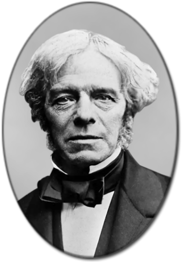
PHY204 Project: The Faraday Effect (P9)



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Introduction

- 1845: Michael Faraday
- evidence of interaction between light and electromagnetism
  → early demonstration of light as an electromagnetic wave

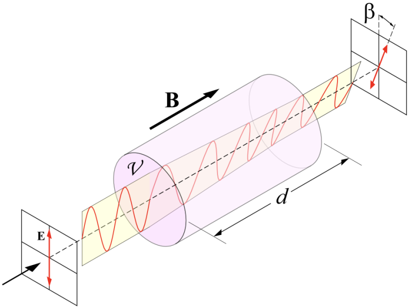

Illustration of the Faraday effect [1]

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A theoretical approach to the Faraday effect

Why is the plane of polarization rotated by the Faraday effect?

Right-handed and left-handed circularly polarized components propagate at different velocity through the medium

$\rightarrow$ **the index of refraction depends on the handedness of the wave**

A theoretical approach to the Faraday effect

Using Lorentz model for the bound electron of charge $-e$ and mass m_e:

$$\ddot{\vec{\underline{r}}} + \omega_0^2 \vec{\underline{r}} = \frac{-e}{m_e} \vec{E} + \frac{-e}{m_e} \vec{v_e} \times \vec{B_0} \tag{1}$$

- $\omega_0^2 = \frac{e^2}{4\pi\epsilon_0 m_e R^3}$ with $R = 1\text{Å}$ the radius of the atom.
- Contribution from the damping term is neglected
- $\vec{E} = Re(\vec{E_0} e^{i(kx-\omega t)})$.
- $\vec{B_0} = B_0 \vec{e_x}$
- Non relativistic motion $v \ll c$

A theoretical approach to the Faraday effect

- Ansatz $\vec{r} = \mathrm{Re}(y_0 e^{-i\omega t}\vec{e_y} + z_0 e^{-i\omega t}\vec{e_z})$
- current density $\vec{j} = -ne\vec{v}$ where n is the electron number density
- introducing plasma frequency $\omega_p^2 = \frac{ne^2}{m_e \epsilon_0}$ and cyclotron frequency $\Omega = \frac{eB_0}{m_e}$

$$\vec{j} = \begin{pmatrix} j_y \\ j_z \end{pmatrix} = \frac{-i\omega\epsilon_0\omega_p^2}{(\omega_0^2-\omega^2)^2-(\omega\Omega)^2} \begin{pmatrix} \omega_0^2 - \omega^2 & i\omega\Omega \\ -i\omega\Omega & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} E_y \\ E_z \end{pmatrix} =: [\sigma] \begin{pmatrix} E_y \\ E_z \end{pmatrix} \tag{2}$$

A theoretical approach to Faraday effect

Maxwell's equation:

$$\Delta\vec{E} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial\vec{j}}{\partial t}$$

Matricial form:

$$\left(-k^2\mathbb{1} + \frac{\omega^2}{c^2}\mathbb{1} + i\omega\mu_0[\sigma]\right)\begin{pmatrix} E_y \\ E_z \end{pmatrix} = 0$$

Not interested in the trivial solution $\vec{E} = \vec{0} \Rightarrow$

$$\left(-k^2\mathbb{1} + \frac{\omega^2}{c^2}\mathbb{1} + i\omega\mu_0[\sigma]\right) = 0$$

A theoretical approach to Faraday effect

Diagonal in the basis: $\left(\begin{pmatrix} E_y \\ -iE_z \end{pmatrix}, \begin{pmatrix} E_y \\ iE_z \end{pmatrix}\right)$ (right-handed and left-handed circular polarized wave).

$$\left(-k^2\mathbb{1} + \frac{\omega^2}{c^2}\mathbb{1} + i\omega\mu_0[\sigma]\right) = 0$$

$\Rightarrow$

$$\begin{cases} k_R^2 = \dfrac{\omega^2}{c^2}\left(1 + \dfrac{\omega_p^2}{(\omega_0^2 - \omega^2) - \omega\Omega}\right) \\ k_L^2 = \dfrac{\omega^2}{c^2}\left(1 + \dfrac{\omega_p^2}{(\omega_0^2 - \omega^2) + \omega\Omega}\right) \end{cases} \qquad \begin{cases} n_R^2 = 1 + \dfrac{\omega_p^2}{(\omega_0^2 - \omega^2) - \omega\Omega} \\ n_L^2 = 1 + \dfrac{\omega_p^2}{(\omega_0^2 - \omega^2) + \omega\Omega} \end{cases}$$

A theoretical approach to the Faraday effect

Linear polarization=superposition of right-handed and left-handed circular components

At the end of the medium, we find:

$$\vec{E}(x=L) = E_0(\cos(\frac{k_R+k_L}{2}L-\omega t)(\cos(\frac{k_R-k_L}{2}L)\vec{e_y}+\sin(\frac{k_R-k_L}{2}L)\vec{e_z}) \tag{3}$$

$$\beta = \frac{k_R-k_L}{2}L$$

A theoretical approach to Faraday effect

Extra assumption:

- ω is far enough from resonance such that $\frac{\omega\Omega}{\omega_0^2-\omega^2} \ll 1$
- $\omega_0 \approx 0$
- $\omega_p \ll \omega$

$$\beta = VB_0L \tag{4}$$

with the Verdet constant $V = \frac{e}{2m_ec}\frac{\omega_p^2}{\omega^2}$.

Modeling the Faraday effect in a plasmonic medium

Consider:

- incident wave: sine wave of frequency ω,modulated by a Gaussian, propagating in the x-direction, y polarized
- $\vec{B_0} = B_0 \vec{e_x}$
- $\vec{H} = \frac{\vec{B}}{\mu_0}$ assuming $\vec{M} = 0$
- $\omega_0 \approx 0$

Recall Maxwell's equations:

- $\nabla \times \vec{E} = -\mu_0 \frac{\vec{H}}{\partial t}$
- $\nabla \times \vec{H} = \vec{J} + \epsilon_0 \frac{\vec{E}}{\partial t}$

as well as the time-evolution equation for the current density $\vec{J}$:

- $\frac{\partial \vec{J}}{\partial t} = -\gamma \vec{J} + \frac{ne^2}{m} \vec{E} - \frac{e}{m} \vec{J} \times \vec{B}_0$

The Finite Difference Time Derivative (FDTD)

Finite Difference Time Derivative (FDTD) method

- $\frac{\partial E_y}{\partial x}\Big|_{x+\frac{\Delta x}{2}} \approx \frac{E_y(x+\Delta x)-E_y(x)}{\Delta x}$
- $\frac{\partial E_y}{\partial t}\Big|_{t+\frac{\Delta t}{2}} \approx \frac{E_y(t+\Delta t)-E_y(t)}{\Delta t}$

and similarly for E_z, H_y and H_z

The Finite Difference Time Derivative (FDTD)

[6] This gives us (setting $x = m$, $t = n$):

- $\frac{E_y(m+1,n)-E_y(m,n)}{\Delta x} \approx \mu_0 \frac{H_z(m+\frac{1}{2},n+\frac{1}{2})-H_z(m+\frac{1}{2},n-\frac{1}{2})}{\Delta t}$
- $\frac{H_z(m+\frac{1}{2},n+\frac{1}{2})-H_z(m-\frac{1}{2},n+\frac{1}{2})}{\Delta x} \approx (J_y(n+\frac{1}{2}) + \epsilon_0 \frac{E_y(m,n+1)-E_y(m,n)}{\Delta t})$

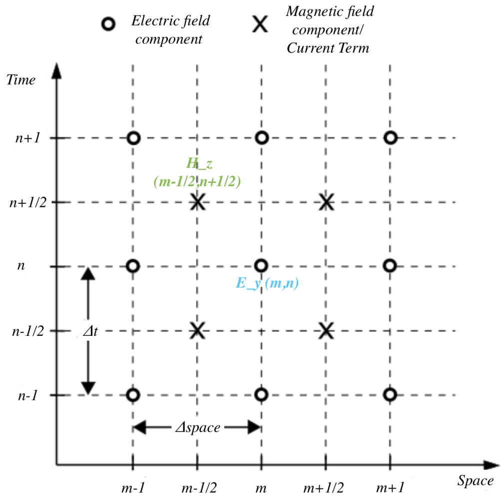
and so

- $H_z(m+\frac{1}{2}, n+\frac{1}{2}) = H_z(m+\frac{1}{2}, n-\frac{1}{2}) + \frac{\Delta t}{\mu_0 \Delta x}(E_y(m+1, n) - E_y(m, n))$
- $E_y(m, n+1) =$
  $E_y(m, n) + \frac{\Delta t}{\epsilon_0 \Delta x}(H_z(m+\frac{1}{2}, n+\frac{1}{2}) - H_z(m-\frac{1}{2}, n+\frac{1}{2})) - \frac{\Delta t}{\epsilon_0} J_y(n+\frac{1}{2})$

Yee-Grid method

Staggered Yee-grid layout[2]:

- E and H are offset by half a grid cell
- Current density J computed at the same point as H



Integer indices for iteration

Python implementation $\implies$ **integer indices** $\implies$ shift

- $m + \frac{1}{2} \mapsto m'$
- $m - \frac{1}{2} \mapsto m' - 1$

Implementation: loop for H: $m = 0$ to $m = m_{max} - 1$

loop for E: $m = 1$ to $m = m_{max}$

- $H_z(m, n) = H_z(m, n-1) + \frac{\Delta t}{\mu_0 \Delta x}(E_y(m+1, n) - E_y(m, n))$
- $E_y(m, n+1) = E_y(m, n) + \frac{\Delta t}{\epsilon_0 \Delta x}(H_z(m+1, n) - H_z(m, n)) - \frac{\Delta t}{\epsilon_0} J_y(n)$

Similarly, for the couple (E_z, H_y)

Boundary conditions

To avoid reflection at the boundaries of our medium, we implement the **Absorbing Boundary condition (ABC)** [4]:

$$E_y(m = 0, n + 1) = E_y(m = 1, n)$$

$$H_z(m_{max}, n) = H_z(m_{max} - 1, n - 1)$$

We impose the analogous condition for the couple $(E_z,\ H_y)$

Results and discussion

Recall that

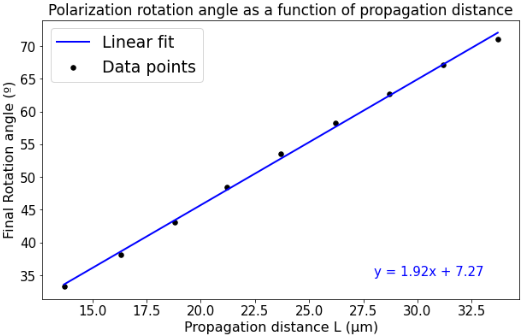
$$\beta = VB_0L$$

Goal of our simulations:

- **qualitative**: Verify
  - $\beta \propto B_0$
  - $\beta \propto L$
- **quantitative**: compare simulated values of β to theoretical ones

Linearity of the rotation: Distance

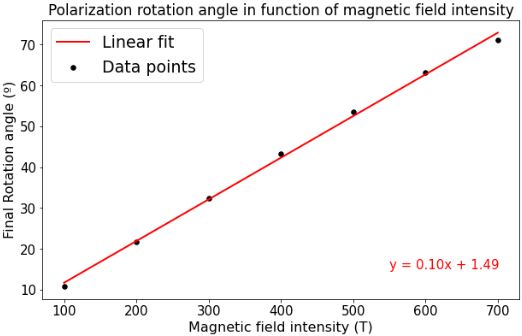
Rotation of polarization: fixed $B_0 = 500\text{T}$ and varying distance L



Theoretical value for the slope: $VB_0 = 2.02 \text{ deg } \mu\text{m}^{-1}$ 5% relative error

Linearity of the rotation: Magnetic field

Rotation of polarization: fixed $L = 23.7\mu$m and varying magnetic field B_0

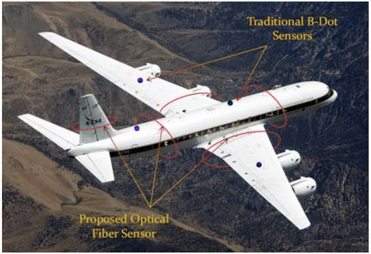


Theoretical value for the slope: $VL = 0.096$ degT^{-1} 4.2% relative error

Fiber Optic Current Sensors (FOCS)

- $\beta = VB_0L = V\int \vec{B}d\vec{l}$

  Ampère's law $\Rightarrow \beta = V\mu_0 I$
- Measurement of β to deduce current value I



FOCS to measure lightning current on planes[3]

Relevance of the Faraday effect in the Ionosphere and satellite communication

Faraday effect in the ionosphere: $\beta \propto \lambda^2,\ TEC,\ B_{Earth}$

Consequences? [5]

- reduction in phase coherence $\Rightarrow$ impairs interferometric data from satellite (useful for topographical maps)
- Faraday effect must be taken into account!

Summary and discussion

Key takeaways: Faraday effect = rotation of the plane of polarization of an electromagnetic wave propagating in a media where we apply a $\vec{B}$ field

- due to a difference in index of refraction for right-handed and left-handed circular polarized wave
- rotation angle $\propto L$ (length of medium) and B_0 (magnitude of the magnetic field along the direction of propagation)

Numerical simulation using **FDTD** and **Yee staggered grid**

Summary and discussion

Simulation: getting rid of the assumption $\omega_0 \approx 0$.

Time-evolution equation for the current becomes:

- $\frac{\partial \vec{J}}{\partial t} = -\gamma \vec{J} + \frac{ne^2}{m}\vec{E} - \frac{e}{m}\vec{J} \times \vec{B}_0 + \omega_0^2 \vec{P}$
  with $\vec{J} = \frac{\partial P}{\partial t}$

References I

[1] Wikipedia contributors. *Faraday effect.* https://en.wikipedia.org/wiki/Faraday_effect. Accessed: 2024-05-27. 2024.

[2] Anand Kumar, Jogesh Dash, and Debdeep Sarkar. *Computational Techniques for Design and Analysis of Time-Varying Capacitor loaded Transmission Lines using FDTD and Simulink*. The Yee-grid template was taken from that reference and edited according to our model. May 2022. DOI: 10.36227/techrxiv.19846576.v1.

References II

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[4] Ansys Optics. *PML boundary conditions in FDTD and MODE*. Accessed: 2024-05-23. 2023. URL: https://optics.ansys.com/hc/en-us/articles/360034382674-PML-boundary-conditions-in-FDTD-and-MODE.

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[5] Eric J. M. Rignot. "Effect of Faraday Rotation on L-Band Interferometric and Polarimetric Synthetic-Aperture Radar Data". In: *IEEE Transactions on Geoscience and Remote Sensing* 38.1 (Jan. 2000), p. 383. DOI: 10.1109/36.823905.

[6] John B. Schneider. *Understanding the Finite-Difference Time-Domain Method*. Pullman, WA: School of Electrical Engineering and Computer Science, Washington State University, Aug. 2023.