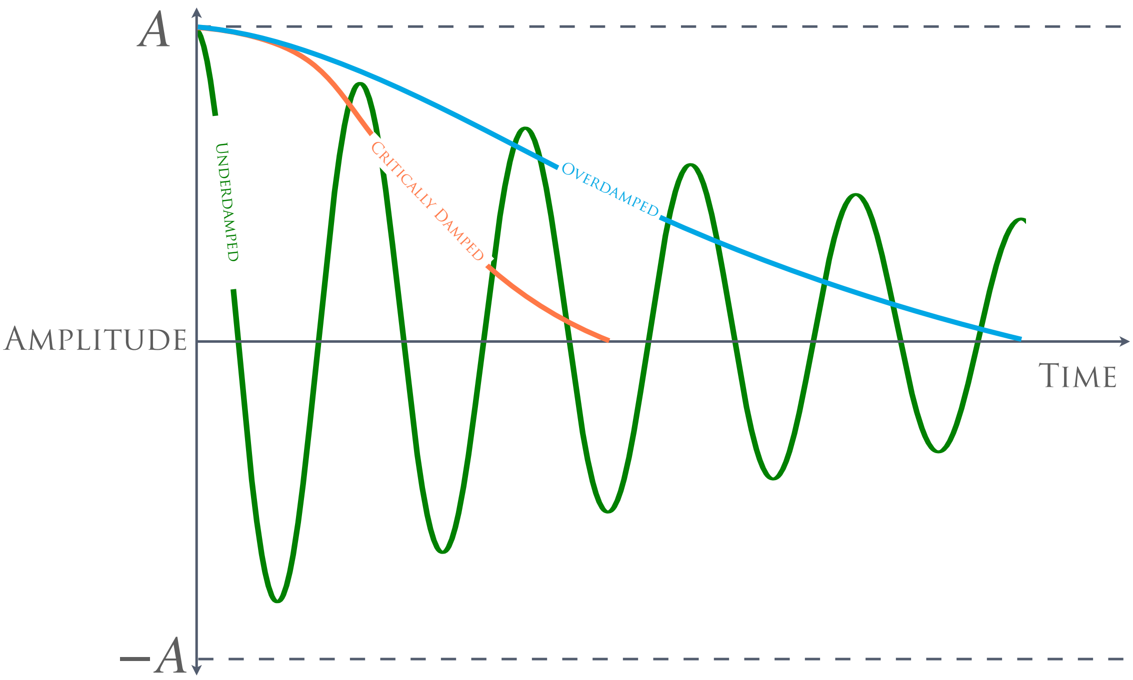
If you were to push against a pendulum bob with your finger once you've let go of it to slowly bring it to the equilibrium position this would be an example of overdamping.

5.3 *Critical Damping*



Critical damping brings the oscillations to 0 amplitude in the fastest possible time *without an overshoot and additional oscillations*.

5.4 *Damping Comparisons*



A **critically damped** brings the amplitude of oscillations to 0 in the shortest amount of time *without any overshoot and continued oscillations*. In practice to achieve critical damping you need a system where the resistive forces start off small and then get larger as the object undergoing simple harmonic motion accelerations and gains speed.

As you can see from the graph above, an **underdamped** system actually brings the amplitude of oscillations to 0 quicker but then the oscillations continue after it reaches zero amplitude. An example of an underdamped system would be a pendulum bob that has a small radius, because there will be drag but it will be small due to the small surface area.

An **overdamped** system also brings the amplitude of the oscillations to 0 but takes much longer than a critically damped system. An example of this would be applying a constant large force with your finger onto a pendulum bob as it hangs freely, slowly bringing it to the equilibrium position.

6 *Forced Oscillations and Resonance*

Learning Objectives and Prior Knowledge

In this section you'll learn:

- The difference between free oscillations and driven oscillations.
- How to describe the conditions for resonance to occur.
- How damping affects resonance.

Imagine a young child on a swing. The child swings backwards and forwards undergoing simple harmonic motion, just like a pendulum bob. If there was no friction or air resistance, they would continue oscillating like this forever. This is called *free oscillations*: the system is oscillating at its *natural frequency* without any interference.

But there is also an opportunity for us to *force* or *drive* the oscillations: an adult or sibling can push the child in the swing, adding energy to the system, to increase the height the child reaches while swinging.

You'll notice that in order to increase the amplitude of the oscillation in the most efficient way, you want to time your energy input with the natural frequency of the swing: you want to push the child when they're at the top of the swinging motion. This is a general property of driven systems.

In this section we will look at general properties of systems that are being *driven*.

6.1 *Natural Frequency*

Every oscillating system has a *natural frequency* which is the frequency the system will oscillate at once it's been displaced from equilibrium.

We have already looked at the pendulum system in detail, where the natural frequency is $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ and the mass-spring system where the natural frequency is $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

Other - more complicated - systems have their own natural frequency that can be measured even if they can't be easily calculated such as in the case of the pendulum or spring-mass system.

6.2 *Driving Frequency*

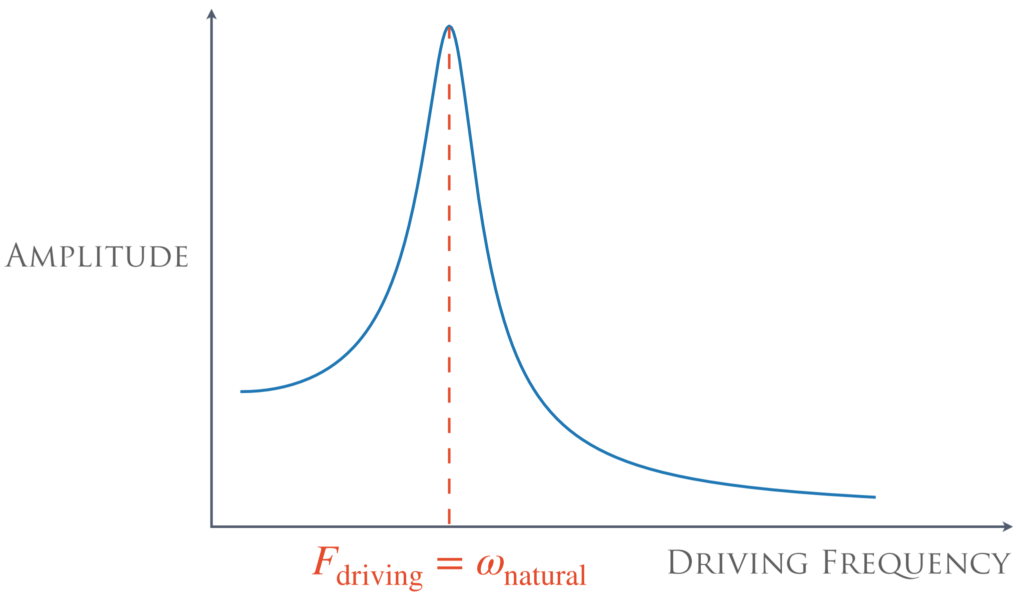
The driving frequency is the frequency of energy inputs: for the child on the swing, it's the frequency that we push the swing. For a guitar string, it might be the frequency with which we pluck the string, and for a bell, it would be how frequently we hit the bell.

6.3 *Resonance*

Resonance is phenomena that occurs whereby the amplitude of oscillations in a system increases.

If there is no damping or minimal damping, maximum resonance occurs when the *driving frequency* equals the *natural frequency* of a system.

It is still possible to achieve significant resonance when the driving frequency does not exactly match the natural frequency: as the graph below shows, there is a broad "peak" where the amplitude of the oscillations are very large even if the two frequencies do not exactly match.



For driving frequencies less than the natural frequency, you sometimes push the system to go faster but sometimes push it to go slower, too, because the frequency aren't matched. So overall, the amplitude doesn't continually build up and reinforce.

When you match the natural frequency, every push or pull matches the natural motion of the object, and so you add energy to the system and the amplitude keeps increasing until the energy losses from resistive forces match your energy inputs.

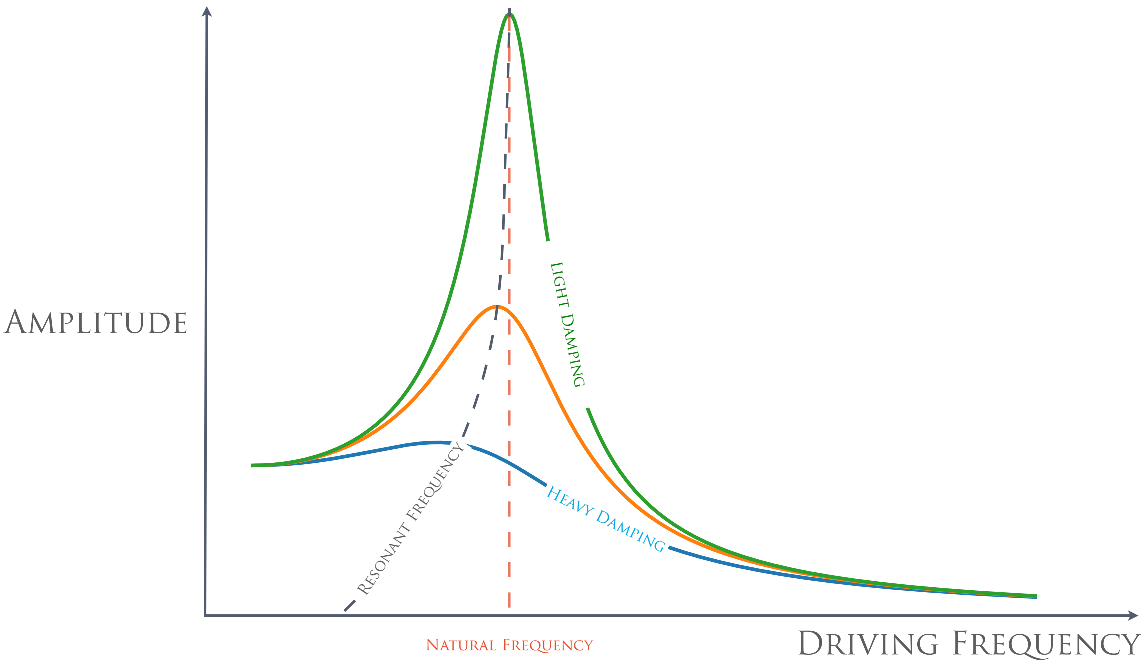
When the driving frequency is greater than the natural frequency, again sometimes you push in the direction of motion and add energy and sometimes against the direction of motion and take energy away, leading to low overall amplitudes. However, the reason the amplitude is overall lower than when the driving frequency is less than the natural frequency is because the system has a very difficult job of "getting going" at all when you are pushing and pulling it so frequently.

6.4 *Damping and Resonance*

Though there are uses and applications of resonance, there can also be problems with resonance. For example, several bridges have become unstable and even been destroyed because of resonance.

One way to *decrease* resonance is to *increase* the damping on the system.

The graph below shows the effect of increasing the damping on the system.



Increasing the damping:

- Reduces the amplitude of the oscillations.
- Increases the range of frequencies that cause resonance (broadens the peak).
- Shifts the resonant frequency to values smaller than the natural frequency of the system.