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Semiregularity maps and deformations of modules over Lie algebroids

Ruggero Bandiera, Emma Lepri, and Marco Manetti

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Abstract: We determine a DG-Lie algebra controlling deformations of a locally free module over a Lie algebroid $\mathcal{A}$. Moreover, for every flat inclusion of Lie algebroids $\mathcal{A} \subset \mathcal{L}$ we introduce semiregularity maps and prove that they annihilate obstructions, provided that the Leray spectral sequence of the pair $(\mathcal{L}, \mathcal{A})$ degenerates at E_1.

Keywords: Curved DG-algebras, L_∞ maps, Atiyah class, Lie algebroids, semiregularity.

1. Introduction

Let X be a separated scheme of finite type over a field $\mathbb{K}$ of characteristic 0 and let $\mathcal{E}$ be a locally free sheaf on X. Following Buchweitz and Flenner [5], the semiregularity maps of $\mathcal{E}$ are defined as

$$\tau_k \colon \operatorname{Ext}^2_X(\mathcal{E}, \mathcal{E}) \to H^{2+k}(\Omega^k_X), \quad \tau_k(x) = \frac{1}{k!} \operatorname{Tr}(\operatorname{At}(\mathcal{E})^k x), \qquad k \geq 0,$$

where $\operatorname{At}(\mathcal{E}) \in \operatorname{Ext}^1_X(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$ is the Atiyah class of $\mathcal{E}$.

After [2, 5, 24] it is known that these semiregularity maps annihilate obstructions to deformations, provided that the Hodge to de Rham spectral sequence of X degenerates at E_1. More generally, writing $\Omega^{\leq k}_X$ for the algebraic de Rham complex truncated in degree $\leq k$, it is known that the composition of τ_k with the natural map $H^{2+k}(\Omega^k_X) \to \mathbb{H}^{2+2k}(\Omega^{\leq k}_X)$ annihilates obstructions, regardless of degeneration properties of the aforementioned spectral sequence.

The main goal of this paper is to extend these results to locally free modules over a Lie algebroid $\mathcal{A}$ on X, see Definition 3.1 below. By definition,

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a locally free $\mathcal{A}$-module is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, and

$$\nabla \colon \mathcal{A} \to \mathcal{H}om_{\mathbb{K}}(\mathcal{E}, \mathcal{E}), \quad l \mapsto \nabla_l,$$

is an $\mathcal{O}_X$-linear map such that:

1. ∇ is an $\mathcal{A}$-connection; by definition, this means that $\nabla_l(fe) = a(l)(f)e + f\nabla_l(e)$ for $l \in \mathcal{A}$, $f \in \mathcal{O}_X$ and $e \in \mathcal{E}$, where $a \colon \mathcal{A} \to \Theta_X$ is the anchor map;
2. the $\mathcal{A}$-connection ∇ is flat, i.e., its curvature $\nabla^2(l, m) = [\nabla_l, \nabla_m] - \nabla_{[l,m]}$ vanishes identically.

When $\mathcal{A} = \Theta_X$ with anchor map the identity, then the notion of $\mathcal{A}$-connection reduces to the usual definition of analytic connection.

Recall also that the Atiyah class of a locally free sheaf can be defined as the obstruction to the existence of an analytic connection. In other words, the Atiyah class of $\mathcal{E}$ can be defined as the obstruction to the lifting of the (unique) 0-connection on $\mathcal{E}$ to a Θ-connection; in view of the generalisation considered in this paper we also write $\mathrm{At}(\mathcal{E}) = \mathrm{At}_{\Theta/0}(\mathcal{E})$.

By a straightforward generalisation, we can replace Θ with $\mathcal{A}$ and define $\mathrm{At}_{\mathcal{A}/0}(\mathcal{E})$ as the obstruction to the existence of an $\mathcal{A}$-connection on $\mathcal{E}$; however, this generalisation does not lead to anything new from the point of view of semiregularity maps and deformation theory.

Instead, we are here interested in the definition of a class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ in the following situation:

1. $\mathcal{A} \subset \mathcal{L}$ is an inclusion of Lie algebroids such that the quotient sheaf $\mathcal{L}/\mathcal{A}$ is locally free;
2. $(\mathcal{E}, \nabla)$ is a locally free $\mathcal{A}$-module.

In the above situation the quotient sheaf $\mathcal{L}/\mathcal{A}$ carries a natural structure of $\mathcal{A}$-module given by the *Bott connection* $\nabla^B \colon \mathcal{A} \to \mathcal{E}nd_{\mathbb{K}}(\mathcal{L}/\mathcal{A}, \mathcal{L}/\mathcal{A})$, $\nabla^B_a(x) = [a, x] \pmod{\mathcal{A}}$. Thus, for every $r \geq 0$, the sheaf $\mathcal{M}_r := \bigwedge^r(\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ carries a natural structure of $\mathcal{A}$-module.

Denoting by $\mathbb{H}^*(\mathcal{A}; \mathcal{M}_r)$ the Lie algebroid cohomology of $\mathcal{A}$ with coefficients in $\mathcal{M}_r$ (see Definition 3.11), in this paper we prove in particular that:

1. $\mathbb{H}^1(\mathcal{A}; \mathcal{M}_0)$ is the space of first order deformations of $\mathcal{E}$ as an $\mathcal{A}$-module;
2. $\mathbb{H}^2(\mathcal{A}; \mathcal{M}_0)$ is a complete obstruction space for deformations of $\mathcal{E}$ as an $\mathcal{A}$-module;
3. the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \in \mathbb{H}^1(\mathcal{A}; \mathcal{M}_1)$ is properly defined.

The first two items above are proved by showing that the DG-Lie algebra of derived sections of the sheaf of DG-Lie algebras $\Omega^*(\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ controls deformations of $\mathcal{E}$ as an $\mathcal{A}$-module, where $\Omega^*(\mathcal{A})$ is the de Rham DG-algebra of $\mathcal{A}$. The Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is the primary obstruction to the extension of ∇ to a flat $\mathcal{L}$-connection. More precisely, $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is the obstruction to the extension of ∇ to an $\mathcal{L}$-connection $\nabla' \colon \mathcal{L} \to \mathcal{H}om_{\mathbb{K}}(\mathcal{E}, \mathcal{E})$ such that $[\nabla'_l, \nabla'_a] = \nabla'_{[l,a]}$ for every $l \in \mathcal{L}$ and $a \in \mathcal{A}$, cf. [6].

By analogy with the classical case, we define the semiregularity maps

$$\tau_k \colon \mathbb{H}^2(\mathcal{A}; \mathcal{M}_0) \to \mathbb{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right), \quad \tau_k(x) = \frac{1}{k!} \operatorname{Tr}(\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),$$

and we use the main result of [2] in order to prove that every τ_k annihilates obstructions, provided that the Leray spectral sequence (Definition 5.2) of the pair $(\mathcal{L}, \mathcal{A})$ degenerates at E_1.

Notation

In this paper we work over a fixed field $\mathbb{K}$ of characteristic 0; unless otherwise specified every (graded) vector space is intended over $\mathbb{K}$.

Unless otherwise specified the term *differential graded* (DG) means graded over the integers and with differential of degree $+1$. The degree of a homogeneous element x in a graded vector space will be denoted $|x|$. We adopt the Grothendieck–Verdier formalism for degree shifting: given a DG-vector space $(V = \oplus_n V^n, d_V)$ and an integer p, we define the DG-vector space $(V[p], d_{V[p]})$ by setting $V[p]^n = V^{n+p}$, $d_{V[p]} = (-1)^p d_V$.

2. Semiregularity maps for curved DG-algebras

We briefly review some definitions and results from [2]. By a graded algebra we intend a unitary graded associative algebra over a fixed field $\mathbb{K}$ of characteristic 0. Every graded associative algebra is also a graded Lie algebra, with the bracket given by the graded commutator $[a, b] = ab - (-1)^{|a||b|} ba$.

Definition 2.1. A curved DG-algebra is the datum $(A, d, \cdot, R)$ of a graded associative algebra $(A, \cdot)$ together with a degree one derivation $d \colon A^* \to A^{*+1}$ and a degree two element $R \in A^2$, called *curvature*, such that

$$d(R) = 0, \qquad d^2(x) = [R, x] = R \cdot x - x \cdot R \quad \forall\, x \in A.$$

For notational simplicity we shall write (A, d, R) in place of $(A, d, \cdot, R)$ when the product $\cdot$ is clear from the context. We denote by $[A, A] \subset A$ the *linear span* of all the graded commutators $[a, b] = ab - (-1)^{|a||b|}ba$. Notice that $[A, A]$ is a homogeneous Lie ideal and then $A/[A, A]$ inherits a natural structure of DG-Lie algebra with trivial bracket.

Definition 2.2. Let $A = (A, d, R)$ be a curved DG-algebra. A *curved ideal* in A is homogeneous bilateral ideal $I \subset A$ such that $d(I) \subset I$ and $R \in I$.

By a *curved DG-pair* we mean the data (A, I) of a curved DG-algebra A equipped with a curved ideal I.

In particular, for every curved DG-pair (A, I), the quotient A/I is a (non-curved) associative DG-algebra, and therefore also a DG-Lie algebra. Writing $I^{(k)}$, $k \geq 0$, for the kth power of I, we have that $I^{(k)}$ is an associative bilateral ideal of A for every k. The differential graded algebra $\operatorname{Gr}_I A = \bigoplus_{k\geq 0} \frac{I^{(k)}}{I^{(k+1)}}$ is non-curved, since $d(I) \subset I$ and $d^2(I) \subset I^{(2)}$, the derivation d factors through differentials

$$d\colon \frac{I^{(k)}}{I^{(k+1)}} \to \frac{I^{(k)}}{I^{(k+1)}}, \qquad d^2 = 0.$$

Definition 2.3. Let $A = (A, d, R)$ be a curved DG-algebra and $I \subset A$ a curved ideal. The *Atiyah cocycle* of the pair (A, I) is the class of R in the DG-vector space $\frac{I}{I^{(2)}}$. The *Atiyah class* of the pair (A, I) is the cohomology class of the Atiyah cocycle:

$$\operatorname{At}(A, I) = [R] \in H^2\left(\frac{I}{I^{(2)}}\right).$$

For every $x \in I$ of degree 1, we can consider the twisted derivation $d_x := d + [x, -]$ with curvature $R_x = R + dx + \frac{1}{2}[x, x]$. Then I remains a curved ideal of the twisted curved DG-algebra (A, d_x, R_x).

Lemma 2.4. *The Atiyah class of the pair (A, d_x, R_x, I) does not depend on the choice of $x \in I$. The Atiyah class $\operatorname{At}(A, I)$ is trivial if and only if there exists $x \in I$ of degree 1 such that R_x belongs to $I^{(2)}$.*

Proof. Firstly, notice that the differential on the algebra $\operatorname{Gr}_I A$ does not depend on the choice of $x \in I$: since x belongs to I the adjoint operator $[x, -]$ sends $I^{(k)}$ to $I^{(k+1)}$, and so $d = d_x := d + [x, -]$ in $\frac{I^{(k)}}{I^{(k+1)}}$. In $\frac{I}{I^{(2)}}$, one has that $[x, x] = 0$, so that

$$R_x - R = R + dx + \frac{1}{2}[x,x] - R = dx,$$

and the cohomology classes of R and R_x in $H^*\big(\frac{I}{I^{(2)}}\big)$ coincide.

Let now $x \in I$ be such that $R_x = R + dx + \frac{1}{2}[x,x]$ belongs to $I^{(2)}$. Then $R + dx$ also belongs to $I^{(2)}$ and $R = -dx$ in $\frac{I}{I^{(2)}}$, so that the Atiyah class is trivial. Conversely, let $R = dx$ in $\frac{I}{I^{(2)}}$, then $R - dx$ belongs to $I^{(2)}$, and so does $R_{-x} = R - dx + \frac{1}{2}[x,x]$. □

Definition 2.5. A *trace map* on a curved DG-algebra (A, d, R) is the data of a complex of vector spaces (C, δ) and a morphism of graded vector spaces $\mathrm{Tr}\colon A \to C$ such that $\mathrm{Tr} \circ d = \delta \circ \mathrm{Tr}$ and $\mathrm{Tr}([A, A]) = 0$.

Assume now there are given a curved DG-algebra (A, d, R), a curved ideal I and a trace map $\mathrm{Tr}\colon A \to C$. Consider the decreasing filtration $C_k = \mathrm{Tr}(I^{(k)})$ of subcomplexes of C. By basic homological algebra, the spectral sequence associated to this filtration degenerates at E_1 if and only if for every k the inclusion $C_k/C_{k+1} \subset C/C_{k+1}$ is injective in cohomology, see e.g. [22, Thm. C.6.6].

In the above situation we can define semiregularity maps

$$\tau_k\colon H^2(A/I) \to H^{2+2k}(C_k/C_{k+1}), \qquad \tau_k(x) = \frac{1}{k!}\,\mathrm{Tr}(\mathrm{At}(A, I)^k x).$$

The composition of τ_k with the natural morphism

$$H^{2+2k}(C_k/C_{k+1}) \to H^{2+2k}(C/C_{k+1})$$

is induced by the morphism of complexes

$$\sigma_k^1\colon \frac{A}{I} \to \frac{C}{C_{k+1}}[2k], \quad \sigma_k^1(x) = \frac{1}{k!}\,\mathrm{Tr}(R^k x).$$

Considering C/C_{k+1} as a DG-Lie algebra with trivial bracket, we can immediately see that σ_k^1 is a morphism of DG-Lie algebras for $k = 0$, while for $k > 0$ we have the following result.

Theorem 2.6 ([2, Corollary 2.10])**.** *In the above situation, the map σ_k^1 is the linear component of an L_∞-morphism $\sigma_k\colon A/I \rightsquigarrow C/C_{k+1}[2k]$. In particular, σ_k^1 annihilates obstructions for the deformation functor associated to the DG-Lie algebra A/I.*

3. Lie algebroid connections

Throughout all this paper, X will denote a smooth separated scheme of finite type over a field $\mathbb{K}$ of characteristic 0.

We denote by Θ_X its tangent sheaf and by Ω_X^k, $k \geq 0$, the sheaves of differential forms. For every pair of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we denote by $\mathcal{H}om_{\mathbb{K}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the sheaves of $\mathbb{K}$-linear morphisms and $\mathcal{O}_X$-linear morphisms respectively. The $\mathcal{O}_X$-module structure on $\mathcal{G}$ induces an $\mathcal{O}_X$-module structure both on $\mathcal{H}om_{\mathbb{K}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. We also write $\mathcal{E}nd_{\mathbb{K}}(\mathcal{F})$ and $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ for $\mathcal{H}om_{\mathbb{K}}(\mathcal{F}, \mathcal{F})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ respectively.

Unless otherwise specified we write $\otimes$ for the tensor product over $\mathcal{O}_X$, in particular for two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we have $\mathcal{F} \otimes \mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Definition 3.1. A *Lie algebroid* over X is the data of $(\mathcal{L}, [-,-], a)$ where:

- $\mathcal{L}$ is a locally free coherent sheaf of $\mathcal{O}_X$-modules;
- $[-,-]$ is a $\mathbb{K}$-linear Lie bracket on $\mathcal{L}$;
- $a \colon \mathcal{L} \to \Theta_X$ is a morphism of sheaves of $\mathcal{O}_X$-modules, called the *anchor map*, commuting with the brackets;
- finally, we require the Leibniz rule to hold

$$[l, fm] = a(l)(f)m + f[l, m], \qquad \forall\, l, m \in \mathcal{L}, f \in \mathcal{O}_X.$$

Example 3.2. The trivial sheaf $\mathcal{L} = 0$ and the tangent sheaf $\mathcal{L} = \Theta_X$, with anchor map equal to the identity, are Lie algebroids. A Lie algebroid over $\operatorname{Spec} \mathbb{K}$ is exactly a Lie algebra over the field $\mathbb{K}$. Every sheaf of Lie algebras with $\mathcal{O}_X$-linear bracket can be considered as a Lie algebroid over X with trivial anchor map.

Example 3.3 (See [16] for details)**.** Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module, then the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol has a natural structure of Lie algebroid. Since Θ_X is the sheaf of $\mathbb{K}$-linear derivations of $\mathcal{O}_X$, we can introduce the sheaf

$$\begin{aligned}&P(\Theta_X, \mathcal{E}) = \\ &\{(\theta, \phi) \in \Theta_X \times \mathcal{E}nd_{\mathbb{K}}(\mathcal{E}) \mid \phi(fe) = f\phi(e) + \theta(f)e,\ f \in \mathcal{O}_X,\ e \in \mathcal{E}\}.\end{aligned}$$

Denoting by $a \colon P(\Theta_X, \mathcal{E}) \to \Theta_X$ the projection on the first factor, we have an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \to P(\Theta_X, \mathcal{E}) \xrightarrow{a} \Theta_X \to 0$$

and it is immediate to check that $P(\Theta_X, \mathcal{E})$ is a Lie algebroid with anchor map a. Moreover, the map $P(\Theta_X, \mathcal{E}) \to \mathcal{E}nd_{\mathbb{K}}(\mathcal{E})$, $(\theta, \phi) \mapsto \phi$, is injective and its image is the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol.

The *de Rham algebra of* $\mathcal{L}$ is defined as the sheaf of commutative graded algebras

$$\Omega^*(\mathcal{L}) = \bigoplus_{k\geq 0}^{\operatorname{rank}\mathcal{L}} \Omega^k(\mathcal{L}), \qquad \Omega^k(\mathcal{L}) = \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{L}[1]^{\odot k}, \mathcal{O}_X),$$

equipped with the convolution product. Notice that $\mathcal{L}[1]$ is just $\mathcal{L}$ considered as a graded sheaf concentrated in degree -1, hence $\Omega^*(\mathcal{L})$ is a locally free graded sheaf with $\Omega^k(\mathcal{L})$ in degree k. By definition the convolution product is the dual of the coproduct Δ on the graded symmetric algebra $S(\mathcal{L}[1]) = \bigoplus_k \mathcal{L}[1]^{\odot k}$, defined by

$$\Delta(l_1, \ldots, l_n) = \sum_{a=0}^{n} \sum_{\sigma \in S(a,n-a)} \epsilon(\sigma)(l_{\sigma(1)}, \ldots, l_{\sigma(a)}) \otimes (l_{\sigma(a+1)}, \ldots, l_{\sigma(n)}),$$

where $\epsilon(\sigma)$ is the Koszul sign and $S(a, n-a)$ is the subset of unshuffles. More concretely, for $\omega \in \Omega^k(\mathcal{L})$ and $\eta \in \Omega^j(\mathcal{L})$ we have

$$(\omega\eta)(l_1, \ldots, l_{k+j}) \sum_{\sigma \in S(k,j)} (-1)^{\sigma} \omega(l_{\sigma(1)}, \ldots, l_{\sigma(k)}) \eta(l_{\sigma(k+1)}, \ldots, l_{\sigma(k+j)}).$$

Notice that the *contraction product*

$$\mathcal{L} \times \Omega^{k+1}(\mathcal{L}) \xrightarrow{\lrcorner} \Omega^k(\mathcal{L}), \qquad (l \lrcorner \omega)(l_1, \ldots, l_k) = \omega(l, l_1, \ldots, l_k),$$

is $\mathcal{O}_X$-bilinear and satisfies the Koszul identity

$$l \lrcorner (\omega\eta) = (l \lrcorner \omega)\eta + (-1)^{|\omega|}\omega(l \lrcorner \eta).$$

More generally, if $\mathcal{C}^*$ is a sheaf of graded associative $\mathcal{O}_X$-algebras, the same holds for

$$\Omega^*(\mathcal{L}, \mathcal{C}^*) = \Omega^*(\mathcal{L}) \otimes \mathcal{C}^* = \bigoplus_{k\geq 0} \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{L}[1]^{\odot k}, \mathcal{C}^*).$$

The *de Rham differential of* $\mathcal{L}$, denoted by $d_{\mathcal{L}}\colon \Omega^k(\mathcal{L}) \to \Omega^{k+1}(\mathcal{L})$, is defined by the formula (see e.g. [20]):

$$\begin{aligned} d_{\mathcal{L}}(\omega)(l_0,\dots,l_k) &= \sum_{i=0}^{n}(-1)^i a(l_i)(\omega(l_0,\dots,\widehat{l_i},\dots,l_k)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([l_i,l_j],l_0,\dots,\widehat{l_i},\dots,\widehat{l_j},\dots,l_k). \end{aligned}$$

In particular for $\omega \in \Omega^0(\mathcal{L}) = \mathcal{O}_X$ we have $l \lrcorner d_{\mathcal{L}}(\omega) = d_{\mathcal{L}}(\omega)(l) = a(l)(\omega)$, for every $l \in \mathcal{L}$. By definition $\Omega^k(\Theta_X)[k] = \Omega^k_X$ is the sheaf of k-differential forms on X and the global formula for the exterior derivative implies that d_Θ is the usual de Rham differential.

For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we denote $\Omega^*(\mathcal{L},\mathcal{F}) = \Omega^*(\mathcal{L}) \otimes \mathcal{F}$ and by

$$\begin{aligned} &\Omega^*(\mathcal{L}) \times \Omega^*(\mathcal{L},\mathcal{F}) \xrightarrow{\cdot} \Omega^*(\mathcal{L},\mathcal{F}) : \\ &\quad \eta \cdot (\sum_i \mu_i \otimes e_i) = \sum_i \eta\mu_i \otimes e_i, \quad \mu_i \in \Omega^*(\mathcal{L}),\ e_i \in \mathcal{F}, \\ &\mathcal{L} \times \Omega^*(\mathcal{L},\mathcal{F}) \xrightarrow{\lrcorner} \Omega^*(\mathcal{L},\mathcal{F}) : \\ &\quad l \lrcorner (\sum_i \mu_i \otimes e_i) = \sum_i l \lrcorner \mu_i \otimes e_i, \quad \mu_i \in \Omega^*(\mathcal{L}),\ e_i \in \mathcal{F}. \end{aligned}$$

Definition 3.4. Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, an $\mathcal{L}$-connection ∇ on $\mathcal{F}$ is a $\mathbb{K}$-linear morphism of graded sheaves of degree 1

$$\nabla\colon \mathcal{F} \to \Omega^1(\mathcal{L},\mathcal{F}) = \Omega^1(\mathcal{L}) \otimes \mathcal{F},$$

such that

$$\nabla(fe) = d_{\mathcal{L}}(f)\cdot e + f\nabla(e), \qquad \forall\ f \in \mathcal{O}_X,\ e \in \mathcal{F}.$$

As in the usual case, every $\mathcal{L}$-connection ∇ admits a unique extension to $\mathbb{K}$-linear morphism of graded sheaves of $\mathcal{O}_X$-modules of degree 1

$$\nabla\colon \Omega^*(\mathcal{L},\mathcal{F}) \to \Omega^*(\mathcal{L},\mathcal{F})$$

such that

$$\nabla(fe) = d_{\mathcal{L}}(f)\cdot e + (-1)^{|f|} f\nabla(e), \qquad \forall\ f \in \Omega^*(\mathcal{L}),\ e \in \Omega^*(\mathcal{L},\mathcal{F}),$$

and the connection is called flat if $\nabla^2 = 0$.

Remark 3.5. Since the contraction product $\lrcorner\colon \mathcal{L}\times\Omega^1(\mathcal{L})\to\mathcal{O}_X$ is nondegenerate, every $\mathbb{K}$-linear morphism of sheaves $\nabla\colon \mathcal{F}\to\Omega^1(\mathcal{L},\mathcal{F})$ is completely determined by the morphism of $\mathcal{O}_X$-modules

$$\mathcal{L}\to\mathcal{H}om_{\mathbb{K}}(\mathcal{F},\mathcal{F}),\quad l\mapsto\nabla_l:\qquad \nabla_l(e)=l\lrcorner\nabla(e),\quad e\in\mathcal{F}.$$

It is straightforward to verify that ∇ is a connection if and only if

$$\nabla_l(fe)=a(l)(f)e+f\nabla_l(e),\qquad \forall\, f\in\mathcal{O}_X,\ l\in\mathcal{L},\ e\in\mathcal{F}.$$

A simple computation shows that the curvature is given by the formula

$$\nabla^2(l,m)(e)=\nabla_l\nabla_m(e)-\nabla_m\nabla_l(e)-\nabla_{[l,m]}(e),\qquad \forall\, l,m\in\mathcal{L},e\in\mathcal{F}.$$

For instance, if $\mathcal{F}$ is locally free and $\mathcal{L}=\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ (with trivial anchor map), then the natural inclusion $\mathcal{L}\to\mathcal{E}nd_{\mathbb{K}}(\mathcal{F})$ is a flat connection.

Since $\mathcal{L}$ is locally free we have natural isomorphisms

$$\begin{aligned}\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))&=\mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{F},\Omega^*(\mathcal{L},\mathcal{F}))\\&=\mathcal{H}om^*_{\Omega^*(\mathcal{L})}(\Omega^*(\mathcal{L},\mathcal{F}),\Omega^*(\mathcal{L},\mathcal{F})\end{aligned}$$

and, therefore, a natural identification of $\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$ with the subset of morphisms of graded sheaves $f\colon\Omega^*(\mathcal{L},\mathcal{F})\to\Omega^*(\mathcal{L},\mathcal{F})$ such that $f(\alpha\cdot\beta)=(-1)^{|f||\alpha|}\alpha\cdot f(\beta)$ for every $\alpha\in\Omega^*(\mathcal{L})$, $\beta\in\Omega^*(\mathcal{L},\mathcal{F})$.

The following lemma is a completely straightforward generalisation of well known facts about connections and curvature.

Lemma 3.6. *Let $\nabla\colon\Omega^*(\mathcal{L},\mathcal{F})\to\Omega^*(\mathcal{L},\mathcal{F})$ be an $\mathcal{L}$-connection, then $\nabla^2\in\Omega^2(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$ and $[\nabla,f]$ belongs to $\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$ for every $f\in\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$.*

In particular, $(\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})),d=[\nabla,-],\nabla^2)$ is a properly defined sheaf of curved DG-algebras over X.

If in addition $\mathcal{F}$ admits a locally free resolution, then the trace map $\mathrm{Tr}\colon\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})\to\mathcal{O}_X$, which is a morphism of sheaves of Lie algebras, is properly defined. By an analogous calculation to that of [18, Lemma 2.6], its extension

$$\begin{aligned}&\mathrm{Tr}\colon\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))\to\Omega^*(\mathcal{L}),\\&\mathrm{Tr}(\omega\cdot f)=\omega\cdot\mathrm{Tr}(f),\quad \omega\in\Omega^*(\mathcal{L}),\ f\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}),\end{aligned}\tag{3.1}$$

is a trace map in the sense of Definition 2.5.

Definition 3.7. An $\mathcal{L}$-module is a pair $(\mathcal{F}, \nabla)$ consisting of a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and a flat $\mathcal{L}$-connection ∇ on $\mathcal{F}$. An $\mathcal{L}$-module $(\mathcal{F}, \nabla)$ is said to be coherent (resp.: torsion free, locally free) if $\mathcal{F}$ is coherent (resp.: torsion free, locally free) as an $\mathcal{O}_X$-module.

Example 3.8. Every $\mathcal{O}_X$-module has a unique structure of module over the trivial Lie algebroid $\mathcal{L} = 0$.

Example 3.9. For every Lie algebroid $\mathcal{L}$, the pair $(\mathcal{O}_X, d_{\mathcal{L}})$ is an $\mathcal{L}$-module. More generally every choice of a basis on a free $\mathcal{O}_X$-module gives an $\mathcal{L}$-module structure.

Every $\mathcal{L}$-connection ∇ on a locally free $\mathcal{O}_X$-module $\mathcal{F}$ naturally induces $\mathcal{L}$-connections on the associated sheaves $\mathcal{F}^\vee, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \mathcal{F}^{\wedge k}$ etc.. If $\mathcal{F}$ is an $\mathcal{L}$-module, then also $\mathcal{F}^\vee$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$, $\mathcal{F}^{\wedge k}$ etc. are $\mathcal{L}$-modules in a natural way.

Example 3.10. Let (X, π) be a smooth Poisson variety, and denote by $\{-,-\}$ the Poisson bracket on the sheaf of functions $\mathcal{O}_X$. The cotangent sheaf Ω^1_X of holomorphic differential 1-forms on X has an induced structure of holomorphic Lie algebroid with the anchor $a(df) := \{f, -\}$ and the bracket $[df, dg] := d\{f, g\}$ for all $f, g \in \mathcal{O}_X$ (this defines a and $[-,-]$ completely since Ω^1_X is generated by exact forms as an $\mathcal{O}_X$-module), see e.g. [11] for more details. An Ω^1_X-module is the same as a coherent sheaf $\mathcal{E}$ together with a sheaf of Poisson modules structure on the sections of $\mathcal{E}$. Namely, continuing to denote by $\{-,-\}$ the Poisson bracket on $\mathcal{E}$, the associated connection is defined by

$$\nabla \colon \Omega^1_X \to \mathcal{E}nd_{\mathbb{K}}(\mathcal{E}), \qquad df \mapsto \nabla_{df}, \qquad \nabla_{df} e := \{f, e\} \quad \forall f \in \mathcal{O}_X,\ e \in \mathcal{E}.$$

The fact that ∇ is an Ω^1_X-connection on $\mathcal{E}$ is equivalent to the Poisson identities

$$\{f, ge\} = \{f, g\}e + g\{f, e\}, \qquad \{fg, e\} = f\{g, e\} + g\{f, e\},$$

while the flatness of ∇ is equivalent to the Jacobi identity

$$\{\{f, g\}, e\} = \{f, \{g, e\}\} - \{g, \{f, e\}\}.$$

Definition 3.11. Let $\mathcal{L}$ be a Lie algebroid over X. The hypercohomology of the complex $(\Omega^*(\mathcal{L}), d_{\mathcal{L}})$ is called the *Lie algebroid cohomology of* $\mathcal{L}$, and it is denoted by $\mathbb{H}^*(\mathcal{L})$,

For an $\mathcal{L}$-module $(\mathcal{F}, \nabla)$ the complex $(\Omega^*(\mathcal{L}, \mathcal{F}), \nabla)$ is called the *standard complex* of $(\mathcal{F}, \nabla)$ and its hypercohomology, denoted by $\mathbb{H}^*(\mathcal{L}; \mathcal{F})$, is called the *Lie algebroid cohomology of* $\mathcal{L}$ *with coefficients in* $\mathcal{F}$.

Notice that $\mathbb{H}^*(\mathcal{L}) = \mathbb{H}^*(\mathcal{L}; \mathcal{O}_X)$, where $\mathcal{O}_X$ carries the $\mathcal{L}$-module structure of Example 3.9. The notion of standard complex is borrowed from [20], while for Lie algebroid cohomology we follow the notation of [1, 4].

Example 3.12. The Lie algebroid cohomology of the tangent sheaf Θ_X is the de Rham cohomology of X. The Lie algebroid cohomology of a Lie algebroid $\mathfrak{g}$ over $\operatorname{Spec}\mathbb{K}$ is the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$.

4. Infinitesimal deformations of locally free $\mathcal{L}$-modules

In this section we describe a DG-Lie algebra controlling the infinitesimal deformations of a locally free $\mathcal{L}$-module. In order to do so, we give a brief review of the Thom–Whitney totalisation.

Let $\mathcal{L}$ be a Lie algebroid over X and let $(\mathcal{E}, \nabla)$ be an $\mathcal{L}$-module, with $\mathcal{E}$ locally free as an $\mathcal{O}_X$-module. Let B be an Artin local $\mathbb{K}$-algebra with residue field $\mathbb{K}$. We denote by $X_B = X \times \operatorname{Spec}(B)$, by $p_X \colon X \times \operatorname{Spec}(B) \to X$ the projection onto the first factor, and by $\imath_X : X \to X \times \operatorname{Spec}(B)$ the inclusion induced by $B \to B/\mathfrak{m}_B = \mathbb{K}$. We notice that the pull-back sheaf $p_X^*\mathcal{L} = \mathcal{L} \otimes_{\mathbb{K}} B$ has a natural structure of Lie algebroid over X_B, with the Lie bracket extending B-bilinearly the one on $\mathcal{L}$. Moreover, it is easy to check that a $p_X^*\mathcal{L}$-module $\mathcal{F}$ on X_B restricts to an $\mathcal{L}$-module $\imath_X^*\mathcal{F}$ on the central fibre X.

Definition 4.1. A deformation of the $\mathcal{L}$-module $(\mathcal{E}, \nabla)$ over $\operatorname{Spec}(B)$ consists of the data of a deformation $\mathcal{E}_B$ of $\mathcal{E}$ over X_B and a $p_X^*\mathcal{L}$-module structure

$$\nabla_B \colon \mathcal{E}_B \to \Omega^1(p_X^*\mathcal{L}, \mathcal{E}_B) = \Omega^1(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{E}_B$$

such that the restriction $\imath_X^*\mathcal{E}_B$ to X, with the naturally induced $\mathcal{L}$-module structure, coincides with $(\mathcal{E}, \nabla)$. An isomorphism of deformations $(\mathcal{E}_B, \nabla_B) \to (\mathcal{E}'_B, \nabla'_B)$ is an isomorphism of deformations of sheaves $\phi \colon \mathcal{E}_B \to \mathcal{E}'_B$ such that $\phi\nabla_B = \nabla'_B\phi$.

We want to describe a DG-Lie algebra controlling the infinitesimal deformations of $(\mathcal{E}, \nabla)$. To this end we first review the definition and some of the main properties of the Thom–Whitney totalisation functor Tot; for more details see e.g. [9, 10, 15, 22]. The Thom–Whitney totalisation is a functor from the category of semicosimplicial DG-vector spaces to the category of DG-vector spaces. For every $n \geq 0$ consider

$$A_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(1 - \sum_i t_i, \sum_i dt_i)}$$

the commutative differential graded algebra of polynomial differential forms on the affine standard n-simplex, and the maps

$$\delta_k^*\colon A_n \to A_{n-1}, \quad 0 \le k \le n \qquad \delta_k^*(t_i) = \begin{cases} t_i & i < k \\ 0 & i = k \\ t_{i-1} & i > k. \end{cases}$$

Definition 4.2. The Thom–Whitney totalisation of a semicosimplicial DG-vector space V

$$V: \qquad V_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} V_1 \underset{\delta_2}{\overset{\delta_0}{\overset{\delta_1}{\Rrightarrow}}} V_2 \;\;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\;\; \cdots$$

is the DG-vector space

$$\operatorname{Tot}(V) = \{(x_n) \in \prod_{n\ge 0} A_n \otimes_{\mathbb{K}} V_n \mid (\delta_k^* \otimes \operatorname{Id})x_n = (\operatorname{Id} \otimes \delta_k)x_{n-1} \ \forall\ 0 \le k \le n\},$$

with differential induced by the one on $\prod_{n\ge 0} A_n \otimes V_n$.

If $f\colon V \to W$ is a morphism of semicosimplicial DG-vector spaces, then $\operatorname{Tot}(f)\colon \operatorname{Tot}(V) \to \operatorname{Tot}(W)$ is defined as the restriction of the map

$$\prod \operatorname{Id} \otimes f\colon \prod_{n\ge 0} A_n \otimes_{\mathbb{K}} V_n \to \prod_{n\ge 0} A_n \otimes_{\mathbb{K}} W_n.$$

The Tot functor is exact: given semicosimplicial DG-vector spaces V, W, Z and morphisms $f\colon V \to W$, $g\colon W \to Z$ such that for every $n \ge 0$ the sequence

$$0 \longrightarrow V_n \xrightarrow{\ f\ } W_n \xrightarrow{\ g\ } Z_n \longrightarrow 0$$

is exact, one obtains an exact sequence

$$0 \longrightarrow \operatorname{Tot}(V) \xrightarrow{\ f\ } \operatorname{Tot}(W) \xrightarrow{\ g\ } \operatorname{Tot}(Z) \longrightarrow 0,$$

see e.g. [8, 22].

Given two semicosimplicial DG-vector spaces V and W, then $\operatorname{Tot}(V \times W)$ is naturally isomorphic to $\operatorname{Tot}(V) \times \operatorname{Tot}(W)$. An important consequence is the preservation of multiplicative structures; in particular, we will use the fact that the functor Tot sends semicosimplicial DG-Lie algebras to DG-Lie algebras.

Example 4.3. Let $(\mathcal{E}^*, \delta)$ be a bounded below complex of quasi-coherent sheaves on X, and let $\mathcal{U} = \{U_i\}$ be an open affine cover of X. Denote by

$U_{i_1\cdots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$, and consider the semicosimplicial DG-vector space of Čech cochains:

$$\mathcal{E}^*(\mathcal{U}): \quad \prod_i \mathcal{E}^*(U_i) \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{E}^*(U_{ij}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\Rrightarrow}}} \prod_{i,j,k} \mathcal{E}^*(U_{ijk}) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots .$$

According to Whitney integration theorem, there exists a natural quasi-isomorphism

$$I\colon \operatorname{Tot}(\mathcal{U}, \mathcal{E}^*) \to C^*(\mathcal{U}, \mathcal{E}^*)$$

where $C^*(\mathcal{U}, \mathcal{E}^*) = \oplus_i C^*(\mathcal{U}, \mathcal{E}^i)[-i]$ is the hypercomplex of Čech cochains (see [26] for the C^∞ version, [12, 19, 22, 23] for the algebraic version used here). Therefore the cohomology of $\operatorname{Tot}(\mathcal{U}, \mathcal{E}^*)$ is isomorphic to the hypercohomology of the complex of sheaves $\mathcal{E}^*$ and then the quasi-isomorphism class of $\operatorname{Tot}(\mathcal{U}, \mathcal{E}^*)$ does not depend on the affine open cover, since $H^i(\operatorname{Tot}(\mathcal{U}, \mathcal{E}^*)) = \mathbb{H}^i(X, \mathcal{E}^*)$ and the map I commutes with refinements of affine covers.

For our later application it is important to point out that there exists a natural inclusion of DG-vector spaces $\Gamma(X, \mathcal{E}^*) \to \operatorname{Tot}(\mathcal{U}, \mathcal{E}^*)$ such that the restriction of I to $\Gamma(X, \mathcal{E}^*)$ is the natural inclusion map

$$i\colon \Gamma(X, \mathcal{E}^*) \to \prod_i \mathcal{E}^*(U_i), \qquad i(s) = \{s_{|U_i}\}.$$

In fact, $\delta_0 i = \delta_1 i$, therefore

$$\delta_{j_k}\delta_{j_{k-1}}\cdots\delta_{i_1} i = \delta_0^k i, \quad \text{for every } 0 \le j_s \le s,$$

and this implies that

$$\iota\colon \Gamma(X, \mathcal{E}^*) \to \operatorname{Tot}(\mathcal{U}, \mathcal{E}^*), \quad \iota(a) = (1 \otimes i(a), 1 \otimes \delta_0 i(a), 1 \otimes \delta_0^2 i(a), \ldots) \tag{4.1}$$

is a properly defined injective morphism of DG-vector spaces.

For later use we point out that for every quasi-coherent sheaf $\mathcal{F}$ and every affine open cover $\mathcal{U}$, the inclusion $\Gamma(X, \mathcal{F}) \subset \operatorname{Tot}(\mathcal{U}, \mathcal{F})$ induces an isomorphism $\Gamma(X, \mathcal{F}) \cong H^0(\operatorname{Tot}(\mathcal{U}, \mathcal{F}))$.

Returning to our initial situation of a locally free $\mathcal{L}$-module $(\mathcal{E}, \nabla)$, since the $\mathcal{L}$-connection ∇ is flat, by Lemma 3.6 $(\Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), d = [\nabla, -])$ is a sheaf of locally free DG-algebras, which gives rise to a sheaf of locally free DG-Lie algebras $(\Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), d = [\nabla, -], [-, -])$.

Theorem 4.4. *In the above situation, for every affine open cover* $\mathcal{U} = \{U_i\}$, *the DG-Lie algebra* $\mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$ *controls the infinitesimal deformations of* $(\mathcal{E}, \nabla)$. *In particular* $\mathbb{H}^1(\mathcal{L}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ *is the space of first order deformations and* $\mathbb{H}^2(\mathcal{L}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ *is an obstruction space.*

Proof. This result is probably well known to experts, at least in the case $\mathcal{L} = \Theta_X$, cf. [13, Thm. 6.8], and follows easily from Hinich's theorem on descent of Deligne groupoids. According to [14], it is sufficient to check that locally the Deligne groupoid of $\Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is equivalent to the groupoid of deformations of $(\mathcal{E}, \nabla)$.

In order to check this, it is not restrictive to assume X affine. Given an Artin ring B as above, up to isomorphism every deformation of $\mathcal{E}$ is trivial, i.e. $\mathcal{E}_B = \mathcal{E} \otimes_{\mathbb{K}} B$ and $\mathcal{H}om_{\mathcal{O}_{X_B}}(\mathcal{E}_B, \mathcal{E}_B) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \otimes_{\mathbb{K}} B$. Denoting by $\nabla_0 \colon \mathcal{E}_B \to \Omega^1(p_X^*\mathcal{L}, \mathcal{E}_B) = \Omega^1(\mathcal{L}, \mathcal{E}) \otimes_{\mathbb{K}} B$ the natural B-linear extension of ∇, every deformation of ∇ over B is of the form $\nabla_0 + x$, with $x \in \Gamma(X, \Omega^1(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))) \otimes_{\mathbb{K}} \mathfrak{m}_B$, and the flatness condition $(\nabla_0 + x)^2 = 0$ is exactly the Maurer–Cartan equation $dx + \frac{1}{2}[x, x] = 0$.

To conclude the proof we only need to show that two solutions of the Maurer–Cartan equation x, y are gauge equivalent if and only if there exists an isomorphism of deformations $\phi \colon \mathcal{E}_B \to \mathcal{E}_B$ such that $\phi(\nabla_0 + x)\phi^{-1} = \nabla_0 + y$. Every ϕ as above is of the form $\phi = e^a$, with $a \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \otimes_{\mathbb{K}} \mathfrak{m}_B$, and then the condition $\phi(\nabla_0 + x)\phi^{-1} = \nabla_0 + y$ is equivalent to

$$\nabla_0 + y = e^{[a,-]}(\nabla_0 + x) = \nabla_0 + x + \sum_{n=0}^{\infty} \frac{[a,-]^n}{(n+1)!}([a, x] - da),$$

which is the same as $y = e^a * x$, where $*$ denotes the gauge action. $\square$

Remark 4.5. One can consider a different deformation problem, namely the deformation of pairs (bundle, $\mathcal{L}$-connection) without requiring the vanishing of the curvature. The same argument as above shows that this deformation problem is controlled by the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \Omega^{\leq 1}(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$, while it is well known that $\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ controls the deformations of $\mathcal{E}$ [9].

5. Lie pairs

Definition 5.1. A *Lie pair* $(\mathcal{L}, \mathcal{A})$ of Lie algebroids over X is a pair consisting of a Lie algebroid $\mathcal{L}$ over X and a Lie subalgebroid $\mathcal{A} \subset \mathcal{L}$ such that the quotient sheaf $\mathcal{L}/\mathcal{A}$ is locally free.

Let $(\mathcal{L},\mathcal{A})$ be a Lie pair. Since $\mathcal{L}/\mathcal{A}$ is assumed locally free we have a surjective restriction map $\varrho\colon \Omega^*(\mathcal{L})\to\Omega^*(\mathcal{A})$, which is a morphism of sheaves of commutative differential graded algebras. The powers of its kernel give a finite decreasing filtration of differential graded ideal sheaves

$$\Omega^*(\mathcal{L})=\mathcal{G}^*_0\supset\mathcal{G}^*_1=\ker(\varrho)\supset\cdots\mathcal{G}^*_r=(\ker(\varrho))^{(r)}\supset\cdots.$$

If we forget the de Rham differential, we can immediately see that $\mathcal{G}^*_p$ is the image of the morphism of graded $\mathcal{O}_X$-modules

$$\bigwedge^p(\mathcal{L}/\mathcal{A})^\vee[-p]\otimes\Omega^*(\mathcal{L})\to\Omega^*(\mathcal{L}),$$

and we have natural isomorphisms of graded sheaves

$$\frac{\mathcal{G}^*_p}{\mathcal{G}^*_{p+1}}[p]\cong\bigwedge^p(\mathcal{L}/\mathcal{A})^\vee\otimes\Omega^*(\mathcal{A}). \tag{5.1}$$

In particular, $\mathcal{G}^i_p\neq 0$ only for pairs (i,p) such that $p\le i\le\operatorname{rank}\mathcal{L}$ and $p\le\operatorname{rank}\mathcal{L}-\operatorname{rank}\mathcal{A}$. For instance, whenever $i=2$ we have $\mathcal{G}^2_0=\Omega^2(\mathcal{L})$, $\mathcal{G}^2_3=0$,

$$\begin{aligned}\mathcal{G}^2_1&=\{\phi\in\Omega^2(\mathcal{L})\mid\phi(a,b)=0\ \forall a,b\in\mathcal{A}\},\\ \mathcal{G}^2_2&=\{\phi\in\Omega^2(\mathcal{L})\mid\phi(a,l)=0\ \forall a\in\mathcal{A},\ l\in\mathcal{L}\}.\end{aligned}$$

Recall that $\mathbb{H}^*(\mathcal{L})=\mathbb{H}^*(X,\Omega^*(\mathcal{L}))$ denotes the Lie algebroid cohomology of $\mathcal{L}$, as in Definition 3.11.

Definition 5.2. In the above notation, the filtration $\Omega^*(\mathcal{L})=\mathcal{G}^*_0\supset\mathcal{G}^*_1\cdots$ is called the *Leray filtration* of the Lie pair $(\mathcal{L},\mathcal{A})$. We shall call the associated spectral sequence in hypercohomology

$$E_1^{p,q}=\mathbb{H}^q\left(X,\mathcal{G}^*_p/\mathcal{G}^*_{p+1}[p]\right)\Rightarrow\mathbb{H}^{p+q}(\mathcal{L})$$

the *Leray spectral sequence* of the Lie pair $(\mathcal{L},\mathcal{A})$.

The name *Leray filtration* is motivated by Example 5.4 below. Notice however that for the Lie pair $(\Theta_X,0)$ the Leray filtration coincides with the Hodge filtration on differential forms.

Given an $\mathcal{A}$-module $(\mathcal{E},\nabla)$, we can also define a filtration $\mathcal{G}^*_r(\mathcal{E})=\mathcal{G}^*_r\otimes\mathcal{E}$ of the graded sheaf $\Omega^*(\mathcal{L},\mathcal{E})$; equivalently, $\mathcal{G}^*_r(\mathcal{E})$ may be defined as the image of the multiplication map

$$\mathcal{G}^*_r \otimes \Omega^*(\mathcal{L},\mathcal{E}) \to \Omega^*(\mathcal{L},\mathcal{E}).$$

If ∇' is an $\mathcal{L}$-connection on $\mathcal{E}$ extending ∇, then by Leibniz rule the filtration $\mathcal{G}^*_r(\mathcal{E})$ is preserved by ∇' and we can immediately see that the maps induced on the quotients $\mathcal{G}^*_r(\mathcal{E})/\mathcal{G}^*_{r+1}(\mathcal{E})$ are independent of ∇' and square-zero operators. Notice also that the curvature of ∇' belongs to $\mathcal{G}^2_2(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ if and only if $[\nabla'_l,\nabla'_a]=\nabla'_{[l,a]}$ for every $l\in\mathcal{L}$ and $a\in\mathcal{A}$.

Since ∇ always admits extensions locally (see Remark 7.3 below), for every r there is a properly defined structure of differential graded sheaf on $\mathcal{G}^*_r(\mathcal{E})/\mathcal{G}^*_{r+1}(\mathcal{E})$.

It is interesting to point out that the groups $E_1^{p,q}=\mathbb{H}^q\left(X,\mathcal{G}^*_p/\mathcal{G}^*_{p+1}[p]\right)$, and more generally the hypercohomology groups of $\mathcal{G}^*_r(\mathcal{E})/\mathcal{G}^*_{r+1}(\mathcal{E})$, are cohomology groups of $\mathcal{A}$ with coefficients in suitable $\mathcal{A}$-modules. In fact, there is a canonical $\mathcal{A}$-module structure on the quotient sheaf $\mathcal{L}/\mathcal{A}$ given by the *Bott connection*: denoting by $\pi\colon\mathcal{L}\to\mathcal{L}/\mathcal{A}$ the projection, the connection is defined by the formula

$$\nabla^B_a\pi(b)=\pi([a,b]),\qquad \forall\, a\in\mathcal{A},\ b\in\mathcal{L}.$$

Therefore, there is a canonical $\mathcal{A}$-module structure on $\bigwedge^r(\mathcal{L}/\mathcal{A})^\vee$ for every r.

Lemma 5.3. *Let $(\mathcal{L},\mathcal{A})$ be a Lie pair and let $\mathcal{E}$ be an $\mathcal{A}$-module. Then for every $r\geq 1$, the differential graded sheaf $\frac{\mathcal{G}^*_r(\mathcal{E})}{\mathcal{G}^*_{r+1}(\mathcal{E})}[r]$ is isomorphic to the standard complex of the $\mathcal{A}$-module $\bigwedge^r(\mathcal{L}/\mathcal{A})^\vee\otimes\mathcal{E}$. In particular, the Leray spectral sequence of the pair $(\mathcal{L},\mathcal{A})$ is*

$$E_1^{p,q}=\mathbb{H}^q\left(\mathcal{A};\bigwedge^p(\mathcal{L}/\mathcal{A})^\vee\right).$$

Proof. For every $r\geq 1$, consider the isomorphism of graded sheaves

$$\varphi\colon \frac{\mathcal{G}^*_r}{\mathcal{G}^*_{r+1}}[r]\to\bigwedge^r(\mathcal{L}/\mathcal{A})^\vee\otimes\Omega^*(\mathcal{A})$$

of (5.1). We begin by showing that this is an isomorphism of complexes, where the differential on the left is induced by $d_{\mathcal{L}}$, and the differential on the right is given by the dual connection to the Bott connection.

Denote by ∇^B the Bott connection on $\mathcal{L}/\mathcal{A}$, and by $\nabla^{B,\vee}$ the induced connection on $\bigwedge^r(\mathcal{L}/\mathcal{A})^\vee$ for every $r \geq 0$. We denote by $a_\mathcal{L}$ and $a_\mathcal{A}$ the anchor maps of $\mathcal{L}$ and $\mathcal{A}$ respectively. Finally, denote by j the inclusion $j\colon \left(\frac{\mathcal{L}}{\mathcal{A}}\right)^\vee[-1] \to \Omega^1(\mathcal{L})$, and by π the projection $\pi\colon \mathcal{L} \to \frac{\mathcal{L}}{\mathcal{A}}$, so that for $m \in \mathcal{L}$ and $\eta \in \left(\frac{\mathcal{L}}{\mathcal{A}}\right)^\vee[-1]$ one has that $m \lrcorner j(\eta) = (j(\eta))(m) = \eta(\pi(m)) = \pi(m) \lrcorner \eta$. For every $\eta \in \mathcal{G}^*_r/\mathcal{G}^*_{r+1}[r]$, we prove that

$$\varphi(d_\mathcal{L}\eta) = \nabla^{B,\vee}\varphi(\eta).$$

Firstly, consider $\omega \in \mathcal{G}^*_1/\mathcal{G}^*_2[1] \cong (\mathcal{L}/\mathcal{A})^\vee \otimes \Omega^*(\mathcal{A})$ of degree zero, so that ω belongs to $\mathcal{G}^1_1/\mathcal{G}^1_2[1] = \mathcal{G}^1_1[1] \cong (\mathcal{L}/\mathcal{A})^\vee$. Then $d_\mathcal{L}\omega$ belongs to $\mathcal{G}^2_1[1]$, but we consider its projection to $\frac{\mathcal{G}^2_1}{\mathcal{G}^2_2}[1] \cong \left(\frac{\mathcal{L}}{\mathcal{A}}\right)^\vee \otimes \Omega^1(\mathcal{A})$. Hence we calculate it on $b \in \mathcal{A}$ and $\pi(l) \in \frac{\mathcal{L}}{\mathcal{A}}$, obtaining

$$\begin{aligned} d_\mathcal{L}\omega(b, \pi(l)) &= a_\mathcal{L}(b)(j(\omega)(l)) - a_\mathcal{L}(l)(j(\omega)(b)) - j(\omega)([b, l]) \\ &= a_\mathcal{A}(b)(\omega(\pi(l))) - a_\mathcal{L}(l)(\omega(\pi(b))) - \omega(\pi([b, l])) \\ &= a_\mathcal{A}(b)(\omega(\pi(l))) - \omega(\pi([b, l])), \end{aligned}$$

since $\pi(b) = 0$. The connection $\nabla^{B,\vee}$ for $\omega \in \left(\frac{\mathcal{L}}{\mathcal{A}}\right)^\vee$, $b \in \mathcal{A}$ and $\pi(l) \in \frac{\mathcal{L}}{\mathcal{A}}$ is given by

$$\begin{aligned} \pi(l) \lrcorner \nabla^{B,\vee}_b \omega = d_\mathcal{L}(\pi(l) \lrcorner \omega)(b) - (\nabla^B_b \pi(l)) \lrcorner \omega &= a_\mathcal{L}(b)(\pi(l) \lrcorner \omega) - (\pi([b, l])) \lrcorner \omega \\ = a_\mathcal{A}(b)(\omega(\pi(l))) - \omega(\pi([b, l])), \end{aligned}$$

therefore $d_\mathcal{L}\omega = \nabla^{B,\vee}\omega$.

Consider now $\eta \in \frac{\mathcal{G}^*_r}{\mathcal{G}^*_{r+1}}[r]$ of degree $k - r \geq 0$, which we can assume to be of the form $\eta = \omega_1 \cdots \omega_k$, with $\omega_i \in \Omega^1(\mathcal{L})[1]$ for $i = 1, \ldots, r$ such that $\varrho(\omega_1) = \cdots = \varrho(\omega_r) = 0$ (i.e., $\omega_i \in (\mathcal{L}/\mathcal{A})^\vee$ for $i = 1, \ldots, r$) and $\omega_j \in \Omega^1(\mathcal{L})$ for $j = r + 1, \ldots, k$ such that $\varrho(\omega_{r+1}), \ldots, \varrho(\omega_k) \neq 0$.

Then we have that

$$\varphi\colon \frac{\mathcal{G}^*_r}{\mathcal{G}^*_{r+1}}[r] \to \bigwedge^r \left(\frac{\mathcal{L}}{\mathcal{A}}\right)^\vee \otimes \Omega^*(\mathcal{A}), \quad \varphi(\eta) = \omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k),$$

and so

$$\begin{aligned} \nabla^{B,\vee}(\varphi(\eta)) &= \nabla^{B,\vee}(\omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k)) \\ &= \sum_{i=1}^r \omega_1 \cdots \nabla^{B,\vee}(\omega_i) \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=r+1}^{k}(-1)^{i-1-r}\omega_1\cdots\omega_r\otimes\varrho(\omega_{r+1})\cdots d_{\mathcal{A}}(\varrho(\omega_i))\cdots\varrho(\omega_k)\\
&=\sum_{i=1}^{r}\omega_1\cdots\nabla^{B,\vee}(\omega_i)\cdots\omega_r\otimes\varrho(\omega_{r+1})\cdots\varrho(\omega_k)\\
&\quad+\sum_{i=r+1}^{k}(-1)^{i-1-r}\omega_1\cdots\omega_r\otimes\varrho(\omega_{r+1})\cdots\varrho(d_{\mathcal{L}}(\omega_i))\cdots\varrho(\omega_k)\\
&=\sum_{i=1}^{r}\omega_1\cdots d_{\mathcal{L}}(\omega_i)\cdots\omega_r\otimes\varrho(\omega_{r+1})\cdots\varrho(\omega_k)\\
&\quad+\sum_{i=r+1}^{k}(-1)^{i-1-r}\omega_1\cdots\omega_r\otimes\varrho(\omega_{r+1})\cdots\varrho(d_{\mathcal{L}}(\omega_i))\cdots\varrho(\omega_k)=\\
&(\mathrm{Id}\otimes\varrho)\left(\sum_{i=1}^{r}\omega_1\cdots d_{\mathcal{L}}(\omega_i)\cdots\omega_k+\sum_{i=r+1}^{k}(-1)^{i-1-r}\omega_1\cdots d_{\mathcal{L}}(\omega_i)\cdots\omega_k\right)\\
&=\varphi(d_{\mathcal{L}}(\omega_1\cdots\omega_k))=\varphi(d_{\mathcal{L}}(\eta)).
\end{aligned}
$$

For every $r\geq 1$, it follows by (5.1) and by the definition of $\mathcal{G}^*_r(\mathcal{E})$ that there is an isomorphism of graded sheaves

$$
\varphi\otimes\mathrm{Id}_{\mathcal{E}}\colon\frac{\mathcal{G}^*_r(\mathcal{E})}{\mathcal{G}^*_{r+1}(\mathcal{E})}[r]\to\bigwedge^r(\mathcal{L}/\mathcal{A})^\vee\otimes\Omega^*(\mathcal{A})\otimes\mathcal{E}.
$$

Denote by ∇ the flat $\mathcal{A}$-connection on $\mathcal{E}$, and by ∇' a local extension of ∇ to an $\mathcal{L}$-connection on $\mathcal{E}$, which is such that $(\varrho\otimes\mathrm{Id})\nabla'=\nabla$ and which induces a differential on $\mathcal{G}^*_r(\mathcal{E})/\mathcal{G}^*_{r+1}(\mathcal{E})[r]$.

Take now $\eta\otimes e\in\mathcal{G}^*_r(\mathcal{E})[r]=(\mathcal{G}^*_r\otimes\mathcal{E})[r]$, then $\nabla'(\eta\otimes e)=d_{\mathcal{L}}\eta\otimes e+(-1)^{|\eta|}\eta\otimes\nabla'(e)$, and

$$
\begin{aligned}
(\varphi\otimes\mathrm{Id}_{\mathcal{E}})(\nabla'(\eta\otimes e))&=\varphi(d_{\mathcal{L}}\eta)\otimes e+(-1)^{|\eta|}\varphi(\eta)\otimes(\varphi\otimes\mathrm{Id}_{\mathcal{E}})\nabla'(e)\\
&=\nabla^{B,\vee}(\varphi(\eta))\otimes e+(-1)^{|\eta|}\varphi(\eta)\otimes(\varphi\otimes\mathrm{Id}_{\mathcal{E}})\nabla'(e).
\end{aligned}
$$

Since

$$
\begin{aligned}
(\nabla^{B,\vee}\otimes\nabla)((\varphi\otimes\mathrm{Id}_{\mathcal{E}})(\eta\otimes e))&=(\nabla^{B,\vee}\otimes\nabla)(\varphi(\eta)\otimes e)\\
&=\nabla^{B,\vee}(\varphi(\eta))\otimes e+(-1)^{|\eta|}\varphi(\eta)\otimes\nabla(e),
\end{aligned}
$$

it remains only to show that $(\varphi\otimes\mathrm{Id}_{\mathcal{E}})\nabla'(e)=\nabla(e)$ for every $e\in\mathcal{E}$, which follows by the definition of φ and by the fact that $(\varrho\otimes\mathrm{Id})\nabla'=\nabla$, since ∇' is a local extension of ∇. $\square$

Example 5.4. Let $f\colon X \to Y$ be a smooth morphism of irreducible smooth schemes. Then a Lie pair on X is given by (Θ_X, Θ_f), where

$$\Theta_f = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the subsheaf of relative vector fields: since f is smooth there exists an exact sequence of sheaves

$$0 \to \Theta_f \to \Theta_X \to f^*\Theta_Y \to 0.$$

In this case $\Omega^*(\mathcal{L}) = \Omega^*_X$ is the usual de Rham complex of X, while $\Omega^*(\mathcal{A}) = \Omega^*_{X/Y}$ is the relative de Rham complex and the filtration $\mathcal{G}^*_r$ is the algebraic analogue of the holomorphic Leray filtration, see [25, 17.2], [27, 2.16].

Since the relative de Rham differential is $f^{-1}\mathcal{O}_Y$-linear and $\mathcal{G}^*_1$ is the ideal sheaf generated by $f^{-1}\Omega^1_Y$, for every r we have a natural isomorphism of differential graded sheaves

$$\frac{\mathcal{G}^*_r}{\mathcal{G}^*_{r+1}} \cong f^{-1}\Omega^r_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega^*_{X/Y}$$

and therefore the first page of the Leray spectral sequence is

$$\begin{aligned} E^p_1 = \mathbb{H}^*\left(X, \mathcal{G}^*_p/\mathcal{G}^*_{p+1}\right) &= \mathbb{H}^*\left(X, f^{-1}\Omega^p_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega^*_{X/Y}\right) \\ &= \mathbb{H}^*\left(Y, \Omega^p_Y \otimes_{\mathcal{O}_Y} Rf_*\Omega^*_{X/Y}\right). \end{aligned}$$

It is an easy consequence of Deligne's results on Hodge theory that if X and Y are complex projective manifolds, then the Leray spectral sequence of the Lie pair (Θ_X, Θ_f) degenerates at E_1. In fact, by Hodge decomposition we have

$$Rf_*\Omega^*_{X/Y} = \oplus_q R^q f_*\Omega^*_{X/Y}[-q] \simeq \oplus_q \mathcal{O}_Y \otimes_{\mathbb{C}} R^q f_*\mathbb{C}[-q],$$

and then $E^p_1 = \oplus_q H^*(Y, \Omega^p_Y \otimes_{\mathbb{C}} R^q f_*\mathbb{C})[-p-q]$. Since $R^q f_*\mathbb{C}$ is a local system with real structure and Y is compact Kähler, according to [27, 2.11] (see also [13, 8.5]), the cohomology of $\Omega^p_Y \otimes_{\mathbb{C}} R^q f_*\mathbb{C}$ is a direct summand of the cohomology of $R^q f_*\mathbb{C}$. Since the (topological) Leray spectral sequence of $Rf_*\mathbb{C}$ degenerates at E_2 [7, 2.6.2], we have that E^p_1 is a direct summand of $\mathbb{H}^*(Y, Rf_*\mathbb{C}) = H^*(X, \mathbb{C}) = \mathbb{H}^*(X, \Omega^*_X)$.

For every locally free sheaf $\mathcal{E}$ on Y its pull-back $f^*\mathcal{E} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E}$ has a natural structure of Θ_f-module with connection

$$\nabla_\eta(g \otimes e) = \eta(g) \otimes e.$$

More generally, every Θ_f-module can be interpreted, as in [3], as a locally free sheaf on X which is endowed with a connection relative to f that is flat.

6. Reduced Atiyah classes

For every Lie algebroid $\mathcal{L}$ and every $\mathcal{O}_X$-module $\mathcal{F}$ we define the sheaf of $\mathcal{O}_X$-modules

$$P(\mathcal{L},\mathcal{F}) = \\ \{(l,\phi)\in\mathcal{L}\times\mathcal{H}om_{\mathbb{K}}(\mathcal{F},\mathcal{F}) \mid \phi(fe)=f\phi(e)+a(l)(f)e,\ f\in\mathcal{O}_X,\ e\in\mathcal{F}\}.$$

If $\mathcal{F}$ is coherent then also $P(\mathcal{L},\mathcal{F})$ is coherent. This has been proved in [16, Prop. 5.1] in the case $\mathcal{L}=\Theta_X$, while for the general case it is sufficient to observe that $P(\mathcal{L},\mathcal{F})=P(\Theta_X,\mathcal{F})\times_{\Theta_X}\mathcal{L}$.

Denoting by $p\colon P(\mathcal{L},\mathcal{F})\to\mathcal{L}$ the projection on the first factor, we have two exact sequences of (graded) $\mathcal{O}_X$-modules

(6.1)
$$\begin{gathered} 0\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})\to P(\mathcal{L},\mathcal{F})\xrightarrow{p}\mathcal{L}\,,\\ 0\to\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})\to\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{F})\xrightarrow{p}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})[-1], \end{gathered}$$

where the second sequence is obtained by applying the exact functor $\Omega^1(\mathcal{L})\otimes-$ to the first, and by noticing that $\Omega^1(\mathcal{L})\otimes\mathcal{L}=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})[-1]$. Now and in the sequel, we will consider $\mathrm{Id}_{\mathcal{L}}$ as a global section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})[-1]$, a graded sheaf concentrated in degree 1.

Lemma 6.1. *In the above setup, there exists a natural bijection between the set of $\mathcal{L}$-connections on $\mathcal{F}$ and global sections $D\in\Gamma(X,\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{F}))$ such that $p(D)=\mathrm{Id}_{\mathcal{L}}$.*

Proof. Let $l_1,\dots,l_r$ be a local frame of $\mathcal{L}$ with dual frame $\phi_1,\dots,\phi_r\in\Omega^1(\mathcal{L})$. Every $\mathbb{K}$-linear morphism $\nabla\colon\mathcal{F}\to\Omega^1(\mathcal{L},\mathcal{F})$ can be written locally as $\nabla=\sum_{i=1}^r\phi_i\cdot D_i$, with $D_i\in\mathcal{H}om_{\mathbb{K}}(\mathcal{F},\mathcal{F})$. By definition, ∇ is a connection if and only if for every $f\in\mathcal{O}_X$, $e\in\mathcal{F}$, and every i we have

$$D_i(fe)=l_i\lrcorner\nabla(fe)=a(l_i)(f)e+fD_i(e)$$

and this is equivalent to the fact that $\sum_{i=1}^r\phi_i\otimes(l_i,D_i)\in\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{F})$. $\square$

Lemma 6.2. *If $\mathcal{F}$ is a locally free sheaf, then the morphism $p\colon P(\mathcal{L},\mathcal{F})\to\mathcal{L}$ is surjective.*

Proof. We show this locally, with a proof similar to [16, Lemma 3.1]. Let R be a $\mathbb{K}$-algebra, let $(L, [-,-], a)$ be a Lie algebroid over R with anchor map $a\colon L \to \mathrm{Der}_{\mathbb{K}}(R,R)$, and let F be a free R-module with basis $\{e_i\}$. We set

$$\begin{aligned}&P(L,F) =\\ &\{(l,\phi) \in L \times \mathrm{Hom}_{\mathbb{K}}(F,F) \mid \phi(re) = r\phi(e) + a(l)(r)e,\ \forall r \in R,\ e \in F\},\end{aligned}$$

and show that the projection $p\colon P(L,F) \to L$ is surjective. For every $x \in L$, consider the derivation $a(x) \in \mathrm{Der}_{\mathbb{K}}(R,R)$, and set

$$w\Bigl(\sum_i r_i e_i\Bigr) := \sum_i a(x)(r_i)e_i, \quad r_i \in R.$$

Then the pair (x, w) belongs to $P(L,F)$. $\square$

Assume now that $\mathcal{F}$ is a locally free sheaf, so that the morphism

$$p\colon P(\mathcal{L},\mathcal{F}) \to \mathcal{L}$$

is surjective and we have an exact sequence of locally free graded sheaves of $\mathcal{O}_X$-modules

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to \Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E}) \xrightarrow{p} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})[-1] \to 0.$$

We can rewrite the above short exact sequence of graded sheaves concentrated in degree 1 as a sequence of sheaves in degree 0:

$$0 \to \Omega^1(\mathcal{L})[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to \Omega^1(\mathcal{L})[1] \otimes P(\mathcal{L},\mathcal{E}) \xrightarrow{p} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L}) \to 0.$$

By Lemma 6.1, there exists an $\mathcal{L}$-connection on $\mathcal{E}$ if and only if the identity on $\mathcal{L}$ lifts to a global section of $\Omega^1(\mathcal{L})[1] \otimes P(\mathcal{L},\mathcal{E})$. Writing

$$\mathrm{At}_{\mathcal{L}}(\mathcal{E}) = \partial(\mathrm{Id}_{\mathcal{L}}) \in H^1(X, \Omega^1(\mathcal{L})[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) = \mathrm{Ext}^1_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E}),$$

where ∂ is the connecting morphism in the cohomology long exact sequence, we have that $\mathrm{At}_{\mathcal{L}}(\mathcal{E}) = 0$ if and only if there exists an $\mathcal{L}$-connection on $\mathcal{E}$.

Equivalently, we can define $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ as the extension class of the short exact sequence

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to Q(\mathcal{L},\mathcal{E}) \xrightarrow{p} \mathcal{O}_X[-1] \to 0,$$

where, by definition, $Q(\mathcal{L},\mathcal{E}) = p^{-1}(\mathcal{O}_X[-1]\cdot \mathrm{Id}_{\mathcal{L}})$. More explicitly, in a local frame $l_1,\dots,l_r$ of $\mathcal{L}$, with dual frame $\phi_1,\dots,\phi_r \in \Omega^1(\mathcal{L})$, the elements of

$Q(\mathcal{L}, \mathcal{E})$ are those of $\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E})$ of the form $\sum_{i=1}^r \phi_i \otimes (fl_i, D_i)$ for some $f \in \mathcal{O}_X$.

Let now $(\mathcal{L}, \mathcal{A})$ be a Lie pair on X. Given an $\mathcal{A}$-connection $\nabla\colon \mathcal{E} \to \Omega^1(\mathcal{A}, \mathcal{E})$ on $\mathcal{E}$ locally free it makes sense to ask whether ∇ lifts to an $\mathcal{L}$-connection or not. We prove that the solution to this problem is completely determined by an obstruction

$$\partial(\nabla) \in \operatorname{Ext}^1_X\left(\frac{\mathcal{L}}{\mathcal{A}} \otimes \mathcal{E}, \mathcal{E}\right) = \operatorname{Ext}^1_X\left(\mathcal{E}, \mathcal{E} \otimes \mathcal{G}^1_1[1]\right).$$

It is possible to prove, by applying the results of [16, Section 3] to an injective resolution, that the same holds also if $\mathcal{E}$ is not locally free; however we don't need this result.

The case $\mathcal{A} = 0$ has been already considered. Suppose $\mathcal{A} \neq 0$, then we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \arrow[r] \arrow[d, "\alpha"] & Q(\mathcal{L}, \mathcal{E}) \arrow[r, "p"] \arrow[d, "\beta"] & \mathcal{O}_X[-1] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1(\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \arrow[r] & Q(\mathcal{A}, \mathcal{E}) \arrow[r, "p"] & \mathcal{O}_X[-1] \arrow[r] & 0
\end{tikzcd}$$

where α, β are the natural restriction maps. In a local frame $l_1, \dots, l_r$ of $\mathcal{L}$, with dual frame $\phi_1, \dots, \phi_r \in \Omega^1(\mathcal{L})$ and such that $l_1, \dots, l_s$ is a local frame for $\mathcal{A}$, we have

$$\alpha\left(\sum_{i=1}^r \phi_i \otimes g_i\right) = \sum_{i=1}^s \phi_i \otimes g_i, \quad \beta\left(\sum_{i=1}^r \phi_i \otimes (fl_i, D_i)\right) = \sum_{i=1}^s \phi_i \otimes (fl_i, D_i).$$

Since α and β are surjective, by the snake lemma we have an exact sequence

$$0 \to \mathcal{G}^1_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to Q(\mathcal{L}, \mathcal{E}) \xrightarrow{\beta} Q(\mathcal{A}, \mathcal{E}) \to 0, \tag{6.2}$$

since $\mathcal{G}^1_1$ is by definition the kernel of the surjective map $\Omega^1(\mathcal{L}) \to \Omega^1(\mathcal{A})$. For simplicity we can rewrite the above short exact sequence of graded sheaves living in degree 1 as a short exact sequence of sheaves in degree 0:

$$0 \to \mathcal{G}^1_1[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to Q(\mathcal{L}, \mathcal{E})[1] \xrightarrow{\beta} Q(\mathcal{A}, \mathcal{E})[1] \to 0.$$

Then the $\mathcal{A}$-connection ∇ is an element of $H^0(Q(\mathcal{A},\mathcal{E})[1])$ such that $p(\nabla)=1$, and the element

$$\begin{aligned}\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E},\nabla) := \partial(\nabla) &\in H^1\left(X, \mathcal{G}^1_1[1]\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right)\\ &= \mathrm{Ext}^1_X\left(\mathcal{E}, \mathcal{E}\otimes \mathcal{G}^1_1[1]\right),\end{aligned}$$

is the obstruction to lifting ∇ to an $\mathcal{L}$-connection. We will call this the *reduced Atiyah class* of $(\mathcal{E},\nabla)$.

7. Simplicial $\mathcal{L}$-connections

In this section, following [17], we define simplicial $\mathcal{L}$-connections for a Lie algebroid $\mathcal{L}$, and simplicial extensions of an $\mathcal{A}$-connection for a Lie pair $(\mathcal{L},\mathcal{A})$. We prove that the adjoint operator of a simplicial $\mathcal{L}$-connection on a locally free sheaf $\mathcal{E}$ induces a curved DG-algebra structure on $\mathrm{Tot}(\mathcal{U},\Omega^*(\mathcal{L})\otimes \mathcal{H}om^*_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. In the case of a Lie pair $(\mathcal{L},\mathcal{A})$ and of a simplicial extension of a flat $\mathcal{A}$-connection ∇ on $\mathcal{E}$, we obtain the data of a curved DG-pair. Simplicial connections allow us to give representatives of the classes $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ and $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E},\nabla)$, and a representative of the obstruction to extending a flat $\mathcal{A}$-connection on $\mathcal{E}$ to a $\mathcal{L}$-connection on $\mathcal{E}$ with curvature in $\mathcal{G}^2_2\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$.

Let $\mathcal{L}$ be a Lie algebroid on X and $\mathcal{E}$ a locally free sheaf. We have seen that $\mathcal{L}$-connections on $\mathcal{E}$ exist locally but in general it does not exist any globally defined connection. However we can define a weaker notion of connection, which always exists and equally gives a significative example of curved DG-algebra.

In the notation of Sections 3 and 6, consider the short exact sequence

$$0\to\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\to\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E})\xrightarrow{p}\Omega^1(\mathcal{L})\otimes\mathcal{L}\to 0, \tag{7.1}$$

and recall that by Lemma 6.1 an $\mathcal{L}$-connection on $\mathcal{E}$ is a global section D of $\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E})$ such that $p(D)=\mathrm{Id}_{\mathcal{L}}$, where $\mathrm{Id}_{\mathcal{L}}$ is considered as a global section of $\Omega^1(\mathcal{L})\otimes\mathcal{L}$. Fix an affine open cover $\mathcal{U}=\{U_i\}$ of X; by the exactness of the Thom–Whitney totalisation functor one obtains a short exact sequence of DG-vector spaces

$$\begin{aligned}0\to\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))&\\ \xrightarrow{p}\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{L})\to 0.&\end{aligned}$$

Because of the natural inclusion (4.1) of global sections in the totalisation, we can consider $\mathrm{Id}_{\mathcal{L}}$ as an element of $\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{L})$.

Definition 7.1. A simplicial $\mathcal{L}$-connection on $\mathcal{E}$ is a lifting

$$\nabla \in \operatorname{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}))$$

of $\mathrm{Id}_{\mathcal{L}}$ in $\operatorname{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{L})$.

It is clear that a simplicial $\mathcal{L}$-connection on $\mathcal{E}$ always exists.

In the case of a Lie pair $(\mathcal{L}, \mathcal{A})$ and of an $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on the locally free sheaf $\mathcal{E}$, we can define an analogous notion of simplicial $\mathcal{L}$-connection extending $\nabla^{\mathcal{A}}$. It is not restrictive to assume $\mathcal{A} \neq 0$; then the exact sequence of locally free graded sheaves (6.2)

$$0 \to \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to Q(\mathcal{L}, \mathcal{E}) \xrightarrow{\beta} Q(\mathcal{A}, \mathcal{E}) \to 0$$

induces the short exact sequence of DG-vector spaces

$$0 \to \operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \operatorname{Tot}(\mathcal{U}, Q(\mathcal{L}, \mathcal{E})) \xrightarrow{\beta} \operatorname{Tot}(\mathcal{U}, Q(\mathcal{A}, \mathcal{E})) \to 0. \tag{7.2}$$

We have already observed that an $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{E}$ is a global section of $Q(\mathcal{A}, \mathcal{E})$ such that $p(\nabla^{\mathcal{A}}) = 1$, where $p\colon Q(\mathcal{A}, \mathcal{E}) \to \mathcal{O}_X[-1]$ is induced by the map p of (7.1). By the inclusion of global sections in the totalisation, $\nabla^{\mathcal{A}}$ belongs to $\operatorname{Tot}(\mathcal{U}, Q(\mathcal{A}, \mathcal{E}))$.

Definition 7.2. By a simplicial extension of an $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{E}$ we mean a lifting ∇ in $\operatorname{Tot}(\mathcal{U}, Q(\mathcal{L}, \mathcal{E}))$ of $\nabla^{\mathcal{A}}$ in $\operatorname{Tot}(\mathcal{U}, Q(\mathcal{A}, \mathcal{E}))$.

Remark 7.3. Notice that the exact sequence (6.2) implies that a local extension of an $\mathcal{A}$-connection to an $\mathcal{L}$-connection always exists.

Since maps on the totalisation are induced locally, with a similar argument to that of Lemma 5.3 on can show that every simplicial extension ∇' of a flat $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{E}$ induces a differential on the complex $\operatorname{Tot}(\mathcal{U}, \mathcal{G}_r^*(\mathcal{E})/\mathcal{G}_{r+1}^*(\mathcal{E})[r])$. We then have that $H^*(\operatorname{Tot}(\mathcal{U}, \mathcal{G}_r^*(\mathcal{E})/\mathcal{G}_{r+1}^*(\mathcal{E})[r]))$ $\cong \mathbb{H}^*(X, \mathcal{G}_r^*(\mathcal{E})/\mathcal{G}_{r+1}^*(\mathcal{E})[r])$ is isomorphic to the Lie algebroid cohomology of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-module $\bigwedge^r(\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{E}$, again by Lemma 5.3.

Lemma 7.4. *For a Lie algebroid $\mathcal{L}$ and a simplicial $\mathcal{L}$-connection ∇ on $\mathcal{E}$, the cohomology class of $d_{\mathrm{Tot}}\nabla$ in $\operatorname{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is the obstruction $\operatorname{At}_{\mathcal{L}}(\mathcal{E})$ to the existence of an $\mathcal{L}$-connection on $\mathcal{E}$.*

For a Lie pair $(\mathcal{L}, \mathcal{A})$ and a simplicial extension ∇ of an $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{E}$, the cohomology class of $d_{\mathrm{Tot}}\nabla$ in $\operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is the obstruction $\overline{\operatorname{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$ to the extension of $\nabla^{\mathcal{A}}$ to an $\mathcal{L}$-connection.

Proof. According to Example 4.3 we have natural isomorphisms

$$H^0(\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E})))=\Gamma(X,\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E})),$$
$$H^0(\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))=\Gamma(X,\Omega^1(\mathcal{L})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).$$

Consider first the case of a simplicial $\mathcal{L}$-connection ∇ on $\mathcal{E}$; notice that $d_{\mathrm{Tot}}\nabla$ belongs to $\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, because

$$p(d_{\mathrm{Tot}}\nabla)=d_{\mathrm{Tot}}p(\nabla)=d_{\mathrm{Tot}}\,\mathrm{Id}_{\mathcal{L}}=0,$$

since $\mathrm{Id}_{\mathcal{L}}$ is a global section. If there exists an $\mathcal{L}$-connection ∇' on $\mathcal{E}$ it belongs to $\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))$ by the inclusion of global sections in the totalisation, and one has that $d_{\mathrm{Tot}}\nabla'=0$. Then for any simplicial connection ∇, the difference $\nabla-\nabla'$ belongs to $\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ and $d_{\mathrm{Tot}}(\nabla-\nabla')=d_{\mathrm{Tot}}\nabla$, so that $d_{\mathrm{Tot}}\nabla$ is trivial in the cohomology of $\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. Conversely, if $d_{\mathrm{Tot}}\nabla = d_{\mathrm{Tot}}\varphi$, with $\varphi\in \mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, then $\nabla-\varphi$ is a global $\mathcal{L}$-connection on $\mathcal{E}$.

In the case of a Lie pair $(\mathcal{L},\mathcal{A})$ and a simplicial extension ∇ of an $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{E}$, notice that $d_{\mathrm{Tot}}\nabla$ is in $\mathrm{Tot}\left(\mathcal{U},\mathcal{G}^1_1\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right)$: in fact, $\beta(d_{\mathrm{Tot}}\nabla)=d_{\mathrm{Tot}}\beta(\nabla)=d_{\mathrm{Tot}}\nabla^{\mathcal{A}}=0$, because $\nabla^{\mathcal{A}}$ is a global section. If $\nabla^{\mathcal{A}}$ extends to an $\mathcal{L}$-connection there exists ∇' in $\Gamma(X,Q(\mathcal{L},\mathcal{E}))$ with $\beta(\nabla')=\nabla^{\mathcal{A}}$, which is such that $d_{\mathrm{Tot}}\nabla'=0$ in $\mathrm{Tot}(\mathcal{U},Q(\mathcal{L},\mathcal{E}))$, because it is a global section. Then for every simplicial connection ∇ lifting $\nabla^{\mathcal{A}}$, $\nabla-\nabla'$ belongs to the kernel of β, which is $\mathrm{Tot}(\mathcal{U},\mathcal{G}^1_1\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, and $d_{\mathrm{Tot}}(\nabla-\nabla')=d_{\mathrm{Tot}}\nabla$, so that $d_{\mathrm{Tot}}\nabla$ is trivial in cohomology. Vice versa, if $d_{\mathrm{Tot}}\nabla=d_{\mathrm{Tot}}\phi$ is trivial in the cohomology of $\mathrm{Tot}(\mathcal{U},\mathcal{G}^1_1\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, it is easy to see that $\nabla-\phi$ is a connection lifting $\nabla^{\mathcal{A}}$. □

A simplicial $\mathcal{L}$-connection on a locally free sheaf $\mathcal{E}$ induces a curved DG-algebra structure on the DG-vector space $\mathrm{Tot}(\mathcal{U},\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$. To see this, the first step is the construction of an adjoint operator for the simplicial connection, which is done via the following lemma.

Lemma 7.5. *In the above situation, the $\mathcal{O}_X$-bilinear map*

$$[-,-]\colon (\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))\times\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\to\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}),$$
$$[\eta\otimes(l,v),g]=\eta\otimes[v,g],\qquad \eta\in\Omega^1(\mathcal{L}),\ (l,v)\in P(\mathcal{L},\mathcal{E}),\ g\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}).$$

is well defined.

Proof. For $r \in \mathcal{O}_X$,

$$\begin{aligned}[v,g](re) &= v(rg(e)) - g(rv(e) + a(l)(r)e) \\ &= rvg(e) + a(l)(r)g(e) - rgv(e) - a(l)(r)g(e) \\ &= r[v,g](e),\end{aligned}$$

so $[v,g]$ belongs to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$. The bracket is well-defined: for $r \in \mathcal{O}_X$,

$$[\eta \otimes (rl, rv), g] = \eta \otimes [rv, g] = \eta \otimes r[v,g] = r\eta \otimes [v,g] = [r\eta \otimes (l,v), g]. \qquad \square$$

The bracket defined in Lemma 7.5 induces a graded Lie bracket on the totalisation

$$\begin{aligned}[-,-]\colon\ &\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E})) \times \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \\ &\to \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})),\end{aligned}$$

which allows to define the adjoint operator to a simplicial $\mathcal{L}$-connection ∇ on $\mathcal{E}$:

(7.3)

$$d_\nabla := [\nabla, -]\colon\ \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \to \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).$$

Recall that since $\Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ is a sheaf of graded algebras and the Tot preserves multiplicative structures, $\mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$ is a differential graded algebra, with differential denoted by d_{Tot}.

Lemma 7.6. *The adjoint operator*

$$d_\nabla = [\nabla, -]\colon\ \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \to \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$$

extends for every $i \geq 0$ to a $\mathbb{K}$-linear operator

$$d_\nabla\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^i(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \to \mathrm{Tot}(\mathcal{U}, \Omega^{i+1}(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).$$

Then $(\mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))), d_{\mathrm{Tot}} + d_\nabla)$ is a curved DG-algebra with curvature $d_{\mathrm{Tot}}\nabla + C$, with $d_{\mathrm{Tot}}\nabla \in \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ and $C \in \mathrm{Tot}(\mathcal{U}, \Omega^2(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ such that $d_\nabla^2 = [C, -]$.

Proof. Consider first the case of a germ of an $\mathcal{L}$-connection, i.e., an element Y of $\Gamma(V, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E}))$ such that $p(Y) = \mathrm{Id}_{\mathcal{L}}|_V$, for some open set $V \subset X$. As usual, Y extends uniquely to a $\mathbb{K}$-linear morphism of degree 1

$$Y\colon \Omega^*(\mathcal{L},\mathcal{E})|_V \to \Omega^*(\mathcal{L},\mathcal{E})|_V$$

such that

$$Y(\eta\otimes e)=d_{\mathcal{L}}(\eta)\otimes e+(-1)^{|\eta|}\eta\otimes Y(e)$$

for all $\eta\in\Omega^*(\mathcal{L})|_V, e\in\mathcal{E}|_V$. It is easy to see that the map Y^2 is $\mathcal{O}_X$-linear, so it can be identified with a section of $\Omega^2(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V$.

One can define an adjoint operator

$$d_Y:=[Y,-]\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})|_V\to\Omega^1(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V,$$

which can be extended for all $i\geq 0$ to an operator

$$d_Y\colon \Omega^i(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V\to\Omega^{i+1}(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V$$

by setting

$$d_Y(\eta\otimes f):=d_{\mathcal{L}}(\eta)\otimes f+(-1)^{|\eta|}\eta\otimes[Y,f], \tag{7.4}$$

where $[Y,f]$ denotes the Lie bracket of Lemma 7.5.

As in the classical case, one can see that

$$d_Y^2(\eta\otimes f)=[Y^2,\eta\otimes f] \tag{7.5}$$

for all $\eta\in\Omega^*(\mathcal{L})|_V$ and $f\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})|_V$.

Let now ∇ be a simplicial $\mathcal{L}$-connection on $\mathcal{E}$, namely an element of $\operatorname{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))$ such that $p(\nabla)=\operatorname{Id}_{\mathcal{L}}\in\operatorname{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{L})$. Then for every $i\geq 0$ the extension of the operator $d_\nabla=[\nabla,-]$, defined in (7.3), to an operator $d_\nabla=[\nabla,-]\colon \operatorname{Tot}(\mathcal{U},\Omega^i(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to\operatorname{Tot}(\mathcal{U},\Omega^{i+1}(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ can be defined by using the map induced by (7.4) on the totalisation, and one obtains a degree one operator

$$d_\nabla=[\nabla,-]\colon \operatorname{Tot}(\mathcal{U},\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))\to\operatorname{Tot}(\mathcal{U},\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))).$$

In detail, let $\nabla=(D_n)$ with $D_n\in A_n\otimes\prod_{i_1,\dots,i_n}(\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))(U_{i_1,\dots,i_n})$ such that $p(D_n)=1\otimes(\operatorname{Id}_{\mathcal{L}}|_{U_{i_1,\dots,i_n}})$ for every $n\geq 0$. Since maps on the totalisation are defined componentwise, it is enough to define the bracket

$$[D_n,\phi_n\otimes(\omega_{i_1,\dots,i_n}\otimes f_{i_1,\dots,i_n})],$$

for $\phi_n\otimes(\omega_{i_1,\dots,i_n}\otimes f_{i_1,\dots,i_n})$ in $A_n\otimes\prod_{i_1,\dots,i_n}(\Omega^i(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})(U_{i_1,\dots,i_n})$. Let

$$D_n=\sum_j\eta_{j,n}\otimes(t_{j,i_1,\dots,i_n}),\ \eta_{j,n}\in A_n,\quad t_{j,i_1,\dots,i_n}\in(\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))(U_{i_1,\dots,i_n}); \tag{7.6}$$

then the bracket can be defined as

$$\begin{aligned}
&[D_n, \phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes f_{i_1,\ldots,i_n})] \\
&= \Big[\sum_j \eta_{j,n} \otimes (t_{j,i_1,\ldots,i_n}), \phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes f_{i_1,\ldots,i_n})\Big] \\
&= p(D_n)(\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes f_{i_1,\ldots,i_n})) \\
&\quad + (-1)^{|\phi_n|+|\omega_{i_1,\ldots,i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1,\ldots,i_n} \otimes [t_{j,i_1,\ldots,i_n}, f_{i_1,\ldots,i_n}]) \\
&= (1 \otimes (\mathrm{Id}_{\mathcal{L}}\,|_{U_{i_1,\ldots,i_n}}))(\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes f_{i_1,\ldots,i_n})) \\
&\quad + (-1)^{|\phi_n|+|\omega_{i_1,\ldots,i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1,\ldots,i_n} \otimes [t_{j,i_1,\ldots,i_n}, f_{i_1,\ldots,i_n}]) \\
&= (-1)^{|\phi_n|} \phi_n \otimes (d_{\mathcal{L}}\omega_{i_1,\ldots,i_n} \otimes f_{i_1,\ldots,i_n}) \\
&\quad + (-1)^{|\phi_n|+|\omega_{i_1,\ldots,i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1,\ldots,i_n} \otimes [t_{j,i_1,\ldots,i_n}, f_{i_1,\ldots,i_n}]),
\end{aligned}$$

where the bracket $[t_{j,i_1,\ldots,i_n}, f_{i_1,\ldots,i_n}]$ is induced by the one of Lemma 7.5.

For every $i \geq 0$ the simplicial $\mathcal{L}$-connection ∇ also induces a map

$$\nabla\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^i(\mathcal{L}) \otimes \mathcal{E}) \to \mathrm{Tot}(\mathcal{U}, \Omega^{i+1}(\mathcal{L}) \otimes \mathcal{E})$$

which allows to define a degree one operator

$$\nabla\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E})) \to \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E})).$$

In fact, let $\nabla = (D_n)$ as in (7.6), and consider $\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes e_{i_1,\ldots,i_n})$ in $A_n \otimes \prod_{i_1,\ldots,i_n} (\Omega^1(\mathcal{L}) \otimes \mathcal{E})(U_{i_1,\ldots,i_n})$. Then the operator can be defined as

$$\begin{aligned}
&D_n(\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes e_{i_1,\ldots,i_n})) \\
&\quad = \Big(\sum_j \eta_{j,n} \otimes (t_{j,i_1,\ldots,i_n})\Big)(\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes e_{i_1,\ldots,i_n})) \\
&\quad = p(D_n)(\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes e_{i_1,\ldots,i_n})) + \\
&\qquad + (-1)^{|\phi_n|+|\omega_{i_1,\ldots,i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1\ldots i_n} \otimes t_{i_1,\ldots,i_n}(e_{i_1,\ldots,i_n})) \\
&\quad = (1 \otimes (\mathrm{Id}_{\mathcal{L}}\,|_{U_{i_1,\ldots,i_n}}))(\phi_n \otimes (\omega_{i_1,\ldots,i_n} \otimes e_{i_1,\ldots,i_n})) + \\
&\qquad + (-1)^{|\phi_n|+|\omega_{i_1,\ldots,i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1\ldots i_n} \otimes t_{i_1,\ldots,i_n}(e_{i_1,\ldots,i_n}))
\end{aligned}$$

$$= (-1)^{|\phi_n|}(\phi_n \otimes (d_{\mathcal{L}}\omega_{i_1,\dots,i_n} \otimes e_{i_1,\dots,i_n})$$
$$+ (-1)^{|\omega_{i_1,\dots,i_n}|}\sum_j \phi_n\eta_{j,n} \otimes (\omega_{i_1\dots i_n} \otimes t_{i_1,\dots,i_n}(e_{i_1,\dots,i_n}))).$$

Since all the maps considered on the totalisation are induced by the ones defined locally on the complexes of sheaves, for $d_\nabla = [\nabla, -]$ one has that, by (7.5),

$$d_\nabla^2 = [C, -], \quad C \in \mathrm{Tot}(\mathcal{U}, \Omega^2(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))).$$

Then $d_{\mathrm{Tot}} + d_\nabla$ is a degree one derivation of $\mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$, with square

$$\begin{aligned}(d_{\mathrm{Tot}} + d_\nabla)^2 = d_{\mathrm{Tot}}^2 + d_{\mathrm{Tot}}[\nabla, -] + [\nabla, d_{\mathrm{Tot}}-] + d_\nabla^2 &= [d_{\mathrm{Tot}}\nabla, -] + [C, -]\\ &= [d_{\mathrm{Tot}}\nabla + C, -],\end{aligned}$$

so the curvature is $d_{\mathrm{Tot}}\nabla + C$. We have already seen in Lemma 7.4 that $d_{\mathrm{Tot}}\nabla$ belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$.

The last thing to prove is that $(d_{\mathrm{Tot}} + d_\nabla)(d_{\mathrm{Tot}}\nabla + C) = 0$. One has that

$$(d_{\mathrm{Tot}}+d_\nabla)(d_{\mathrm{Tot}}\nabla+C) = d_{\mathrm{Tot}}^2\nabla + d_\nabla d_{\mathrm{Tot}}\nabla + d_{\mathrm{Tot}}C + d_\nabla C = d_\nabla d_{\mathrm{Tot}}\nabla + d_{\mathrm{Tot}}C.$$

Then

$$d_\nabla d_{\mathrm{Tot}}\nabla = [\nabla, d_{\mathrm{Tot}}\nabla] = -[d_{\mathrm{Tot}}\nabla, \nabla] = -\frac{1}{2}d_{\mathrm{Tot}}[\nabla, \nabla] = -d_{\mathrm{Tot}}C,$$

so that $(d_{\mathrm{Tot}} + d_\nabla)(d_{\mathrm{Tot}}\nabla + C) = 0$. $\square$

In the case of a Lie pair $(\mathcal{L}, \mathcal{A})$ and a locally free sheaf $\mathcal{E}$, the natural surjective restriction maps

$$\varrho\colon \Omega^*(\mathcal{L}) \to \Omega^*(\mathcal{A}), \quad \varrho \otimes \mathrm{Id}\colon \Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \to \Omega^*(\mathcal{A}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})),$$

induce morphisms on the totalisation

$$\varrho\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L})) \to \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{A})),$$
$$\varrho \otimes \mathrm{Id}\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))) \to \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{A}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))),$$

whose kernels define bilateral ideals

$$\mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^*) = \ker(\varrho) \subset \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L})),$$
$$\mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^* \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = \ker(\varrho \otimes \mathrm{Id}) \subset \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))).$$

Lemma 7.7. *Let $(\mathcal{E},\nabla^{\mathcal{A}})$ be a locally free $\mathcal{A}$-module, and let ∇ be a simplicial extension of $\nabla^{\mathcal{A}}$ to an $\mathcal{L}$-connection. Then $I := \mathrm{Tot}(\mathcal{U},\mathcal{G}_1^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ is a curved ideal of the curved DG-algebra*

$$(\mathrm{Tot}(\mathcal{U},\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))), d_{\mathrm{Tot}}+d_{\nabla}, d_{\mathrm{Tot}}\nabla + C),$$

where C, the curvature of the simplicial connection ∇, belongs to $\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^2\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ and $d_{\mathrm{Tot}}\nabla$ belongs to $\mathrm{Tot}\left(\mathcal{U},\mathcal{G}_1^1\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right)$.

Proof. It is clear that the ideal $I=\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ is d_{Tot}-closed. Let x be an element of I, so that $(\varrho\otimes \mathrm{Id})(x)=0$, then

$$(\varrho\otimes \mathrm{Id})(d_{\nabla}x) = d_{\nabla^{\mathcal{A}}}(\varrho\otimes \mathrm{Id})(x)=0,$$

so I is also d_{∇}-closed. Since the $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ is flat, the curvature C of ∇ belongs to $\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^2\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\subset I$, which is the kernel of the surjective map

$$\varrho\otimes \mathrm{Id}\colon\ \mathrm{Tot}(\mathcal{U},\Omega^2(\mathcal{L})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to \mathrm{Tot}(\mathcal{U},\Omega^2(\mathcal{A})\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).$$

By Lemma 7.4, $d_{\mathrm{Tot}}\nabla$ belongs to $\mathrm{Tot}\left(\mathcal{U},\mathcal{G}_1^1\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right)$, therefore it belongs to the ideal I. □

For the ideal $I=\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ we have that

$$I^{(n)} = \mathrm{Tot}(\mathcal{U},\mathcal{G}_n^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})). \tag{7.7}$$

In fact, the inclusion $I^{(n)}\subset \mathrm{Tot}(\mathcal{U},\mathcal{G}_n^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ is clear. For the other one, it suffices to notice that the multiplication map $\underbrace{\mathcal{G}_1^*\otimes\cdots\otimes\mathcal{G}_1^*}_{n}\to\mathcal{G}_n^*$ is surjective on all affine open sets.

According to Definition 2.3, the Atiyah cocycle of the curved DG-pair

$$(A=\mathrm{Tot}(\mathcal{U},\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))), I=\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$$

is the class of the curvature $R=d_{\mathrm{Tot}}\nabla + C$ in

$$\frac{I}{I^{(2)}} = \mathrm{Tot}\left(\mathcal{U},\frac{\mathcal{G}_1^*}{\mathcal{G}_2^*}\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right).$$

Theorem 7.8. *Given a Lie pair $(\mathcal{L},\mathcal{A})$ and a locally free $\mathcal{A}$-module $(\mathcal{E},\nabla^{\mathcal{A}})$, the Atiyah class $\mathrm{At}(A,I)$ of the curved DG-pair*

$$(A=\mathrm{Tot}(\mathcal{U},\Omega^*(\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))), I=\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^*\otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$$

does not depend on the choice of the simplicial $\mathcal{L}$-connection extending $\nabla^{\mathcal{A}}$. Moreover, it is the obstruction to the existence of a $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^{\mathcal{A}}$ with curvature in $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$.

Proof. Let ∇ and ∇' be two simplicial extensions of the $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$; their difference belongs to the ideal I. In fact, considering the short exact sequence (7.2),

$$0 \to \operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to \operatorname{Tot}(\mathcal{U}, Q(\mathcal{L},\mathcal{E})) \xrightarrow{\beta} \operatorname{Tot}(\mathcal{U}, Q(\mathcal{A},\mathcal{E})) \to 0,$$

we have that $\beta(\nabla - \nabla') = \nabla^{\mathcal{A}} - \nabla^{\mathcal{A}} = 0$ and therefore, writing $\phi := \nabla - \nabla'$, we have $\phi \in \operatorname{Tot}\left(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right) \subset I$. Then $d_\nabla = d_{\nabla'} + [\phi, -]$ and the first claim follows from Lemma 2.4.

Next, we show that the Atiyah class $\operatorname{At}(A, I)$ of the curved DG-pair is the obstruction to the existence of a $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^{\mathcal{A}}$, with curvature in $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$. By Lemma 2.4, $\operatorname{At}(A, I)$ is the obstruction to existence of $x \in I = \operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^* \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ of degree 1 such that $R + (d_{\operatorname{Tot}} + d_\nabla)x$ belongs to $I^{(2)} = \operatorname{Tot}(\mathcal{U}, \mathcal{G}_2^* \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. Assume that there exists such x, and notice that by degree reasons it belongs to $\operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, since $\mathcal{G}_1^0 = 0$. Then, since $\mathcal{G}_2^1 = 0$,

$$\begin{aligned} &d_{\operatorname{Tot}}\nabla + d_{\operatorname{Tot}}x \ \in I^{(2)} \cap \operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) = 0 \\ &C + d_\nabla x \ \in I^{(2)} \cap \operatorname{Tot}(\mathcal{U}, \mathcal{G}_1^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) = \operatorname{Tot}(\mathcal{U}, \mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})), \end{aligned}$$

and by the first equation $\nabla + x$ is a global $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^{\mathcal{A}}$.

We denote by $R_x = d_{\operatorname{Tot}}(\nabla + x) + C_x = C_x$ the curvature of the curved DG-algebra $(A, d_{\operatorname{Tot}} + d_{\nabla + x})$. Then

$$\begin{aligned} R_x = R + (d_{\operatorname{Tot}} + d_\nabla)x + \frac{1}{2}[x,x] &= d_{\operatorname{Tot}}\nabla + C + d_{\operatorname{Tot}}x + d_\nabla x + \frac{1}{2}[x,x] \\ &= C + d_\nabla x + \frac{1}{2}[x,x], \end{aligned}$$

so that the curvature of $\nabla + x$ is equal to $C_x = C + d_\nabla x + \frac{1}{2}[x,x]$, which belongs to $\operatorname{Tot}(\mathcal{U}, \mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. Finally, since $d_{\nabla + x}(C_x) = 0$, one has that

$$0 = (d_{\operatorname{Tot}} + d_{\nabla+x})(R_x) = (d_{\operatorname{Tot}} + d_{\nabla+x})(C_x) = d_{\operatorname{Tot}}C_x,$$

and C_x is a global section of $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$.

The converse is clear. $\square$

By the above, the Atiyah class $\mathrm{At}(A, I)$ of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^* \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))$$

is well-defined:

$$\mathrm{At}(A, I) \in \mathbb{H}^2\left(X, \frac{\mathcal{G}_1^*}{\mathcal{G}_2^*} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

Definition 7.9. In the above situation, via the isomorphisms of Lemma 5.3, we call

$$\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) := \mathrm{At}(A, I) \in \mathbb{H}^1\left(\mathcal{A}; (\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

the $(\mathcal{L}, \mathcal{A})$-Atiyah class of $\mathcal{E}$.

Remark 7.10. Recalling that $\mathcal{G}_2^1 = 0$, the morphism of graded sheaves $t\colon \frac{\mathcal{G}_1^*}{\mathcal{G}_2^*} \to \mathcal{G}_1^1$ with kernel $\frac{\mathcal{G}_1^{\geq 2}}{\mathcal{G}_2^*}$ induces a morphism of DG-vector spaces

$$t\colon \mathrm{Tot}\left(\mathcal{U}, \frac{\mathcal{G}_1^*}{\mathcal{G}_2^*} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) \to \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})),$$

which sends the class of $R = d_{\mathrm{Tot}}\nabla + C$ to $d_{\mathrm{Tot}}\nabla$. The reduced Atiyah class $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$ is then the image of the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^* \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))$$

via the map induced by t in hypercohomology

$$\begin{aligned} t\colon \mathbb{H}^*\left(X, \frac{\mathcal{G}_1^*}{\mathcal{G}_2^*} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) &\to \mathbb{H}^*(X, \mathcal{G}_1^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \\ \mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) &\mapsto \overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}}). \end{aligned}$$

In particular if $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is trivial, then so is $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$.

If we consider the Lie pair $(\mathcal{L}, 0)$, both the obstructions $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ and $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$ reduce to the obstruction $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ to the existence of an $\mathcal{L}$-connection on $\mathcal{E}$.

Corollary 7.11. *Let $(\mathcal{L}, \mathcal{A})$ be a Lie pair on X such that there exists an $\mathcal{O}_X$-linear projection $p\colon \mathcal{L} \to \mathcal{A}$ which commutes with anchor maps and with adjoint Lie actions of $\mathcal{A}$. Then for every $\mathcal{A}$-module $\mathcal{E}$ the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is trivial.*

Proof. The assumption that $p\colon \mathcal{L} \to \mathcal{A}$ commutes with adjoint Lie actions of $\mathcal{A}$ means that $p([x,y]) = [x,p(y)]$ for every $x \in \mathcal{A}$ and $y \in \mathcal{L}$.

Let $\nabla\colon \mathcal{A} \to \mathcal{E}nd_{\mathbb{K}}(\mathcal{E})$ be a flat $\mathcal{A}$-connection on $\mathcal{E}$. The existence of an $\mathcal{O}_X$-linear projection $p\colon \mathcal{L} \to \mathcal{A}$ commuting with anchor maps ensures that the composition $\widetilde{\nabla} := \nabla p\colon \mathcal{L} \to \mathcal{E}nd_{\mathbb{K}}(\mathcal{E})$ is a connection. In fact, for $l \in \mathcal{L}$, $f \in \mathcal{O}_X$ and $e \in \mathcal{E}$,

$$\widetilde{\nabla}_l(fe) = \nabla_{p(l)}(fe) = a_{\mathcal{A}}(p(l))(f)e + f\nabla_{p(l)}(e) = a_{\mathcal{L}}(l)(f)e + f\widetilde{\nabla}_l(e).$$

For every $a \in \mathcal{A}$ and every $l \in \mathcal{L}$ we have

$$[\widetilde{\nabla}_a, \widetilde{\nabla}_l] = [\nabla_a, \nabla_{p(l)}] = \nabla_{[a,p(l)]} = \nabla_{p[a,l]} = \widetilde{\nabla}_{[a,l]},$$

and this implies that the curvature of $\widetilde{\nabla}$ belongs to $\mathcal{G}_2^2 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$, so that by Theorem 7.8 the Atiyah class of $\mathcal{E}$ is trivial. $\square$

Notice that Corollary 7.11 applies in particular in the case $X = \operatorname{Spec}(\mathbb{K})$ and $\mathcal{A}$ a semisimple Lie algebra. On the other hand, the Examples 2.10 and 2.11 of [6] give explicit situations where X is a single point and the Atiyah class does not vanish.

8. Semiregularity maps and obstructions

Let $(\mathcal{L}, \mathcal{A})$ be a Lie pair on a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic 0. Given a locally free $\mathcal{A}$-module $(\mathcal{E}, \nabla^{\mathcal{A}})$ we introduced the Atiyah class

$$\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \in \mathbb{H}^1(\mathcal{A}; (\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})),$$

which is the *primary obstruction* to the extension of the $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ to a flat $\mathcal{L}$-connection; more precisely the Atiyah class is a complete obstruction to the extension of $\nabla^{\mathcal{A}}$ to an $\mathcal{L}$-connection with curvature in $\mathcal{G}_2^2 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

Taking exterior cup products in $\mathcal{A}$-cohomology it makes sense to consider the exterior powers

$$\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k \in \mathbb{H}^k\left(\mathcal{A}; \bigwedge\nolimits^k (\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right)$$

together with the morphisms of graded vector spaces

$$\begin{aligned}\mathbb{H}^*\left(\mathcal{A}; \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) &\to \mathbb{H}^*\left(\mathcal{A}; \bigwedge\nolimits^k (\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right)[k] \\ &\to \mathbb{H}^*\left(\mathcal{A}; \bigwedge\nolimits^k (\mathcal{L}/\mathcal{A})^\vee\right)[k],\end{aligned}$$

$$x \mapsto \frac{1}{k!}\operatorname{Tr}(\operatorname{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x).$$

The following definition is a clear natural extension of the definition of semiregularity maps for coherent sheaves [2, 5].

Definition 8.1. In the above situation, for every $k \geq 0$ the map

$$\begin{aligned} \tau_k \colon \mathbb{H}^2\left(\mathcal{A}; \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) &\to \mathbb{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right), \\ \tau_k(x) &= \frac{1}{k!}\operatorname{Tr}(\operatorname{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x), \end{aligned}$$

is called the *k-semiregularity map of the $\mathcal{A}$-module* $(\mathcal{E}, \nabla^{\mathcal{A}})$, (with respect to the Lie pair $(\mathcal{L}, \mathcal{A})$).

If $\mathcal{G}^*_*$ is the Leray filtration of the Lie pair $(\mathcal{L}, \mathcal{A})$ we have proved in Lemma 5.3 that there exist canonical isomorphisms $\mathbb{H}^{2+k}(\mathcal{A}; \bigwedge^k(\mathcal{L}/\mathcal{A})^\vee) \cong \mathbb{H}^{2+2k}\left(X, \mathcal{G}^*_k/\mathcal{G}^*_{k+1}\right)$ and therefore there exist natural maps

$$i_k \colon \mathbb{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right) \to \mathbb{H}^{2+2k}\left(X, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}\right),$$

which are injective whenever the Leray spectral sequence degenerates at E_1.

We are now ready to apply the abstract general results of [2] to our situation in order to obtain the following result.

Theorem 8.2. *Let $(\mathcal{L}, \mathcal{A})$ be a Lie pair on a smooth separated scheme X of finite type over a field $\mathbb{K}$ of characteristic 0. Given a locally free $\mathcal{A}$-module $(\mathcal{E}, \nabla^{\mathcal{A}})$, for every $k \geq 0$ the composite map*

$$i_k\tau_k \colon \mathbb{H}^2\left(\mathcal{A}; \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) \to \mathbb{H}^{2+2k}\left(X, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}\right)$$

annihilates every obstruction to deformations of $(\mathcal{E}, \nabla^{\mathcal{A}})$ as an $\mathcal{A}$-module. In particular, if the Leray spectral sequence of the Lie pair $(\mathcal{L}, \mathcal{A})$ degenerates at E_1, then every semiregularity map annihilates obstructions.

Proof. We take an affine cover $\mathcal{U}$ of X and we choose a simplicial connection $\nabla \in \operatorname{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}))$ extending $\nabla^{\mathcal{A}}$. By Lemma 7.7, the ideal $I := \operatorname{Tot}(\mathcal{U}, \mathcal{G}^*_1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ is a curved ideal of the curved DG-algebra

$$A := \left(\operatorname{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}) \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})), d_{\operatorname{Tot}} + d_\nabla, d_{\operatorname{Tot}}\nabla + C\right),$$

so that the quotient

$$B := A/I = \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{A}) \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$$

is a non-curved DG-Lie algebra, with differential given by $d_{\mathrm{Tot}} + d_{\nabla^{\mathcal{A}}}$. This is precisely the DG-Lie algebra controlling deformations of the $\mathcal{A}$-module $(\mathcal{E}, \nabla^{\mathcal{A}})$ of Theorem 4.4.

The trace morphism

$$\mathrm{Tr}\colon \Omega^*(\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \to \Omega^*(\mathcal{L})$$

of (3.1) induces

$$\mathrm{Tr}\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))) \to \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L})),$$

which is a trace map in the sense of Definition 2.5. It is plain that

$$\mathrm{Tr}(\mathrm{Tot}(\mathcal{U}, \mathcal{G}^*_k \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))) \subset \mathrm{Tot}(\mathcal{U}, \mathcal{G}^*_k),$$

for every $k \geq 0$. Finally, according to (7.7) and the exactness properties of Tot, for every $i \leq j$ we have

$$\frac{I^{(i)}}{I^{(j)}} = \frac{\mathrm{Tot}(\mathcal{U}, \mathcal{G}^*_i \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))}{\mathrm{Tot}(\mathcal{U}, \mathcal{G}^*_j \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))} = \mathrm{Tot}\left(\mathcal{U}, \frac{\mathcal{G}^*_i}{\mathcal{G}^*_j} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

Now, by Theorem 2.6, there exists an L_∞ morphism between DG-Lie algebras

$$\sigma^k\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{A}) \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \rightsquigarrow \mathrm{Tot}\left(\mathcal{U}, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}[2k]\right)$$

whose linear component is given by

$$\sigma^k_1\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{A}) \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \to \mathrm{Tot}\left(\mathcal{U}, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}[2k]\right), \sigma^k_1(x) = \frac{1}{k!}\,\mathrm{Tr}(R^k x),$$

where $R = d_{\mathrm{Tot}}\nabla + C$ denotes the curvature of the DG-algebra A.

In cohomology the above maps σ^k_1 may be written as

$$\sigma^k_1\colon \mathbb{H}^2(\mathcal{A}; \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \to \mathbb{H}^{2k+2}\left(X, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}\right), \quad \sigma^k_1(x) = \frac{1}{k!}\,\mathrm{Tr}(\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),$$

and then $\sigma^k_1 = i_k \tau_k$.

Then the theorem is a consequence of the fact that the DG-Lie algebra $\mathrm{Tot}\left(\mathcal{U}, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}[2k]\right)$ is abelian and then, by general facts (see e.g. [21, 22]), every obstruction of the deformation functor associated to the DG-Lie algebra B is annihilated by the maps σ_1^k. $\square$

Remark 8.3. The induced map in hypercohomology σ_1^k depends only on the $\mathcal{A}$-module $(\mathcal{E}, \nabla^{\mathcal{A}})$ and not on the choice of a simplicial $\mathcal{L}$-connection ∇ extending $\nabla^{\mathcal{A}}$. In fact, σ_1^k depends only on the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}^*_1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))),$$

which we proved in Theorem 7.8 does not depend on the choice of ∇.

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Ruggero Bandiera
Università degli studi di Roma La Sapienza
Dipartimento di Matematica Guido
Castelnuovo, P.le Aldo Moro 5
I-00185 Roma
Italy
E-mail: bandiera@mat.uniroma1.it

Emma Lepri
School of Mathematics and Statistics
University of Glasgow University Place
Glasgow G12 8QQ
UK
E-mail: Emma.Lepri@glasgow.ac.uk

Marco Manetti
Università degli studi di Roma La Sapienza
Dipartimento di Matematica Guido
Castelnuovo, P.le Aldo Moro 5
I-00185 Roma
Italy
E-mail: manetti@mat.uniroma1.it