Math 1000 - The Art of Writing Mathematics

Final Portfolio

Emma Zou

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Chapter 1

Reflections

1.1 Reflection 1

Prompt: *The main purpose of this course was to develop your proof writing skills and develop some of the fundamental knowledge and skills required to succeed in higher level mathematics courses. The reflections you wrote in this course provide a time-lapse of your thoughts and feelings towards the subject matter and challenges throughout the course.*

1. *What are a few things that stand out to you as you look back on your work this term?*
2. *Which concepts or skills did you find particularly interesting / boring / challenging / useful?*
3. *Do you feel that you've grown in ways that you were hoping to?*
4. *Are there other ways you've grown this term you weren't expecting?*
5. *Do you feel prepared to continue pursuing mathematics—which courses are you thinking about taking in the future?*

After all the proofs, structures, and concepts I have absorbed this semester, I conclude that the most important of my new mathematical habits is not writing clearer proofs, making better use of feedback, or discussing more successfully with peers: it is the simple confidence to say "I'm not convinced." In the past, I tended to just accept proofs and theorems as they were, and didn't think more deeply about how they're constructed and validated. I was never one to question a proof. Even when I did notice inconsistencies, I was scared to point them out. I believed more in the rightness of teachers than in the rightness of whatever I had noticed.

Now, my better understanding of proofs and mathematical logic gives me the knowledge and confidence to question what I am given. More importantly, I feel like my mind itself has been rewired to be more inquisitive about the axioms around me, until I believe them enough to incorporate it into my personal truth.

This course promised to teach us the value of practice and of establishing our own truths, and I think it's fulfilled that promise. Even outside of math and academics, I find myself questioning authority more (in a good way)! I ponder how my own values and opinions interact with the world more, and I express them too.

This class has also taught me that sometimes there is no stone-hard truth or absolute right way to do something. Most memorably, while preparing my curated exercises section for this portfolio, I couldn't find any explicit existence proofs in the problem sets. However, there was a theorem that stated every odd integer is the difference of two squares. I wondered if I could rephrase this theorem and my proof of it to create an existence proof; after all, I was basically proving that for every odd integer, there exists two squares such that their difference equals that number. After confirming with Jordan that this was a reasonable action, I have done exactly that in my portfolio. This made me ponder how every theorem and every proof has a little bit of every structure in it, and there is no one way to approach anything. In many ways, this is a little scary, but I've learned to make the most of this truth.

It is precisely because of my experiences and growth in this class that I believe learning math is not just an academic endeavor, but a way of life. No other subject so closely entwines with our natural systems of logic, communication, and truth. No other classes have reframed my ways of thinking in such a way as my math classes have. I will absolutely continue taking pure math courses, even if my loyalties still lie with applied math-computer science! My dream is to take Abstract Algebra, although I may need to postpone it until I take a few more concentration requirements.

However, I also feel that math is uniquely intimidating as a subject. It requires a different way of thinking from memory-dependent subjects like biology, or even problem-solving subjects like computer science. So, while I do feel more prepared to take on more math classes, I am also cautious. I know that if I continue taking math, I will continue having those moments of extreme confusion and doubt too. This class, however, has given me the confidence and humility to ask for help when I need it. Because of that, I think I will survive whatever I decide my math journey will look like.

1.2 Reflection 2

Prompt: *One of our main goals was providing and responding to peer feedback. Based on your summative experience over the semester, what peer feedback was most helpful to you? What did you find helpful about giving/receiving feedback in your own written? If you had a time machine, what are a few short lessons you might share with your former self about good mathematical writing based on this experience?*

Although I have been saved by many suggestions for ways to approach difficult proofs or mathematical concepts, I found that the most impactful feedback isn't related to the math I do: it's based on the words I write. One of the first pieces of feedback I received was wonderfully detailed and pointed out some very specific words and phrases in my proofs. The common thread between suggestions to "remove the word 'actually'" and "explicitly say x and y are both in R" was that my reviewer believed many points in my writing could be misinterpreted. For my first few problem sets, I continued receiving feedback like this, and it made me rethink how I actually write out my proofs.

Because a lot of proof-writing involves "scratch work" that never makes it into the final product, it can be hard to figure out exactly how much of the foundational work to include, and how much to spell things out for the readers. It's very easy to write it in a way that makes sense to you but may not be clear to others, and I often fell into this trap. Because of this, peer feedback was invaluable to me–it gave me an outsider's perspective of my writing and helped me gauge what wordings and phrases worked best in my proofs. Over time, I became more mindful about my word choices and argument structures, and the amount of grammatical/semantic feedback I received dwindled.

Unexpectedly, I also learned a lot just by looking at the work that my peers submitted for review. I came in contact with countless different styles of writing and problem-solving, all of which taught me something. Constructively criticizing the more muddled and confusing proofs inspired me to pick at the unclear parts of my own work. The really elegant and intuitive proofs acted as my exemplars. Furthermore, through reading so many different styles, my own proof-writing style evolved to incorporate bits and pieces from every around me. Just like how everyone maintains their own world of truth that changes with experience, I think my problem-solving style now reflects the trials and triumphs my peers and I have experienced throughout this class, and that alone makes my proofs better and more true to myself.

If I had a time machine to the start of the semester, I would tell myself to think more about making my line of reasoning explicit, and to ignore the pain of repeating the same few transitional phrases ("therefore", "as such", "it follows"...) throughout my writing. The elegance of mathematical writing is not flowery language and interesting sentence structure: it's all about clarity and simple, clean logic. A literary scholar might not be impressed by my writing, but as long as math-enjoyers can clearly understand my arguments, I'm happy.

In a way, however, I don't think talking to my past self would change anything. Ultimately, hearing feedback from outside perspectives (which doesn't quite include advice from my future self!) was the only way I could've recognized the weaknesses and played into the strengths of my proof-writing.

Chapter 2

Curated Problems

2.1 Use of proper notation and LaTeX

1. Example from Problem Set 2 (Week 3)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: Although I received a 4/4 on this problem, looking back, there were several formatting and logic errors. Some of the feedback Mac gave included adding periods at the end of proofs, suggesting I elaborate more, and pointing out other small mistakes. I revised my answers to take this feedback into account.

Other Information: This exercise is taken from one of the first problem sets in this class–back then, I had only used LaTeX once and never individually. I've included my first version as well as my revised version to show my growth in using proper notation and LaTeX.

Problem Statement: Suppose $A = \{b, c, d\}$ and $B = \{a, b\}$. Find the sets representing the following.

Link to my Initial Draft

Revised Version: 4a) $(A \times B) \cap (B \times B)$

Proof. The Cartesian product of A and B is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$, resulting in the set

$$\{(b, a), (c, a), (d, a), (b, b), (c, b), (d, b)\}.$$

The Cartesian product of B with itself is the set of ordered pairs where both terms are elements of B, resulting in the set

$$\{(a, a), (b, a), (a, b), (b, b)\}.$$

The intersection of these two sets contains all elements shared between the two. This gives us

$$(A \times B) \cap (B \times B) = \{(b, a), (b, b)\}.$$

□

4b) $(A \times B) \cup (B \times B)$

Proof. As seen in the previous problem, the Cartesian product of A and B is

$$\{(b, a), (c, a), (d, a), (b, b), (c, b), (d, b)\},$$

while the Cartesian product of B with itself is the set

$$\{(a, a), (b, a), (a, b), (b, b)\}.$$

The union of these products contains all the elements that are in at least one of $A \times B$ and $B \times B$. This results in the set

$$(A \times B) \cup (B \times B) = \{(b, a), (c, a), (d, a), (b, b), (c, b), (d, b), (a, a), (a, b)\}.$$

□

4c) $(A \times B) - (B \times B)$

Proof. As seen previously, the Cartesian product of A and B is the set

$$\{(b, a), (c, a), (d, a), (b, b), (c, b), (d, b)\},$$

while the Cartesian product of B with itself is the set

$$\{(a, a), (b, a), (a, b), (b, b)\}.$$

The difference, $(A \times B) - (B \times B)$, contains all elements of $A \times B$ that are not in $B \times B$. This results in the set

$$(A \times B) - (B \times B) = \{(c, a), (d, a), (c, b), (d, b)\}.$$

□

4d) $(A \cap B) \times A$

Proof. The intersection of sets A and B is composed of all elements that are in both sets. Since b is the only such element,

$$A \cap B = \{b\}.$$

This set's Cartesian product with set A results in the set of all ordered pairs (x, y) where $x = b$ and $y \in A$. This gives us

$$(A \cap B) \times A = \{(b, b), (b, c), (b, d)\}.$$

□

2. Example from Problem Set 3 (Week 4)

Graded by: Mac

Initial Grade: 3/4

Explanation for Revision: The feedback I received was to change the formatting of the table to have the label in the caption. I decided to change the formatting altogether to include the problem statement, meaning I don't need the label at all!

Other Information: This exercise was my first exposure to tabular structures in LaTeX, and shows how I use them to represent truth tables.

Problem Statement: Write a truth table for three out of the nine choices in the exercises for §2.5, being sure that at least two involve the three statements P, Q, R.

1. $P \vee (Q \Rightarrow R)$

P	Q	R	$Q \Rightarrow R$	$P \vee (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

Table 2.1

2. $(P \wedge \sim P) \vee Q$

P	Q	$P \wedge \sim P$	$(P \wedge \sim P) \vee Q$
T	T	F	T
T	F	F	F
F	T	F	T
F	F	F	F

Table 2.2

3. $P \vee (Q \wedge \sim R)$

P	Q	R	$Q\wedge \sim R$	$P \vee (Q\wedge \sim R)$
T	T	T	F	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	T
F	T	T	F	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	F

Table 2.3

2.2 Use of direct proof

1. Example from Problem Set 6 (Week 7)

Graded by: Jordan

Initial Grade: 3/4

Explanation for Revision: Initially, instead of defining a as an integer between 4 and n inclusive, I defined it as $a \in [2, n]$. Jordan pointed out that this notation made it seem like a could be a non-integer in the range, so I changed it to be more clearly defined as an integer.

Other Information: N/A

Problem Statement: If $n \in \mathbb{N}$, and $n \geq 2$, show that the numbers $n!+2, n!+3, \ldots, n!+n$ are all composite.

Proof. Suppose that $n \in \mathbb{N}$ and $n \geq 2$. For all integers a that are between 2 and n inclusive, the number $n! = 1 \cdot 2 \cdot 3 \cdot ...n$ is divisible by a; that is, $a|n!$

Therefore, for any a, we see that

$$n! + a = a \cdot (1 \cdot 2 \cdot ...(a-1) \cdot (a+1) \cdot ...n) + a.$$

Let $b = (1 \cdot 2 \cdot ...(a-1) \cdot (a+1) \cdot ...n)$ be an integer. Thus, $n! + a = a(b+1)$, showing that $n! + a$ can be expressed as a multiple of two numbers not itself or 1, meaning it is composite. Because this holds true for all values of a between 2 and n, it follows that the numbers $n!+2, n!+3, \ldots, n!+n$ are all composite. □

2. Example from Problem Set 6 (Week 7)

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: N/A

Problem Statement:

(a) Show that if $n \in \mathbb{Z}$, then $3n^2 + 7n + 5$ is odd.

Proof. Suppose $n \in \mathbb{Z}$. There are two cases for the parity of n.

Case 1: n is even and can be expressed as $n = 2k$ for some $k \in \mathbb{Z}$. Using algebra, we see that

$$3n^2 = 12k^2$$
$$3n^2 + 7n = 12k^2 + 14k$$
$$3n^2 + 7n + 5 = 12k^2 + 14k + 5 = 2(6k^2 + 7k + 2) + 1$$

Let $a = 6k^2 + 7k + 2$ be an integer. Thus, $3n^2 + 7n + 5 = 2a + 1$ is odd.

Case 1: n is odd and can be expressed as $n = 2p + 1$ for some $p \in \mathbb{Z}$. Using algebra, we see that

$$3n^2 = 3(2p + 1)^2 = 12p^2 + 12p + 3$$
$$3n^2 + 7n = 12p^2 + 12p + 3 + 7(2p + 1) = 12p^2 + 26p + 10$$
$$3n^2 + 7n + 5 = 12p^2 + 26p + 15 = 2(6p^2 + 13p + 7) + 1.$$

Let $b = 6p^2 + 13p + 7$ be an integer. Thus, $3n^2 + 7n + 5 = 2b + 1$ is odd.

Since $3n^2 + 7n + 5$ is odd in both cases, we see that this holds true for all values of n. □

(b) Show that if $n \in \mathbb{Z}$, then $n^2 + 11n + 6$ is even.

Proof. Suppose $n \in \mathbb{Z}$. There are two cases for the parity of n.

Case 1: n is even and can be expressed as $n = 2k$ for some $k \in \mathbb{Z}$. Using algebra, we see that

$$n^2 = 4k^2$$
$$n^2 + 11n = 4k^2 + 11(2k) = 4k^2 + 22k$$
$$n^2 + 11n + 6 = 4k^2 + 22k + 6 = 2(2k^2 + 11k + 3).$$

Let $a = 2k^2 + 11k + 3$ be an integer. Thus, $n^2 + 11n + 6 = 2a$ is an even number.

Case 2: n is odd and can be expressed as $n = 2p + 1$ for some $p \in \mathbb{Z}$. Using algebra, we see that

$$n^2 = (2p+1)^2 = 4p^2 + 4p + 1$$
$$n^2 + 11n = 4p^2 + 4p + 1 + 11(2p+1) = 4p^2 + 26p + 2$$
$$n^2 + 11n + 6 = 4p^2 + 26p + 8 = 2(p^2 + 13p + 4).$$

Let $b = p^2 + 13p + 4$ be an integer. Thus, $n^2 + 11n + 6 = 2b$ is even.

Since $n^2 + 11n + 6$ is even in both cases, we see that this holds true for all values of n. □

2.3 Use of contrapositive proof

1. Example from Problem Set 7 (Week 8)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: N/A

Problem Statement: Prove the following using contrapositive proof.

(a) Suppose $x, y \in \mathbb{Z}$. If $x^2(y+3)$ is even, then x is even or y is odd.

Proof. We will prove this using the contrapositive. Suppose that x is odd and y is even. By definition, $x = 2n+1$ for some $n \in \mathbb{Z}$ and $y = 2m$ for some $m \in \mathbb{Z}$. It follows that $x^2 = 4n^2 + 4n + 1$ and $y + 3 = 2m + 3$. Thus,

$$x^2(y+3) = (4n^2+4n+1)(2m+3) = 8mn^2 + 8mn + 2m + 12n^2 + 12n + 3.$$

We rearrange the right side to get:

$$x^2(y+3) = 2(4mn^2 + 4mn + m + 6n^2 + 6n + 1) + 1.$$

Since $4mn^2 + 4mn + m + 6n^2 + 6n + 1$ is an integer, it follows that $x^2(y+3)$ is odd.

We have proven that if x is odd and y is even, then $x^2(y+3)$ is odd. Through contrapositive, it follows that if $x^2(y+3)$ is even, then x is even or y is odd. □

(b) Suppose $x \in \mathbb{R}$. If $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.

Proof. We will prove this using the contrapositive. Suppose that $x \in \mathbb{R}$ and $x < 0$. By definition of even exponents, $x^4 + x^2 + 8$ will be positive. However, because x is negative, x^5, $7x^3$, and $5x$ will all be negative numbers too. It follows that $x^5 + 7x^3 + 5x$ will be negative as well. Thus,

$$x^5 + 7x^3 + 5x < x^4 + x^2 + 8$$

for all $x < 0$.

Through contrapositive, we see that if $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$. □

2. Example from Problem Set 7 (Week 8)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This is part B of question 3 in the problem set, which allowed us to use any proof structure to prove the theorem. I used contradiction for part A, and you can find it in Section 2.4!

Problem Statement: Prove the following: Let $n \in \mathbb{N}$. If $2^n - 1$ is a prime number, then n itself is a prime number.

Proof. We will prove this using the contrapositive. Suppose that n is not a prime number. This means that $n = ab$ for some $a, b \in \mathbb{Z}$ and $a, b \neq 1, n$. So, $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1^b$. By the difference of powers, this can be expressed as

$$(2^a - 1) \cdot (2^{a(b-1)} + 2^{a(b-2)} + ...1).$$

Thus, $2^n - 1 = (2^a - 1) \cdot (2^{a(b-1)} + 2^{a(b-2)} + ...1)$ has factors and is not a prime number. By contrapositive, therefore, if $2^n - 1$ is a prime number then n itself is a prime number. □

2.4 Use of proof by contradiction

1. Example from Problem Set 7 (Week 8)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This is part A of question 3 in the problem set, which allowed us to use any proof structure to prove the theorem. I used the contrapositive for part B, and you can find it in Section 2.3!

Problem Statement: Prove the following: The number $\log_2 3$ is irrational.

Proof. Suppose for the sake of contradiction that $\log_2 3$ is a rational number. Then, $\log_2 3 = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. By algebra and log rules,

$$b \log_2 3 = a$$

$$\log_2 3^b = a.$$

$$2^a = 3^b$$

Since 3 is an odd number, 3^b will always be odd. Thus, 2^a is odd too. However, this is a contradiction, because powers of 2 are always even!

Therefore, $\log_2 3$ must be irrational. □

2. Example from Textbook Chapter 6, #17

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: N/A

Problem Statement: For every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.

Proof. Suppose, for the sake of contradiction, that for all $n \in \mathbb{Z}$ it is true that $4|(n^2 + 2)$. In other words, there exists $k \in \mathbb{Z}$ such that $4k = n^2 + 2$ Observe that there are two cases for the value of n:

Case 1: n is even. By definition, n can be written as $n = 2x$ for some $x \in \mathbb{Z}$. As such, $n^2 = 4x^2$, and it follows that $4k = 4x^2 + 2$. We can divide both sides by 2 to obtain the equation

$$2k = 2x^2 + 1.$$

This means that the left side is an even number while the right side is odd, producing a contradiction. Therefore, when n is even, it's untrue that $4|(n^2+2)$.

Case 2: n is odd. By definition, n can be written as $n = 2y + 1$ for some $y \in \mathbb{Z}$. As such, $n^2 = (2y+1)^2 = 4y^2+4y+1$. This means that $4k = 4y^2+4y+3$. We can rewrite this as

$$2(2k) = 2(2y^2 + 2y + 1) + 1.$$

We can see that the left side is an even number while the right side is odd. This produces a contradiction; as such, when n is odd, it cannot be that $4|(n^2 + 2)$.

For all cases of n, it is untrue that $4|(n^2 + 2)$. Therefore, for every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$. □

2.5 Use of existence proof

1. Example from Problem Set 6 (Week 7)

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: Initially, the problem statement was as follows:

> *Prove that every odd integer is a difference of two square integers.*
>
> I decided that, when interpreted differently, this is actually an existence proof. See below for the new, revised problem statement. When rephrased in this way, this theorem clearly becomes a question of existence!

Problem Statement: Prove that for every odd integer n, there exists two square integers such that their difference is n.

Proof. Suppose that n is an odd integer. As such, it can be expressed as $n = 2k + 1$ for some $k \in \mathbb{Z}$. We want to show that there exist $a, b \in \mathbb{Z}$ such that $a^2 - b^2 = n$.

Let $a = k + 1$. It follows that $a^2 = (k + 1)^2 = k^2 + 2k + 1$. Next, let $b = k$, so that $b^2 = k^2$.

Consider the difference $a - b$. Using our definitions of a and b, we see that

$$a - b = k^2 + 2k + 1 - k^2$$

$$= 2k + 1 = n\,.$$

Therefore, for an arbitrary odd integer n, there exist two square integers such that their difference is n. □

2.6 Use of uniqueness proof

1. Example given by Jordan

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: This was an example introduced in office hours (and later on Canvas!)

Problem Statement: Suppose that $f : X \to Y$ is a bijective function. Prove that its inverse is the unique function g for which both $f \circ g = id_Y$ and $g \circ f = id_X$.

Proof. We know that for any bijective function $f : X \to Y$, there exists an inverse function $f^{-1} : Y \to X$ such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Suppose there is a function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. We want to show that if this is true, then $g = f^{-1}$.

Given the equation $f \circ g = id_Y$, we can obtain an equivalent statement by composing both sides with f^{-1}. This gives us

$$f^{-1} \circ (f \circ g) = f^{-1} \circ id_Y\ .$$

Consider the right side of the equation. For any $y \in Y$, we see that

$$f^{-1} \circ id_Y(y) = f^{-1}(id_Y(y)).$$

By definition, the function id_Y returns its input y. As such,

$$f^{-1}(id_Y(y)) = f^{-1}(y).$$

Because of this, $f^{-1} \circ id_Y = f^{-1}$.

Furthermore, because function composition is associative, we can rewrite the left side as $(f^{-1} \circ f) \circ g$.

By definition, $f^{-1} \circ f = id_X$, so this simplifies to $f^{-1} \circ g$. For any $x \in X$, we see that

$$f^{-1} \circ g(x) = f^{-1}(g(x)).$$

The function id_X returns its input set, so $f^{-1}(g(x))$ is equivalent to just $g(x)$ alone. As such, $g \circ id_X = g$.

Returning to our equation, we see that $f^{-1} \circ (f \circ g) = f^{-1} \circ id_Y$ simplifies to $g = f^{-1}$. This means that any function g fulfilling $f \circ g = id_Y$ and $g \circ f = id_X$ will be equal to the inverse function f^{-1}. In other words, f has a unique inverse g such that $f \circ g = id_Y$ and $g \circ f = id_X$. □

2.7 Proof by induction

1. Example from Problem Set 8 (Week 9)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This is part 2 of question 3 in the problem set, which specifically asked for a proof by induction for this theorem. Part 1 asked for a combinatorial proof, and you can find it in Section 2.8!

Problem Statement: Prove the following formula using induction:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

Proof. **Base Case:** Let n=1. Then,

$$\sum_{k=1}^{n} k\binom{n}{k} = 1\binom{1}{1} = 1 = 1 \cdot 2^0 = n2^{n-1}.$$

Thus, the formula holds for the base case.

Induction Hypothesis: Assume that $\sum_{k=1}^{a} k\binom{a}{k} = a2^{a-1}$ for $a \geq 1$.

Inductive Step: We want to show that $\sum_{k=1}^{a+1} k\binom{a+1}{k} = (a+1)2^{(a+1)-1}$. We know that $\binom{a+1}{k} = \binom{a}{k-1} + \binom{a}{k}$. So, we can rewrite the left-hand side as

$$\sum_{k=1}^{a+1} k\binom{a+1}{k} = \sum_{k=1}^{a+1} k\binom{a}{k-1} + k\binom{a}{k}$$

$$= \sum_{k=1}^{a+1} k\binom{a}{k-1} + \sum_{k=1}^{a+1} k\binom{a}{k}.$$

Using our inductive hypothesis, we rewrite this as

$$\sum_{k=1}^{a+1} k\binom{a}{k-1} + a2^{a-1}.$$

$$= (\sum_{k=1}^{a+1} k\binom{a}{k-1} - \binom{a}{k-1} + \binom{a}{k-1}) + a2^{a-1}$$

$$= \sum_{k=1}^{a+1} (k-1)\binom{a}{k-1} + \sum_{k=1}^{a+1} \binom{a}{k-1} + a2^{a-1}.$$

Consider the sum $\sum_{k=1}^{a+1}(k-1)\binom{a}{k-1}$. If we subtract the base case $k = 1$, the expression will be the same as $\sum_{k=1}^{a} k\binom{a}{k}$. The value of the base case is $0 \cdot \binom{a}{0} = 0$. Thus, we rewrite the expression as

$$\sum_{k=1}^{a} k\binom{a}{k} + \sum_{k=1}^{a+1} \binom{a}{k-1} + a2^{a-1}.$$

Using our inductive hypothesis again,

$$2 \cdot a2^{a-1} + \sum_{k=1}^{a+1} \binom{a}{k-1} = a2^a + \sum_{k=1}^{a+1} \binom{a}{k-1}.$$

We can re-index the expression $\sum_{k=1}^{a+1} \binom{a}{k-1}$ to $\sum_{k=0}^{a} \binom{a}{k}$. Our expression becomes:

$$a2^a + \sum_{k=0}^{a} \binom{a}{k}$$

The expression $\sum_{k=0}^{a} \binom{a}{k}$ represents the total number of possible subsets of integers 1 to a, regardless of size. In other words, it is the cardinality of the power set of integers 1 to a, which can also be written as 2^a. Thus,

$$\sum_{k=1}^{a+1} k\binom{a+1}{k} = a2^a + 2^a = (a+1)2^{(a+1)-1}.$$

Therefore, $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for all natural numbers n. □

2. Example from Problem Set 9 (Week 10)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: N/A

Problem Statement: Suppose that n straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point.

Prove that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

Proof. We will induct on the number n of lines with the conditions stated above.

Base Case: Let $n = 1$. When there is only one line in a plane, it will divide it into 2 regions. We see that $\frac{n^2+n+2}{2} = \frac{1^2+1+2}{2} = 2$ also gives the same answer, meaning that the theorem holds true for $n = 1$.

Inductive Hypothesis: Suppose the theorem holds true for some $k \geq 1$, meaning that k lines fulfilling the conditions on a plane will divide it into $\frac{k^2+k+2}{2}$ regions. We want to show that adding an additional line, such that $n = k + 1$, results in $\frac{(k+1)^2+(k+1)+2}{2}$ regions.

Induction: Let us add another line to a plane originally with k lines such that no two lines are parallel and no three lines intersect at a single point. We will call this line l. Because of the lack of parallelism, l must intersect with each of the k lines at some point. Furthermore, l must have exactly k intersections total, since it must intersect each original line independently.

This means that l is split into $k + 1$ segments by its intersections, which in turn creates $k + 1$ new regions.

Adding this to the previous number of regions, we see:

$$\begin{aligned}
&\frac{k^2+k+2}{2}+k+1 \\
&=\frac{k^2+k+2}{2}+\frac{2k+2}{2} \\
&=\frac{k^2+3k+4}{2} \\
&=\frac{k^2+2k+1+k+1+2}{2} \\
&=\frac{(k+1)^2+(k+1)+2}{2}
\end{aligned}$$

which matches our inductive hypothesis. We see that if k lines dividing a plane in such a way results in $\frac{k^2+k+2}{2}$ regions, then $k+1$ lines result in $\frac{(k+1)^2+(k+1)+2}{2}$ regions.

Therefore, by induction, n straight lines where none are parallel and no three lines intersect at a single point will divide a plane into $\frac{n^2+n+2}{2}$ regions. $\square$

2.8 Combinatorial proof

1. Example from Problem Set 5 (Week 6)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: N/A

Problem Statement: Choose two of the exercises in §3.10 between 1 and 6.

Exercise 2: Using combinatorial proof, show that $1=2+3+...+n=\binom{n+1}{2}$.

Proof. Let $S=\{0,1,...n\}$ be a set of size $n+1$. The number of 2-element subsets of S can be computed using $\binom{n+1}{2}$.

Alternatively, let k represent the larger element in any 2-element subset of S. For each value of k, there are k possible values for the smaller of the two elements. By accounting for every possible value of k and its counterpart, we find that the number of 2-element subsets of S is $\sum_{k=0}^{n} k=1+2+...n$.

Since this can also be computed using $\binom{n+1}{2}$, we find that

$$\sum_{k=0}^{n} k=1+2+...n=\binom{n+1}{2}.$$

$\square$

Exercise 6: Using combinatorial proof, show that $\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$.

Proof. Let $S = \{1, 2, ...n, n+1, ...2n, 2n+1, ...3n\}$. The number of 3-element subsets of S can be computed using $\binom{3n}{3}$.

Alternatively, we can partition S into three disjoint sets: $S_1 = \{1, 2, ...n\}$, and $S_2 = \{n+1, n+2, ...2n\}$, and $S_3 = \{2n+1, 2n+2, ...3n\}$. Each of these sets is size n. There are three ways to create 3-element subsets of S using S_1, S_2, and S_3.

Case 1: Each element is from each of S_1, S_2, and S_3. Since each set is size n, the number of such 3-element subsets is n^3.

Case 2: Two elements are from one of S_1, S_2, and S_3, and the third is from one of the other sets. There are $3 \cdot 2 = 6$ ways to choose two sets for this purpose. There are $\binom{n}{2}$ ways to choose two elements from one set, and n ways to choose one element from the other. Therefore, the number of such 3-element subsets is $6n\binom{n}{2}$.

Case 3: All three elements are from one of S_1, S_2, and S_3. There are 3 ways to choose one set for this purpose, and $\binom{n}{3}$ ways to choose three elements from that set. Therefore, the number of such 3-element subsets is $3\binom{n}{3}$.

Adding together the subsets from all three cases, we see that there are $3\binom{n}{3} + 6n\binom{n}{2} + n^3$ ways to create a 3-element subset of S. Since this can also be computed using $\binom{3n}{3}$, we find that

$$\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3.$$

□

2. Example from Problem Set 8 (Week 9)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This is part 1 of question 3 in the problem set, which specifically asked for a combinatorial proof for this theorem. Part 2 asked for a proof by induction, and you can find it in Section 2.7!

Problem Statement: Prove the following formula using a combinatorial proof:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

Proof. Let $S = \{1, 2, ...n\}$ be a set with n elements. Suppose we want to form a subset where one particular element is highlighted. The number of ways to form a subset of S with size at least 1 is $\sum_{k=1}^{n} \binom{n}{k}$.

The number of ways to highlight one element of each subset is k, where k is the size. Thus, the number of different subsets with one element highlighted is $\sum_{k=1}^{n} k\binom{n}{k}$.

Alternatively, consider choosing the element to be highlighted first. There are n ways to do this. After selecting this element, we have $n - 1$ elements left to choose from. We can then create arbitrary subsets to put the element into. These subsets can be size 0, since with the element added, that would bring the total size to 1.

The number of ways to make any-size subsets of a $n - 1$ size set is simply the cardinality of the power set of a $n - 1$ size set: 2^{n-1}. Thus, the number of ways to choose a highlighted element then form a subset around it is $n2^{n-1}$.

Both approaches result in the number of arbitrary size subsets with a "highlighted" element. Therefore, $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. □

2.9 Understanding of relations

1. Example from Problem Set 10 (Week 11)

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: N/A

Problem Statement:

(a) Suppose that $A \neq \varnothing$. Since $\varnothing \subseteq A \times A$, it follows that $R = \varnothing$ is a relation on A. Determine if R is symmetric, reflexive, or transitive.

Proof. **1 - Reflexivity:** For a relation R to be reflexive, it must be that for every $a \in A$, $(a, a) \in R$. Since $R = \varnothing$ contains nothing, for all $a \in A$, the ordered pair (a, a) will never be an element of R. Thus, the relation $R = \varnothing$ is not reflexive.

2 - Symmetry: For a relation R to be symmetric, it is required that if $x, y \in A$ and xRy (which can also be written as $(x, y) \in R$), then yRx and $(y, x) \in R$. Since $R = \varnothing$ contains nothing, there are no elements $x, y \in A$ such that $(x, y) \in R$, meaning that no elements fulfill the first part of this conditional statement.

Because the first part is always false, it follows that the entire conditional is always true. (This is because when given a statement $P \Rightarrow Q$ where P is always false, then regardless of the value of Q, the entire statement is always true.) Thus, $R = \varnothing$ is symmetric.

3 - Transitivity: For a relation R to be transitive, it must be that if $x, y, z \in A$ and xRy and yRz, then zRx. In other words, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Similarly to the last property, since $\varnothing$ contains nothing, there are no elements $x, y, z \in A$ that fulfill the conditions of $(x, y) \in \varnothing$ and $(y, z) \in \varnothing$. Since the first part of the conditional is false, it follows that the entire statement is true. So, $R = \varnothing$ is transitive. □

(b) Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for each $x \in A$. Prove R is reflexive.

Proof. Assume R is symmetric and transitive. Because of transitivity, we see that for $k, m, n \in A$, if kRm and mRn then kRn. We know that aRx for every $x \in A$, and R is symmetric, so it is also true that xRa.

Using transitivity, we see that because xRa and aRx, it follows that xRx. Because this applies to every $x \in A$, by definition, this means R is reflexive. □

2. Example from Problem Set 10 (Week 11)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: N/A

Problem Statement:

(a) Suppose that $[a], [b] \in \mathbb{Z}_5$ and $[a] \cdot [b] = [0]$. Is it necessarily true that $[a] = [0]$ or $[b] = [0]$?

Proof. Suppose that $[a] \cdot [b] = [0]$. The equivalence class $[0]$ contains multiples of 5. Since 5 is a prime number, it is in the prime factorization of every element of $[0]$. Therefore, in order for $[a] \cdot [b]$ to equal $[0]$, it must be that either $[a]$ or $[b]$ contains elements that have 5 as a factor. In other words, the elements must be multiples of 5.

If an equivalence class contains multiples of 5, it must be $[0]$. So, it follows that either $[a]$ or $[b]$ must be equal to $[0]$. □

(b) Same question but now with $\mathbb{Z}_6$ and then $\mathbb{Z}_7$.

Proof. **For $\mathbb{Z}_6$:** Let $[a] = [2]$ and $[b] = [3]$. We observe that $[a] \cdot [b] = [2] \cdot [3] = [6] = [0]$, but clearly $[a] \neq [0]$ and $[b] \neq [0]$. Thus, it is NOT always necessary that $[a] = [0]$ or $[b] = [0]$.

For $\mathbb{Z}_7$: Similar to the case of $\mathbb{Z}_7$, because 7 is a prime number, it must be in the prime factorization of every element of $[0]$. Therefore, in order for $[a] \cdot [b]$ to equal $[0]$, one of the two equivalence classes must contain multiples of 7. By definition, this means that one of the classes must equal $[0]$. So, in this case, it is necessary that $[a] = [0]$ or $[b] = [0]$. $\square$

(c) Prove or disprove: if p is prime, and $[a] \cdot [b] = [0]$ in $\mathbb{Z}_p$, then $[a] = [0]$ or $[b] = [0]$.

Proof. Suppose that p is prime and $[a] \cdot [b] = [0]$ in $\mathbb{Z}_p$. In this set, $[0]$ contains all multiples of p. Since p is prime, all factors of these elements must be divisible by p as well. It follows that either $[a]$ or $[b]$ must be an equivalence class containing elements divisible by p.

By definition, an equivalence class containing elements divisible by p is equal to $[0]$. Therefore, if p is prime and $[a] \cdot [b] = [0]$ in $\mathbb{Z}_p$, then it must be that $[a] = [0]$ or $[b] = [0]$. $\square$

2.10 Understanding of functions

1. Example from Problem Set 12 (Week 11)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: Though I received a 4/4 on this problem, I got a comment recommending that I elaborate just a little more on part B. I made edits to my proof to provide a little more justification!

Other Information: N/A

Problem Statement:

(a) Consider the sets $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \colon 3x + y = 4\}$ and $g = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \colon x + 3y = 4\}$. Which, if any, of f and g are functions from $\mathbb{Z}$ to $\mathbb{Z}$?

Proof. First, we want to show that f maps each input to exactly one output, and that such a mapping exists for every input. Consider an arbitrary element $a \in \mathbb{Z}$ where a is the input. There will only be one b such that $3a + b = 4$, or $b = 4 - 3a$, and that b will always be an integer. Thus, for every $a \in \mathbb{Z}$, there is a unique ordered pair (a, b) that is part of the set. It follows that f is a function.

Next, we want to show the same for g. Consider an arbitrary element $c \in \mathbb{Z}$ where c is the input. We want to find an element $d \in \mathbb{Z}$ such that $c + 3d = 4$, or $d = \frac{4}{3} - \frac{c}{3}$. It can be seen that d will only be an integer for certain values of c: for example, when $c = 2$, it must be that $d = \frac{2}{3}$ and is therefore not part of the output set. So, g does not map every element of its input set $\mathbb{Z}$ to an output, meaning it is not a function. □

(b) Consider the sets $f = \{(x^2, x) \colon x \in \mathbb{R}\}$ and $g = \{(x^3, x) \colon x \in \mathbb{R}\}$. Which, if any, of f, g are functions from $\mathbb{R}$ to $\mathbb{R}$?

Proof. First, we'll consider set f. We immediately see that the input x^2 can only be positive (because all squares must be positive), meaning that not all elements of the domain $\mathbb{R}$ can be input into this set. Therefore, f cannot be a function.

Next, we'll consider set g. Think of g as the inverse relation of the set $g^{-1} = \{(x, x^3) : x \in \mathbb{R}\}$. By definition, we know that g is a function if and only if g^{-1} is a bijective function.

We know that g^{-1} allows a mapping for every x in the input set $\mathbb{R}$. We also know that if $a, b \in \mathbb{R}$ and $a \neq b$, then $a^3 \neq b^3$, meaning there is a unique mapping x^3 for every input. Because of these two properties, g^{-1} is a function.

Furthermore, g^{-1} is injective because given outputs $a^3 = (a')^3$, taking the cube root of both sides gives $a = a'$. g^{-1} is also surjective because for every $a^3 \in \mathbb{R}$, there exists an $a \in \mathbb{R}$. Thus, g^{-1} is a bijective function.

Because g^{-1} is bijective, it follows that its inverse $g = \{(x^3, x) \colon x \in \mathbb{R}\}$ is a function. □

2. Example from Problem Set 11 (Week 12)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: N/A

Problem Statement:

(a) Prove that $f \colon \mathbb{N} \to \mathbb{Z}$ given by $f(n) = \frac{(-1)^n(2n-1)+1}{4}$ is bijective.

Proof. First, we want to show that f is injective. Consider $f(a)$ and $f(b)$ such that $a, b \in \mathbb{N}$ and $f(a) = f(b)$. This means that

$$\frac{(-1)^a(2a-1)+1}{4} = \frac{(-1)^b(2b-1)+1}{4}$$

$$(-1)^a(2a-1)+1 = (-1)^b(2b-1)+1$$

$$(-1)^a(2a-1) = (-1)^b(2b-1)$$

$$(-1)^{a-b}(2a-1) = 2b-1$$

$$(-1)^{a-b}(2a-1) = 2b-1.$$

Because the input is $n \in \mathbb{N}$, we know that $2n-1$ must be always positive. Therefore, $(-1)^a$ and $(-1)^b$ must have the same sign in order for $f(a) = f(b)$ to be true. This means a and b must have the same parity. Because of this, the difference $a-b$ will be even. It follows that $(-1)^{a-b} = 1$. So,

$$(-1)^{a-b}(2a-1) = 2a-1 = 2b-1$$

$$a = b.$$

This means that for any input values $a, b \in \mathbb{N}$, if $f(a) = f(b)$ then $a = b$. By definition, f is injective.

Next, we want to show that f is surjective. Consider any integer b in the codomain set. If f is surjective, then there exists an $a \in \mathbb{N}$ such that

$$b = \frac{(-1)^a(2a-1)+1}{4}$$

$$4b = (-1)^a(2a-1)+1$$

$$4b-1 = (-1)^a(2a-1)$$

$$\frac{4b-1}{(-1)^a} = 2a-1$$

$$\frac{4b-1}{(-1)^a} + 1 = 2a$$

$$\frac{4b-1}{2(-1)^a} + \frac{1}{2} = a.$$

There are two cases for the value of a.

Case 1: a is even. Then, $(-1)^a = 1$. Our equation becomes

$$\frac{4b-1}{2} + \frac{1}{2} = a$$

$$\frac{4b}{2} = a$$

$$2b = a.$$

Case 2: a is odd. Then, $(-1)^a = -1$. Our equation becomes

$$\frac{1-4b}{2} + \frac{1}{2} = a$$
$$\frac{2-4b}{2} = a$$
$$1 - 2b = a.$$

We know that a must be a positive integer, and it may equal either $2b$ or $1-2b$. If b is negative, then $a = 1-2b$ will make a a positive odd integer. If b is positive, $a = 2b$ will make a a positive even integer. This covers all cases of what b can be while keeping a in the domain set, showing that for any element $b \in \mathbb{Z}$, there exists an $a \in \mathbb{N}$ such that $f(a) = b$. Thus, f is surjective.

Because f is both injective and surjective, it follows that f is bijective. □

(b) Prove that $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m,n) = 2^{m-1}(2n-1)$ is bijective.

Proof. First, we want to show that f is injective. Consider $a, b, c, d \in \mathbb{N}$ such that $f(a,b) = f(c,d)$. This means

$$2^{a-1}(2b-1) = 2^{c-1}(2d-1)$$
$$2^{a-c}(2b-1) = 2d-1.$$

The value of $2b-1$ will be odd regardless of the value of b. Additionally, because the right side $2d-1$ is an integer, the left side of the equation must also be an integer. So, it cannot be that $c > a$: otherwise, 2^{a-c} is a fraction of the form $\frac{1}{k}$ where k is a multiple of 2. In this case, $\frac{1}{k} \cdot (2b-1)$ will not be an integer. Therefore, it must be that $a \geq c$.

We can also rearrange this equation to get $2b-1 = 2^{c-a}(2d-1)$. The value of $2d-1$ will be odd regardless of the value of d. Additionally, because the left side $2b-1$ is an integer, the right side of the equation must also be an integer. So, it cannot be that $c < a$: otherwise, 2^{c-a} is a fraction of the form $\frac{1}{k}$ where k is a multiple of 2. In this case, $\frac{1}{k} \cdot (2d-1)$ will not be an integer. Therefore, it must be that $a \leq c$.

Because $a \geq c$ and $a \leq c$, it follows that $a = c$. Returning to the equation,

$$2b - 1 = 2d - 1$$
$$b = d.$$

Thus, if $f(a,b) = f(c,d)$ then $(a,b) = (c,d)$, meaning f is injective.

Next, we want to show that f is surjective. Consider any element $c \in \mathbb{N}$. If f is surjective, then there exists an ordered pair $(a, b) \in \mathbb{N} \times \mathbb{N}$ such that $c = 2^{a-1}(2b-1)$. By the fundamental theorem of algebra, there exists a unique prime factorization for c. Knowing this, consider the two cases for the value of c:

Case 1: c is odd. The prime factorization of c will therefore contain no powers of 2, meaning that a must equal 1. Then, $c = 2b-1$, or $2b = c+1$. Note that both $2b$ and $c+1$ are even, so there will exist a natural number b that fulfills this. Thus, there exist an a and b such that $c = 2^{a-1}(2b-1)$.

Case 2: c is even. The prime factorization of c must contain a power of 2, which we can fulfill by setting $a > 1$. Suppose that the prime factorization of c takes the form of $2^k \cdot p$, where $p \in \mathbb{Z}$ is the product of the odd prime factors of c.

It follows that if $c = 2^{a-1}(2b-1)$, then $p = 2b-1$, or $2b = p+1$. Note that both $2b$ and $p+1$ are even, so there will exist a natural number b that fulfills this. If we let $a = k+1$, we see that there exists (a, b) such that $c = 2^k \cdot p$ and $c = 2^{a-1}(2b-1)$.

Because there exists a pair (a, b) that fulfills $c = 2^{a-1}(2b-1)$ regardless of the value of c, it follows that f is surjective. Being both injective and surjective, f is therefore bijective. □

2.11 Understanding of calculus

1. Example from Textbook Chapter 13.2, #5

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: N/A

Problem Statement: Prove that $\lim_{x \to 3}(x^2 - 2) = 7$.

Proof. We want to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x-3| < \delta$ forces $|(x^2-2) - 7| < \varepsilon$ to be true.

Fix $\varepsilon > 0$, and let $\delta = min(1, \frac{\varepsilon}{7})$. Suppose $|x-3| < \delta$. Observe that

$$|(x^2-2) - 7| = |x^2 - 9| = |(x+3)(x-3)| \leq 7|x-3|.$$

This inequality holds true because δ will never be larger than 1 and we assume $|x-3| < \delta$, so

$$|x+3| = |x-3+6| \leq |x-3| + 6 \leq 1 + 6 = 7.$$

Because $|x-3| < \delta$, we see that

$$7|x-3| < 7\delta \leq 7(\frac{\varepsilon}{7}) = \varepsilon.$$

As such, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x-3| < \delta$, then $|(x^2-2)-7| < \varepsilon$. By definition of limits, this proves that $\lim_{x\to 3}(x^2-2) = 7$. $\square$

2. Example from Textbook Chapter 13.4, #2

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: Theorem 13.7, which is referenced in the problem statement and used in the proof, states:

If both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x)) \cdot (\lim_{x\to c} g(x)).$$

Problem Statement: Given two or more functions $f_1, f_2, ...f_n$, suppose that $\lim_{x\to c} f_i(x)$ exists for each $1 \leq i \leq n$. Prove that

$$\lim_{x\to c}(f_1(x)f_2(x)...f_n(x)) = \lim_{x\to c} f_1(x) \cdot \lim_{x\to c} f_2(x) \cdot \ ... \cdot \lim_{x\to c} f_n(x).$$

Use induction on n, with Theorem 13.7 serving as the base case.

Proof. We will induct on n, the number of functions.

Base Case: Let $n = 2$, meaning there are only two functions. By Theorem 13.7, given $\lim_{x\to c} f_1(x)$ and $\lim_{x\to c} f_2(x)$, then

$$\lim_{x\to c} f_1(x)f_2(x) = \lim_{x\to c} f_1(x) \cdot \lim_{x\to c} f_2(x).$$

In this case, the theorem holds true.

Inductive Hypothesis: Assume that given $n = k$ functions (where $k \geq 2$), where $\lim_{x\to c} f_i(x)$ exists for all $1 \leq i \leq k$,

$$\lim_{x\to c}\left(\prod_{i=1}^{k} f_i(x)\right) = \prod_{i=1}^{k}\left(\lim_{x\to c} f_i(x)\right).$$

We want to show that this implies the theorem holds for $n = k+1$ as well; in other words, we want

$$\lim_{x\to c}\left(\prod_{i=1}^{k+1} f_i(x)\right) = \prod_{i=1}^{k+1}\left(\lim_{x\to c} f_i(x)\right).$$

Induction: Consider $\lim_{x\to c}\left(\prod_{i=1}^{k+1} f_i(x)\right)$. We can also write this as

$$\lim_{x\to c}\left(\left(\prod_{i=1}^{k} f_i(x)\right)\cdot f_{k+1}(x)\right).$$

Let $g(x)=\prod_{i=1}^{k} f_i(x)$. By Theorem 13.7,

$$\lim_{x\to c}(g(x)\cdot f_{k+1}(x)) = \lim_{x\to c} g(x)\cdot \lim_{x\to c} f_{k+1}(x).$$

We apply our inductive hypothesis:

$$\lim_{x\to c} g(x) = \lim_{x\to c}\left(\prod_{i=1}^{k} f_i(x)\right) = \prod_{i=1}^{k}\left(\lim_{x\to c} f_i(x)\right).$$

Returning to our original equation, we see that

$$\prod_{i=1}^{k}\left(\lim_{x\to c} f_i(x)\right)\cdot \lim_{x\to c} f_{k+1}(x) = \prod_{i=1}^{k+1}\left(\lim_{x\to c} f_i(x)\right).$$

Thus,

$$\lim_{x\to c}\left(\prod_{i=1}^{k+1} f_i(x)\right) = \prod_{i=1}^{k+1}\left(\lim_{x\to c} f_i(x)\right).$$

By induction, this proves that for any $n\geq 2$,

$$\lim_{x\to c}\left(\prod_{i=1}^{n} f_i(x)\right) = \prod_{i=1}^{n}\left(\lim_{x\to c} f_i(x)\right).$$

In other words,

$$\lim_{x\to c}(f_1(x)f_2(x)...f_n(x)) = \lim_{x\to c} f_1(x)\cdot \lim_{x\to c} f_2(x)\cdot\ ...\ \cdot \lim_{x\to c} f_n(x).$$

□

2.12 Understanding of cardinality

1. Example from Textbook Chapter 14.2, #4

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: This proof makes use of the following theorem:

Theorem 14.6: If A and B are both countably infinite, then so is $A\cup B$.

Problem Statement: Prove that the set of all irrational numbers is uncountable.

Proof. Consider Theorem 14.6. Taking the contrapositive of the theorem gives us the following statement: if $A \cup B$ is uncountable, then either A or B must be uncountable.

The set of all irrational numbers is $\mathbb{R} - \mathbb{Q}$. Consider the union between the set of irrational numbers and the set of rational numbers, $(\mathbb{R} - \mathbb{Q}) \cup \mathbb{Q}$. This simply leaves us with $\mathbb{R}$, which is an uncountable set.

By the contrapositive of Theorem 14.6, it follows that either $\mathbb{R} - \mathbb{Q}$ or $\mathbb{Q}$ must be uncountable. We know that $\mathbb{Q}$ is countable, so it must be that $\mathbb{R} - \mathbb{Q}$, the set of all irrational numbers, is uncountable. □

2. Example from Textbook Chapter 14.2, #6-9

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: This exercise makes use of the following theorems:

Theorem 14.3: A set A is countably infinite if and only if its elements can be arranged in an infinite list $a_1, a_2, a_3, ...$

Theorem 14.5: If A and B are countably infinite, then so is $A \times B$.

Problem Statement:

6. Prove or disprove: There exists a bijective function $f : \mathbb{Q} \to \mathbb{R}$.

   Proof. We know that $\mathbb{Q}$ is a countably infinite set while $\mathbb{R}$ is uncountably infinite. This means $|\mathbb{Q}| < |\mathbb{R}|$, which in turn implies that there exists an injection $f : \mathbb{Q} \to \mathbb{R}$ but not a bijection. Therefore, there does not exist a bijective function $f : \mathbb{Q} \to \mathbb{R}$. □

7. Prove or disprove: The set $\mathbb{Q}^{100}$ is countably infinite.

   Proof. We will use induction to show that not only is $\mathbb{Q}^{100}$ countably infinite, but that $\mathbb{Q}^n$ for any integer $n > 0$ is too!

   Base Case: Let $n = 1$. We know that $\mathbb{Q}^1 = \mathbb{Q}$ is countably infinite. Thus, the theorem holds for the base case.

   Inductive Hypothesis: Assume that for an integer $k \geq 1$, $\mathbb{Q}^k$ is countably infinite. We want to show that this implies $\mathbb{Q}^{k+1}$ is also countably infinite.

   Induction: Note that $\mathbb{Q}^{k+1}$ is another way of writing $\mathbb{Q}^k \times \mathbb{Q}$. The inductive hypothesis tells us that $\mathbb{Q}^k$ is countably infinite, and our base case tells us that $\mathbb{Q}$ is also countably infinite. By Theorem 14.5, this means $\mathbb{Q}^k \times \mathbb{Q} = \mathbb{Q}^{k+1}$ is countably infinite as well.

Therefore, by induction, the set $\mathbb{Q}^n$ for any integer $n > 0$ is countably infinite. It follows, therefore, that $\mathbb{Q}^{100}$ is countably infinite. □

8. Prove or disprove: The set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

   Proof. Both $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite sets. Therefore, by Theorem 14.5, $\mathbb{Z} \times \mathbb{Q}$ is also countably infinite. □

9. Prove or disprove: The set $\{0, 1\} \times \mathbb{N}$ is countably infinite.

   Proof. We cannot apply Theorem 14.5 because $\{0, 1\}$ is not infinite. However, note that $\{0, 1\} \times \mathbb{N}$ results in ordered pairs:

   $$(0, 1), (1, 1), (0, 2), (1, 2), (0, 3), (1, 3), ...$$

   These elements can be arranged into an ordered, infinite list. By Theorem 14.3, this means $\{0, 1\} \times \mathbb{N}$ is countably infinite. □

2.13 Other notable mentions

1. Example from Problem Set 3 (Week 4)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This exercise is a good example of working backwards from a given statement to obtain the truth values of its "sub-statements".

Problem Statement:

(a) Suppose that the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false. Find the truth values of P, Q, R, S—this can be done without a truth table, you may find some of them helpful.

Proof. A conditional $A \Rightarrow B$ is only false if A is true and B is false. We see, therefore, that $((P \wedge Q) \vee R)$ is true and $(R \vee S)$ is false.

A statement $A \vee B$ is only false if both A and B are false. $(R \vee S)$ is of that form, so R and S are both false.

Since $((P \wedge Q) \vee R)$ is true, we know that only one of $(P \wedge Q)$ and R can be false, by definition of the $\vee$ operator. We know that R is false, and can therefore conclude that $(P \wedge Q)$ must be true. A statement $A \wedge B$ is only true if both A and B are true. So, both P and Q are true.

Therefore P and Q are true, while R and S are false. □

(b) Suppose that P is false and that the statement $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true. Find the truth values of R and S–same comments as above.

Proof. Since P is false, the statement $(P \wedge Q)$ cannot be true. An iff statement can only be true if both sides of the conditional are true, or if both are false. Since $(P \wedge Q)$ is false, $(R \Rightarrow S)$ must also be.

The only way for a conditional $A \Rightarrow B$ to be false is for A to be true and B to be false. So, R is true and S is false. □

2. Example from Problem Set 8 (Week 9)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This example showcases how to prove that a set of statements are equivalent, using a ”circular” proof structure of conditionals between statements.

Problem Statement: Suppose that A and B are sets. Prove that the following statements are equivalent:

(a) $A \subseteq B$;

(b) $A - B = \varnothing$;

(c) $A \cap B = A$

Proof. First, we prove that if $A \subseteq B$, then $A - B = \varnothing$. Suppose hat $A \subseteq B$. This means that for every element $a \in A$, a is also an element of B.

The operation $A - B$ results in a set containing all elements of A that are not also in B. However, there are no elements of A that are not in B. Therefore, $A - B = \varnothing$.

Second, we prove that if $A - B = \varnothing$, then $A \cap B = A$. Suppose that $A - B = \varnothing$. This implies that A is entirely composed of common elements with B; in other words, A is the set of their common elements. The operation $A \cap B$ gives us every element in common between A and B. Because $A - B = \varnothing$, we see that $A \cap B = A$ itself.

Finally, we prove that if $A \cap B = A$, then $A \subseteq B$. Suppose that $A \cap B = A$. This means the entirety of set A is composed of common elements between A and B. It follows that every element of A is also an element of B. By definition of subsets, this means that $A \subseteq B$.

We have proven that if (a) then (b), if (b) then (c), and if (c) then (a). This cycle of implications shows that these three statements are logically equivalent. □

3. Example from Problem Set 5 (Week 6)

Graded by: Mac

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This proof shows how I applied the division principle to a geometric problem which, at first glance, doesn't seem to be at all related.

Problem Statement: Select any five points on a square whose side-length is one unit. Show that at least two of these points are within $\frac{\sqrt{2}}{2}$ units of each other.

Proof. Divide the square into 4 identical square regions. The maximum distance between any two points within the same mini-square is the length of the diagonal. Since the sides of each mini-square are $\frac{1}{2}$ unit long, the length of the diagonal is

$$\sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}.$$

Therefore, within each mini-square, the maximum distance between points is $\frac{\sqrt{2}}{2}$.

Let us populate the large square with 5 random points. By the division principle, at least one of the 4 mini-squares contain $\lceil \frac{5}{4} \rceil = 2$ or more points.

As such, at least 2 of the 5 random points will have a maximum distance of $\frac{\sqrt{2}}{2}$ units between them. □

4. Example from Problem Set 9 (Week 10)

Graded by: Jordan

Initial Grade: 3/4

Explanation for Revision: My initial mistake was that I only accounted for one base case, when I should've had two consecutive base cases because I used two "steps" in the induction. I have since revised my proof to address this.

Other Information: This exercise is an example of working with Fibonacci numbers, which in turn requires induction with multiple base cases.

Problem Statement: Prove that the number of n-digit binary strings that have no consecutive 1s is the Fibonacci number F_{n+2}.

For example, when $n = 2$, there are three such: $00, 10, 01$.

Proof. We will induct on the size n of the binary strings.

Base Case: Let $n = 1$. There are two 1-digit binary string without consecutive 1s (and only two 1-digit binary strings in general). The term F_{1+2} is 2, matching our answer. Therefore the theorem holds for our first base case.

Now, consider $n = 2$. There are three 1-digit binary strings without consecutive 1s: 00, 10, and 01. The term F_{2+2} is 3, matching our answer. As such, the theorem also holds for our second base case.

Inductive Hypothesis: Assume the theorem holds for all k-digit binary strings where $k \geq 1$; that is, the number of k-digit binary strings without consecutive 1s is F_{k+2}. We want to show that the number of $k + 1$ digit binary strings without consecutive 1s is $F_{(k+1)+2} = F_{k+3}$.

Induction: We want to form $k+1$ digit binary strings without consecutive 1s. These strings take two forms.

Case 1: The string begins with a 0. In this case, we can create a valid $k+1$ digit string by simply adding a 0 to the start of a valid k-digit string. Since there are F_{k+2} valid k-digit strings, there are also F_{k+2} different valid $k+1$ digit strings starting with a 0.

Case 2: The string begins with a 1. In order to avoid consecutive 1s, it follows that the second digit must be a 0. In this case, we can create such a string by adding a 10 to the start of a valid $k - 1$ digit string. Since there are F_{k+1} valid $k - 1$ digit strings, there are also F_{k+1} different valid $k + 1$ digit strings starting with a 10.

Since these two cases perfectly partition all valid $k + 1$ digit binary strings, the total number of $k + 1$ digit binary strings without consecutive 1s is

$$F_{k+2} + F_{k+1} = F_{k+3} = F_{(k+1)+2}.$$

We have shown that if the number of k digit binary strings without consecutive 1s is F_{k+2}, then the number of $k+1$ digit strings without consecutive 1s is F_{k+3}. Therefore, by induction, the number of n-digit binary strings that have no consecutive 1s is F_{n+2}. $\square$

5. Example from Problem Set 10 (Week 11)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This problem involves exploring multiple cases to cover all possible scenarios for a problem (relations) that is more abstract than other applications of casework.

Problem Statement: Suppose that R is an equivalence relation on a set A, with four equivalence classes. How many different equivalence relations S on A are there for which $R \subseteq S$?

Proof. By definition, the equivalence classes associated with R form a partition of A. The elements of R can be constructed by forming every possible pair between elements within each class. Thus, R consists of pairs formed within each "section" of A that the four equivalence classes represent.

In order for R to be a subset of an equivalence relation S, it must be that every ordered pair $(x, y) \in R$ must also be an element of S. Furthermore, it is possible that there are ordered pairs in S that are not in R.

Since S is also an equivalence relation, its elements can be constructed in the same way as the elements of R. Its equivalence classes will also partition A. In order to preserve all the ordered pairs in R, the equivalence classes of S must keep together all elements that were equivalent under R.

In other words, the equivalence classes of S must be unions of the 4 equivalence classes of R.

Case 1: 1 equivalence class. If S has only 1 equivalence class, then by definition, $S = A \times A$. Thus, there is only one relation S that has 1 equivalence class.

Case 2: 2 equivalence classes. If S has 2 equivalence classes, then either each class combines 2 of R's classes, or one class is the same as one of 4's classes and the other combines the remaining 3. In the first sub-case, there are $\frac{\binom{4}{2}}{2} = 3$ ways to group the classes. (We divide by 2 to account for overcounting, since each counted 2-subset also creates another 2-subset with the remaining elements that is later double-counted.)

In the second sub-case, there are $\binom{4}{3} = 4$ ways to choose which three of R's classes will be combined. Thus, there are 7 ways for S to have 2 equivalence classes and have R as a subset.

Case 3: 3 equivalence classes. If S has 3 equivalence classes, then it must be that 2 of them are the same as 2 of R's classes, and the third combines the remaining 2 classes of R. There are $\binom{4}{2} = 6$ ways to choose which of the 2 classes of R will be combined in S.

Case 4: 4 equivalence classes. If S has 4 equivalence classes, and has R as a subset, then its classes must be equivalent to R's, making the two relations equivalent too. So, there is only 1 way for S to have 4 equivalence classes.

In total, there are $1 + 7 + 6 + 1 = 15$ different equivalence relations S on A such that $R \subseteq S$. $\square$

6. Example from Problem Set 1 (Week 2)

Graded by: Jordan

Initial Grade: 4/4

Explanation for Revision: N/A

Other Information: This problem shows how I interpret set notation and Cartesian product, and how I am able to rewrite sets in different, expanded forms.

Problem Statement: Of the exercises in Section 1.2 A, choose four from numbers 3 through 8 to write solutions for, explaining your reasoning.

A3) Write out the indicated sets by listing their elements between braces:
$\{x \in \mathbb{R} \colon x^2 = 2\} \times \{a, c, e\}$

Proof. The only two elements of the set $\{x \in \mathbb{R} \colon x^2 = 2\}$ are $-\sqrt{2}$ and $\sqrt{2}$, meaning it can also be written as $\{-\sqrt{2}, \sqrt{2}\}$.

The Cartesian product of $\{-\sqrt{2}, \sqrt{2}\}$ and $\{a, c, e\}$ is the set containing all pairings of elements from the two sets. This results in the set

$$\{(-\sqrt{2}, a), (-\sqrt{2}, c), (-\sqrt{2}, e), (\sqrt{2}, a), (\sqrt{2}, c), (\sqrt{2}, e)\}.$$

□

A4) Write out the indicated sets by listing their elements between braces:
$\{n \in \mathbb{Z} \colon 2 < n < 5\} \times \{n \in \mathbb{Z} \colon |n| = 5\}$.

Proof. The only integers that are between 2 and 5 (noninclusive) are 3 and 4. Thus, the set $\{n \in \mathbb{Z} \colon 2 < n < 5\}$ can be written as $\{3, 4\}$. Furthermore, the set $\{n \in \mathbb{Z} \colon |n| = 5\}$ contains only the elements -5 and 5, and can be rewritten as $\{-5, 5\}$.

The Cartesian product of these two sets results in

$$\{(3, -5), (4, -5), (3, 5), (4, 5)\}.$$

□

A6) Write out the indicated sets by listing their elements between braces:
$\{x \in \mathbb{R} \colon x^2 = x\} \times \{x \in \mathbb{N} \colon x^2 = x\}$

Proof. The first set, $\{x \in \mathbb{R} \colon x^2 = x\}$, can be written as $\{0, 1\}$. The second set has the same "rule" as the first, but all elements must be natural numbers. While 0 fulfills the requirements of the first set, it is not a natural number, and therefore not a member of the second set. Thus, the second set only contains element 1.

The Cartesian product of these two sets is

$$\{(0,1),(1,1)\}.$$

□

A7) Write out the indicated sets by listing their elements between braces:
$\{\varnothing\} \times \{0, \varnothing\} \times \{0, 1\}$

Proof. In the ordered triples that result from this Cartesian product, the first element will always be $\varnothing$, since that is the only element in the first given set $\{\varnothing\}$. The second element can be either 0 or $\varnothing$, and the third can be either 0 or 1.

Thus, the Cartesian product will be:

$$\{(\varnothing,0,0),(\varnothing,\varnothing,0),(\varnothing,0,1),(\varnothing,\varnothing,1)\}.$$

□

7. Example from Problem Set 3 (Week 4)

Graded by: Mac

Initial Grade: 1/4

Explanation for Revision: I initially interpreted the problem statement wrong, and converted the sentences into non-grammatically correct versions. I have since revised my answers to reflect expectations!

Other Information: This problem shows how I draw connections between ordinary English statements and the logic statements/structures we practiced in class.

Problem Statement: Without changing the meaning, convert the following sentences into sentences of the form "If P, then Q" or "P if and only if Q" as appropriate.

(a) Whenever a surface has only one side, it is non-orientable.

Solution. If a surface has only one side, then it is non-orientable. □

(b) A geometric series with ratio r converges if $|r| < 1$.

Solution. If a geometric series has ratio r where $|r| < 1$, then it converges. □

(c) People will generally accept facts as truth only if the facts agree with what they already believe. (Andy Rooney)

Solution. If people generally accept facts as truth, then the facts agree with what they already believe. □

(d) If $xy = 0$, then $x = 0$ or $y = 0$, and conversely.

Solution. $xy = 0$ if and only if $x = 0 \vee y = 0$. □

(e) If $a \in \mathbb{Q}$, then $5a \in \mathbb{Q}$, and if $5a \in \mathbb{Q}$, then $a \in \mathbb{Q}$.

Solution. $a \in Q$ if and only if $5a \in Q$. □

(f) For an odd prime number p to be a sum of two square integers, it is necessary and sufficient for p to leave a remainder of 1 when divided by 4.

Solution. The odd prime number p is a sum of two square integers if and only if it has a remainder of 1 when divided by 4. □

8. Example from Problem Set 5 (Week 6)

Graded by: N/A

Initial Grade: N/A

Explanation for Revision: N/A

Other Information: This exercise shows how I use concepts related to multisets in order to solve combinatorial problems.

Problem Statement:

(a) A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, and one white ball. You reach in and grab 15 balls. How many different outcomes are possible?

Solution. Let S be the set of all ball colors: red, blue, green, and white. The number of possible outcomes can be represented as the number of cardinality-15 multisets of S.

There are more balls in the red, blue, and green categories than there are elements in the multiset, meaning we don't need to worry about running out of them. However, there is only one white ball. The outcomes can be split into two cases:

Case 1: The white ball is selected. There are 14 balls remaining to be drawn, and three possible color categories (red, blue, and green). As a stars and bars problem, this means there are 14 stars and 2 bars. Therefore, there are $\binom{14+2}{2} = \binom{16}{2}$ different outcomes in this case.

Case 2: The white ball is not selected. There are 15 balls to be drawn and, again, three possible color categories. This means there are 15 stars and 2 bars, and that there are $\binom{15+2}{2} = \binom{17}{2}$ outcomes in this case.

Adding together all the outcomes of both cases, we see that there are $\binom{16}{2} + \binom{17}{2}$ total outcomes of grabbing 15 balls. $\square$

(b) A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, one white ball, and one black ball. You reach in and grab 20 balls. How many different outcomes are possible?

Solution. Let S be the set of all ball colors: red, blue, green, white, and black. The number of possible outcomes can be represented as the number of cardinality-20 multisets of S.

Again, since there are the same amount of balls in the red, blue, and green categories individually as there are elements in the multiset, we don't need to worry about them running out. There are only one white ball and one black ball, however. The outcomes can be split into three cases:

Case 1: Neither are selected. There are 20 balls remaining to be drawn, and three color categories. As a stars and bars problem, this results in 20 stars and 2 bars. Thus there are $\binom{20+2}{2} = \binom{22}{2}$ different outcomes of this case.

Case 2: One of either the white or black ball is selected, while the other isn't. There are 19 balls remaining to be drawn, and three color categories. This results in 19 stars and 2 bars. Thus there are $\binom{19+2}{2} = \binom{21}{2}$ different outcomes of this case. Since this case can happen in two ways (the white ball is drawn or the black ball is drawn), we multiply this number by 2 to get $2\binom{21}{2}$.

Case 3: Both are selected. There are 18 balls remaining to be drawn, and three color categories. This results in 18 stars and 2 bars. Thus there are $\binom{18+2}{2} = \binom{20}{2}$ different outcomes of this case.

Adding together all the possible outcomes, we see there are $\binom{22}{2} + 2\binom{21}{2} + \binom{20}{2}$ total different outcomes of grabbing 20 balls. $\square$

Chapter 3

Exam Reflection

Prompt: *Reflecting on your three exam experiences in Math 1000, what are your biggest takeaways as you think about your next math course? Were you successful in implementing specific study changes to improve your exam experience? What, in your opinion, needs continued improvement?*

One big takeaway from my three exams is how well I fare with more versus less intuitive proofs, as well as proofs that require more out-of-the-box thinking versus those that rely more on abstract understanding.

Proofs such as the combinatorial problems on Exam 1 and the problems regarding relations and fucntions on Exam 2 came very easily to me, because they were more intuitive. Those sorts of problems can only be solved with an internalized understanding of the concepts: once I achieved that, I didn't even need to study the proofs—I could reproduce them on my own easily enough. On the other hand, I needed to spend time reviewing proofs such as the epsilon-delta in Exam 3 and the perfect numbers proof in Exam 1. I struggled with the epsilon-delta proofs in particular, although I did eventually grow to (somewhat) understand and appreciate them. I actually did the best on Exam 3, which contained more proofs I was more uncomfortable with!

I now have an incredulous sort of admiration for analysis techniques and epsilon-delta. A lot of them feel like the writers are just saying things and making assumptions to manipulate the problem environment to lead to the truth, which is very impressive to see, but extremely unintuitive, hard to reproduce, and a little silly at times. I never would've thought of the method of notating decimal digits to prove $/mathbbR$ is uncountable, but I can appreciate the ingenuity of it.

All this is to say that I realize my strengths lie in solving problems that rely on baseline logic, intuition, and a deep internalized understanding of concepts. My weaknesses are problems that require unexpected assumptions, variables, and frameworks. In the future, I'll know I can rest easy when it comes to logic and intuition, and I will have to learn how to think outside the box more creatively.

As for specific study methods and changes in them, I found that I actually studied less and less for each exam. This was definitely partially because of the relaxed exam policy (thanks Jordan!) but also because I realized I didn't necessarily need to pour over the textbook and grind out problems in order to succeed. For the first exam, I wrote down every definition and proof, and reviewed each one several times. I also looked to the textbook for practice applying the concepts, because I was slightly worried for "part 3" of the exam.

For the next two exams, however, I realized it sufficed to write down every definition and proof once (with references if needed), then reproduce each one once (or twice if needed!) without notes. I didn't bother to look for practice problems, although I did also review relevant theorems/definitions that weren't included in the exam information sheet. I found that this method worked just as well and took much less time and brainpower—as long as I'm able to reproduce something without reference, that means that I understand it enough to do it on an exam, and that I'll be able to apply it to new problems too! In future classes, though their exams will almost certainly be worth more grade-wise than in Math 1000, I will continue using this study method and hope that it keeps working.

Chapter 4

Proof Analyses

4.1 Theorem 8.2 - Perfect Numbers

Theorem: If $A = \{2^{n-1}(2^n - 1) : n \in \mathbb{N} \text{ and } 2^n - 1 \text{ is prime}\}$ and $E = \{p \in \mathbb{N} : p \text{ is perfect and even}\}$, then $A = E$.

Definition: A perfect number is a number $p \in \mathbb{N}$ that is equal to the sum of its divisors, excluding itself.

Lemma 1: If $A = \{2^{n-1}(2^n - 1) : n \in \mathbb{N} \text{ and } 2^n - 1 \text{ is prime}\}$ and $P = \{p \in \mathbb{N} : p \text{ is perfect}\}$, then $A \subseteq P$.

Lemma 2: $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$

4.1.1 Toulmin Analysis

Link to my Toulmin Analysis

Proof. To show that $A = E$, we need to show $A \subseteq E$ and $E \subseteq A$.

First we will show that $A \subseteq E$. Suppose $p \in A$. This means p is even, because the definition of A shows that every element of A is a multiple of a power of 2. Also, p is a perfect number because Theorem 8.1 states that every element of A is also an element of P, hence perfect. Thus p is an even perfect number, so $p \in E$. Therefore $A \subseteq E$.

Next we show that $E \subseteq A$. Suppose $p \in E$. This means p is an even perfect number. Write the prime factorization of p as $p = 2^k 3^{n_1} 5^{n_2} 7^{n_3} \ldots$, where some of the powers $n_1, n_2, n_3 \ldots$ may be zero. But, as p is even, the power k must be greater than zero. It follows $p = 2^k q$ for some positive integer k and an odd integer q. Now, our aim is to show that $p \in A$, which means we must show p has form $p = 2^{n-1}(2^n - 1)$. To get our current $p = 2^k q$ closer to this form, let $n = k + 1$, so we now have

$$p = 2^{n-1} q. \qquad (8.3)$$

List the positive divisors of q as $d_1, d_2, d_3, \ldots, d_m$. (Where $d_1 = 1$ and $d_m = q$.) Then the divisors of p are:

$$\begin{array}{ccccc}
2^0 d_1 & 2^0 d_2 & 2^0 d_3 & \ldots & 2^0 d_m \\
2^1 d_1 & 2^1 d_2 & 2^1 d_3 & \ldots & 2^1 d_m \\
2^2 d_1 & 2^2 d_2 & 2^2 d_3 & \ldots & 2^2 d_m \\
2^3 d_1 & 2^3 d_2 & 2^3 d_3 & \ldots & 2^3 d_m \\
\vdots & \vdots & \vdots & & \vdots \\
2^{n-1} d_1 & 2^{n-1} d_2 & 2^{n-1} d_3 & \ldots & 2^{n-1} d_m.
\end{array}$$

Since p is perfect, these divisors add up to $2p$. By Equation (8.3), their sum is $2p = 2(2^{n-1}q) = 2^n q$. Adding the divisors column-by-column, we get

$$\sum_{k=0}^{n-1} 2^k d_1 + \sum_{k=0}^{n-1} 2^k d_2 + \sum_{k=0}^{n-1} 2^k d_3 + \cdots + \sum_{k=0}^{n-1} 2^k d_m = 2^n q.$$

Applying Equation (8.1), this becomes

$$\begin{aligned}
(2^n - 1)d_1 + (2^n - 1)d_2 + (2^n - 1)d_3 + \cdots + (2^n - 1)d_m &= 2^n q \\
(2^n - 1)(d_1 + d_2 + d_3 + \cdots + d_m) &= 2^n q \\
d_1 + d_2 + d_3 + \cdots + d_m &= \frac{2^n q}{2^n - 1},
\end{aligned}$$

so that

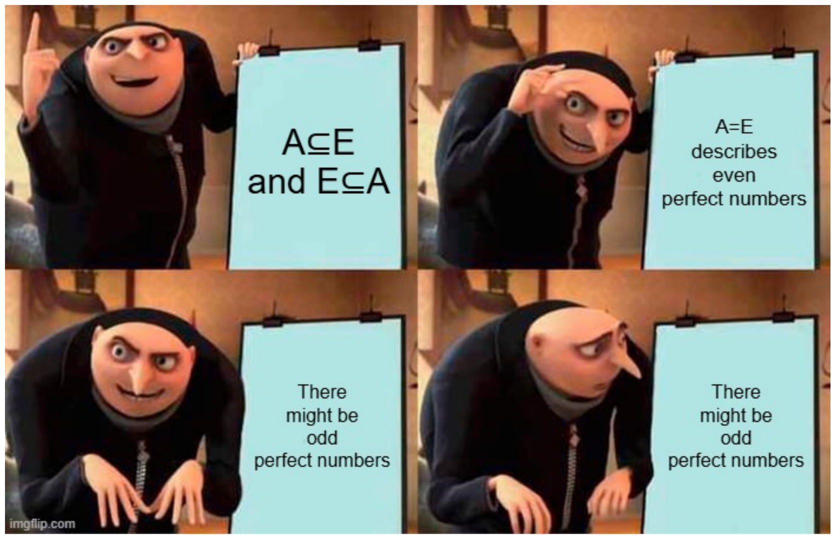
$$d_1 + d_2 + d_3 + \cdots + d_m = \frac{(2^n - 1 + 1)q}{2^n - 1} = \frac{(2^n - 1)q + q}{2^n - 1} = q + \frac{q}{2^n - 1}.$$

From this we see that $\frac{q}{2^n-1}$ is an integer. It follows that both q and $\frac{q}{2^n-1}$ are positive divisors of q. Since their sum equals the sum of *all* positive divisors of q, it follows that q has only two positive divisors, q and $\frac{q}{2^n-1}$. Since one of its divisors must be 1, it must be that $\frac{q}{2^n-1} = 1$, which means $q = 2^n - 1$. Now a number with just two positive divisors is prime, so $q = 2^n - 1$ is prime. Plugging this into Equation (8.3) gives $p = 2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. This means $p \in A$, by definition of A. We have now shown that $p \in E$ implies $p \in A$, so $E \subseteq A$.

Since $A \subseteq E$ and $E \subseteq A$, it follows that $A = E$. ■

Figure 4.1: Taken from Hammack: The Book of Proof

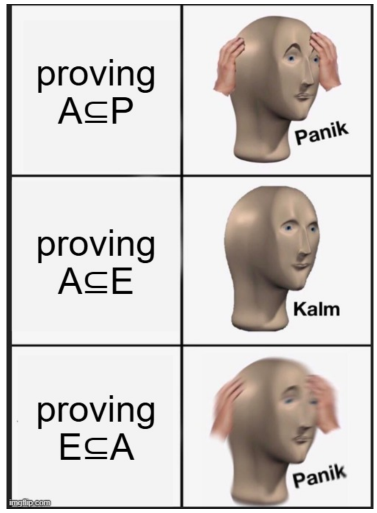
4.1.2 Meme



The thing that struck me most about Theorem 8.2 is the fact that, while its proof is very neat and has a satisfactory line of reasoning, the theorem itself is not as satisfyingly "complete" as some of the others I've seen. It's ingenious to relate perfect numbers and numbers of the form $2^{n-1}(2^n - 1)$, as it makes the concept of perfect numbers a little more grounded in real, visualizable numbers. However, this isn't a complete equivalence, because perfect numbers are so mysterious that no one knows whether there exist any odd ones. We can't say that all perfect numbers take the form $2^{n-1}(2^n - 1)$; we can only say this is true for even perfect numbers.

As such, while this was a very cool theorem and proof, one of my main takeaways is that even such an ingenious truth leaves much room for debate and uncertainty. This meme expresses how it feels like this theorem is an elaborate step towards understanding perfect numbers a little more, and yet, our plan to completely and satisfactorily comprehend perfect numbers is foiled by the fact that we have no idea whether there are odd ones.

Here's a bonus meme to show my experience with the actual proof-writing process for this theorem:



4.1.3 My Proof

Proof. We want to show that $A \subseteq E$ and $E \subseteq A$ in order to prove that $A = E$. By our first lemma, because all elements of A are even (due to the 2^{n-1} term), it follows that $A \subseteq P$ implies $A \subseteq E$.

Now, consider an element $p \in E$. Because of the fundamental theorem of arithmetic, p can be expressed as $p = 2^{n-1} \cdot q$ where $n \geq 2$ and q is odd. We wish to show that p is an element of A as well, meaning that q must be prime and equal to $2^n - 1$.

Consider the divisors of p. Let $\{d_1, d_2, ..., d_m\}$ be the set of all divisors of q, where $d_1 = 1$ and $d_m = q$. As such, the divisors of p are:

$$\begin{array}{cccc} 2^0 d_1 & 2^0 d_2 & ... & 2^0 d_m \\ 2^1 d_1 & 2^1 d_2 & ... & 2^1 d_m \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 2^{n-1} d_1 & 2^{n-1} d_2 & ... & 2^{n-1} d_m \end{array}$$

The sum of all divisors of p is

$$\sum_{i=0}^{n-1} 2^i d_1 + \sum_{i=0}^{n-1} 2^i d_2 + ... + \sum_{i=0}^{n-1} 2^i d_m \,.$$

Using our second lemma, we can rewrite this expression as

$$= (2^n - 1)d_1 + (2^n - 1)d_2 + ... + (2^n - 1)d_m = (2^n - 1) \sum_{i=1}^{m} d_i \,.$$

Let us return to the definition of p as an even perfect number. By definition, this means that the sum of all its divisors (including p itself) will equal $2p$. Furthermore, because $p = 2^{n-1}q$, it follows that $2p = 2(2^{n-1}q) = 2^n q$.

As such, we see that $(2^n - 1)\sum_{i=1}^{m} d_i = 2p = 2^n q$. So,

$$\sum_{i=1}^{m} d_i = \frac{2^n q}{(2^n - 1)} = \frac{(2^n - 1 + 1)q}{(2^n - 1)} = q + \frac{q}{2^n - 1}\,.$$

The left-side expression $\sum_{i=1}^{m} d_i$ represents the sum of all divisors of q. Since all divisors are integers, this sum must be an integer too. As q is also an integer by definition, we see that $\frac{q}{2^n-1}$ must also be an integer in order to ensure the right-side expression $q + \frac{q}{2^n-1}$ evaluates to an integer too.

By definition, q is a divisor of itself. Additionally, the integer $\frac{q}{2^n-1}$ is also a divisor of q, because there exists a number $2^n - 1$ that, when multiplied with $\frac{q}{2^n-1}$, results in q. Because $q + \frac{q}{2^n-1}$ is the sum of two divisors of q and is also equal to $\sum_{i=1}^{m} d_i$, the sum of *all* divisors of q, it follows that q and $\frac{q}{2^n-1}$ are the *only* divisors of q.

Prime numbers are defined as numbers whose only divisors are itself and 1; since one of the only two divisors of q is itself, it follows that q is prime. Furthermore, this means the other divisor $\frac{q}{2^n-1}$ must equal 1. As such, $q = 2^n - 1$.

We have shown that for an arbitrary element $p \in E$, it can be written as $p = 2^{n-1}q$ where $q = 2^n - 1$ and is prime, meaning that $p = 2^{n-1}(2^n - 1)$ is an element of A as well. This means $E \subseteq A$, and ultimately, $A = E$. □

4.2 Theorem 11.2 - Equivalence Class Partition

Theorem: Suppose R is an equivalence relation on a set A. Then the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

Lemma: Given equivalence relation R on set A, and $a, b \in A$, $[a] = [b]$ if and only if aRb.

4.2.1 Toulmin Analysis

Link to my Toulmin Analysis

Proof. **To show that $\{[a] : a \in A\}$ is a partition of A we need to show two things: We need to show that the union of all the sets $[a]$ equals A, and we need to show that if $[a] \neq [b]$, then $[a] \cap [b] = \emptyset$.**

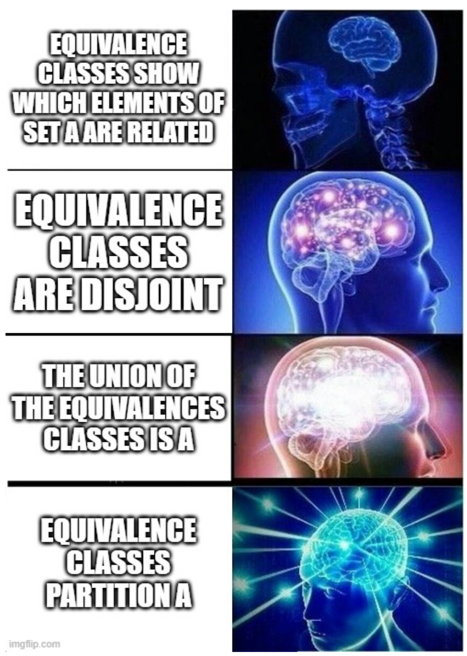
Notationally, the union of all the sets $[a]$ is $\bigcup_{a \in A}[a]$, so we need to prove $\bigcup_{a \in A}[a] = A$. Suppose $x \in \bigcup_{a \in A}[a]$. This means $x \in [a]$ for some $a \in A$. Since $[a] \subseteq A$, it then follows that $x \in A$. Thus $\bigcup_{a \in A}[a] \subseteq A$. On the other hand, suppose $x \in A$. As $x \in [x]$, we know $x \in [a]$ for some $a \in A$ (namely $a = x$). Therefore $x \in \bigcup_{a \in A}[a]$, and this shows $A \subseteq \bigcup_{a \in A}[a]$. Since $\bigcup_{a \in A}[a] \subseteq A$ and $A \subseteq \bigcup_{a \in A}[a]$, it follows that $\bigcup_{a \in A}[a] = A$.

Next we need to show that if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$. Let's use contrapositive proof. Suppose it's not the case that $[a] \cap [b] = \emptyset$, so there is some element c with $c \in [a] \cap [b]$. Thus $c \in [a]$ and $c \in [b]$. Now, $c \in [a]$ means cRa, and then aRc since R is symmetric. Also $c \in [b]$ means cRb. Now we have aRc and cRb, so aRb (because R is transitive). By Theorem 11.1, aRb implies $[a] = [b]$. Thus $[a] \neq [b]$ is not true.

We've now shown that the union of all the equivalence classes is A, and the intersection of two different equivalence classes is $\emptyset$. Therefore the set of equivalence classes is a partition of A. ∎

Figure 4.2: Taken from Hammack: The Book of Proof

4.2.2 Meme



The significance of Theorem 11.2 is that it relates two seemingly unrelated topics: equivalence classes, which seem very specifically tied to relations at first glance, and partitions, which are a more general set topic. Partitions are useful for breaking down sets into subsets and cases, and knowing that equivalence classes are partitions means that equivalence relations can be used for this purpose. My meme emphasizes how building up facts about equivalence classes (from the basic definition to the fact that their union is the set A itself) creates this very cool and applicable result!

4.2.3 My Proof

Proof. A partition of a set A is defined as a set of subsets of A such that their union is A and they are each disjoint. As such, we want to show that:

1) the union of all equivalence classes, denoted $\bigcup_{a\in A}[a]$, equals A, and
2) all different equivalence classes are disjoint, meaning that if $[a] \neq [b]$ for $a, b \in A$, then $[a] \cap [b] = \varnothing$.

1) First, in order to show that $\bigcup_{a\in A}[a] = A$, we must show that $\bigcup_{a\in A}[a] \subseteq A$ and $A \subseteq \bigcup_{a\in A}[a]$. Consider an arbitrary $x \in \bigcup_{a\in A}[a]$. This means that $x \in [a]$ for some $a \in A$. By definition of equivalence classes, $[a] \subseteq A$, and by definition of subsets, it follows that $x \in A$. Because an arbitrary element of $\bigcup_{a\in A}[a]$ will also be an element of A, we see that $\bigcup_{a\in A}[a] \subseteq A$.

Next, consider an arbitrary $y \in A$. Because the relation R is equivalent and therefore reflexive, we know that yRy and thus $y \in [y]$. As $y \in A$, it follows that $[y]$ is one of the sets in the union $\bigcup_{a\in A}[a]$; in other words, $[y] \in \bigcup_{a\in A}[a]$. It follows that $y \in \bigcup_{a\in A}[a]$. Because an arbitrary element of A will also be an element of $\bigcup_{a\in A}[a]$, we see that $A \subseteq \bigcup_{a\in A}[a]$.

We have shown that $\bigcup_{a\in A}[a] \subseteq A$ and $A \subseteq \bigcup_{a\in A}[a]$; therefore, $\bigcup_{a\in A}[a] = A$.

2) Now, we want to show that if $[a] \neq [b]$ for $a, b \in A$, then $[a] \cap [b] = \varnothing$.

For the sake of contrapositive, suppose that $[a] \cap [b] \neq \varnothing$. Then, there exists a $z \in [a]$ such that $z \in [b]$ as well. In other words, zRa and zRb.

Because R is equivalent and therefore symmetric, we see that aRz. Furthermore, because R is also transitive, the fact that aRz and zRb implies that aRb. By our lemma, this means $[a] = [b]$. Ultimately, this proves that if $[a] \neq [b]$ then $[a] \cap [b] = \varnothing$, meaning that all equivalence classes are disjoint.

We have proven that the union of all equivalence classes equals A, and that all classes are disjoint. By definition of a partition, this means the set of equivalence classes of A forms a partition A. □

4.3 Euler's Formula for Planar Graphs

Theorem: If G is a connected planar graph with n vertices, q edges, and r regions, then

$$n - q + r = 2.$$

Background Information:

- A *planar graph* is a graph that can be drawn/embedded in a plane such that pairs of edges intersect only at vertices.
- A *region* or *face* of a planar representation G is an area bounded by a cycle of edges. The external area outside G is also a region.
- A *connected graph* is a graph where there exists a path between every pair of vertices.
- A *tree* is a connected graph that does not contain cycles (closed paths that connect vertices to themselves).

Lemma: The number of edges in a tree is $n-1$, where n is the number of vertices.

4.3.1 Toulmin Analysis

Link to my Toulmin Analysis

Proof. We induct on q, the number of edges. If $q = 0$, then G must be K_1, a graph with 1 vertex and 1 region. The result holds in this case. Assume that the result is true for all connected planar graphs with fewer than q edges, and assume that G has q edges.

Case 1. Suppose G is a tree. We know from our work with trees that $q = n-1$; and of course, $r = 1$, since a planar representation of a tree has only one region. Thus $n - q + r = n - (n-1) + 1 = 2$, and the result holds.

Case 2. Suppose G is not a tree. Let C be a cycle in G, let e be an edge of C, and consider the graph $G - e$. Compared to G, this graph has the same number of vertices, one edge fewer, and one region fewer, since removing e coalesces two regions in G into one in $G - e$. Thus the induction hypothesis applies, and in $G - e$,

$$n - (q-1) + (r-1) = 2,$$

implying that $n - q + r = 2$.

The result holds in both cases, and the induction is complete. $\square$

Figure 4.3: Taken from Harris, Hirst, and Mossinghoff: Combinatorics and Graph Theory

4.3.2 Meme

This meme shows just how important Euler's formula is for anything related to planar graphs. When I was learning about planarity, I truly didn't go a single day without using the formula in some way! Countless other ideas, from simple corollaries to complex algorithms for planarity testing, use Euler's formula as a foundation. It's because Euler's formula makes use of all the basic properties of any graph: the vertices, edges, and faces. As such, it's easy to apply to any graph and in many contexts.

4.3.3 My Proof

Proof. In order to show that the equation $n - q + r = 2$ holds for all possible connected planar graphs G, we will induct on q, the number of edges.

Base Case: Let $q = 0$ for some G. Because we know that G is connected, it must be that $n = 1$: if there were more than 1 vertex, without any edges to join the vertices, G would not be connected. It also follows that there is only one region, so $r = 1$. We see that $n - q + r = 1 - 0 + 1 = 2$. Thus, the base case holds true.

Inductive Hypothesis: Assume that $n - q + r = 2$ is true for all connected planar graphs G with $q = k - 1 \geq 0$ edges. We want to show that this implies $n - q + r = 2$ is true for G with $q = k$ as well.

Induction: There are two cases for the type of graph that G is.

Case 1: G is a tree. By the lemma, G has $q = n - 1$ edges. Additionally, since G is a tree and has no cycles, the only region is the external region. Thus, $r = 1$. We see that

$$n - q + r = n - (n - 1) + 1) = 2$$

and the equation holds true.

Case 2: G is not a tree. This means that there exists at least one cycle C in G. Let e be an edge of C, and consider the graph $G' = G - e$ formed by deleting e from the original graph (but preserving its endpoints).

This results in a graph with the same number of vertices n, but one less edge. We assumed that G had $q = k$, so G' will have $q = k - 1$. The graph G' will also have one less region due to the cycle being disrupted, meaning the number of regions is $r - 1$.

We can use our inductive hypothesis because $q = k - 1$. So, it is true that $n - q + (r - 1) = 2$, or $n - (k - 1) + (r - 1) = 2$. Rearranging, we see that $n - k + r = 2$. The original graph G has $q = k$, n vertices, and r regions, meaning the equation applies to a non-tree graph G.

As such, we see that in all cases of G, the equation applies. Therefore, $n - q + r = 2$ is true for all connected planar graphs. $\square$