Stats 120B Homework 1: Due Wed, Jan. 18

1. Let X and Y be two independent random variables following beta distributions Beta$(120, 2017)$.
   (a) [5 pts] What's $\mathrm{P}(X = 0.5)$?
   (b) [5 pts] What's $\mathrm{P}(X + 3 < 2Y)$?
   (c) [5 pts] What's $\mathrm{P}(X > Y)$?

2. Suppose that X has the gamma distribution with parameters α and β.
   (a) [10 pts] Determine the mode of X. (Be careful about the range of α)
   (b) [5 pts] Let c be a positive constant. Show that cX has the gamma distribution with parameters α and β/c.

3. Suppose that $X_i \sim \text{Gamma}(\alpha_i, \beta)$ independently for $i = 1, \ldots, N$. The mgf of X_i is $M_{X_i}(t) = (1 - \frac{t}{\beta})^{-\alpha_i}$.
   (a) [5 pts] Use the mgf of X_i to derive the mgf of $\sum_{i=1}^N X_i$.
   (b) [5 pts] Determine the distribution of $\sum_{i=1}^N X_i$ based on its mgf.
   (c) [5 pts] Suppose that $Y_1, \ldots, Y_N \overset{\text{iid}}{\sim} \text{Exp}(\beta)$, determine the distribution of $\sum_{i=1}^N Y_i$ using results from part (b).
   (d) [5 pts] Determine the distribution of sample mean $\bar{Y} = \sum_{i=1}^N Y_i/N$, using results from problem 2.

4. Let X follow beta distribution Beta(α, β).
   (a) [10 pts] Determine the mode of X. (Be careful about the range of α and β)
   (b) [5 pts] Show that $1 - X$ has the beta distribution with parameters β and α.

5. The Cauchy distribution, named after great mathematician Augustin Cauchy, has been widely used in statistics and physics. Let X follow a standard Cauchy distribution, its pdf is given by

$$f(x) = \frac{1}{\pi(1 + x^2)}, x \in (-\infty, \infty).$$

   (a) [5 pts] Verify that $f(x)$ is a valid probability density function.
   (b) [5 pts] Show that $\int_{-\infty}^{\infty} x f(x) dx$ does not exist. Hence conclude that the mean of Cauchy distribution does not exist.
   (c) [5 pts] Read examples 4.1.8 and 4.1.9 on your textbook. Explain in words, why the mean of Cauchy distribution does not exist.

6. [10 pts] Suppose that a random variable X can take each of the six values -2, -1, 0, 1, 2, 3 with equal probability. Determine the probability mass function of $Y = X^2 - X$.

7. [10 pts] Suppose that $X_1, \ldots, X_k$ are independent random variables, and $X_i \sim \text{Exp}(\beta_i)$ for $i = 1, \ldots, k$. Let $Y = \min\{X_1, \ldots, X_k\}$. Show that $Y \sim \text{Exp}(\sum_{i=1}^{k} \beta_i)$.