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CIRCULAR MOTION

NAME: CLASS:

Contents

1 *Uniform Circular Motion*

Learning Objectives and Prior Knowledge

In this section you'll learn:

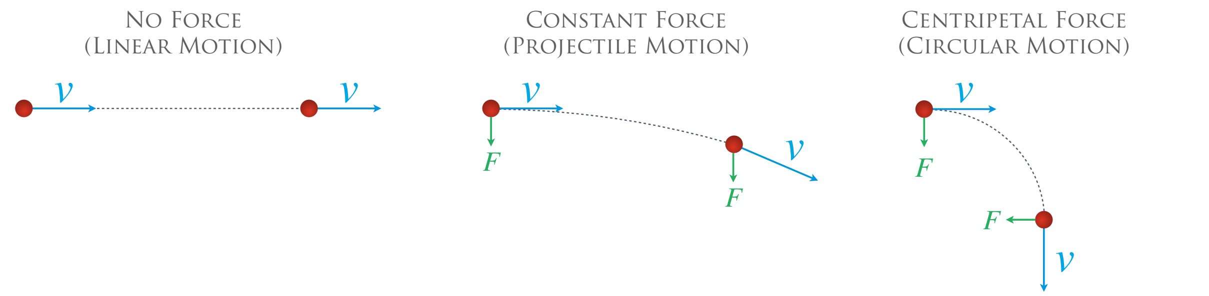
- What a centripetal force is and why it is required to keep an object moving in a circle.
- What tangential velocity is and how it is related to the centripetal force
- What the time-period T and frequency $f = \frac{1}{T}$ of circular motion refer to.
- What the radius r of circular motion refers to.

Imagine you're walking down the street side-by-side with a friend. You're walking in a straight line at a constant speed, we say we have a *constant velocity*. Unless something changes (like collapsing in exhaustion) you'll keep walking at this constant velocity forever.

Now imagine that your evil friend plays a prank and slightly leans into your shoulder as you're walking. You'll keep moving forward at the same speed, but now you'll veer slightly to the side as you walk, tracing out a parabolic arc. If you're not careful, you'll end up in the road!

Now finally, imagine that your friend leans into you as you walk, but as you drift slightly to the side, they change the angle at which they're pushing you so they push you slightly backwards and try to slow you down. If you kept walking and your friend kept adjusting their push to constantly try to push you sideways *and* backwards, you would end up walking in a circle at constant speed. This is what we call *uniform circular motion*.

The diagram below illustrates this:



With no force applied to the object, the object continues moving in a straight line with constant speed (i.e. constant velocity)

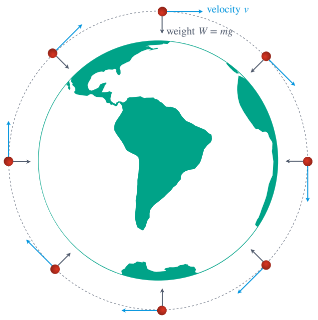
With a constant force initially at 90° to the velocity, the horizontal component of velocity stays constant but the vertical component increases, causing projectile/parabolic motion.

If the force changes direction so that it always points towards the centre of a circle (centripetal force), the velocity is always 90° to the centripetal force so we call it **tangential** velocity.

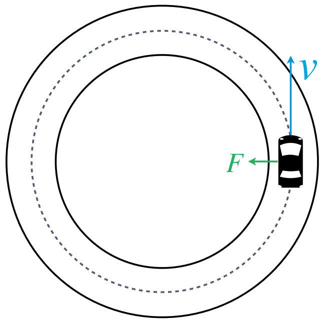
So this is how we can get circular motion to occur: we simply apply a force that is always perpendicular to the velocity and directed towards the centre of path the object travels. This is the origin of the word *centripetal* force: centripetal means "centre-seeking".

Although it might seem to you that the chances of a force being applied that is always perpendicular to the velocity are low, the reason we are so interested in studying circular motion in detail in physics is because this situation - in which a force is perpendicular to the velocity of an object - occurs surprisingly frequently. The diagram below shows some examples:

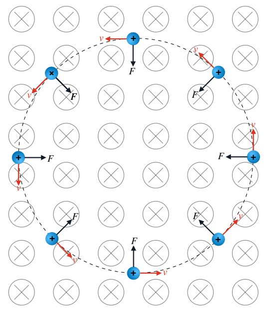
Orbits



Cars turning around bends



Charged Particles in magnetic fields



For other situations, even if the object doesn't move in a perfect circle we can often *approximate* its path as a circle, and we can apply the tools of circular motion to parts of its motion.

The study of circular motion involves developing some mathematical tools and formulae for describing and understanding circular motion, and then applying those to handful of specific scenarios.

1.1 *Frequency*

In circular motion, an object follows a circular path that has some radius *radius* r. Eventually, the object will get back to where it started, and the time it takes to do this is called the time-period T. We can therefore define a frequency of motion:

$$\boxed{f = \frac{1}{T}}$$

which is simply the number of trips around the circle the object makes each second.

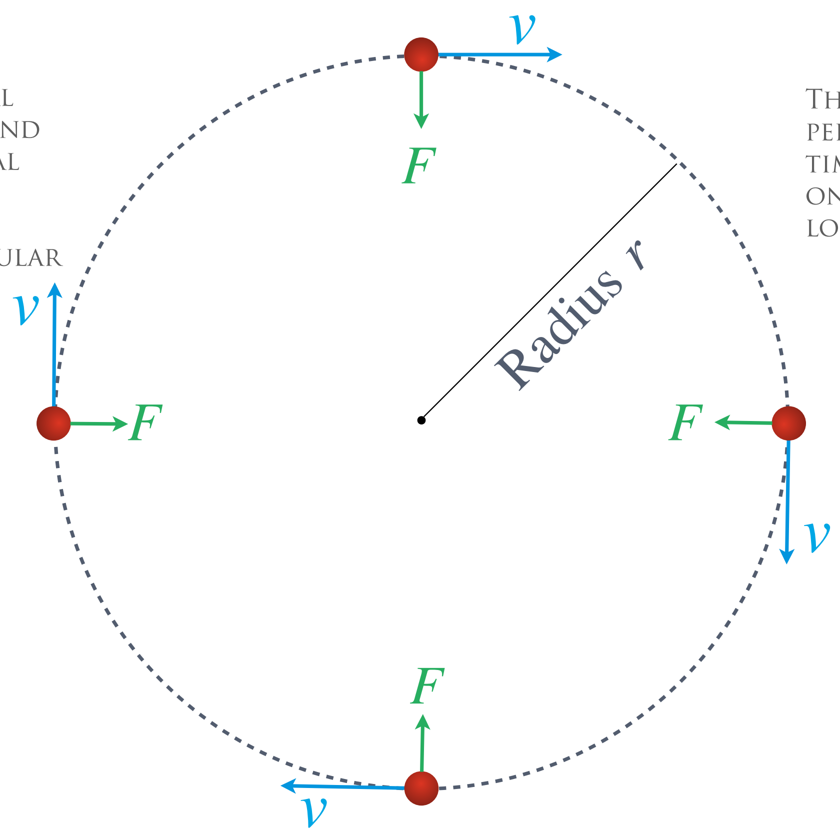
A common scenario is to be given the time taken for a number of full circles to occur, for example "A car travels around a roundabout 10 times in 1 minute". To find the frequency in this case, which is the time taken for a single complete circle, you can simply divide the number of complete loops N by the total time taken t:

$$\boxed{f = \frac{N}{t}}$$

1.2 Velocity and Speed

THE VELOCITY OF THE OBJECT CONSTANTLY CHANGES, BUT THE SPEED REMAINS CONSTANT

TANGENTIAL VELOCITY AND CENTRIPETAL FORCE ARE ALWAYS PERPENDICULAR



THE TIME-PERIOD T IS THE TIME TAKEN FOR ONE COMPLETE LOOP OCCUR

Finally, we need to understand a bit more about the velocity and speed of an object undergoing circular motion.

When an object moves in a circle, it has a constant force applied to it, and therefore its *velocity* must *constantly change*. However, we remember that velocity is the combination of *speed* **and** *direction*.

In *uniform* circular motion, the *speed* is *uniform* (constant) but the direction of the object constantly changes, so it is the change in direction that means the velocity is constantly changing, not the change in speed.

We can understand this further by thinking about work. We remember from year 12 the work energy theorem: any work done on an isolated object must increase its kinetic energy. But work is defined as $W = Fd$ where the force F is *in the direction* of the displacement. Since the force is always perpendicular to the tangential velocity and therefore also the displacement of the object, no work is done on the object, and therefore its kinetic energy and hence speed cannot change.

There's a physics joke about this that you can add to your repertoire.

A student is on her driving test, when the examiner says: "Once you exit the roundabout, hit the accelerator." The student, dismayed, says "Which one? There are three!"

If the speed is somehow changing while an object is moving in a path, it is called *non-uniform circular motion*, which sometimes crops up in A-level questions, particular those involving an object moving in a vertical circle under the influence of gravity, and we will deal with those later in this book.

2 The Angular Language of Circular Motion

Learning Objectives and Prior Knowledge

In this section you'll learn:

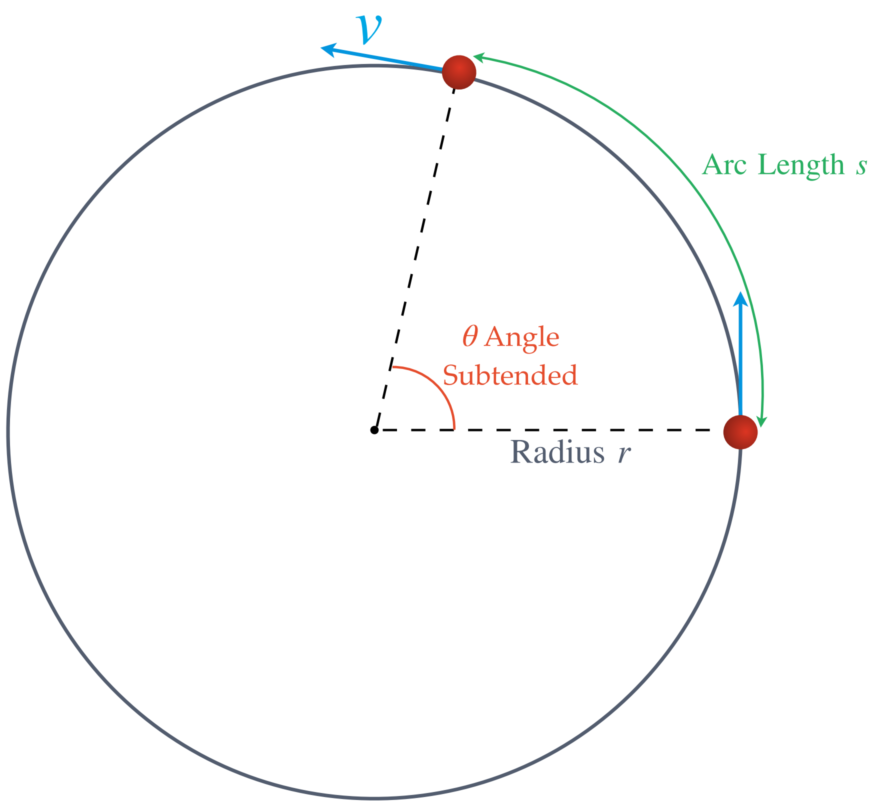
- What the term angular displacement θ means.
- What the term angular velocity ω means.

When dealing with problems in circular motion, we are often interested in how far around the circle the object has moved and also how fast it is moving.

It turns out that expressing these quantities in terms of how much *angle* it has rotated through (angular displacement) and how much angle it moves through *per second* (angular velocity) can often be more useful than dealing with tangential velocity v and arc-length s.

2.1 Angular Displacement

When an object covers a distance s between two points on a circular arc with radius r, it *subtends* or "moves through" a certain angle as shown in the image below.



We can calculate this angle using the arc length formula $s = r\theta$ you have learnt in maths:

$$\theta = \frac{s}{r}$$

This angle is measured in *radians*

2.2 Radians

If you've never encountered them before, or only learnt them in A-level maths, the radian seems like a mysterious unit. However, just think of radians as the number of times you have moved through a distance equal to the radius of the circular path r.

You can fit 3.14 lines of length r around a semi-circle and 6.28 around a full circle, which is why we have the relationship $\pi^c = 180°$ and $2\pi^c = 180°$.

2.3 Angular Velocity

This then allows us to define an *angular velocity* ω as follows. We know that the velocity is just distance over time $v = \frac{s}{t}$ but now we can write the arc-length distance s in terms of the angular displacement θ to get:

$$\begin{aligned} v &= \frac{s}{t} \\ v &= \frac{r\theta}{t} \\ v &= r\omega \end{aligned}$$

where we define

$$\omega = \frac{\theta}{t}$$

as the *angular velocity*.

Intuitively this is simply the amount of angle (in radians) that the object has "moved through" or subtended each second, and therefore has units of *radians per second* or rad s^{-1}.

Finally, we can link the angular velocity to the frequency f and time-period T.

Since we know that a complete circle takes T seconds and we also know that it travels 2π radians or $2\pi r$ in that time, we know that its velocity is $v = \frac{2\pi r}{T}$. But we also know that $v = r\omega$ and so it follows that:

$$\omega = \frac{2\pi}{T}$$

This simply tells us that the angular speed is the amount of a circle (measured in radians) the object travels through each second, which is the definition of angular velocity we discussed earlier.

3 Forces and Acceleration in Circular Motion

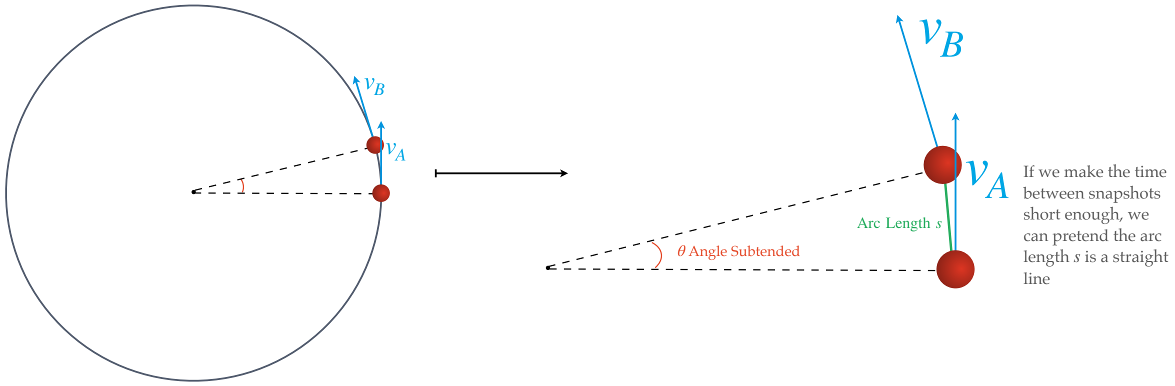
Learning Objectives and Prior Knowledge

In this section you'll learn:

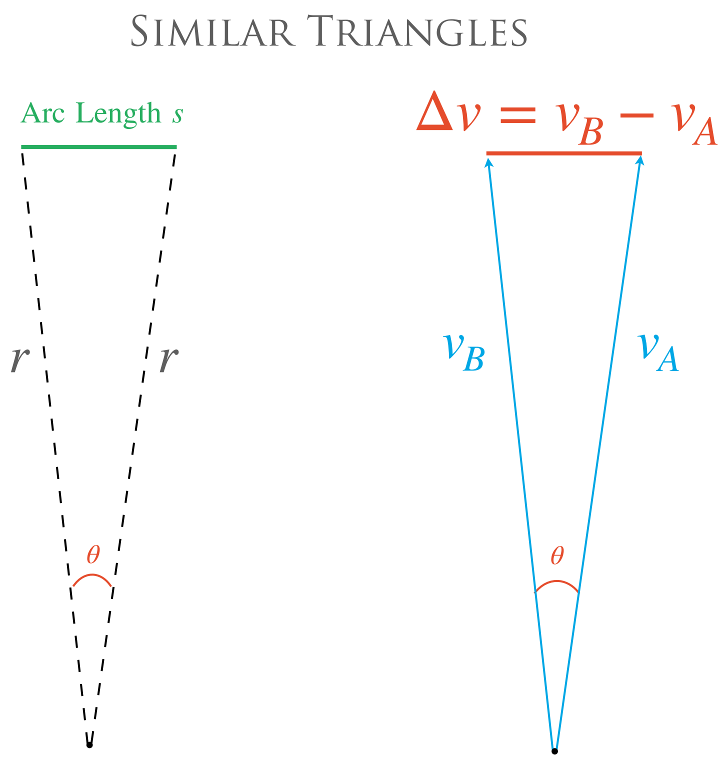
- How to derive the relationship between the acceleration a to the tangential velocity v when an object moves in a circle.
- How to apply $F_{\text{net}} = ma$ to circular motion problems.

3.1 Acceleration

We know that acceleration is the rate of change of velocity, $a = \frac{\Delta v}{\Delta t}$, so we need to see how the velocity changes as an object moves around a circle. Let's look at our angular displacement diagram again but this time make it so that we look at snapshots of the velocity a very short time.



If we make the time period Δt between snapshots short enough, we can approximate the arc length s as a straight line. This allows us to set up two similar isosceles triangles as follows:



It is very hard to visualise why these two triangles are similar. If you struggle to see it, just accept it for now

Because the two triangles are similar, the ratio $\frac{s}{r}$ is equal to the ratio $\frac{\Delta v}{v_A} = \frac{\Delta v}{v_B}$ which we can just simplify to $\frac{\Delta v}{v}$ since the speed isn't changing (the magnitude of $v_A = v_B$). Therefore:

$$\frac{s}{r} = \frac{\Delta v}{v}$$

Finally, since we know that the distance travelled is related to the velocity via $s = v\Delta t$ we obtain:

$$\begin{aligned} \frac{s}{r} &= \frac{\Delta v}{v} \\ \implies \frac{v\Delta t}{r} &= \frac{\Delta v}{v} \\ \implies \frac{\Delta v}{\Delta t} = a &= \frac{v^2}{r} \end{aligned}$$

We can get this relationship in terms of the angular velocity ω by substituting $v = r\omega$ to obtain:

$$a = \frac{v^2}{r} = r\omega^2$$

3.2 Force

Now that we know how the acceleration relates to the tangential velocity v and the angular velocity ω we can easily related this to the force using Newton's second law $F_{\text{net}} = ma$:

$$F = ma = m\frac{v^2}{r} = mr\omega^2$$

It is important to realise what this force represents: **it is not an additional force** that is present for circular motion. It is simply new way of writing $F_{\text{net}} = ma$ specifically for the case of circular motion.

In order to use this equation for an object moving in a circle, you still need to find what is causing the net force F_{net} on the object that causes it to move in a circle - the centripetal force - in the first place.

4 Problem Patterns in Circular Motion

While the formulae used in circular motion are short and straightforward to derive - we have covered every formula we need to solve any circular motion problem already - the difficulty in circular motion problems is in *using* and *applying* them. Most textbooks will just give you the formulae and then say "off you go", assuming that it is obvious how you apply the formulae in each situation, which it most definitely is not.

In this section, we'll go through many specific problems in detail: all of the ones I have seen in A level past paper questions, and several other interesting ones that will help you understand and practice solving circular motion problems. Some of these problems may themselves one day be used in your A-level paper.

The main problem solving strategy for solving circular motion problems is the following:

1. Draw a **free-body force diagram** for the scenario given and **identify the centripetal force** that causes circular motion.
2. Apply the circular motion formulae to calculate the quantities asked for.

The simplest scenarios to deal with are when an object is moving in a *horizontal plane* perpendicular to the force of gravity. We'll start with these to get familiar with the basic problem solving techniques before moving on to circular motion in the *vertical plane* where weight plays a role. Finally, we'll look at some of the most challenging circular motion problems in the vertical plane that combine conservation of energy with forces.

4.1 Masses on Turn-Tables

A common piece of equipment used to demonstrate circular motion is the turn-table. Essentially it's just a disc that rotates. Exam questions will often feature questions to do with masses on turn-tables because it is a relatively simple circular motion problem and it is easy to change the speed at which the turn-table rotates.

The first thing we must do is to think in terms of forces: the mass is moving in a circle, which means it is constantly accelerating because its velocity is constantly changing: remember, the speed stays constant but the *direction* of the velocity changes.

Therefore, there must be a *centripetal* force acting on the mass.

In this particular case, the force that holds the mass on the turn-table as it rotates is (static) friction.

A simple demonstration of a mass on a turn-table reveals that if the turn-table rotates too fast, the mass will "fly off" in a straight line rather than undergo circular motion. We might be interested in asking questions such as what is the maximum angular speed the turn-table can rotate for the mass to stay on.

Worked Example 4-1 - Calculating the frictional force on a mass on a turntable

Q: Calculate the frictional force on two 100 g masses on a turntable with angular velocity $\omega = 2$ rad s^{-1}. One of the masses is $r = 5$ cm from the centre and the other is $r = 10$ cm.

A: Since the angular velocity is how much *angle* the masses travel through each second, the angular velocity is the same for both masses $\omega = 2$ rad s^{-1}. The masses $m = 0.1$ kg are also the same, and so the only difference between the two masses is that distance from the centre of the turn-table.

Using $F = mr\omega^2$ we can see that for the 5 cm mass:

$$\begin{aligned} F &= mr\omega^2 \\ &= 0.1 \times 0.05 \times 2^2 \\ &= 0.02 \text{ N} \end{aligned}$$

and for the 10 cm mass:

$$\begin{aligned} F &= mr\omega^2 \\ &= 0.1 \times 0.1 \times 2^2 \\ &= 0.04 \text{ N} \end{aligned}$$

We can see that the further away mass has a higher frictional force acting on it.

It is instructive to ask *why* this is the case: the reason is that the tangential velocity $v = r\omega$ is higher for the further away mass, and therefore a larger centripetal force is required to *change* that velocity.

Practice Questions 4-1 - Calculating the frictional force on a mass on a turntable

1. During an experiment, it is found that a 5 kg mass on a turn-table can withstand a maximum angular velocity of $\omega = 10$ rad/s when it is at a distance $r = 5$ cm from the centre of the turntable before sliding off the turntable.
   (a) Calculate the tangential velocity of the mass at this distance.
   (b) Calculate the frictional force acting on the mass at this distance.

   (c) Draw the trajectory of the mass if the angular velocity is increased to $\omega = 11$ rad/s

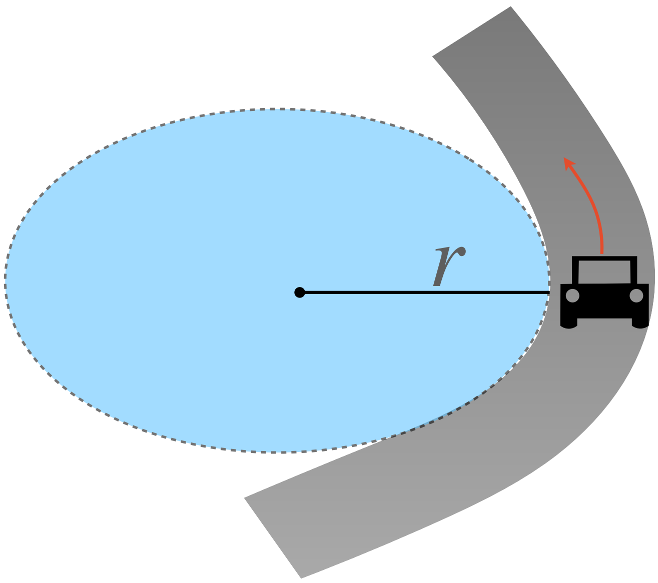
2. The turntable suddenly gets wet and the frictional force between the mass and the turntable halves.
   (a) Calculate the distance from the centre at which the mass will not slide off.
   (b) Alternatively, the turntable can be slowed down: calculate the angular velocity ω at which the mass will not slide off at the original distance of $r = 5$ cm.
3. With reference to the equation $v = r\omega$, explain why a higher frictional force is required to keep the mass on the turn-table as it is moved outwards from the centre.
4. A friend suggests that adding another 5 kg mass on top of the 5 kg mass at a distance $r = 5$ cm from the centre will allow the masses to reach an angular velocity $\omega = 20$ rad/s before sliding off. With reference to the equation $f_s = \mu_s N$, explain whether your friend is right.

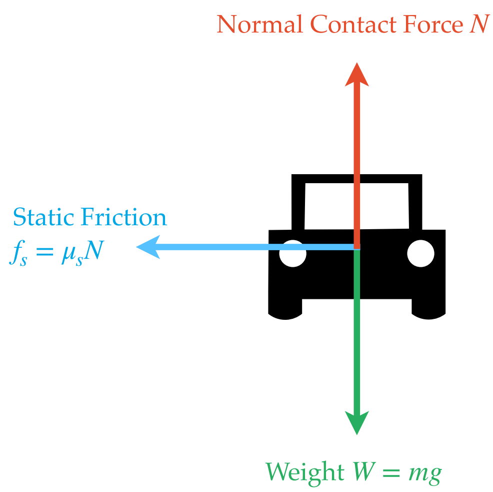
4.2 *Vehicles Turning around Bends*

Let's start with a situation we are all familiar with: a car turning around a bend.

Remember, anything turning in a circle is constantly accelerating due to its changing velocity and therefore must have a force acting on it.

It may not be obvious that this problem is related to circular motion, but a quick look at the diagram below shows that a car follows a curved path when turning a bend. We can draw a "kissing circle" in which we draw the full circle the car might travel if it carried on around the path of the same radius. Now we can use all the mathematical and analytical tools of circular motion that we developed in the previous sections.





First of all, we must realise that in order to turn the bend and travel in a circle, there must be a *centripetal force* that causes the car to turn.

The centripetal force in this case comes from friction between the tyres and the road: turning the steering wheel causes the tyres to turn slightly which provides a horizontal component of friction, as shown in the free body force diagram below.

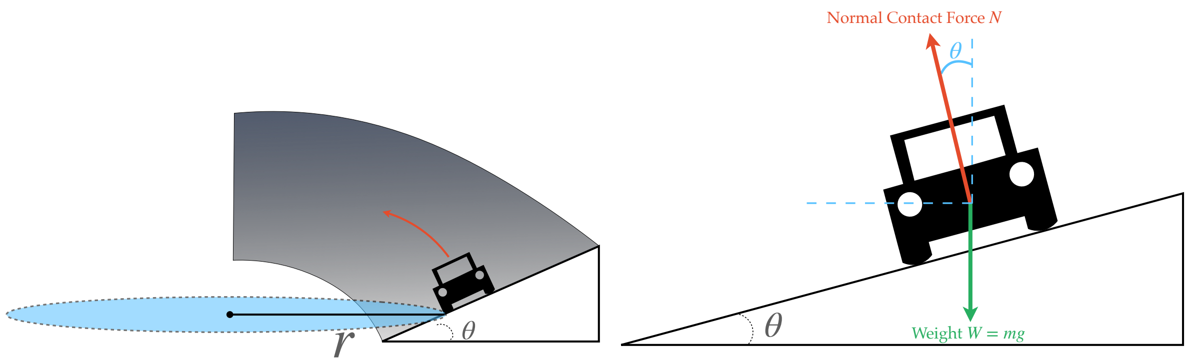
We know that the normal force N and the frictional force f_s are related via $f_s = \mu_s N$ where μ_s is the coefficient of static friction and also that since the car is not accelerating in the vertical direction, the normal force is equal to the weight $N = W = mg \implies m = \frac{N}{g}$. Therefore using our centripetal force equation:

$$\begin{aligned} f_s &= \frac{mv^2}{r} \\ \implies \mu_s N &= \frac{Nv^2}{rg} \\ \implies v &= \sqrt{\mu_s r g} \end{aligned}$$

4.3 Vehicles Turning around Banked Bends

Now imagine a car turning around a bend but the bend is banked by an angle θ as shown in the diagram.

The normal force now has a *horizontal* component, and this provides an additional centripetal force. In fact, it is possible for the car to turn in a circle without any friction between the tyres and the road at all, purely because of this horizontal component of the normal force, as shown in the image below.



Let's consider the situation where there is no frictional force from the tyres to begin with. The horizontal component of the normal force $N = \sin\theta$. Since we also know that the forces are balanced in the vertical direction, we have $N\cos\theta = mg$. Utilising our centripetal force equation we find:

$$\begin{aligned} N\sin\theta &= \frac{mv^2}{r} \\ \implies \frac{mg\sin\theta}{\cos\theta} &= \frac{mv^2}{r} \\ \implies v &= \sqrt{rg\tan\theta} \end{aligned}$$

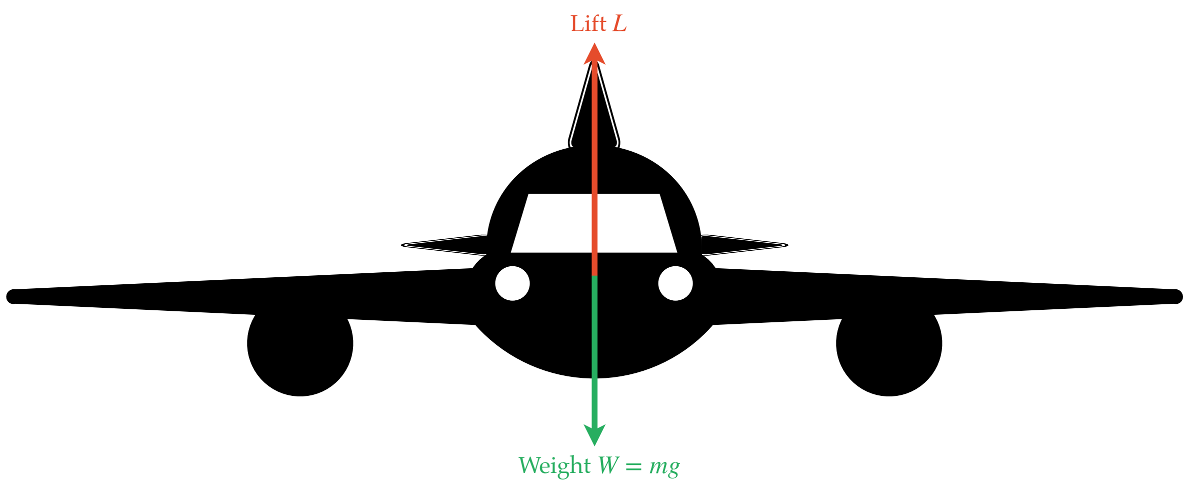
Challenge Questions 4-1 - Vehicles on Banked Turns

1. We can include a static friction force f_s pointing down the slope due to the tyres turning.
   (a) Show that the velocity is given by $v = \sqrt{\frac{rg(\tan\theta + \mu_s)}{1 - \mu_s \tan\theta}}$ in the presence of friction.
   (b) Show that if the frictional force is 0 N, we recover our previous expression $v = \sqrt{rg\tan\theta}$.
   (c) Find the banking angle θ that allows the car to travel around the bend as fast as possible.
   (d) Show that the height h of the outer edge of the road must be $h = l\sin\theta$ where l is the length of the road.

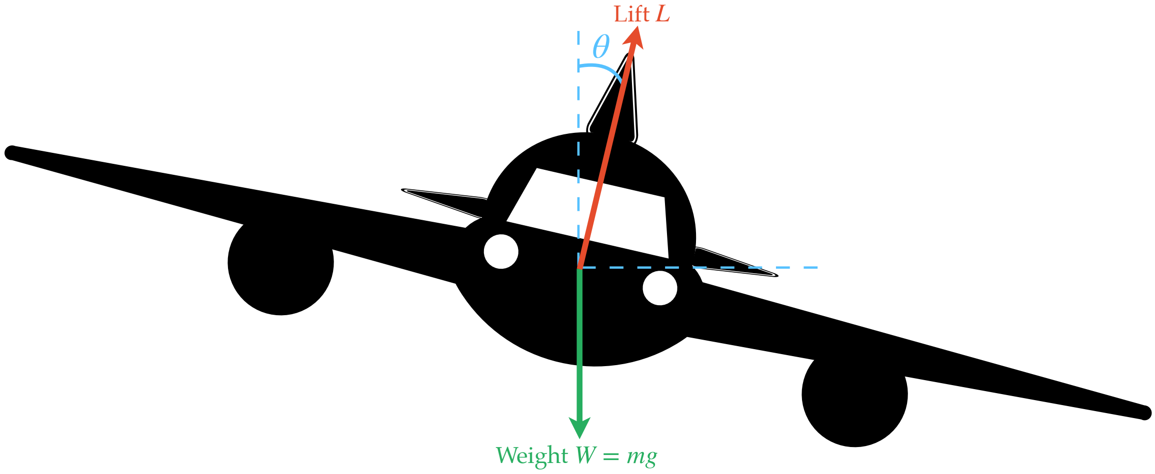
4.4 *Aircraft Turning*

If you've ever been on a plane or watched some war films you may have noticed that when a plane turns, it tilts in the direction in wants to turn.

This is because when a plane is flying straight and level there are no horizontal forces acting on the plane at all, so there's no way it can accelerate in the horizontal direction: only the lift L and weight W act on the plane on the vertical direction as shown in the image below.



If the plane wants to turn, it must somehow produce a horizontal force. By tilting the plane, the lift force acts at an angle θ to the vertical plane and so also creates a horizontal component, allowing the plane to turn horizontally, as shown in the image below.



The horizontal component of the lift force therefore provides the centripetal force:

$$L \sin \theta = \frac{mv^2}{r}$$

We also know that the forces are balanced in the vertical direction, since even though the plane tilts it doesn't move up or down and so $L \cos \theta = mg$.

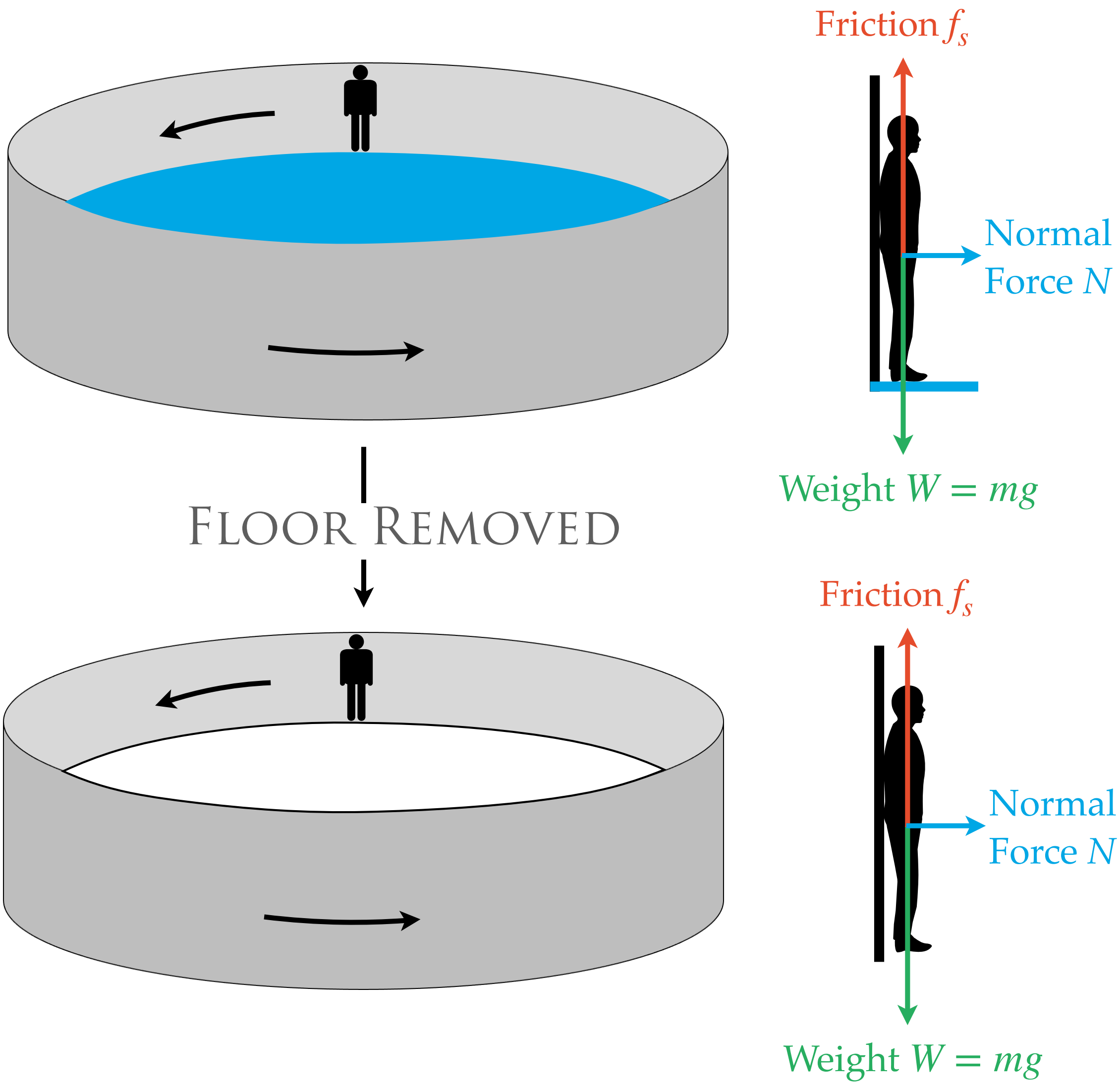
Substituting into our equation above yields:

$$\begin{aligned} L \sin \theta &= \frac{mv^2}{r} \\ \implies \frac{mg \sin \theta}{\cos \theta} &= \frac{mv^2}{r} \\ \implies v &= \sqrt{rg \tan \theta} \end{aligned}$$

This tells us that the faster we want to turn for a given radius r, the larger the angle we need to tilt θ the aircraft.

4.5 *Fairground Rides - Rotor*

The rotor is a thrilling fairground ride: a person stands on a platform with their back against the wall; the platform rotates faster and faster and then the floor is removed. The rider remains stuck against the wall despite the floor disappearing, as shown in the image below.



However, we know that if we were to stand against a wall that wasn't rotating and the floor was removed we would fall down, so something about the fact that the platform is rotating in a circle is clearly playing a part. We might be interested in finding out what speed v the rider needs to go in order for them to remain stuck against the wall.

The only force acting on the rider in the horizontal direction is the normal force N and this force only exists if the rider is in constant with the wall: if the rider starts sliding down the wall this is basically the limiting case where the normal force $N = 0$, which is what we want to avoid.

In the vertical direction, the forces are weight $W = mg$ acting downwards and static friction f_s acting upwards, and in the case where the rider is stuck to the wall we know that $mg = f_s$.

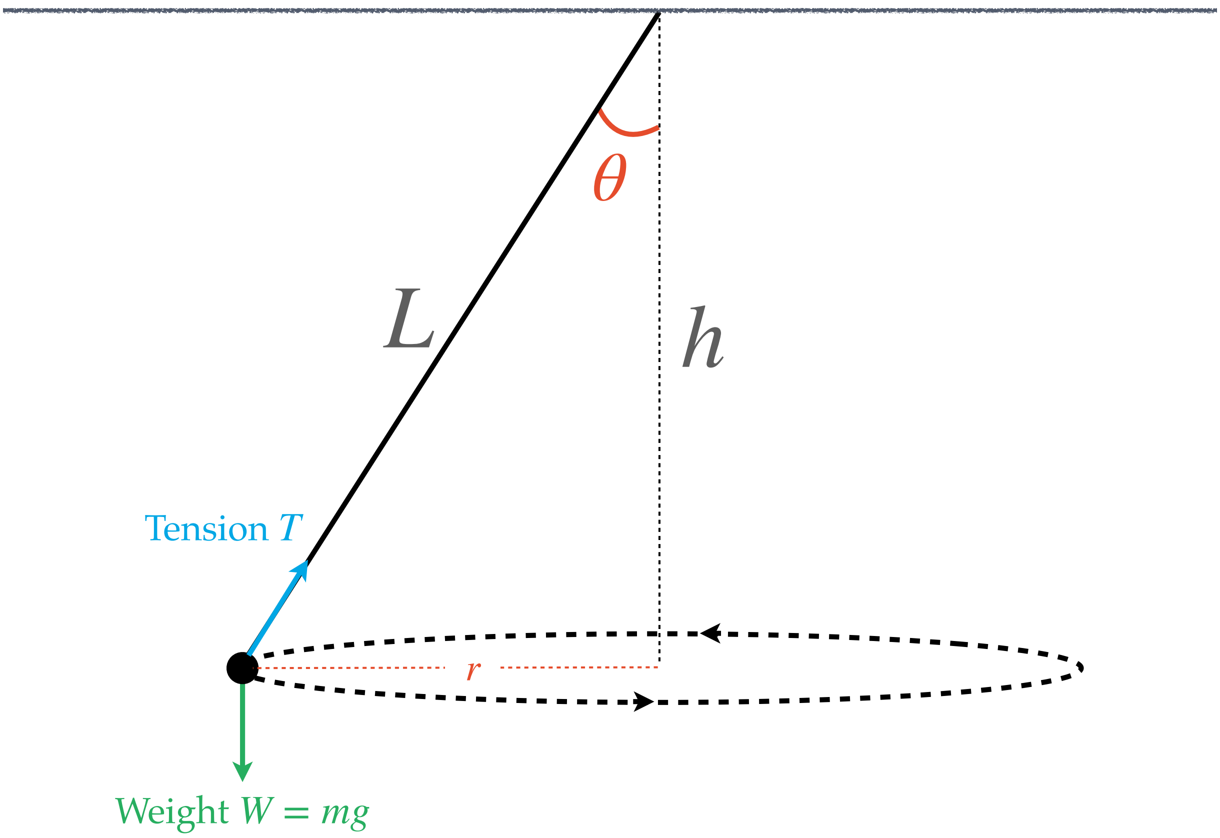
However, we know that static friction and the normal force are linked via $N = \mu f_s$.

Therefore, using our centripetal force equation $F = \frac{mv^2}{r}$ we can write:

$$
\begin{aligned}
N &= \frac{mv^2}{r} \\
\implies \mu f_s &= \frac{mv^2}{r} \\
\implies \mu f_s &= \frac{f_s v^2}{rg} \\
\implies v &= \sqrt{\mu r g}
\end{aligned}
$$

4.6 *Conical Pendulum*

A conical pendulum is a pendulum that is given some tangential velocity v so that the mass traces out a cone (hence *conical*) at its base rather than swinging backwards and forwards, as shown in the image below.



Since we know the pendulum bob is undergoing circular motion, we know that there must be a net centripetal force acting on it.

The centripetal force must be provided by the tension T, since it has a horizontal component: the weight $W = mg$ acts vertically and not in the plane of motion, so it cannot be causing the bob to move in a circle.

Using z-angles we know that the angle between the string and the horizontal plane is θ, and the centripetal force must therefore be $F_{\text{net}} = T \sin\theta$. We can then equate this to $F = \dfrac{mv^2}{r}$ in the usual way:

$$T \sin\theta = \frac{mv^2}{r}$$

However, we can also use force balance in the vertical direction - the pendulum bob is not accelerating in the vertical direction and so the net vertical force acting on it must be 0 N - to obtain $T\cos\theta = mg \implies T = \frac{mg}{\cos\theta}$ which we can substitute into our equation above:

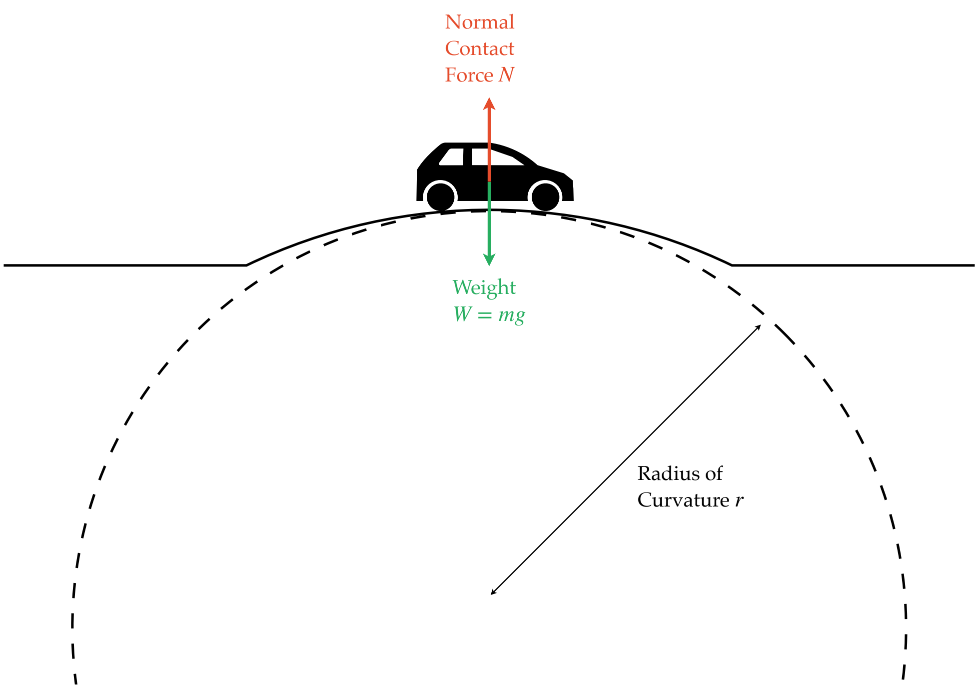
$$\begin{aligned} T \sin\theta &= \frac{mv^2}{r} \\ \implies \frac{mg\sin\theta}{\cos\theta} &= \frac{mv^2}{r} \\ \implies v &= \sqrt{rg\tan\theta} \end{aligned}$$

5 *Non-Uniform Circular Motion*

An interesting class of circular motion problems occur in the vertical plane: this is where gravity acts in addition to other centripetal forces such as the normal force. In this case, we can often take an energy approach in addition to a force approach, and the hardest circular motion problems - which we'll save for the end - include both.

5.1 *A Car Travelling over a Hill*

To get a feel for the kind of physics involved, let's start with the simple scenario of a car travelling over a small hill in the road, as shown in the diagram below.



By drawing the kissing circle we can see that this is a circular motion problem: we have to identify the centripetal force. Unlike in the horizontal plane where there is usually only one force providing the centripetal acceleration, in this scenario we have two forces at play:

the normal force N and the weight $W = mg$ of the car.

The normal contact force N only exists if the car maintains contact with the road: if it momentarily flies off the road because it's going too fast then the contact force $N = 0$ N.

Since the car is following the curve, we know that the net force must be towards the centre of the kissing circle of radius of curvature r.

Therefore:

$$F_{\text{net}} = mg - N = \frac{mv^2}{r}$$

At some point, we might go fast enough over the slope that the car momentarily loses contact with the road. In other words, the normal contact force $N = 0$ N.

When this happens the only force acting on the car will be its weight $W = mg$. What is the maximum speed we could drive over the hill without losing contact with the road? Let's re-arrange the above equation:

$$\begin{aligned} N &= mg - \frac{mv^2}{r} \\ \implies 0 &= mg - \frac{mv^2}{r} \\ \implies mg &= \frac{mv^2}{r} \\ \implies v_{\text{max}} &= \sqrt{rg} \end{aligned}$$

5.2 Weightlessness and Apparent Weight

The previous example of a car travelling over a hill leads us nicely into a discussion of weightlessness: what would a passenger in the car feel as the car went over the hill?

You may have experienced this situation yourself and noticed a strange feeling in your stomach as the car you're travelling in goes over the bump: this is because you momentarily feel lighter as the normal force pushing up on you decreases.

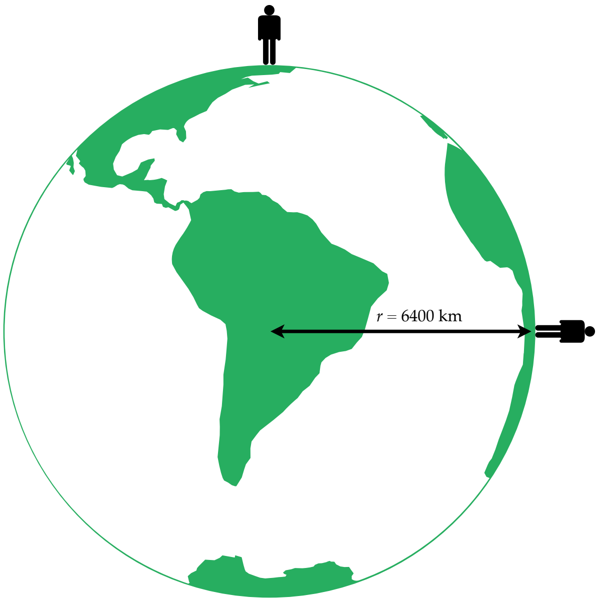
When we talk about "feeling" weight what we're actually referring to is the normal force: that's the force that pushes back on us when we're in contact with some surface and that's the force we actually feel. If we were just free-falling due to weight we'd in fact feel *nothing*: think about the feeling you get when jumping off a diving board before you hit the water. You don't feel as if you have any weight at all.

So paradoxically, the term *weightlessness* actually refers to the scenario when the only force acting is weight! The sensation of weightlessness occurs when nothing pushes back on us to counteract our weight.

The normal force N is often called the *apparent weight*.

Worked Example 5-1 - Losing weight at the Equator

Q: The image shows a person at two points on the Earth: at the North pole, and the Equator.



A person at the equator will actually experience a lower apparent weight due to the rotation of the Earth.

1. Calculate the angular velocity ω of Earth knowing that the time-period $T = 24$ hours.
2. Calculate the tangential velocity of someone at the Equator.
3. Now consider the forces acting on a 100 kg person standing on the North pole and the Equator
   (a) Calculate the weight W and hence the apparent weight (i.e. the normal force N) of a person on the North Pole.
   (b) Calculate the weight W and hence the apparent weight (i.e. the normal force N) of a person on the Equator.

A:

1. The time-period $T = 24$ hours $= 86,400$ s and so the angular velocity $\omega = \dfrac{2\pi}{T} = 7.3 \times 10^{-5}$ rad/s.
2. The tangential velocity at the equator is given by $v = r\omega = 6.4 \times 10^{7} \times 7.3 \times 10^{-5} = 4650$ m/s.

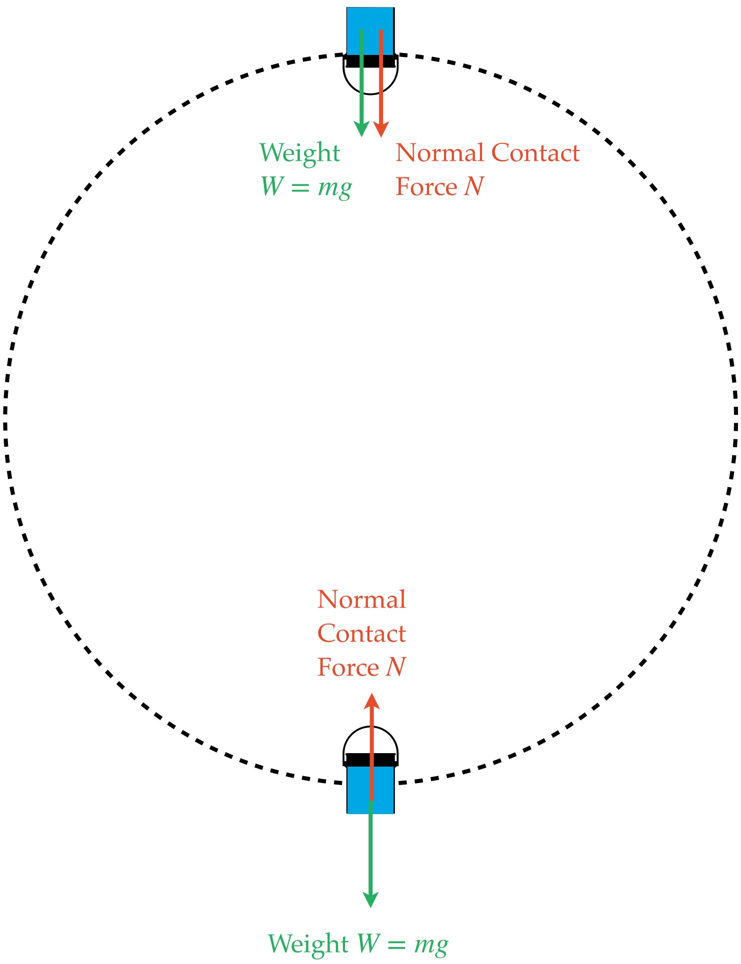
3.(a) At the North Pole, there is no circular motion and hence there is no net force acting on the person: $W - N = 0 \implies N = W = 9.81 \times 100 = 981$ N.

(b) At the Equator, since the person is rotating in a circle, this means there must be a net centripetal force acting towards the centre of the circle $F_{\text{net}} = W - N = mr\omega^2 \implies N = W - mr\omega^2 = 981\text{ N} - 100 \times 6.4 \times 10^{7} \times (7.3 \times 10^{-5}\text{ rad/s})^2 = 981\text{ N} - 34\text{ N} = 947$ N.
In other words, the person will feel the same as a person of mass $m = 947/9.81 = 97$ kg standing at the North Pole.

5.3 *Rotating Buckets of Water*

If you take a bucket of water and swing in vertical circles, you'll notice that the water doesn't fall out of the bucket. Yet if you hold the bucket above your head, the water falls out. Therefore, there must be some speed at which we can swing the bucket that stops the water falling out. What is that speed?

The diagram below shows the forces acting on the water inside the bucket as it rotates in a circle. Let's look at the forces acting on the water at the top of the swing.



Notice that at the top of the swing, the normal contact force points *down*. So at this point the net centripetal force is:

$$F_{\text{net}} = N + mg = \frac{mv^2}{r}$$

We want the water to remain in contact with the bottom of the bucket, and so we want to swing the bucket fast enough that the normal contact force N still exists and is greater than or just equal to 0, $N \geq 0$.

In this case:

$$\begin{aligned}
F_{\text{net}} = {} & N + mg = \frac{mv^2}{r} \\
\implies N = {} & \frac{mv^2}{r} - mg \\
\implies 0 = {} & \frac{mv^2}{r} - mg \\
\implies mg = {} & \frac{mv^2}{r} \\
\implies v_{\min} = {} & \sqrt{rg}
\end{aligned}$$

Notice that this is basically the same formula we derived for a car going over the hill, except there's a subtle difference: the bucket needs to be swung at a minimum speed $\sqrt{rg}$ otherwise the water will fall out, and the car needs to travel at a speed *less than* $\sqrt{rg}$, otherwise it will lose contact with the road.

One final point is that we have considered only the forces acting on the water: in order to get the water and bucket to swing overhead, we clearly need some other force to provide the centripetal acceleration, and in this case that force comes from tension T in the person's arms.

If we then draw a free-body force diagram on the bucket, the situation looks a little different: the normal force acts *outwards* at the top of the swing and so we have $T = N - W$ and at the bottom of the swing it acts *downwards* (notice that in both cases the normal force on the bucket acts in the opposite direction to the normal force on the water, as we expect from Newton's third law) and so we have $T = N + W$.

Worked Example 5-2 - Swinging Buckets

Q: A person swings a bucket of water filled with 1 L of water around in a vertical circle. When fully stretched, a person has an arm span of 80 cm.

1. Calculate the minimum speed v the must be swing the bucket for the water not to fall out.
2. Calculate the minimum angular velocity ω needed for the water to stay in the bucket.
3. Calculate the tension T in the person's arm at the top of the swing.
4. Calculate the tension T in the person's arm at the bottom of the swing.

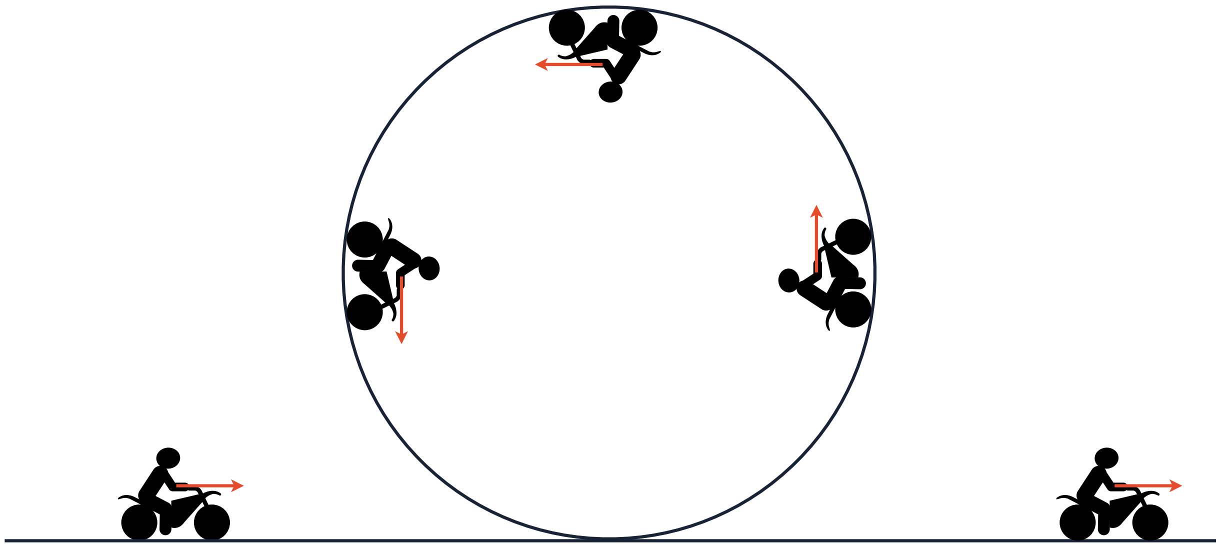
A:

1. The time-period $T = 24$ hours $= 86,400$ s and so the angular velocity $\omega = \dfrac{2\pi}{T} = 7.3 \times 10^{-5}$ rad/s.
2. The tangential velocity at the equator is given by $v = r\omega = 6.4 \times 10^{7} \times 7.3 \times 10^{-5} = 4650$ m/s.
3. (a) At the North Pole, there is no circular motion and hence there is no net force acting on the person: $W - N = 0 \implies N = W = 9.81 \times 100 = 981$ N.

   (b) At the Equator, since the person is rotating in a circle, this means there must be a net centripetal force acting towards the centre of the circle $F_{\text{net}} = W - N = mr\omega^2 \implies N = W - mr\omega^2 = 981\text{ N} - 100 \times 6.4 \times 10^{7} \times (7.3 \times 10^{-5}\text{ rad/s})^2 = 981\text{ N} - 34\text{ N} = 947$ N.
   In other words, the person will feel the same as a person of mass $m = 947/9.81 = 97$ kg standing at the North Pole.

5.4 *Loop the Loop*

Let's extend our knowledge from the previous two problems to a general *loop the loop* problem. Imagine a motorbike rider driving around a loop as shown below.

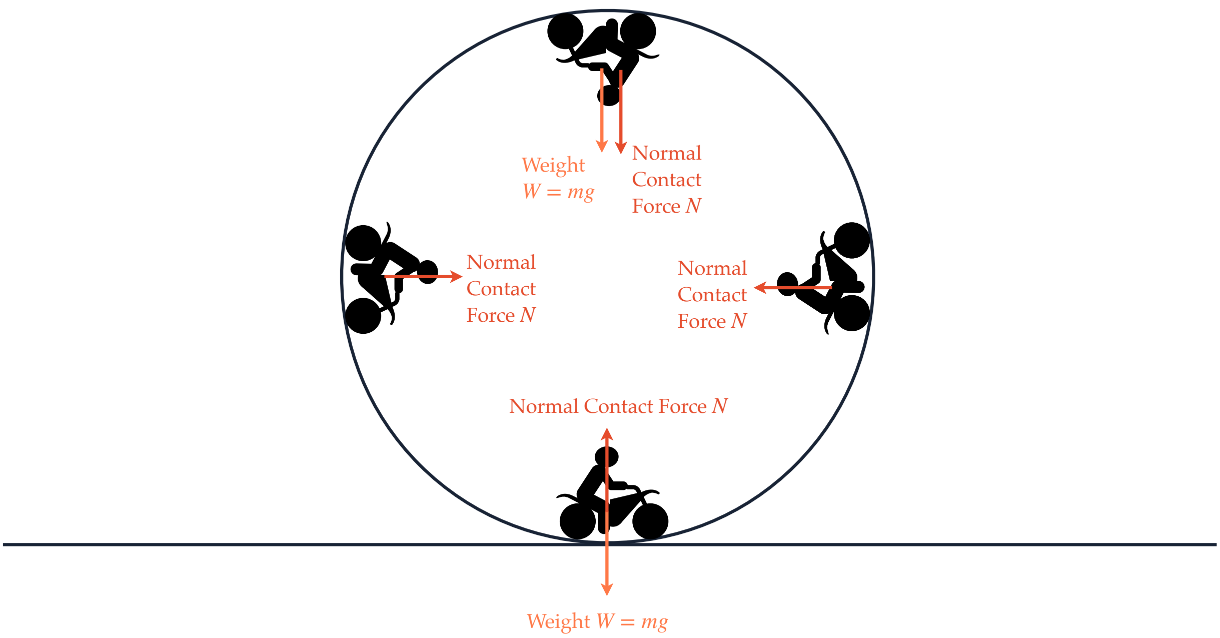


Analysing the forces at various stages around the loop using a free body force diagram, we see that the normal contact force (apparent weight) acting on the rider at the top of the loop is:

$$N = \frac{mv^2}{r} - mg$$

and at the bottom of the loop must therefore be:

$$N = \frac{mv^2}{r} + mg$$



This tells us that the rider feels lighter at the top of the loop but much heavier at the bottom of the loop. This analysis is general and applies equally well to e.g. a rollercoaster ride, which you may have experienced yourself.

At the sides of the loop, only the normal force N provides the centripetal force since the weight acts in the vertical direction only and so:

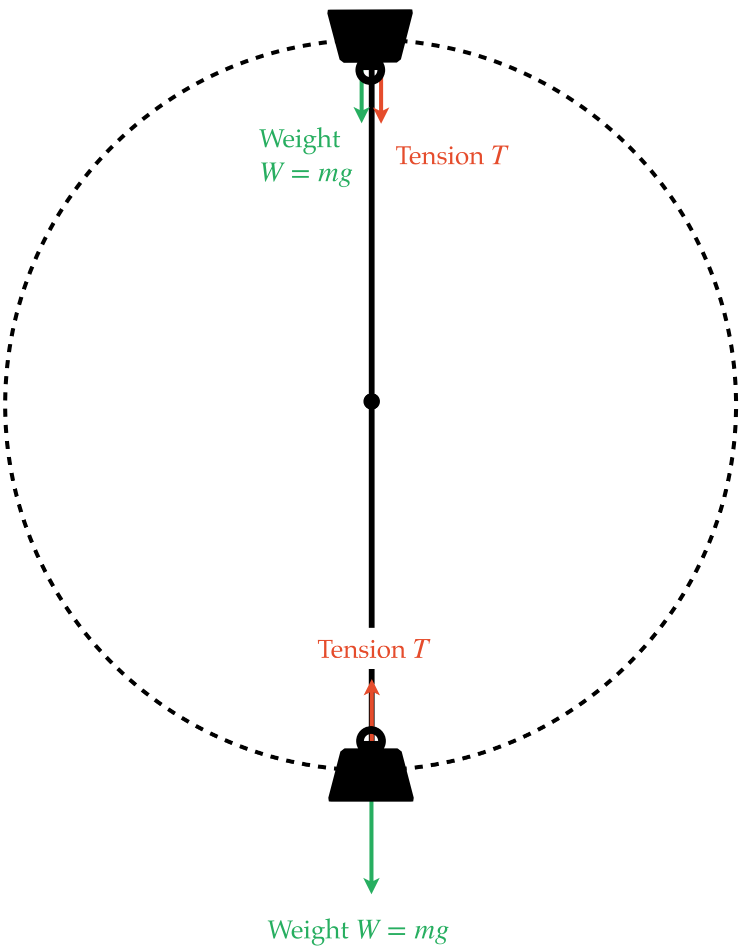
$$N = \frac{mv^2}{r}$$

5.5 *Vertical Mass on a String*

There are scenarios similar to the loop the loop problem but the centripetal force is provided by something other than the normal force.

A common example is a string or a spring with mass on the end being swung in a vertical circle.

In this case, there is no normal force: a mixture of weight and *tension* provide the centripetal acceleration. The diagram below show the forces on the mass as it is swung in a vertical circle.



At the top of the swing, the net centripetal force is:

$$F_{\text{net}} = T + mg = \frac{mv^2}{r}$$

In order for the string to stay *taut* and not *slack* at the top of the motion the tension force must be greater than or just equal to 0, $T \geq 0$, and therefore:

$$\begin{aligned} T + mg &= \frac{mv^2}{r} \\ \implies T &= \frac{mv^2}{r} - mg \geq 0 \\ \implies \frac{mv^2}{r} &\geq mg \\ \implies v_{\text{min}} &= \sqrt{rg} \end{aligned}$$

At the bottom of the motion, the net centripetal force is:

$$\begin{aligned} F_{\text{net}} &= T - mg = \frac{mv^2}{r} \\ \implies T &= \frac{mv^2}{r} + mg \end{aligned}$$

So we can see that the string experiences greater tension at the bottom of the motion. A common exam-style question would be to find the maximum velocity we can swing the mass before the string breaks given some maximum tension force T_{max}, which we will explore in the worked example below.

6 *Combining energy and circular motion*

Sometimes we have to combine conservation of energy with forces in order to understand the motion of an object. This frequently happens when an object moves in a vertical circle.

6.1 *A mass on a string in a vertical circle*

We already looked at how the forces change on a mass attached to a string moving in a vertical circle. Now we're going to look at how its speed changes, which is best illustrated with an example.

Worked Example 6-1 - Calculating the velocity of the ball at the bottom of the loop

Q: A mass $m = 100$ g is attached to a string of length $l = 0.5$ m and is swung in a vertical circle. At the top of the circle, its velocity is 10 m/s. What is its velocity at the bottom of the circle?

A: Here we have to use the fact that the mass converts GPE into KE as it moves from the top of the circle to the bottom. If we choose the bottom of the circle to be the point with zero gravitational potential energy, then at the top the mass's gravitational potential energy must be:

$$\begin{aligned} GPE &= mgh \\ &= 0.1 \text{ kg} \times 10 \text{ N/kg} \times 0.5 \text{ m} \\ &= 0.5 \text{ J} \end{aligned}$$

and its kinetic energy must be:

$$\begin{aligned} KE &= \frac{1}{2}mv^2 \\ &= 0.5 \times 0.1 \times 10^2 \\ &= 5 \text{ J} \end{aligned}$$

giving a total energy $E = GPE + KE = 0.5\text{ J} + 5\text{ J} = 5.5\text{ J}$

This total energy must be the same at the bottom of the loop, but the $GPE = 0$ at this point.

Therefore, the kinetic energy must be $KE = 5.5$ J:

$$\begin{aligned} 5.5 &= \frac{1}{2}mv^2 \\ \implies v &= \sqrt{\frac{2 \times 5.5}{0.1}} \\ &= 10.5 \text{ m/s} \end{aligned}$$

6.2 *Loop the Loop Revisited*

We investigated the loop the loop problem from a forces perspective already, but now we can add an energy dimension: imagine letting go of a ball on a track that rolls down the track before attempting to do the loop.

Challenge Questions 6-1 - Calculating the velocity of the ball at the bottom of the loop

Q: Threaded bead.

A: Here we have to use the fact that the mass converts GPE into KE as it moves from the top of the circle to the bottom. If we choose the bottom of the circle to be the point with zero gravitational potential energy, then at the top the mass's gravitational potential energy must be:

$$\begin{aligned} GPE &= mgh \\ &= 0.1\text{ kg} \times 10\text{ N/kg} \times 0.5\text{ m} \\ &= 0.5\text{ J} \end{aligned}$$

and its kinetic energy must be:

$$\begin{aligned} KE &= \frac{1}{2}mv^2 \\ &= 0.5 \times 0.1 \times 10^2 \\ &= 5\text{ J} \end{aligned}$$

giving a total energy $E = GPE + KE = 0.5\text{ J} + 5\text{ J} = 5.5\text{ J}$

This total energy must be the same at the bottom of the loop, but the $GPE = 0$ at this point.

Therefore, the kinetic energy must be $KE = 5.5$ J:

$$\begin{aligned} 5.5 &= \frac{1}{2}mv^2 \\ \implies v &= \sqrt{\frac{2 \times 5.5}{0.1}} \\ &= 10.5\text{ m/s} \end{aligned}$$

Challenge Questions 6-2 - Calculating the height a ball will reach in loop the loop

1. Show that the height h a ball will reach above the ground when doing loop the loop for a loop of radius r when dropped down a track from height H is $h = \dfrac{2H + r}{3}$

7 *Applied Circular Motion*

7.1 *Gravitation*

In this section you'll learn:

- How to model orbits using the tools of circular motion.
- Kepler's Third Law.

You should already be familiar with:

- Gravitational Fields.

The gravitational force between two objects of mass M and m is $F = \dfrac{GMm}{r^2}$, where r is the centre-centre separation and G is the universal gravitational constant.

If one object orbits the other in a circular orbit - we pretend that the large mass M is fixed in place and the smaller mass m orbits around the larger mass M - we know that the resultant force must be $F_{\text{net}} = \frac{mv^2}{r}$. Here, the mass m is the mass of the smaller body that is orbiting the larger mass M. Therefore:

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$
$$\implies v = \sqrt{\frac{GM}{r}}$$

this equation tells us the tangential velocity v required for an object a distance r from the fixed mass M to orbit in a circle. It is essentially a *constraint* of motion.

If an object is at some distance r and its velocity is higher or lower than this velocity, then it will not orbit in a circle but in an ellipse instead.

Since it is quite unlikely that a planet will have exactly this velocity v during the formation of a planetary system such as our solar system, planets actually orbit in ellipses rather than circles. This is what led Kepler to postulate his first law as "All planets orbit in ellipses with the sun at one focus". However, we can usually approximate the orbits as circular and still learn a lot about their motion.

7.2 *Charged particles in magnetic fields*

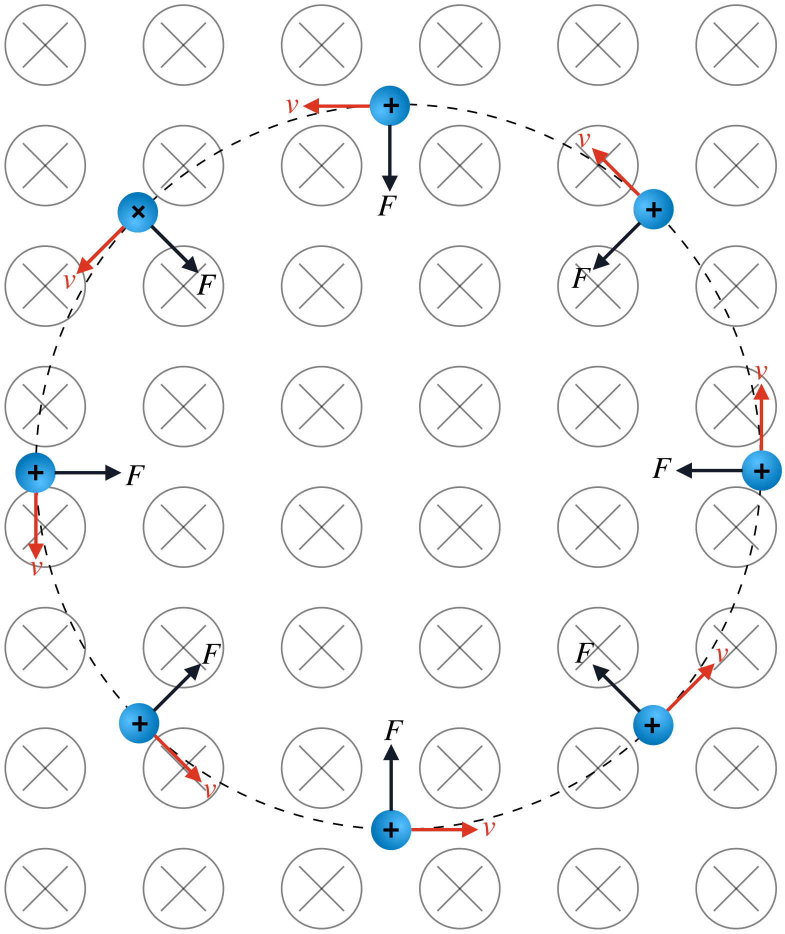
In this section you'll learn:

- That charged particles in magnetic fields orbit in circles.

You should already be familiar with:

- Magnetic Fields.

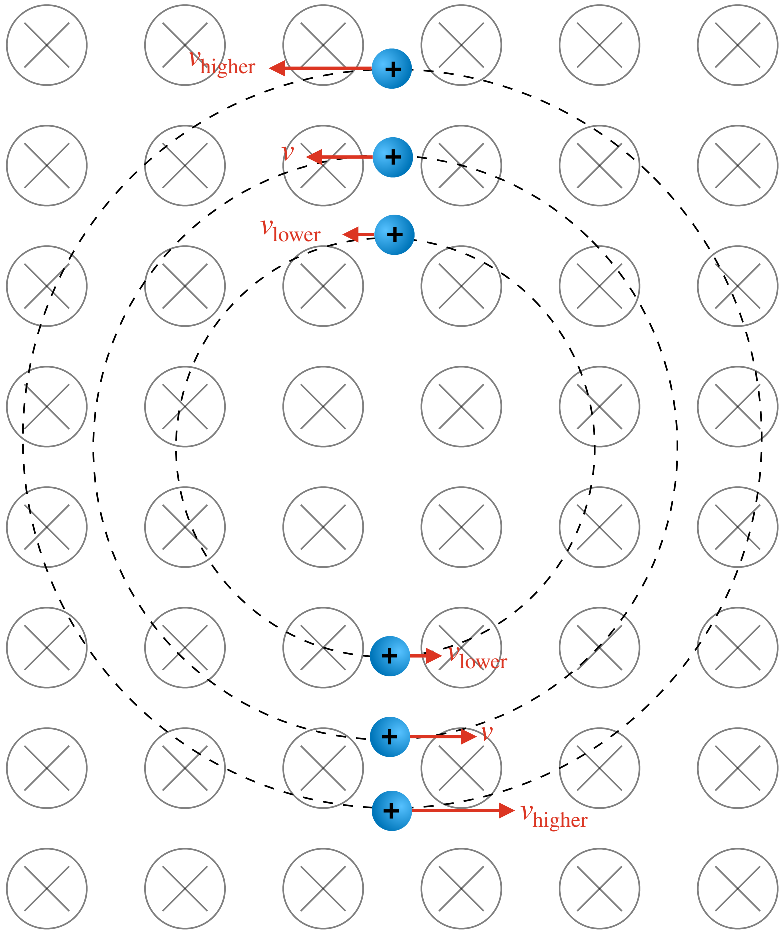
When a charged particle of charge q moves with velocity v perpendicular to a magnetic field of strength B, it experiences force $F = Bqv$ that acts perpendicular to its velocity.



This means the particle will move in a circle with the magnetic force $F = Bqv$ providing the centripetal force:

$$Bqv = \frac{mv^2}{r}$$
$$\implies r = \frac{mv}{Bq}$$

This equation tells us that the radius of the path of the particle depends on the **velocity** v of the particle, provided the magnetic field strength B is constant, and that the particles are all of the same type (and therefore the mass m and charge q of the particles are the same).



Let's now look at how long it takes for a particle to a complete circle - in other words, its time period T of circular motion. The time period will be $T = \dfrac{2\pi r}{v}$. We can substitute this into our equation for the radius of the particle's path r and solve for the time-period T:

$$
\begin{aligned}
r &= \frac{mv}{Bq} \\
\implies r &= \frac{m}{Bq}\left(\frac{2\pi r}{T}\right) \\
\implies T &= \frac{m2\pi r}{Bqr} \\
\implies T &= \frac{2\pi m}{Bq}
\end{aligned}
$$

This is quite a remarkable result: it tells us that the time it takes for a particle to complete its circular path **does not** depend on its velocity, only on its basic properties such as mass m, charge q, and the magnetic field strength B in which it moves. We can assume all these quantities stay constant.

These results will be useful in the particle physics section of the course.

7.3 *The Earth's bulge*