Bayesian Networks

Introduction to Graphical Models

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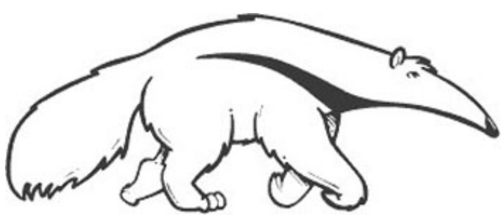

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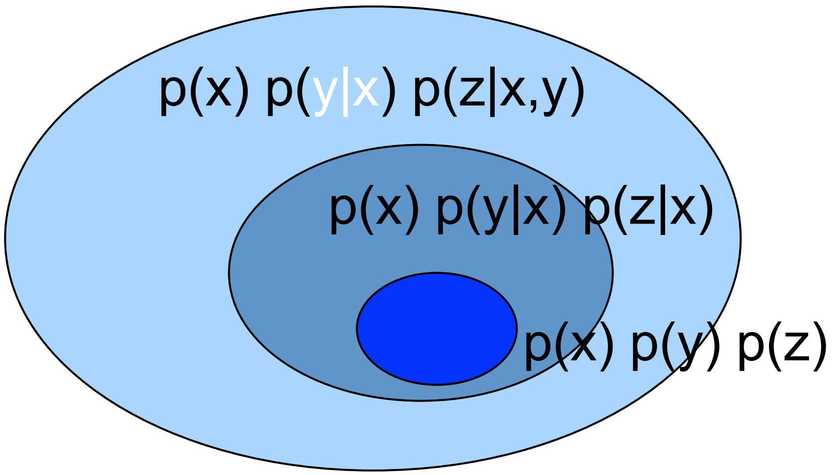
Conditional independence

- Recall: chain rule of probability
  - p(x,y,z) = p(x) p(y|x) p(z|x,y)

- *Some* of these models will be conditionally independent
  - e.g., p(x,y,z) = p(x) p(y|x) p(z|x)

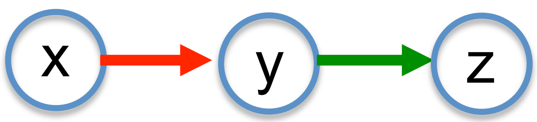
- *Some* models may have even *more* independence
  - E.g., p(x,y,z) = p(x) p(y) p(z)



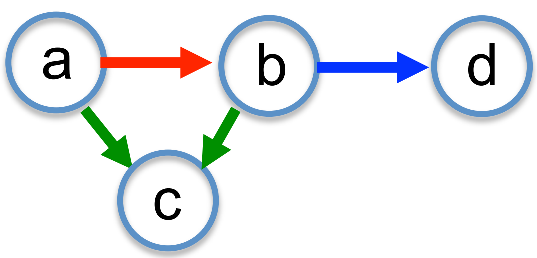
Bayesian networks

- Directed graphical model
- Nodes associated with variables
- "Draw" independence in conditional probability expansion
  - Parents in graph are the RHS of conditional

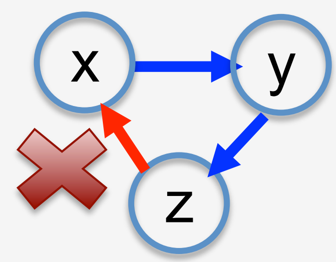
- Ex: $p(x, y, z) = p(x)\ p(y \mid x)\ p(z \mid y)$



- Ex: $p(a, b, c, d) = p(a)\ p(b \mid a)\ p(c \mid a, b)\ p(d \mid b)$



*Graph must be **acyclic***



Corresponds to an order over the variables (chain rule)

Example

- Consider the following 5 binary variables:
  - B = a burglary occurs at your house
  - E = an earthquake occurs at your house
  - A = the alarm goes off
  - J = John calls to report the alarm
  - M = Mary calls to report the alarm

  - What is P(B | M, J) ? (for example)

  - We can use the full joint distribution to answer this question
    - Requires $2^5 = 32$ probabilities

    - Can we use prior domain knowledge to come up with a Bayesian network that requires fewer probabilities?

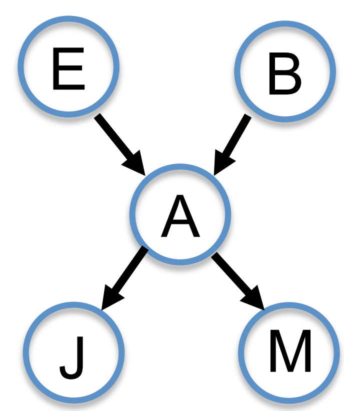
Constructing a Bayesian network

- Order the variables in terms of causality (may be a partial order)
  - e.g., $\{\ E,\ \ B\ \} \longrightarrow \{\ A\ \} \longrightarrow \{\ J,\ \ M\ \}$

- Now, apply the chain rule, and simplify based on assumptions

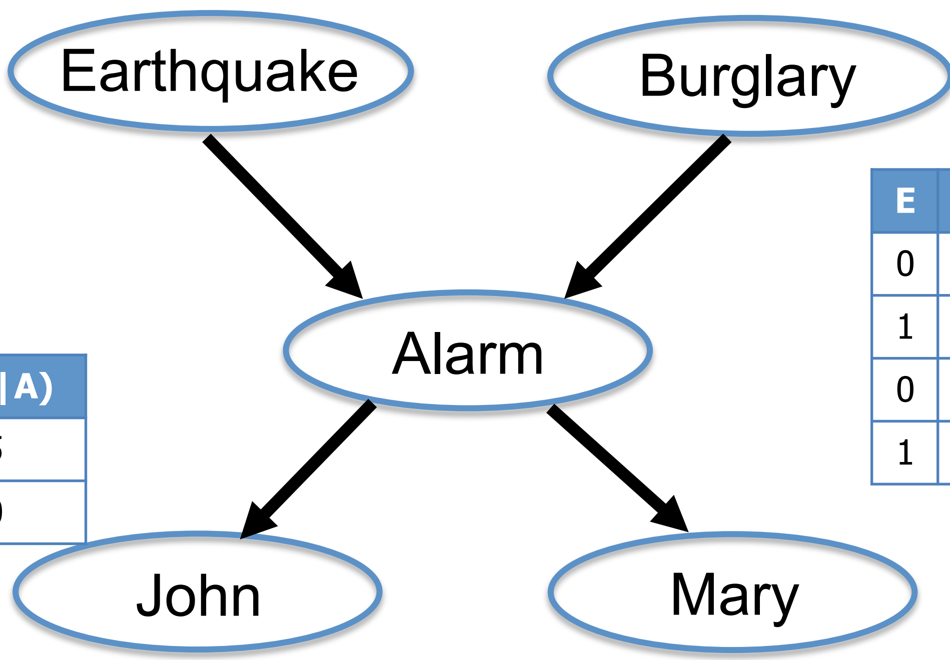
$$\begin{aligned} p(J, M, A, E, B) &= p(E, B)\ p(A \,|\, E, B)\ p(J, M \,|\, A, E, B) \\ &= p(E)\ p(B)\ p(A \,|\, E, B)\ p(J, M \,|\, A) \\ &= p(E)\ p(B)\ p(A \,|\, E, B)\ p(J \,|\, A)\ p(M \,|\, A) \end{aligned}$$

  - These assumptions are reflected in the graph structure of the Bayesian network



Constructing a Bayesian network

- Given $p(J, M, A, E, B) = p(E)\ p(B)\ p(A \mid E, B)\ p(J \mid A)\ p(M \mid A)$
- Define probabilities: 1 + 1 + 4 + 2 + 2
- Where do these come from?
  - Expert knowledge; estimate from data; some combination



P(E=1)
0.002

P(B=1)
0.001

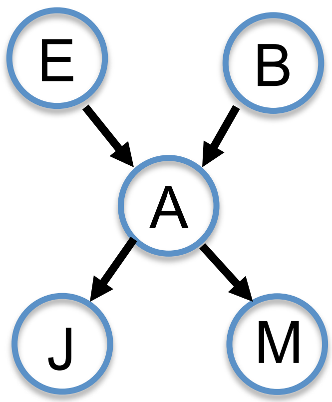
E	B	P(A=1 | E,B)
0	0	0.001
1	0	0.29
0	1	0.94
1	1	0.95

A	P(J=1 | A)
0	0.05
1	0.90

A	P(M=1 | A)
0	0.01
1	0.70

Constructing a Bayesian network

- Joint distribution



Full joint distribution:
2^5 = 32 probabilities

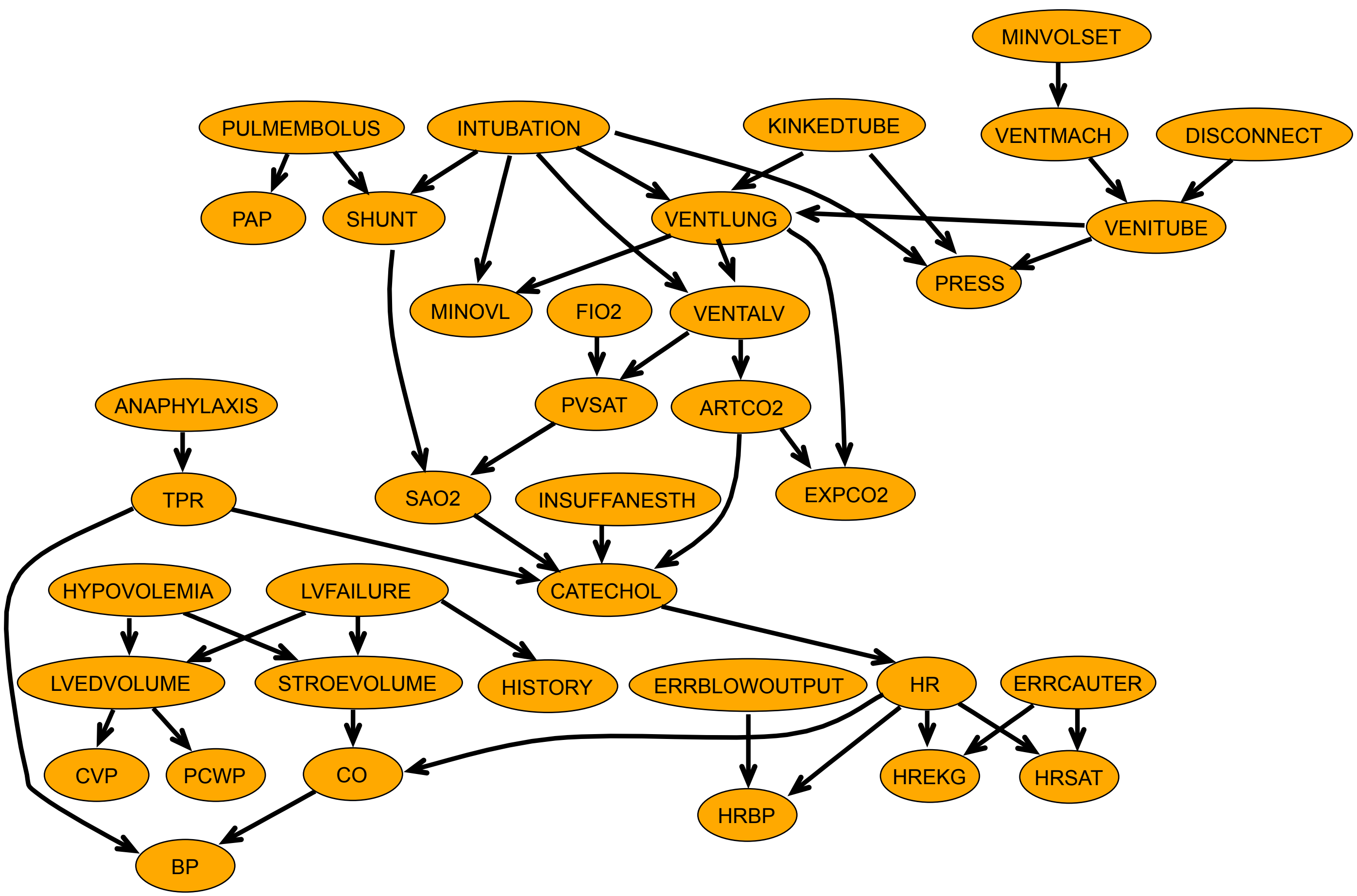
Structured distribution:
specify 10 parameters

E	B	A	J	M	P(…)
0	0	0	0	0	.93674
1	0	0	0	0	.00133
0	1	0	0	0	.00006
1	1	0	0	0	.00000
0	0	1	0	0	.00003
1	0	1	0	0	.00002
0	1	1	0	0	.00003
1	1	1	0	0	.00000
0	0	0	1	0	.04930
1	0	0	1	0	.00007
0	1	0	1	0	.00000
1	1	0	1	0	.00000
0	0	1	1	0	.00027
1	0	1	1	0	.00016
0	1	1	1	0	.00025
1	1	1	1	0	.00000

E	B	A	J	M	P(…)
0	0	0	0	1	.00946
1	0	0	0	1	.00001
0	1	0	0	1	.00000
1	1	0	0	1	.00000
0	0	1	0	1	.00007
1	0	1	0	1	.00004
0	1	1	0	1	.00007
1	1	1	0	1	.00000
0	0	0	1	1	.00050
1	0	0	1		.00000
0	1	0	1	1	.00000
1	1	0	1	1	.00000
0	0	1	1	1	.00063
1	0	1	1	1	.00037
0	1	1	1	1	.00059
1	1	1	1	1	.00000

Alarm network [Beinlich et al., 1989]

The "alarm" network: 37 variables, 509 parameters (rather than $2^{37} = 10^{11}$!)

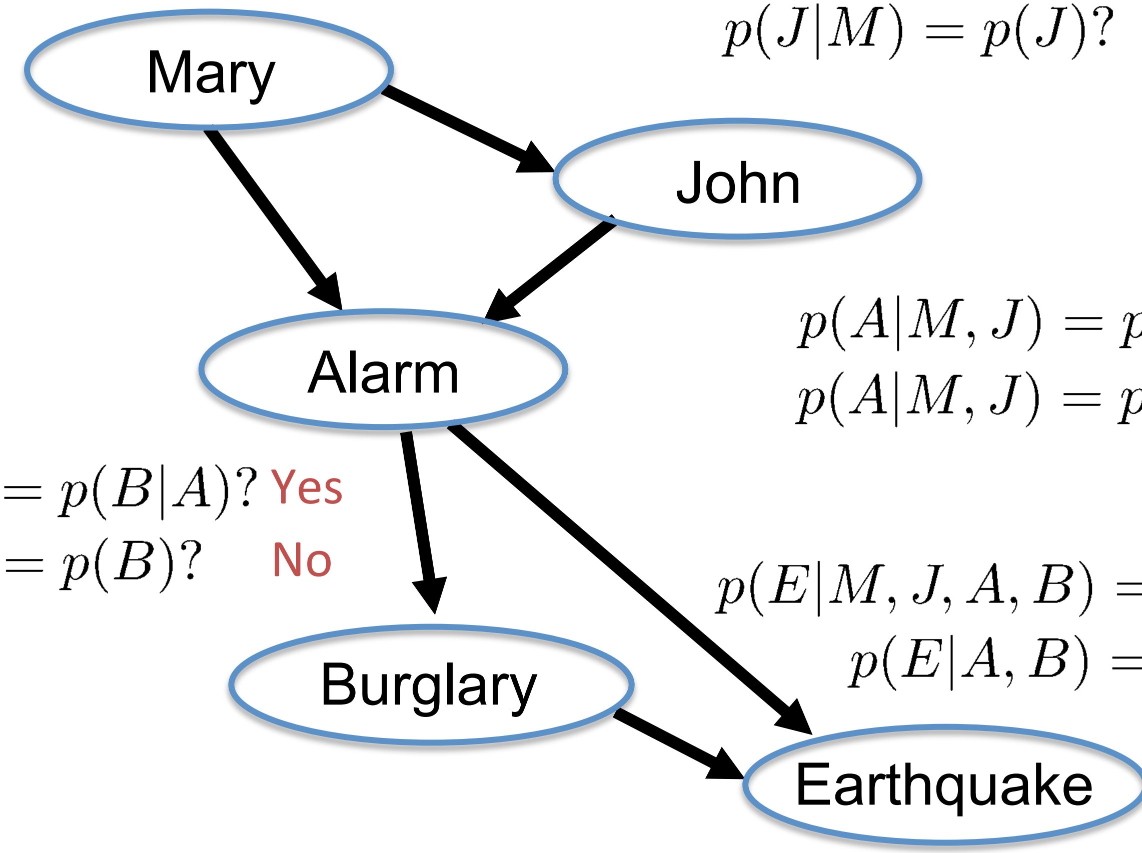


Network structure and ordering

- The network structure depends on the conditioning order
  - Suppose we choose ordering M, J, A, B, E

$p(M)$

$p(J|M) = p(J)?$ No



$p(A|M, J) = p(A|M)?$ No

$p(A|M, J) = p(A|J)?$ No

$p(B|M, J, A) = p(B|A)?$ Yes

$p(B|A) = p(B)?$ No

$p(E|M, J, A, B) = p(E|A, B)?$ Yes

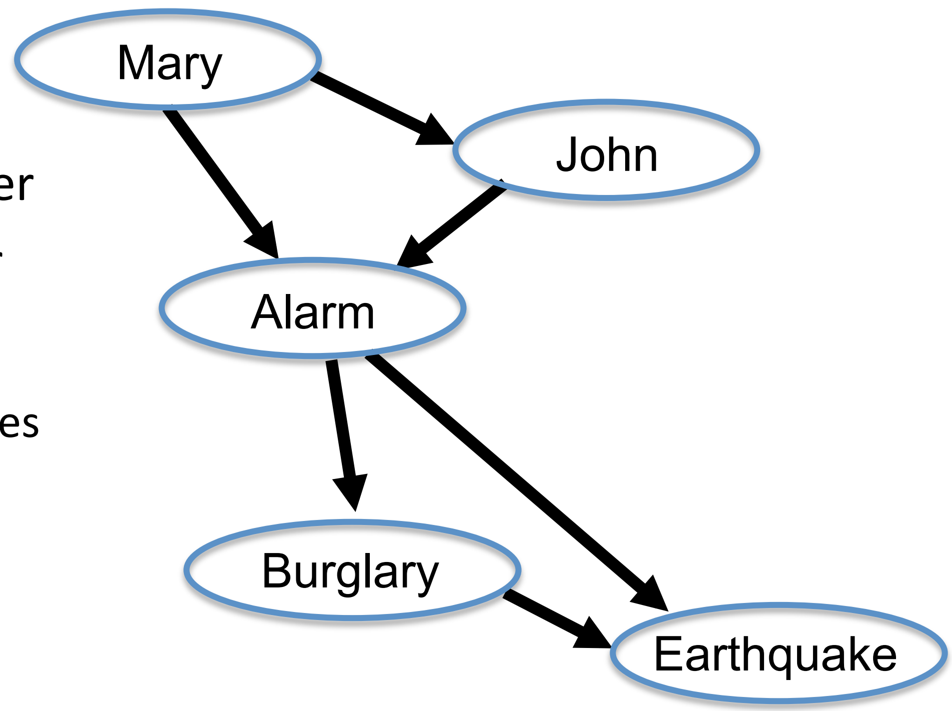
$p(E|A, B) = p(E|A)?$ No

Network structure and ordering

- The network structure depends on the conditioning order
  - Suppose we choose ordering M, J, A, B, E

- "Non-causal" ordering
  - Deciding independence is harder
  - Selecting probabilities is harder
  - Representation is less efficient

    1 + 2 + 4 + 2 + 4 = 13 probabilities



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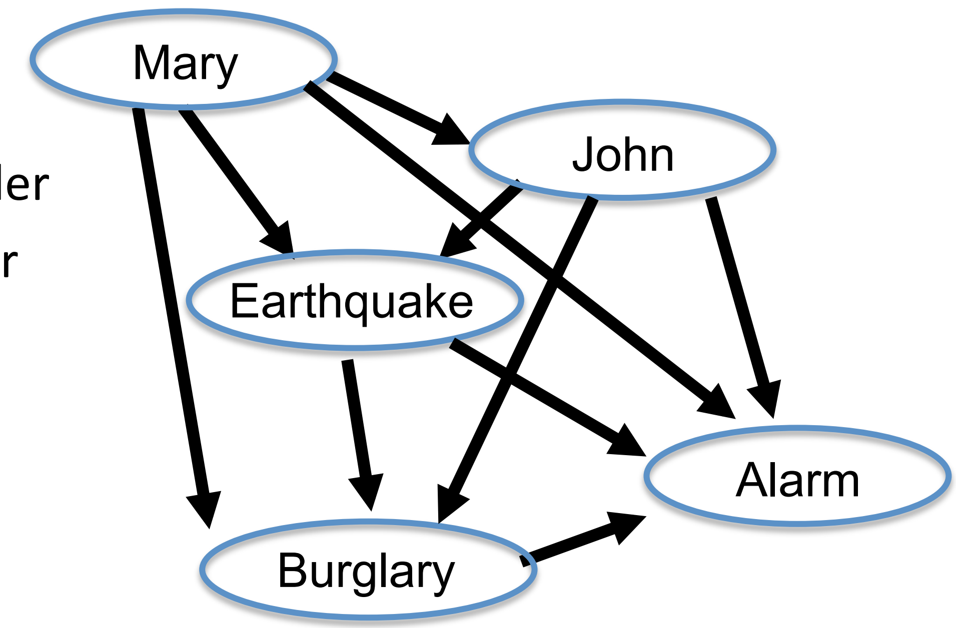
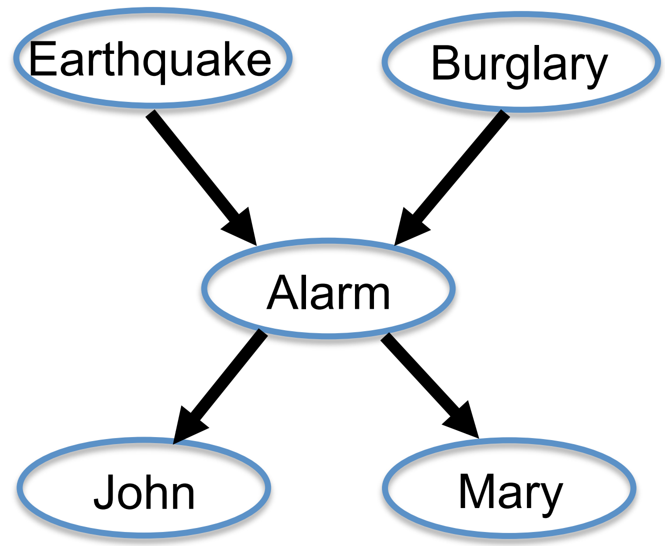


  - Some orders may not reveal any independence!

$$p(J, M, A, E, B) = p(M)\, p(J|M)\, p(E|M, J)\, p(B|M, J, E)\, p(A|M, J, E, B)$$

Reasoning in Bayesian networks

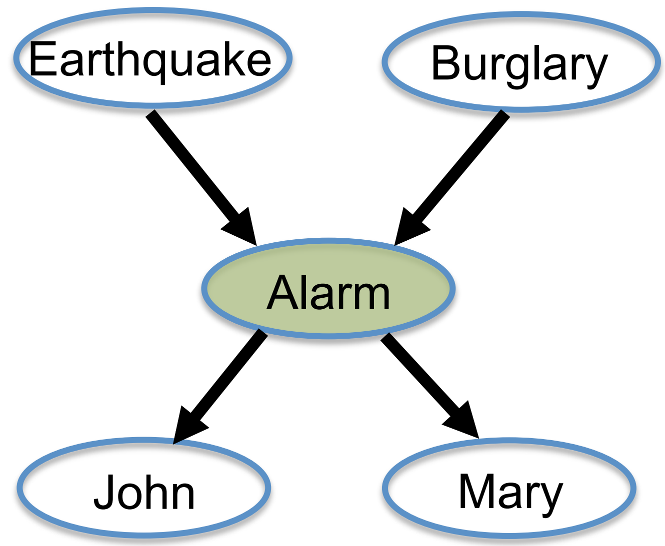


- Suppose we observe J
  - Observing J makes A more likely
  - A being more likely makes B more likely
- Suppose we observe A
  - Makes M more likely
- Observe A and J?
  - J doesn’t add anything to M
  - Observing A makes J, M independent
- How can we read independence directly from the graph?

Reasoning in Bayesian networks

- How are J,M related given A?
  - P(M) = 0.0117
  - P(M|A) = 0.7
  - P(M|A,J) = 0.7
  - Conditionally independent

    (we actually know this by construction!)



- Proof:

$$p(J, M|a) \propto \sum_{e,b} p(e)\ p(b)\ p(a|e,b)\ p(J|a)\ p(M|a)$$

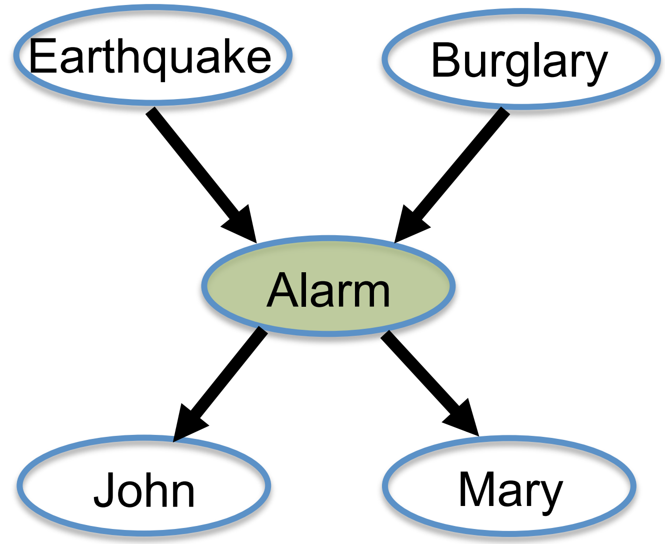
$$= \Big(\sum_{e,b} p(e,b,a)\Big)\ p(J|a)\ p(M|a)$$

$$= p(a)\ p(J|a)\ p(M|a)$$

$$= \ c_a \ \ f_a(J)\ g_a(M)$$

Reasoning in Bayesian networks

- How are J,B related given A?
  - P(B) = 0.001
  - P(B|A) = 0.3735
  - P(B|A,J) = 0.3735
  - Conditionally independent

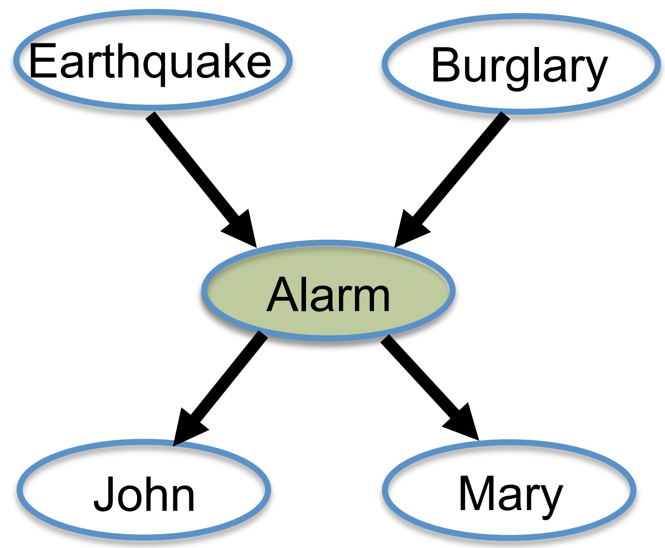


- Proof:

$$
\begin{aligned}
p(J, B|a) &\propto \sum_{e,m} p(e)\ p(B)\ p(a|e, B)\ p(J|a)\ p(m|a) \\
&= \left(\sum_{e} p(e, B, a)\right) p(J|a) \left(\sum_{m} p(m|a)\right) \\
&= p(B, a)\ p(J|a) \\
&= \ f_a(B)\ \ g_a(J)
\end{aligned}
$$

Reasoning in Bayesian networks

- How are E,B related?
  - P(B) = 0.001
  - P(B|E) = 0.001
  - (Marginally) independent

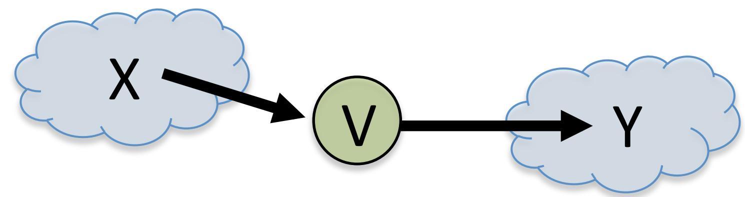


- What about given A?
  - P(B|A) = 0.3735
  - P(B|A,E) = 0.0032
  - Not conditionally independent!
  - The "causes" of A become coupled by observing its value
  - Sometimes called "explaining away"

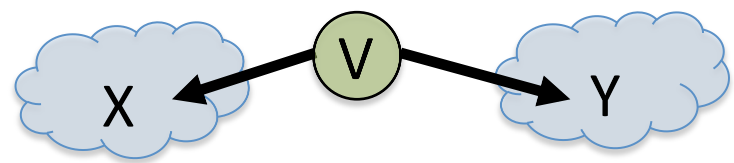
D-Separation

- Prove sets X,Y independent given Z?
- Check all *undirected* paths from X to Y
- A path is “inactive” if it passes through:

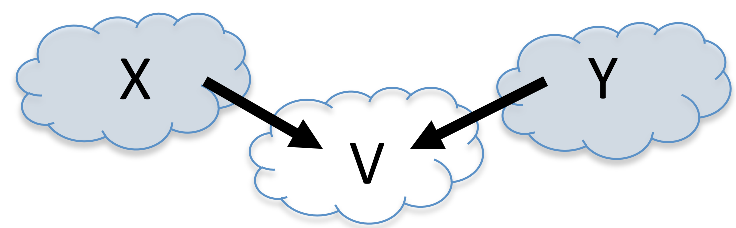
  (1) A “chain” with an observed variable



  (2) A “split” with an observed variable



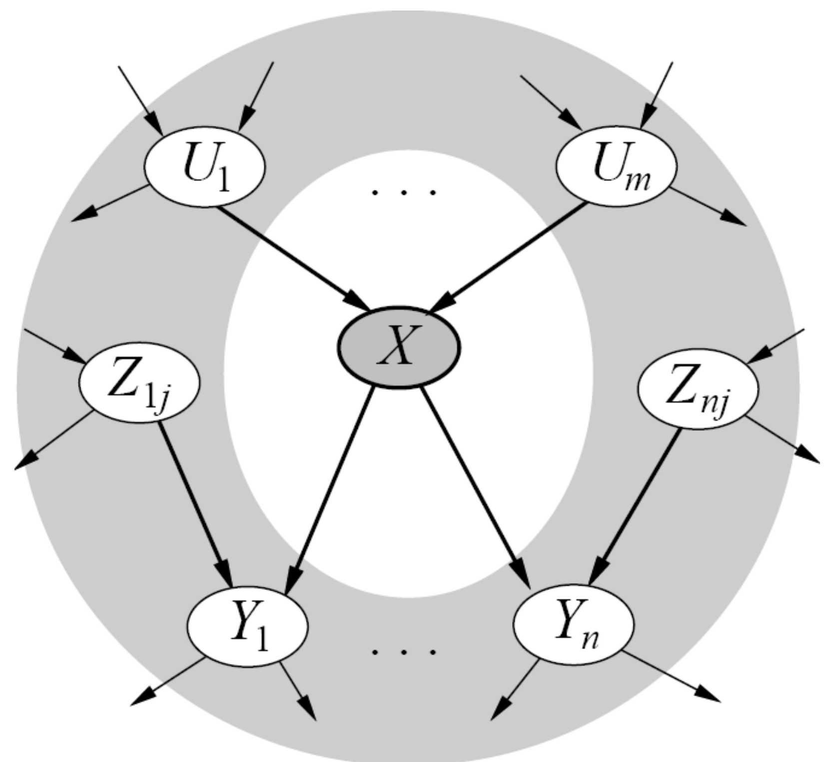
  (3) A “vee” with **only unobserved** variables below it



- If all paths are inactive, conditionally independent!

Markov blanket

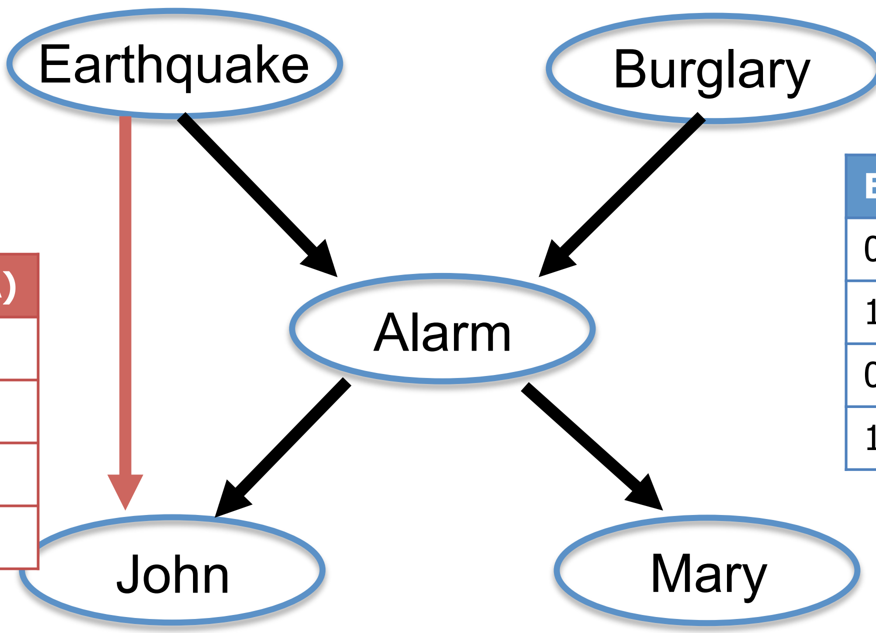
A node is conditionally independent of all other nodes in the network given its Markov blanket (in gray)



Graphs and Independence

- Graph structure allows us to infer independence in p(.)
  - X,Y d-separated given Z?
- Adding edges
  - Fewer independencies inferred, but still valid to represent p(.)
  - Complete graph: can represent any distribution p(.)

P(E)
0.002



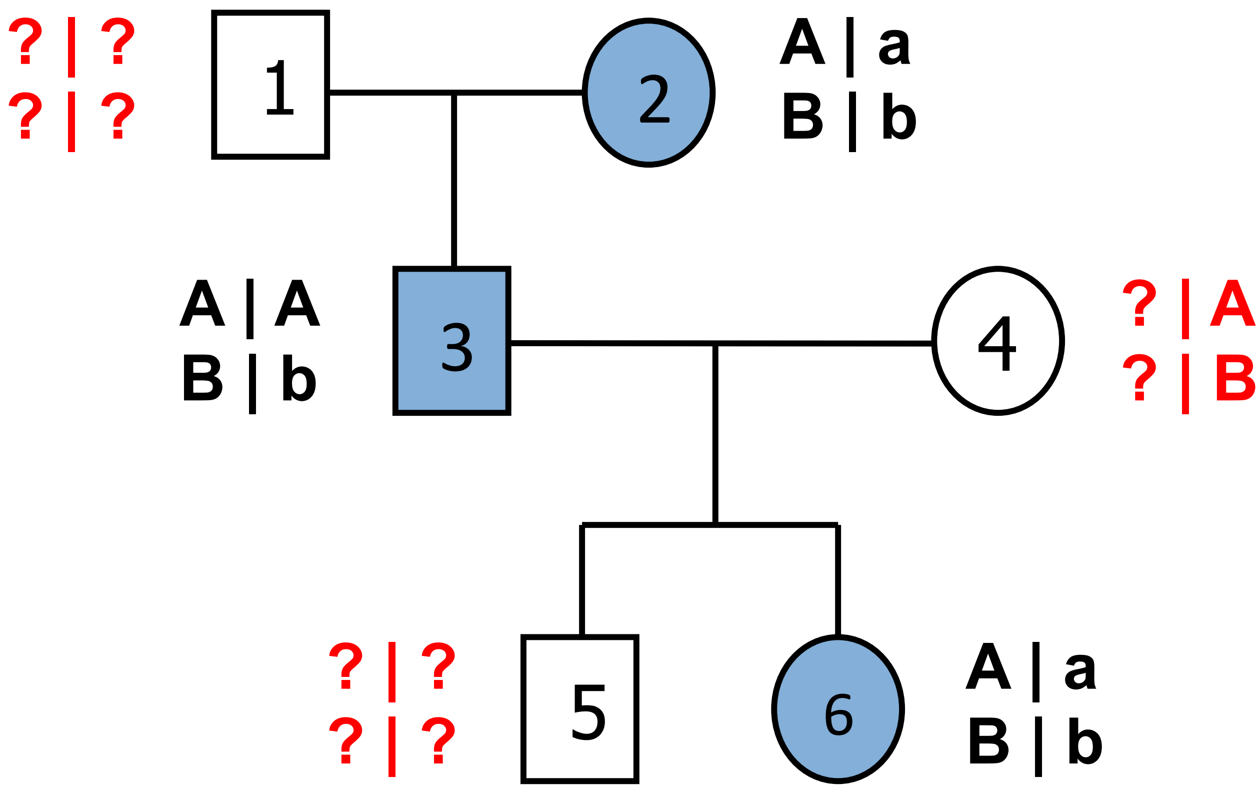
P(B)
0.001

E	B	P(A|E,B)
0	0	0.001
1	0	0.29
0	1	0.94
1	1	0.95

E	A	P(J|E,A)
0	0	0.05
1	0	0.05
0	1	0.90
1	1	0.90

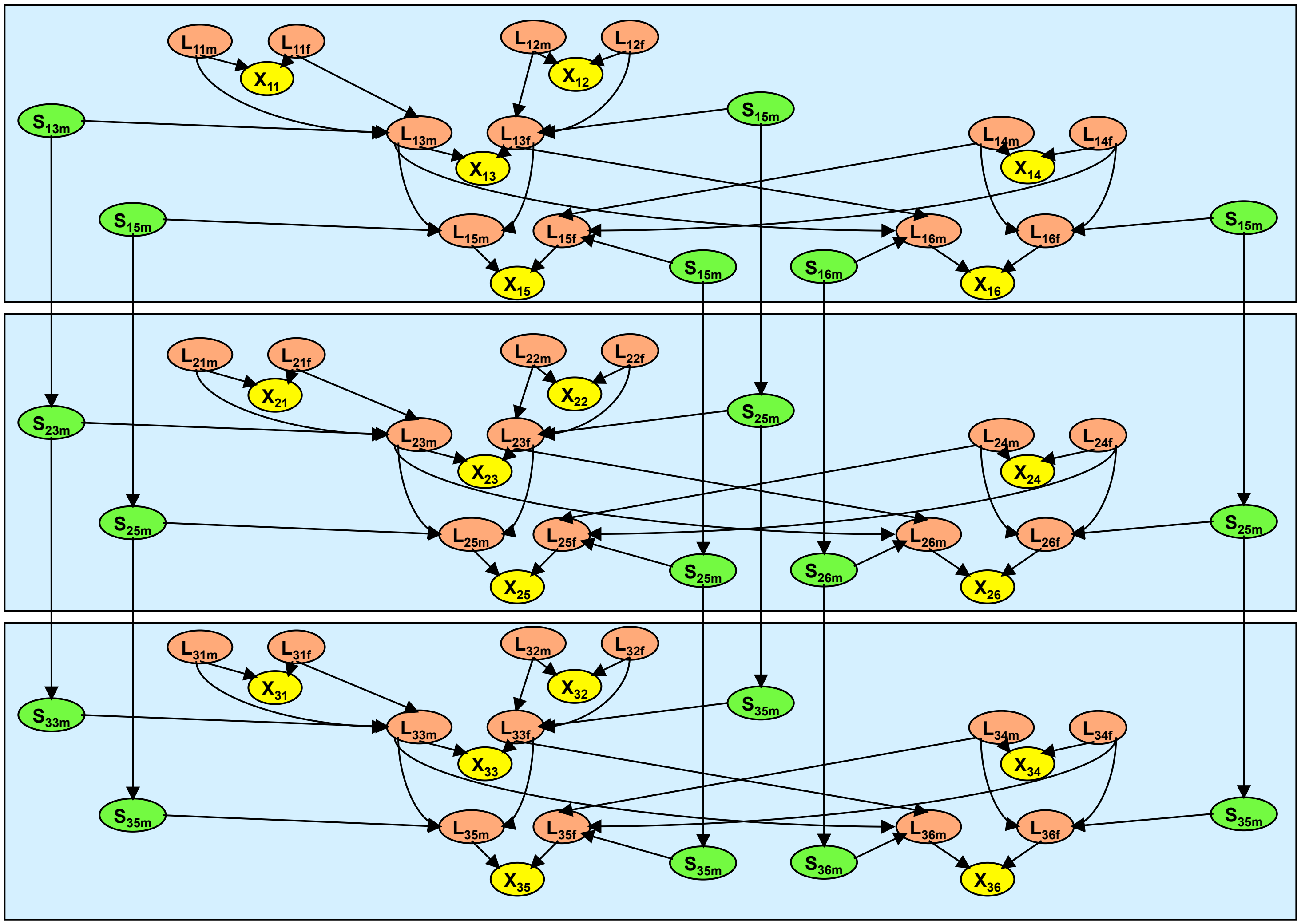
A	P(M|A)
0	0.01
1	0.70

Genetic linkage analysis



- 6 individuals
- Haplotype: {2, 3}
- Genotype: {6}
- Unknown

Pedigree model: 6 people, 3 markers

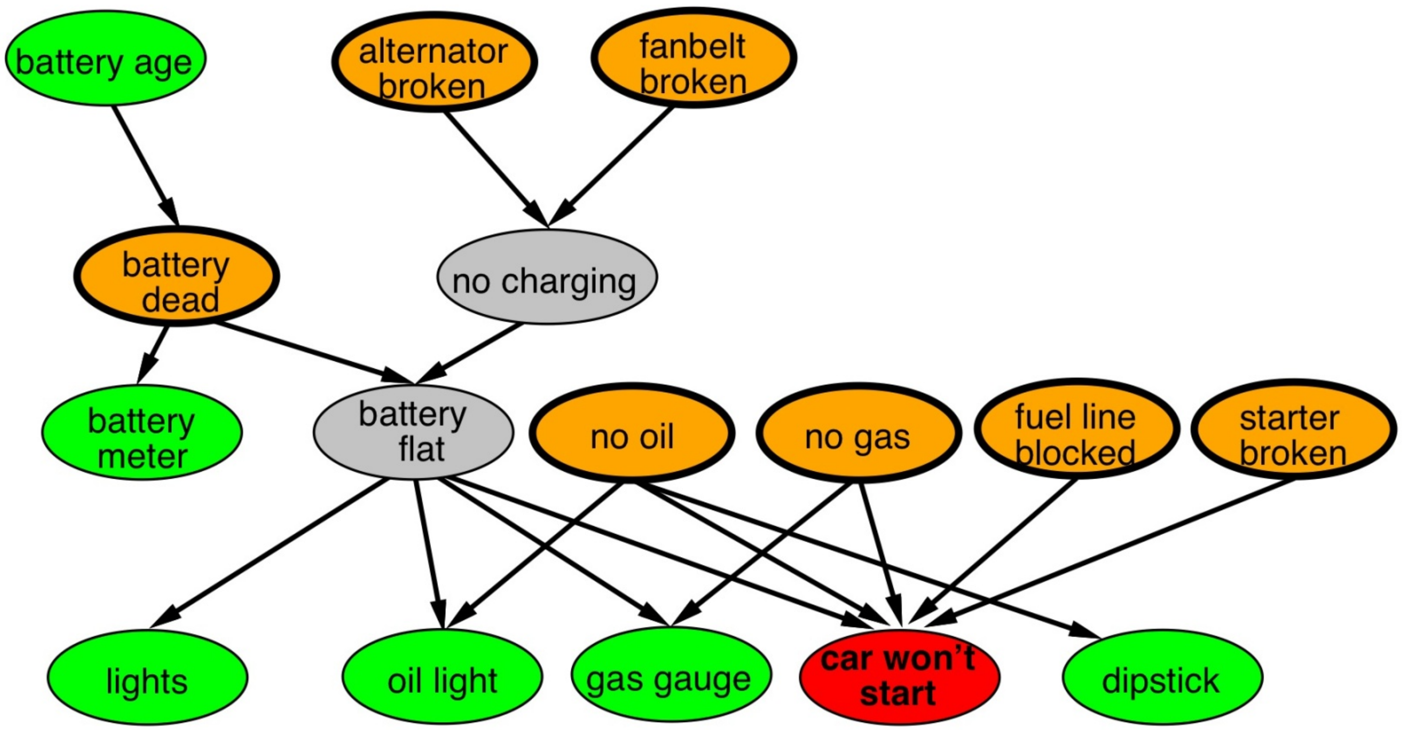


Example: Car diagnosis

Initial evidence: car won't start

Testable variables (green), "broken, so fix it" variables (orange)

Hidden variables (gray) ensure sparse structure, reduce parameters



Compact conditional distributions contd.

Noisy-OR distributions model multiple noninteracting causes
1) Parents $U_1 \ldots U_k$ include all causes (can add leak node)
2) Independent failure probability q_i for each cause alone
$\Rightarrow P(X|U_1 \ldots U_j, \neg U_{j+1} \ldots \neg U_k) = 1 - \prod_{i=1}^{j} q_i$

$Cold$	Flu	$Malaria$	$P(Fever)$	$P(\neg Fever)$
F	F	F	**0.0**	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

Number of parameters **linear** in number of parents

Summary

- Bayesian networks
  - Directed, acyclic graphs
  - Encode chain rule + conditional independence
  - Efficient representation
- Building a Bayesian network
  - Select ordering
  - Evaluate independence relations
  - Assign probabilities
- Reasoning about independence in the graph
  - D-separation
  - Markov blanket