Introduction to Linear Algebra

Week 1 Assignment

Part I: Mathematical Solutions

1. Operations on Vectors
   This question involves component-wise operations on vectors and calculations of vector norms.
   (a) Given vectors $\mathbf{v} = [2, 4, 6, 8, 10]$ and $\mathbf{u} = [1, 3, 5, 7, 9]$.
   - Component-wise Division $\dfrac{\mathbf{v}}{\mathbf{u}}$
     This requires dividing each component of $\mathbf{v}$ by the corresponding component of $\mathbf{u}$.
     $$\begin{aligned}\mathbf{v}/\mathbf{u} &= \left[\frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \frac{10}{9}\right] \\ &= [2, 1.33, 1.2, 1.14, 1.11]\end{aligned}$$
   - Vector Operations $3\mathbf{v} - 2\mathbf{u} + \frac{1}{2}\mathbf{v}$
     We calculate the operation using scalar multiplication and addition:
     $$\begin{aligned}3\mathbf{v} &= [6, 12, 18, 24, 30] \\ 2\mathbf{u} &= [2, 6, 10, 14, 18] \\ \frac{1}{2}\mathbf{v} &= [1, 2, 3, 4, 5] \\ 3\mathbf{v} - 2\mathbf{u} + \frac{1}{2}\mathbf{v} &= [6+1, 12+2, 18+3, 24+4, 30+5] - [2, 6, 10, 14, 18] \\ &= [5, 8, 11, 14, 17]\end{aligned}$$

   (b) Euclidean Norm of a Vector
   Given $\mathbf{v} = [8, 0, 5, 2, 1]$, the Euclidean norm $\|\mathbf{v}\|$ is calculated as follows:
   $$\|\mathbf{v}\| = \sqrt{8^2 + 0^2 + 5^2 + 2^2 + 1^2} = \sqrt{64 + 0 + 25 + 4 + 1} = \sqrt{94} = 9.69535971$$

2. Determinant of a Matrix
   Given the matrix
   $$A = \begin{pmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{pmatrix}$$
   We need to calculate the determinant of A. Using the formula for the determinant of a 3x3 matrix:

   For a matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $\quad \det(A) = a(e \cdot i - f \cdot h) - b(d \cdot i - f \cdot g) + c(d \cdot h - e \cdot g)$

Plugging in the values:

$$
\begin{aligned}
\det(A) &= 2(0 \cdot 0 - 6 \cdot 5) - 3(-4 \cdot 0 - 6 \cdot 1) + 7(-4 \cdot 5 - 0 \cdot 1) \\
&= 2(-30) - 3(0 - 6) + 7(-20) \\
&= -60 + 18 - 140 \\
&= -182
\end{aligned}
$$

3. (a) Eigenvalue and Eigenvector Calculation for Matrix B

Given the matrix:

$$
B = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}
$$

- Finding Eigenvalues
  To find the eigenvalues, we solve the characteristic equation:

  $$\det(B - \lambda I) = 0$$

  where I is the identity matrix.

  Characteristic Polynomial

  Set up $B - \lambda I$:

  $$
  B - \lambda I = \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}
  $$

  Calculate the determinant:

  $$
  \begin{aligned}
  \det(B - \lambda I) &= (3-\lambda)\left[(-\lambda)(3-\lambda) - 4\right] - 2\left[2(3-\lambda) - 8\right] + 4\left[4 + 2\lambda\right] \\
  &= -\lambda^3 + 6\lambda^2 + 5\lambda - 16
  \end{aligned}
  $$

  Solve the characteristic equation:

  $$
  \begin{aligned}
  -\lambda^3 + 6\lambda^2 + 5\lambda - 16 =& 0 \\
  -(\lambda + 1)(\lambda^2 - 7\lambda - 8) =& 0 \\
  (\lambda + 1)(\lambda + 1)(\lambda - 8) =& 0 \quad \text{I divided both sides of the equation by -1 first}
  \end{aligned}
  $$

  The solutions, found by using the cubic formula gives us:

  $$\lambda_1 = -1,\ \lambda_2 = -1,\ \lambda_3 = 8$$

- Finding Eigenvectors

  For $\lambda_1 = -1$

  Substitute $\lambda = -1$ into $B - \lambda I$:

$$B + I = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

$$B\mathbf{v} = \lambda\mathbf{v} \Rightarrow (\mathbf{B} - \lambda I)\cdot\mathbf{v} = 0 \quad \Rightarrow \left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right) \quad \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

  So we have a homogeneous system of linear equations, we solve by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right) \Rightarrow R_1/4 \to R_1 \quad \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right) \Rightarrow R_2 - 2\cdot R_1 \to R_2 \quad \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right) R_3 - 4\cdot R_1 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{1}$$

$$\left\{ x_1 + \frac{1}{2}x_2 + x_3 = 0 \right\} \tag{2}$$

We would now solve for the variables (x_1, x_2, x_3) in the equation of the system (2):

$$x_1 = -\frac{1}{2}\cdot x_2 - x_3 \tag{3}$$

$$x_2 = x_2 \tag{4}$$

$$x_3 = x_3 \tag{5}$$

General Solution: $X = \begin{pmatrix} -\frac{1}{2}\cdot x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$

The solution set: $\left[x_2 \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right]$

Let $x_2 = 1, x_3 = 0, \mathbf{v_1} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$; Let $x_2 = 0, x_3 = 1, \mathbf{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 8$

Substitute $\lambda = 8$ into $B - \lambda I$:

$$B - 8I = 0 \quad \Rightarrow \left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \quad \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$$

A homogeneous system of linear equations that can be solved by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \Rightarrow R_1/(-5) \to R_1 \quad \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) \Rightarrow R_2 - 2 \cdot R_1 \to R_2 \quad \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 4 & 2 & -5 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 4 & 2 & -5 & 0 \end{array}\right) R_3 - 4 \cdot R_1 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right) \Rightarrow R_2/\left(-\frac{36}{5}\right) \to R_2 \quad \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & \frac{18}{5} & -\frac{9}{5} & 0 \end{array}\right) R_3 - \left(\frac{18}{5}\right) \cdot R_2 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & -\frac{36}{5} & \frac{18}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & -\frac{4}{5} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow R_1 - \left(-\frac{2}{5}\right) R_2 \to R_1 \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{6}$$

$$\begin{cases} x_1 \qquad - x_3 & = 0 \\ \qquad x_2 - \frac{1}{2} \cdot x_3 & = 0 \end{cases} \tag{7}$$

We have to solve for x_1, x_2 and x_3 with the following change of subject:

- x_1 from the first equation of (7):

$$x_1 = x_3$$

- x_2 from the second equation of (7):

$$x_2 = \frac{1}{2} \cdot x_3$$

General Solution: $X = \begin{pmatrix} x_3 \\ \frac{1}{2} \cdot x_3 \\ x_3 \end{pmatrix}$

The solution set: $x_3 \cdot \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$

Let $x_3 = 1, \quad \mathbf{v_3} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$

Therefore, the eigenvectors and their corresponding eigenvalues of the matrix B:

- $\mathbf{v_1} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$, eigenvalue $\lambda_1 = -1$
- $\mathbf{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, eigenvalue $\lambda_2 = -1$
- $\mathbf{v_3} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$, eigenvalue $\lambda_3 = 8$

(b) To verify if vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of matrix D, we need to check if there exists a scalar λ such that $D\mathbf{v} = \lambda\mathbf{v}$ for each vector $\mathbf{v}$.

Verification for $\mathbf{v}_1 = [1, 1, 1]$

- Computing $D\mathbf{v}_1$:

$$D\mathbf{v}_1 = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \\ 1 \cdot 1 + 3 \cdot 1 + 1 \cdot 1 \\ 2 \cdot 1 + 1 \cdot 1 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$$

- Checking if $D\mathbf{v}_1 = \lambda\mathbf{v}_1$: We need a scalar λ such that $\begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

  Observing the components:

$$\begin{aligned} 6 = \lambda \cdot 1 &\implies \lambda = 6, \\ 5 = \lambda \cdot 1 &\implies \lambda = 5, \\ 6 = \lambda \cdot 1 &\implies \lambda = 6 \end{aligned}$$

  The equations are inconsistent as λ cannot simultaneously be 5 and 6. Therefore, $\mathbf{v}_2$ is **not** an eigenvector of D.

Verification for $\mathbf{v}_2 = [-1, 1, -1]$

- Compute $D\mathbf{v}_2$:

$$D\mathbf{v}_2 = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \cdot (-1) + 1 \cdot 1 + 2 \cdot (-1) \\ 1 \cdot (-1) + 3 \cdot 1 + 1 \cdot (-1) \\ 2 \cdot (-1) + 1 \cdot 1 + 3 \cdot (-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -4 \end{pmatrix}$$

- Check if $D\mathbf{v}_2 = \lambda \mathbf{v}_2$: We need a scalar λ such that $\begin{pmatrix} -4 \\ 1 \\ -4 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

  Observing the components:

$$\begin{aligned} -4 &= \lambda \cdot (-1) \implies \lambda = 4, \\ 1 &= \lambda \cdot 1 \implies \lambda = 1, \\ -4 &= \lambda \cdot (-1) \implies \lambda = 4. \end{aligned}$$

  The equations are inconsistent as λ cannot simultaneously be 1 and 4. Therefore, $\mathbf{v}_2$ is **not** an eigenvector of D.

Neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ are eigenvectors of the matrix D.

4. (a) To perform Principal Component Analysis (PCA) and address your query, we'll follow these steps:

   i. Standardize the data (to have mean 0 and standard deviation 1 for each feature).

   ii. Compute the covariance matrix of the standardized data.

   iii. Find the eigenvalues and eigenvectors of the covariance matrix.

   iv. Create Feature Vector.

   v. Recast the data along the principal component axes

Step 1: Standardization
Given the dataset $X = \{(4, 2, 3), (3, 13, 4), (9, 4, 5), (0, 5, 7)\}$, let's compute the mean and standard deviation for each feature and then standardize:

Calculating Mean and Standard Deviation

For each feature j, the mean μ_j and standard deviation σ_j are computed as follows:

$$\mu_j = \frac{\sum_{i=1}^{n} x_{ij}}{n}, \quad \sigma_j = \sqrt{\frac{\sum_{i=1}^{n} (x_{ij} - \mu_j)^2}{n}}$$

Where n is the number of samples (4 in this case).

- $\mu_1 = \dfrac{4+3+9+0}{4} = \dfrac{16}{4} = 4$

  $\sigma_1 = \sqrt{\dfrac{(4-4)^2 + (3-4)^2 + (9-4)^2 + (0-4)^2}{4}} = \sqrt{\dfrac{42}{4}} = 3.24$

- $\mu_2 = \dfrac{2+13+4+5}{4} = \dfrac{24}{4} = 6$

  $\sigma_1 = \sqrt{\dfrac{(2-6)^2 + (13-6)^2 + (4-6)^2 + (5-6)^2}{4}} = \sqrt{\dfrac{70}{4}} = 4.18$

- $\mu_3 = \dfrac{3+4+5+7}{4} = \dfrac{19}{4} = 4.75$

  $\sigma_1 = \sqrt{\dfrac{(3-4.75)^2 + (4-4.75)^2 + (5-4.75)^2 + (7-4.75)^2}{4}} = \sqrt{\dfrac{35}{16}} = \sqrt{2.1875}$

Standardizing the Dataset

Standardized value z_{ij} is calculated by:

$$z_{ij} = \frac{x_{ij} - \mu_j}{\sigma_j} \qquad \text{or} \qquad z_{ij} = x_{ij} - \mu_j$$

Either standardization is acceptable for this assignment

$$Z_1 : z_{ij} = \frac{x_{ij} - \mu_j}{\sigma_j} = \begin{pmatrix} \frac{4-4}{3.24} & \frac{2-6}{4.18} & \frac{3-4.75}{1.48} \\ \frac{3-4}{3.24} & \frac{13-6}{4.18} & \frac{4-4.75}{1.48} \\ \frac{9-4}{3.24} & \frac{4-6}{4.18} & \frac{5-4.75}{1.48} \\ \frac{0-4}{3.24} & \frac{5-6}{4.18} & \frac{7-4.75}{1.48} \end{pmatrix} = \begin{pmatrix} 0 & -0.96 & -1.18 \\ -0.31 & 1.67 & -0.51 \\ 1.54 & -0.48 & 0.17 \\ -1.23 & -0.24 & 1.52 \end{pmatrix}$$

$$Z_2 : z_{ij} = x_{ij} - \mu_j = \begin{pmatrix} 4-4 & 2-6 & 3-4.75 \\ 3-4 & 13-6 & 4-4.75 \\ 9-4 & 4-6 & 5-4.75 \\ 0-4 & 5-6 & 7-4.75 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -1.75 \\ -1 & 7 & -0.75 \\ 5 & -2 & 0.25 \\ -4 & -1 & 2.25 \end{pmatrix}$$

Step 2: Covariance Matrix Computation

The covariance matrix S for the standardized data Z is calculated as:

$$\Sigma_2 = \frac{1}{n-1} Z_1^\top Z_1 = \frac{1}{3} \left[\begin{pmatrix} 0 & -0.31 & 1.54 & -1.23 \\ -0.96 & 1.67 & -0.48 & -0.24 \\ -1.18 & -0.51 & 0.17 & 1.52 \end{pmatrix} \cdot \begin{pmatrix} 0 & -0.96 & -1.18 \\ -0.31 & 1.67 & -0.51 \\ 1.54 & -0.48 & 0.17 \\ -1.23 & -0.24 & 1.52 \end{pmatrix} \right]$$

$$= \frac{1}{3} \begin{pmatrix} 3.9806 & -0.9617 & -1.4497 \\ -0.9617 & 3.9985 & -0.1653 \\ -1.4497 & -0.1653 & 3.9918 \end{pmatrix} = \begin{pmatrix} 1.326 & -0.321 & -0.483 \\ -0.321 & 1.333 & -0.0551 \\ -0.483 & -0.0551 & 1.3306 \end{pmatrix}$$

OR

$$\Sigma_2 = \frac{1}{n-1} Z_2^\top Z_2 = \frac{1}{3} \left[\begin{pmatrix} 0 & -1 & 5 & -4 \\ -4 & 7 & -2 & -1 \\ -1.75 & -0.75 & 0.25 & 2.25 \end{pmatrix} \cdot \begin{pmatrix} 0 & -4 & -1.75 \\ -1 & 7 & -0.75 \\ 5 & -2 & 0.25 \\ -4 & -1 & 2.25 \end{pmatrix} \right]$$

$$= \frac{1}{3} \begin{pmatrix} 42 & -13 & -7 \\ -13 & 70 & -1 \\ -7 & -1 & 8.75 \end{pmatrix} = \begin{pmatrix} 14 & -4.333 & -2.333 \\ -4.333 & 23.333 & -0.333 \\ -2.333 & -0.333 & 2.917 \end{pmatrix}$$

Step 3: Eigenvalue & Eigenvector Computation

Using the steps in Question 3a, we have the eigenvalue and eigenvectors of S given :

$$\Sigma_1 = \begin{pmatrix} 1.326 & -0.321 & -0.483 \\ -0.321 & 1.333 & -0.0551 \\ -0.483 & -0.0551 & 1.3306 \end{pmatrix}$$

Eigenvalues: $\begin{bmatrix} 0.72256029 & 1.88404507 & 1.38299464 \end{bmatrix}$

Eigenvectors: $\begin{bmatrix} -0.6927592 & -0.72029568 & 0.0354799 \\ -0.41737305 & 0.36032433 & -0.83424584 \\ -0.5881194 & 0.59273984 & 0.55025 \end{bmatrix}$

$$\Sigma_2 = \begin{pmatrix} 14 & -4.333 & -2.333 \\ -4.333 & 23.333 & -0.333 \\ -2.333 & -0.333 & 2.917 \end{pmatrix}$$

Eigenvalues: $\begin{bmatrix} 25.04803994 & 12.82788219 & 2.37407787 \end{bmatrix}$

Eigenvectors: $\begin{bmatrix} 0.3695733 & -0.90325876 & 0.21803483 \\ -0.92886567 & -0.36543401 & 0.06055212 \\ -0.02498311 & 0.22490351 & 0.9740607 \end{bmatrix}$

Step 4: Create Feature Vector

Two Largest Eigenvalues with corresponding Eigenvectors:

$$\Sigma_1 : \text{Eigenvalues: } \begin{bmatrix} 1.88 & 1.38 \end{bmatrix}, \quad \text{Eigenvectors: } \begin{bmatrix} -0.72 & 0.04 \\ 0.36 & -0.83 \\ 0.59 & 0.55 \end{bmatrix}$$

$$\Sigma_2 : \text{Eigenvalues: } \begin{bmatrix} 25.04 & 12.83 \end{bmatrix}, \quad \text{Eigenvectors: } \begin{bmatrix} 0.37 & -0.90 \\ -0.93 & -0.37 \\ -0.02 & 0.22 \end{bmatrix}$$

Step 5: Recast Data Along the Principal Component Axes

PCA transformed dataset $= Z \cdot$ eigenvectors

- $\text{pca}_1 = \begin{pmatrix} 0 & -0.96 & -1.18 \\ -0.31 & 1.67 & -0.51 \\ 1.54 & -0.48 & 0.17 \\ -1.23 & -0.24 & 1.52 \end{pmatrix} \cdot \begin{pmatrix} -0.72 & 0.04 \\ 0.36 & -0.83 \\ 0.59 & 0.55 \end{pmatrix} = \begin{pmatrix} -1.0418 & 0.1478 \\ 0.5235 & -1.679 \\ -1.1813 & 0.5535 \\ 1.696 & 0.986 \end{pmatrix}$
- $\text{pca}_2 = \begin{pmatrix} 0 & -4 & -1.75 \\ -1 & 7 & -0.75 \\ 5 & -2 & 0.25 \\ -4 & -1 & 2.25 \end{pmatrix} \cdot \begin{pmatrix} 0.37 & -0.90 \\ -0.93 & -0.37 \\ -0.02 & 0.22 \end{pmatrix} = \begin{pmatrix} 3.755 & 1.095 \\ -6.865 & -1.855 \\ 3.705 & -3.705 \\ -0.595 & 4.465 \end{pmatrix}$

  NB: pca_2 is the results you would get using `PCA` from `sklearn.decomposition` with the the signs of the values in the first column different because of the order the eigenvalues and eigenvectors are computed in the library. Either pca_1 or pca_2 is correct.

(b) The percentage of variance explained by each principal component is given by the ratio of the corresponding eigenvalue to the sum of all eigenvalues:

$$\text{Percentage of Variance} = \frac{\lambda_i}{\sum \lambda} \times 100$$

Let's calculate the percentage of variance explained by each of the selected principal components.

- Σ_1 : Total Variance $= 0.72256029 + 1.88404507 + 1.38299464 \approx 4$
  - $\frac{0.72256029}{4} = 18\%$
  - $\frac{1.88404507}{4} = 47\%$
  - $\frac{1.38299464}{4} = 35\%$
- Σ_2 : Total Variance $= 25.04803994 + 12.82788219 + 2.37407787 \approx 40$
  - $\frac{25.04803994}{40} = 62\%$
  - $\frac{12.82788219}{40} = 32\%$
  - $\frac{2.37407787}{40} = 6\%$