

- 1 For any two functions $f(n)$ and $g(n)$, $f(n) = \Theta(g(n))$ means there exists constants c_1 and c_2 such that $g(n)$ creates an upper AND a lower bound for $f(n)$, or $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$ for all values of n greater than some n_0 . Thus, $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ AND $f(n) = \Omega(g(n))$. $f(n) = O(g(n))$ states that $0 \leq f(n) \leq cg(n)$ or that some constant, c , times $g(n)$ creates an upper bound for $f(n)$. $f(n) = \Omega(g(n))$ states that $0 \leq cg(n) \leq f(n)$ or that for some constant, c , times $g(n)$ creates a lower bound for $f(n)$. Thus, in order to have $f(n) = \Theta(g(n))$ you must have $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. Additionally, if you have $f(n) = \Theta(g(n))$ you know you have both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

- 2.a $n^2 + 3n - 20 = O(n^2)$
 $0 \leq n^2 + 3n - 20 \leq cn^2$
 $n^2 + 3n - 20 \leq 2n^2 \Rightarrow 0 \leq -16 \leq 2X$
 \dots
 $n = 3 \Rightarrow 0 \leq -2 \leq 9X$
 $n = 4 \Rightarrow 0 \leq 8 \leq 32\sqrt{}$
 $c = 2$
 $n_0 = 4$

- 2.b $n - 2 = \Omega(n)$
 $0 \leq cn \leq n - 2$
 $0 \leq \frac{1}{2}n \leq n - 2$
 $n = 1 \Rightarrow 0 \leq \frac{1}{2} \leq -1X$
 $n = 2 \Rightarrow 0 \leq 1 \leq 0X$
 \dots
 $n = 5 \Rightarrow 0 \leq \frac{5}{2} \leq 3\sqrt{}$
 $c = \frac{1}{2}$
 $n_0 = 5$

- 2.c $\log_{10}n + 4 = \Theta(\log_2n)$
 $0 \leq c_1\log_2n \leq \log_{10}n + 4 \leq c_2\log_2n$
 $0 \leq c_1\log_2n \leq \frac{\log_2n}{\log_210} + 4 \leq c_2\log_2n$
 $0 \leq \frac{1}{\log_210}\log_2n \leq \frac{\log_2n}{\log_210} + 4 \leq \frac{2}{\log_210}\log_2n$
 $c_1 = \frac{1}{\log_210}$
 $c_2 = \frac{2}{\log_210}$
 $n_0 = 1$

- 2.d $2^{n+1} = O(2^n)$
 $0 \leq 2^{n+1} \leq c2^n$
 $0 \leq 2^n + 1 \leq 3 * 2^n$
 $n = 1 \Rightarrow 0 \leq 4 \leq 6\sqrt{}$
 $n = 2 \Rightarrow 0 \leq 8 \leq 12\sqrt{}$
 $c = 3$
 $n_0 = 1$

2.e $\ln(n) = \Theta(\log_2 n)$

$$0 \leq c_1 \log_2 n \leq \ln(n) \leq c_2 \log_2 n$$

$$0 \leq c_1 \log_2 n \leq \log_e n \leq c_2 \log_2 n$$

$$0 \leq c_1 \log_2 n \leq \frac{\log_2 n}{\log_2 e} \leq c_2 \log_2 n$$

$$c_1 = c_2 = \frac{1}{\log_2 e}$$

$$n_0 = 1$$

2.f $n^\epsilon = \Omega(\lg n)$ for any $\epsilon > 0$

$$0 \leq c * \lg n \leq n^\epsilon$$

$$n = 1 \Rightarrow 0 \leq 0 \leq 1, 1, \dots, 1$$

$$n = 2 \Rightarrow 0 \leq 1 \leq 2, 4, 8, \dots, 2^\epsilon$$

$$c = 1$$

$$n_0 = 1$$

3.a $T(n) = 2T(\frac{n}{2}) + n^3$

$$= 2[2T(\frac{n/2}{2}) + (\frac{n}{2})^3] + n^3$$

$$= 4T(\frac{n}{4}) + 2(\frac{n}{2})^3 + n^3$$

$$= 4[2T(\frac{n/4}{2}) + (\frac{n}{4})^3] + 2(\frac{n}{2})^3 + n^3$$

$$= 8T(\frac{n}{8}) + 4(\frac{n}{4})^3 + 2(\frac{n}{2})^3 + n^3$$

$$= 8[2T(\frac{n/8}{2}) + (\frac{n}{8})^3] + 4(\frac{n}{4})^3 + 2(\frac{n}{2})^3 + n^3$$

$$= 16T(\frac{n}{16}) + (\frac{n}{8})^3 + 4(\frac{n}{4})^3 + 2(\frac{n}{2})^3 + n^3$$

$$= 2^i T(\frac{n}{2^i}) + \sum_{j=0}^{i-1} 2^j (\frac{n}{2^j})^3$$

$$\frac{n}{2^i} = 2$$

$$n = 2 * 2^i$$

$$\frac{n}{2} = 2^i$$

$$\log_2 \frac{n}{2} = i$$

$$2^{\log_2 \frac{n}{2}} T(\frac{n}{2^{\log_2 \frac{n}{2}}}) + \sum_{j=0}^{(\log_2 \frac{n}{2})-1} \frac{2^j n^3}{2^{3j}}$$

$$= \frac{n}{2} T(2) + n^3 \sum_{j=0}^{(\log_2 \frac{n}{2})-1} \frac{1}{2^{2j}}$$

$$\frac{n}{2} T(2) + n^3 \sum_{j=0}^{\infty} (\frac{1}{2})^{2j}$$

$$< \frac{n}{2} T(2) + n^3 \frac{1}{1-\frac{1}{2}}$$

$$= \frac{n}{2} a + 2n^3$$

Proof

Prove that $T(n) < \frac{n}{2} a + 2n^3$

For the base case, let $n = 2$. We need to show that $T(2) < \frac{n}{2} a + 2n^3$. $T(2) = a < \frac{n}{2} a + 2n^3$ which is correct by definition.

Now assume that $T(k) < \frac{n}{2} a + 2n^3$ for all values $k = 2, 3, \dots, n-1$.

To show that $T(n) < \frac{n}{2} a + 2n^3$, we start with the definition of $T(n) = 2T(\frac{n}{2}) + n^3$.

Since $\frac{n}{2} < n-1$ we can use the previous assumption to get the equation

$$T(n) < 2[\frac{n}{4}]a + 2(\frac{n}{2})^3$$

$$= \frac{n}{4} a + 2(\frac{n^3}{8})$$

$$= \frac{n}{4} a + \frac{n^3}{4}$$

which is less than $\frac{n}{2} a + 2n^3$ ■

3.b $T(n) = T(\frac{9n}{10}) + n$

$$= [T(\frac{9}{10}(\frac{9}{10})n + \frac{9}{10}n)] + n$$

$$= T(\frac{81}{100}n) + \frac{9}{10}n + n$$

$$\begin{aligned}
&= [T(\frac{9}{10}(\frac{81}{100})n + \frac{81}{100}n) + \frac{9}{10}n + n \\
&= T(\frac{729}{1000}n) + \frac{81}{100}n + \frac{9}{10}n + n \\
&= T((\frac{9}{10})^i n) + \sum_{j=0}^{i-1} (\frac{9}{10})^j n \\
&(\frac{9}{10})^i n = 2 \\
&(\frac{9}{10})^i = \frac{2}{n} \\
&(\frac{10}{9})^i = \frac{n}{2} \\
&\log_{\frac{10}{9}} \frac{n}{2} = i
\end{aligned}$$

$$\begin{aligned}
&T((\frac{9}{10})^{\log_{\frac{10}{9}} \frac{n}{2}}) + \sum_{j=0}^{\log_{\frac{10}{9}} \frac{n}{2} - 1} (\frac{9}{10})^j n \\
&< T(2) + n \sum_{j=0}^{\infty} (\frac{9}{10})^j \\
&= a + n(\frac{1}{1-\frac{9}{10}}) \\
&= a + 10n
\end{aligned}$$

Proof

Prove that $T(n) < a + 10n$.

For the base case, let $n = 2$. We need to show that $T(2) < a + 10n$. $T(2) = a < a + 10n$ which is correct by definition.

Now assume that $T(k) < a + 10n$ for all values of $k = 2, 3, \dots, n-1$.

To show that $T(n) < a + 10n$, we can start with the definition of $T(n) = T(\frac{9n}{10}) + n$.

Since $\frac{9}{10}n < n-1$ we can use the previous assumption to get the equation

$$\begin{aligned}
T(n) &< a + \frac{9}{10}(10n) + n \\
&= a + 9n + n \\
&= a + 10n.
\end{aligned}$$

Thus, $T(n) < a + 10n$ ■

$$\begin{aligned}
3.c \quad T(n) &= 7T(\frac{n}{3}) + n^2 \\
&= 7[7T(\frac{n/3}{3}) + (\frac{n}{3})^2] + n^2 \\
&= 49T(\frac{n}{9}) + 7(\frac{n}{3})^2 + n^2 \\
&= 49[7T(\frac{n/9}{3}) + (\frac{n}{9})^2] + 7(\frac{n}{3})^2 + n^2 \\
&= 7^3T(\frac{n}{3^3}) + 7^2(\frac{n}{3^2})^2 + 7(\frac{n}{3})^2 + n^2 \\
&= 7^3[7T(\frac{n/3^3}{3}) + (\frac{n}{3^3})^2] + 7^2(\frac{n}{3^2})^2 + 7(\frac{n}{3})^2 + n^2 \\
&= 7^4T(\frac{n}{3^4}) + 7^3(\frac{n}{3^3})^2 + 7^2(\frac{n}{3^2})^2 + 7(\frac{n}{3})^2 + n^2 \\
&= 7^i T(\frac{n}{3^i}) + \sum_{j=0}^{i-1} 7^j (\frac{n}{3^j})^2 \\
&\frac{n}{3^i} = 2 \\
&\frac{n}{2} = 3^i \\
&\log_3 \frac{n}{2} = i
\end{aligned}$$

$$\begin{aligned}
&= 7^{\log_3 \frac{n}{2}} T(\frac{n}{3^{\log_3 \frac{n}{2}}}) + \sum_{j=0}^{(\log_3 \frac{n}{2})-1} 7^j (\frac{n}{3^j})^2 \\
&= 7^{\log_3 \frac{n}{2}} T(2) + n^2 \sum_{j=0}^{(\log_3 \frac{n}{2})-1} \frac{7^j}{3^{2j}} \\
&< 7^{\log_3 \frac{n}{2}} a + 7n^2 \sum_{j=0}^{\infty} (\frac{1}{3})^j \\
&= 7^{\log_3 \frac{n}{2}} a + 7n^2 (\frac{1}{1-\frac{1}{3}}) \\
&= 7^{\log_3 \frac{n}{2}} a + 7n^2 (\frac{3}{2}) \\
&= 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2
\end{aligned}$$

Proof Prove that $T(n) < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2$.

For the base case, let $n = 2$. We need to show that $T(2) < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2$. $T(2) = a < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2$ which is correct by definition.

Now assume that $T(k) < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2$ for all values of $k = 2, 3, \dots, n-1$.

To show that $T(n) < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2$, we can start with the definition of $T(n) = 7T(\frac{n}{3}) + n^2$. Since $\frac{n}{3} < n - 1$ we can use the previous assumption to get the equation

$$\begin{aligned} T(n) &< 7[7^{\log_3 \frac{n/3}{2}} a + \frac{21}{2} (\frac{n}{3})^2] + n^2 \\ &= 7 * 7^{\log_3 \frac{n/3}{2}} a + \frac{21}{2} * \frac{n^2}{9} + n^2 \\ &= 7a * 7^{\log_3 \frac{n/3}{2}} + \frac{39n^2}{18} < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2 \text{ because } \frac{21}{2} > \frac{39}{18}. \text{ Thus, } T(n) < 7^{\log_3 \frac{n}{2}} a + \frac{21}{2} n^2 \blacksquare \end{aligned}$$

3.d $T(n) = T(n^{\frac{1}{2}}) + 1$

$$\begin{aligned} &= T(n^{\frac{1}{4}}) + 2 \\ &= T(n^{\frac{1}{8}}) + 3 \\ &= T(n^{\frac{1}{16}}) + 4 \\ &= T(2^{\frac{1}{2^i}}) + i \\ n^{\frac{1}{2^i}} &= 2 \\ &= \log(n^{\frac{1}{2^i}}) = \log(2) \\ &= \frac{1}{2^i} \log(n) = \log(2) \\ &= \frac{\log(n)}{\log(2)} = 2^i \\ &= \log(\frac{\log(n)}{\log(2)}) = \log(2^i) \\ &= \log(\frac{\log(n)}{\log(2)}) = i(\log(2)) \\ &= \frac{\log(\frac{\log(n)}{\log(2)})}{\log(2)} = i \\ &= \log_2(\frac{\log(n)}{\log(2)}) = i \\ &= \log_2(\log_2(n)) = i \\ T(n^{\frac{1}{2^{\log_2(\log_2(n))}}}) &+ \log_2(\log_2(n)) \\ &= T(2) + \log_2(\log_2(n)) \\ &= a + \log_2(\log_2(n)) \end{aligned}$$

Proof

Prove $T(n) = a + \log_2(\log_2(n))$.

For the base case, let $n = 2$. We need to show that $T(2) = a + \log_2(\log_2(n))$.

$$\begin{aligned} T(2) &= a + \log_2(\log_2(2)) \\ &= a + \log_2(1) \\ &= a \end{aligned}$$

which is true by definition.

Now assume that $T(k) < a + \log_2(\log_2(n))$ for all values of $k = 2, 3, \dots, n - 1$.

To show that $T(n) < a + \log_2(\log_2(n))$ we start with the definition of $T(n) = T(n^{\frac{1}{2}}) + 1$.

Since $n^{\frac{1}{2}}$ we can use the previous assumption to get the equation

$$\begin{aligned} T(n) &= a + \log_2(\log_2(n^{\frac{1}{2}})) + 1 \\ &= a + \log_2(\frac{1}{2} * \log_2(n)) + 1 \\ &= a + \log_2(\frac{1}{2}) + \log_2(\log_2(n)) + 1 \\ &= a + (-1) + \log_2(\log_2(n)) + 1 \\ &= a + \log_2(\log_2(n)) + 1 \end{aligned}$$

which proves $T(n) = a + \log_2(\log_2(n)) \blacksquare$

3.e $T(n) = T(n - 1) + \lg(n)$

$$\begin{aligned} &= T(n - 2) + \lg(n - 1) + \lg(n) \\ &= T(n - 3) + \lg(n - 2) + \lg(n - 1) + \lg(n) \\ &= T(n - 4) + \lg(n - 3) + \lg(n - 2) + \lg(n - 1) + \lg(n) \end{aligned}$$

$$\begin{aligned}
&= T(n-i) + \sum_{j=0}^{i-1} lg(n-j) \\
n-i &= 2 \\
&= n-2 = i \\
T(n-(n-2)) &+ \sum_{j=0}^{(n-2)-1} lg(n-j) \\
&= T(2) + \sum_{j=0}^{n-3} lg(n-j) \\
&\leq a + lg(n!)
\end{aligned}$$

Proof

Prove that $T(n) \leq a + lg(n!)$

For the base case, let $n = 2$. We need to show that $T(2) < a + lg(n!)$. $T(2) = a \leq a + lg(n!)$.

Now assume that $T(k) \leq a + lg(n!)$ for all values of $k = 2, 3, \dots, n-1$.

To show that $T(n) \leq a + lg(n!)$ we can start with the definition of $T(n) = T(n-1) + lg(n)$. Since $n-1$ is in the range of k , we can use the previous assumption to get the equation

$$\begin{aligned}
T(n) &\leq a + lg((n-1)!) + lg(n) \\
&= a + lg((n-1)! + n) \\
&= a + lg(n!) \text{ which is equal.} \blacksquare
\end{aligned}$$

4 Prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$.

For the base case, let $i = 1$ and 2 .

$$\begin{aligned}
F_1 &= \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \\
F_2 &= \frac{\phi^2 - \hat{\phi}^2}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} = \frac{(1+2\sqrt{5}+5) - (1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1 \\
F_3 &= \frac{\phi^3 - \hat{\phi}^3}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^3 - (\frac{1-\sqrt{5}}{2})^3}{\sqrt{5}} = \frac{(16+8\sqrt{5}) - (16-8\sqrt{5})}{8\sqrt{5}} = \frac{16\sqrt{5}}{8\sqrt{5}} = 2
\end{aligned}$$

To find the Fibonacci numbers you sum the previous two Fibonacci numbers. The sequence always begins with 1 so F_1 should equal 1, as it does above. To get F_3 , you sum $F_1 + F_2 = 1 + 1 = 2$ which is what the formula yielded for F_3 .

Now assume that $F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}}$ for all values of $k = 1, 2, 3, \dots, i-1$.

To prove that $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ you need to show the previous two Fibonacci numbers sum to it.

$$\begin{aligned}
F_{i-2} + F_{i-1} &= F_i = \frac{(\frac{1+\sqrt{5}}{2})^{i-2} - (\frac{1-\sqrt{5}}{2})^{i-2}}{\sqrt{5}} \\
F_{i-2} + F_{i-1} &= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} - (\frac{1-\sqrt{5}}{2})^{i-2}}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{i-1} - (\frac{1-\sqrt{5}}{2})^{i-1}}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} + (\frac{1+\sqrt{5}}{2})^{i-1} + (\frac{1-\sqrt{5}}{2})^{i-2} + (\frac{1-\sqrt{5}}{2})^{i-1}}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} * (1 + \frac{1+\sqrt{5}}{2}) + (\frac{1-\sqrt{5}}{2})^{i-2} * (1 + \frac{1-\sqrt{5}}{2})}{\sqrt{5}} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} * (\frac{1+\sqrt{5}}{2})^2 + (\frac{1-\sqrt{5}}{2})^{i-2} * (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \text{ (by properties of the conjugate)} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^i - (\frac{1-\sqrt{5}}{2})^i}{\sqrt{5}} = F_i \blacksquare
\end{aligned}$$