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1 For any two functions $f(n)$ and $g(n)$, $f(n) = \Theta(g(n))$ means there exists constants c_1 and c_2 such that $g(n)$ creates an upper AND a lower bound for $f(n)$, or $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all values of n greater than some n_0. Thus, $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ AND $f(n) = \Omega(g(n))$. $f(n) = O(g(n))$ states that $0 \leq f(n) \leq cg(n)$ or that some constant, c, times $g(n)$ creates an upper bound for $f(n)$. $f(n) = \Omega(g(n))$ states that $0 \leq cg(n) \leq f(n)$ or that for some constant, c, times $g(n)$ creates a lower bound for $f(n)$. Thus, in order to have $f(n) = \Theta(g(n))$ you must have $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. Additionally, if you have $f(n) = \Theta(g(n))$ you know you have both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

2.a $n^2 + 3n - 20 = O(n^2)$
$0 \leq n^2 + 3n - 20 \leq cn^2$
$n^2 + 3n - 20 \leq 2n^2 n = 1 \Rightarrow 0 \leq -16 \leq 2X$
...
$n = 3 \Rightarrow 0 \leq -2 \leq 9X$
$n = 4 \Rightarrow 0 \leq 8 \leq 32\surd$
$c = 2$
$n_0 = 4$

2.b $n - 2 = \Omega(n)$
$0 \leq cn \leq n - 2$
$0 \leq \frac{1}{2}n \leq n - 2$
$n = 1 \Rightarrow 0 \leq \frac{1}{2} \leq -1X$
$n = 2 \Rightarrow 0 \leq 1 \leq 0X$
...
$n = 5 \Rightarrow 0 \leq \frac{5}{2} \leq 3\surd$
$c = \frac{1}{2}$
$n_0 = 5$

2.c $log_{10}n + 4 = \Theta(log_2 n)$
$0 \leq c_1 log_2 n \leq log_{10}n + 4 \leq c_2 log_2 n$
$0 \leq c_1 log_2 n \leq \frac{log_2 n}{log_2 10} + 4 \leq c_2 log_2 n$
$0 \leq \frac{1}{log_2 10} log_2 n \leq \frac{log_2 n}{log_2 10} + 4 \leq \frac{2}{log_2 10} log_2 n$
$c_1 = \frac{1}{log_2 10}$
$c_2 = \frac{2}{log_2 10}$
$n_0 = 1$

2.d $2^{n+1} = O(2^n)$
$0 \leq 2^{n+1} \leq c2^n$
$0 \leq 2^n + 1 \leq 3 * 2^n$
$n = 1 \Rightarrow 0 \leq 4 \leq 6\surd$
$n = 2 \Rightarrow 0 \leq 8 \leq 12\surd$
$c = 3$
$n_0 = 1$

2.e $ln(n) = \Theta(log_2 n)$

$0 \leq c_1 log_2 n \leq ln(n) \leq c_2 log_2 n$
$0 \leq c_1 log_2 n \leq log_e n \leq c_2 log_2 n$
$0 \leq c_1 log_2 n \leq \frac{log_2 n}{log_2 e} \leq c_2 log_2 n$
$c_1 = c_2 = \frac{1}{log_2 e}$
$n_0 = 1$

2.f $n^\epsilon = \Omega(lgn)$ for any $\epsilon > 0$

$0 \leq c * lgn \leq n^\epsilon$
$n = 1 \Rightarrow 0 \leq 0 \leq 1, 1, ..., 1$
$n = 2 \Rightarrow 0 \leq 1 \leq 2, 4, 8...2^\epsilon$
$c = 1$
$n_0 = 1$

3.a $T(n) = 2T(\frac{n}{2}) + n^3$

$= 2[2T(\frac{n/2}{2}) + (\frac{n}{2})^3] + n^3$
$= 4T(\frac{n}{4}) + 2(\frac{n}{2})^3 + n^3$
$= 4[2T(\frac{n/4}{2}) + (\frac{n}{4})^3] + 2(\frac{n}{2})^3 + n^3$
$= 8T(\frac{n}{8}) + 4(\frac{n}{4})^3 + 2(\frac{n}{2})^3 + n^3$
$= 8[2T(\frac{n/8}{8}) + (\frac{n}{8})^3] + 4(\frac{n}{4})^3 + 2(\frac{n}{2})^3 + n^3$
$= 16T(\frac{n}{16}) + (\frac{n}{8})^3 + 4(\frac{n}{4})^3 + 2(\frac{n}{2})^3 + n^3$
$= 2^i T(\frac{n}{2^i}) + \sum_{j=0}^{i-1} 2^j (\frac{n}{2^j})^3$
$\frac{n}{2^i} = 2$
$n = 2 * 2^i$
$\frac{n}{2} = 2^i$
$log_2 \frac{n}{2} = i$
$2^{log_2 \frac{n}{2}} T(\frac{n}{2^{log_2 \frac{n}{2}}}) + \sum_{j=0}^{(log_2 \frac{n}{2})-1} \frac{2^j n^3}{2^{3j}}$
$= \frac{n}{2} T(2) + n^3 \sum_{j=0}^{(log_2 \frac{n}{2})-1} \frac{1}{2^{2j}}$
$\frac{n}{2} T(2) + n^3 \sum_{j=0}^{\infty} (\frac{1}{2})^{2j}$
$< \frac{n}{2} T(2) + n^3 \frac{1}{1-\frac{1}{2}}$
$= \frac{n}{2} a + 2n^3$

Proof

Prove that $T(n) < \frac{n}{2} a + 2n^3$

For the base case, let $n = 2$. We need to show that $T(2) < \frac{n}{2} a + 2n^3$. $T(2) = a < \frac{n}{2} a + 2n^3$ which is correct by definition.

Now assume that $T(k) < \frac{n}{2} a + 2n^3$ for all values $k = 2, 3, ..., n-1$.

To show that $T(n) < \frac{n}{2} a + 2n^3$, we start with the definition of $T(n) = 2T(\frac{n}{2}) + n^3$. Since $\frac{n}{2} < n - 1$ we can use the previous assumption to get the equation

$T(n) < 2[\frac{n}{4}] a + 2(\frac{n}{2})^3$
$= \frac{n}{4} a + 2(\frac{n^3}{8})$
$= \frac{n}{4} a + \frac{n^3}{4}$

which is less than $\frac{n}{2} a + 2n^3 \blacksquare$

3.b $T(n) = T(\frac{9n}{10}) + n$

$= [T(\frac{9}{10}(\frac{9}{10})n + \frac{9}{10} n] + n$
$= T(\frac{81}{100} n) + \frac{9}{10} n + n$

$= [T(\frac{9}{10}(\frac{81}{100})n + \frac{81}{100}n] + \frac{9}{10}n + n$
$= T(\frac{729}{1000}n) + \frac{81}{100}n + \frac{9}{10}n + n$
$= T((\frac{9}{10})^i n) + \sum_{j=0}^{i-1}(\frac{9}{10})^j n$
$(\frac{9}{10})^i n = 2$
$(\frac{9}{10})^i = \frac{2}{n}$
$(\frac{10}{9})^i = \frac{n}{2}$
$log_{\frac{10}{9}}\frac{n}{2} = i$
$T((\frac{9}{10})^{log_{\frac{10}{9}}\frac{n}{2}}) + \sum_{j=0}^{log_{\frac{10}{9}}\frac{n}{2}-1}(\frac{9}{10})^j n$
$< T(2) + n\sum_{j=0}^{\infty}(\frac{9}{10})^j$
$= a + n(\frac{1}{1-\frac{9}{10}})$
$= a + 10n$
Proof
Prove that $T(n) < a + 10n$.
For the base case, let $n = 2$. We need to show that $T(2) < a + 10n.T(2) = a < a + 10n$ which is correct by definition.
Now assume that $T(k) < a + 10n$ for all values of $k = 2, 3, ..., n-1$.
To show that $T(n) < a + 10n$, we can start with the definition of $T(n) = T(\frac{9n}{10}) + n$. Since $\frac{9}{10}n < n - 1$ we can use the previous assumption to get the equation
$T(n) < a + \frac{9}{10}(10n) + n$
$= a + 9n + n$
$= a + 10n.$
Thus, $T(n) < a + 10n\blacksquare$

3.c $T(n) = 7T(\frac{n}{3}) + n^2$
$= 7[7T(\frac{n/3}{3}) + (\frac{n}{3})^2] + n^2$
$= 49T(\frac{n}{9}) + 7(\frac{n}{3})^2 + n^2$
$= 49[7T(\frac{n/9}{3}) + \frac{n}{9})^2] + 7(\frac{n}{3})^2 + n$
$= 7^3T(\frac{n}{3^3}) + 7^2(\frac{n}{3^2})^2 + 7(\frac{n}{3})^2 + n^2$
$= 7^3[7T(\frac{n/3^3}{3}) + (\frac{n}{3^3})^2] + 7^2(\frac{n}{3^2})^2 + 7(\frac{n}{3})^2 + n^2$
$= 7^4T(\frac{n}{3^4}) + 7^3(\frac{n}{3^3})^2 + 7^2(\frac{n}{3^2})^2 + 7(\frac{n}{3})^2 + n^2$
$= 7^iT(\frac{n}{3^i}) + \sum_{j=0}^{i-1} 7^j(\frac{n}{3^j})^2$
$\frac{n}{3^i} = 2$
$\frac{n}{2} = 3^i$
$log_3\frac{n}{2} = i$
$= 7^{log_3\frac{n}{2}}T(\frac{n}{3^{log_3\frac{n}{2}}} + \sum_{j=0}^{(log_3\frac{n}{2})-1} 7^j(\frac{n}{3^j})^2$
$= 7^{log_3\frac{n}{2}}T(2) + n^2\sum_{j=0}^{(log_3\frac{n}{2})-1}\frac{7^j}{3^2j}$
$< 7^{log_3\frac{n}{2}}a + 7n^2\sum_{j=0}^{\infty}(\frac{1}{3})^j$
$= 7^{log_3\frac{n}{2}}a + 7n^2(\frac{1}{1-\frac{1}{3}})$
$= 7^{log_3\frac{n}{2}}a + 7n^2(\frac{3}{2})$
$= 7^{log_3\frac{n}{2}}a + \frac{21}{2}n^2$
Proof Prove that $T(n) < 7^{log_3\frac{n}{2}}a + \frac{21}{2}n^2$.
For the base case, let $n = 2$. We need to show that $T(2) < 7^{log_3\frac{n}{2}}a + \frac{21}{2}n^2$. $T(2) = a < 7^{log_3\frac{n}{2}}a + \frac{21}{2}n^2$ which is correct by definition.
Now assume that $T(k) < 7^{log_3\frac{n}{2}}a + \frac{21}{2}n^2$ for all values of $k = 2, 3, ..., n-1$.

To show that $T(n) < 7^{log_3 \frac{n}{2}} a + \frac{21}{2} n^2$, we can start with the definition of $T(n) = 7T(\frac{n}{3}) + n^2$. Since $\frac{n}{3} < n - 1$ we can use the previous assumption to get the equation
$T(n) < 7[7^{log_3 \frac{n/3}{2}} a + \frac{21}{2} (\frac{n}{3})^2] + n^2$
$= 7 * 7^{log_3 \frac{n/3}{2}} a + \frac{21}{2} * \frac{n^2}{9} + n^2$
$= 7a * 7^{log_3 \frac{n/3}{2}} + \frac{39n^2}{18} < 7^{log_3 \frac{n}{2}} a + \frac{21}{2} n^2$ because $\frac{21}{2} > \frac{39}{18}$. Thus, $T(n) < 7^{log_3 \frac{n}{2}} a + \frac{21}{2} n^2 \blacksquare$

3.d $T(n) = T(n^{\frac{1}{2}}) + 1$
$= T(n^{\frac{1}{4}}) + 2$
$= T(n^{\frac{1}{8}}) + 3$
$= T(n^{\frac{1}{16}}) + 4$
$= T(2^{\frac{1}{2^i}}) + i$
$n^{\frac{1}{2^i}} = 2$
$= log(n^{\frac{1}{2^i}}) = log(2)$
$= \frac{1}{2^i} log(n) = log(2)$
$= \frac{log(n)}{log(2)} = 2^i$
$= log(\frac{log(n)}{log(2)}) = log(2^i)$
$= log(\frac{log(n)}{log(2)}) = i(log(2))$
$= \frac{log(\frac{log(n)}{log(2)})}{log(2))} = i$
$= log_2(\frac{log(n)}{log(2)}) = i$
$= log_2(log_2(n)) = i$
$T(n^{\frac{1}{2^{log_2(log_2(n))}}}) + log_2(log_2(n))$
$= T(2) + log_2(log_2(n))$
$= a + log_2(log_2(n))$
Proof
Prove $T(n) = a + log_2(log_2(n))$.
For the base case, let $n = 2$. We need to show that $T(2) = a + log_2(log_2(n))$.
$T(2) = a + log_2(log_2(2))$
$= a + log_2(1)$
$= a$
which is true by definition.
Now assume that $T(k) < a + log_2(log_2(n))$ for all values of $k = 2, 3, ..., n - 1$.
To show that $T(n) < a + log_2(log_2(n))$ we start with the definition of $T(n) = T(n^{\frac{1}{2}}) + 1$.
Since $n^{\frac{1}{2}}$ we can use the previous assumption to get the equation
$T(n) = a + log_2(log_2(n^{\frac{1}{2}})) + 1$
$= a + log_2(\frac{1}{2} * log_2(n)) + 1$
$= a + log_2(\frac{1}{2}) + log_2(log_2(n)) + 1$
$= a + (-1) + log_2(log_2(n)) + 1$
$= a + log_2(log_2(n)) + 1$
which proves $T(n) = a + log_2(log_2(n)) \blacksquare$

3.e $T(n) = T(n - 1) + lg(n)$
$= T(n - 2) + lg(n - 1) + lg(n)$
$= T(n - 3) + lg(n - 2) + lg(n - 1) + lg(n)$
$= T(n - 4) + lg(n - 3) + lg(n - 2) + lg(n - 1) + lg(n)$

$= T(n-i) + \sum_{j=0}^{i-1} lg(n-j)$
$n-i = 2$
$= n-2 = i$
$T(n-(n-2)) + \sum_{j=0}^{(n-2)-1} lg(n-j)$
$= T(2) + \sum_{j=0}^{n-3} lg(n-j)$
$\leq a + lg(n!)$
Proof
Prove that $T(n) \leq a + lg(n!)$
For the base case, let $n = 2$. We need to show that $T(2) < a + lg(n!)$. $T(2) = a \leq a + lg(n!)$.
Now assume that $T(k) \leq a + lg(n!)$ for all values of $k = 2, 3, ..., n-1$.
To show that $T(n) \leq a + lg(n!)$ we can start with the definition of $T(n) = T(n-1) + lg(n)$. Since $n-1$ is in the range of k, we can use the previous assumption to get the equation
$T(n) \leq a + lg((n-1)!) + lg(n)$
$= a + lg((n-1)! + n)$
$= a + lg(n!)$ which is equal.$\blacksquare$

4 Prove $F_i = \frac{\phi^i - \hat{\phi^i}}{\sqrt{5}}$.
For the base case, let $i = 1 and 2$.
$F_1 = \frac{\phi^1 - \hat{\phi^1}}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$
$F_2 = \frac{\phi^2 - \hat{\phi^2}}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} = \frac{(1+2\sqrt{5}+5)-(1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1$
$F_3 = \frac{\phi^3 - \hat{\phi^3}}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^3 - (\frac{1-\sqrt{5}}{2})^3}{\sqrt{5}} = \frac{(16+8\sqrt{5})-(16-8\sqrt{5})}{8\sqrt{5}} = \frac{16\sqrt{5}}{8\sqrt{5}} = 2$
To find the Fibonacci numbers you sum the previous two Fibonacci numbers. The sequence always begins with 1 so F_1 should equal 1, as it does above. To get F_3, you sum $F_1 + F_2 = 1 + 1 = 2$ which is what the formula yielded for F_3.
Now assume that $F_k = \frac{\phi^k - \hat{\phi^k}}{\sqrt{5}}$ for all values of $k = 1, 2, 3, ..., i-1$.
To prove that $F_i = \frac{\phi^i - \hat{\phi^i}}{\sqrt{5}}$ you need to show the previous two Fibonacci numbers sum to it.
$F_{i-2} + F_{i-1} = F_i = \frac{(\frac{1+\sqrt{5}}{2})^i - (\frac{1-\sqrt{5}}{2})^i}{\sqrt{5}}$
$F_{i-2} + F_{i-1} = \frac{(\frac{1+\sqrt{5}}{2})^{i-2} - (\frac{1-\sqrt{5}}{2})^{i-2}}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{i-1} + (\frac{1-\sqrt{5}}{2})^{i-1}}{\sqrt{5}}$
$= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} + (\frac{1+\sqrt{5}}{2})^{i-1} + (\frac{1-\sqrt{5}}{2})^{i-2} + (\frac{1-\sqrt{5}}{2})^{i-1}}{\sqrt{5}}$
$= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} * (1 + \frac{1+\sqrt{5}}{2}) + (\frac{1-\sqrt{5}}{2})^{i-2} * (1 - \frac{1+\sqrt{5}}{2})}{\sqrt{5}}$
$= \frac{(\frac{1+\sqrt{5}}{2})^{i-2} * (\frac{1+\sqrt{5}}{2})^2 + (\frac{1-\sqrt{5}}{2})^{i-2} * (\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}}$(by properties of the conjugate)
$= \frac{(\frac{1+\sqrt{5}}{2})^i - (\frac{1-\sqrt{5}}{2})^i}{\sqrt{5}} = F_i\blacksquare$