Week 2 Assignment Marking Scheme

PART I: Probability Distributions

`Question ID: 1`

Given that $X \sim \texttt{Binom}(n, p)$ and $P(X = 2) = P(X = 3)$, it follows that

$$\binom{n}{2} p^2 (1-p)^{n-2} = \binom{n}{3} p^3 (1-p)^{n-3}$$

$$\frac{n(n-1)}{1 \times 2} p^2 (1-p)^{n-2} = \frac{n(n-1)(n-2)}{1 \times 2 \times 3} p^3 (1-p)^{n-3}$$

Divide both sides by $n, n-1, p^2, (1-p)^{n-3}$ which are all nonzero

$$\frac{1}{2}(1-p) = \frac{1}{6} p(n-2) \quad \Rightarrow \quad 3(1-p) = p(n-2)$$

$$3 - 3p = np - 2p \quad \Rightarrow \quad 3 - p = np$$

$$np = \mathbb{E}(X) = 3 - p$$

Quod Erat Demonstrandum

Question ID: 2

(a)

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

We need to find the expected value $\mathbb{E}[X]$.

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x P(X = x)$$

Substituting the pmf of the Poisson distribution, we get:

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} \frac{x \lambda^x e^{-\lambda}}{x!} = \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!}$$

Let $y = x - 1$. When $x = 1$, $y = 0$. Therefore, the sum becomes:

$$\mathbb{E}[X] = \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = e^{-\lambda} e^{\lambda} = 1$$

Thus,

$$\mathbb{E}[X] = \lambda \cdot 1 = \lambda$$

Therefore, the expected value of a Poisson random variable X with parameter λ is:

$$\mathbb{E}[X] = \lambda$$

(a*)

$$P(X = x) = \frac{\lambda^x e^{\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

We need to find the expected value $\mathbb{E}[X]$.

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Thus,

$$\mathbb{E}[X] = \lambda \cdot e^{2\lambda} = \lambda e^{2\lambda}$$

(b)

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

The expected value $\mathbb{E}[X^2]$ is defined as:

$$\mathbb{E}[X^2] = \sum_{x=0}^{\infty} x^2 P(X = x)$$

Substituting the pmf of the Poisson distribution, we get:

$$\mathbb{E}[X^2] = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!}$$

To simplify this sum, we use the identity $x^2 = x(x-1) + x$, and split the sum:

$$\mathbb{E}[X^2] = \sum_{x=0}^{\infty} (x(x-1) + x) \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} + \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}$$

For the first term, notice that when $x = 0$ or $x = 1$, the term is zero.

$$\sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=2}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-2)!}$$

Change the index $y = x - 2$:

$$= \sum_{y=0}^{\infty} \frac{\lambda^{y+2} e^{-\lambda}}{y!} = \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = \lambda^2$$

For the second term, we can start the sum from $x = 1$:

$$\sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = \lambda$$

Combining the two terms, we get:

$$\mathbb{E}[X^2] = \lambda^2 + \lambda$$

(b*)

$$P(X = x) = \frac{\lambda^x e^\lambda}{x!}, \quad x = 0, 1, 2, \ldots$$

The expected value $\mathbb{E}[X^2]$ is defined as:

$$\mathbb{E}[X^2] = \sum_{x=0}^{\infty} x^2 P(X = x)$$

Substituting the pmf of the Poisson distribution, we get:

$$\mathbb{E}[X^2] = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^\lambda}{x!}$$

To simplify this sum, we use the identity $x^2 = x(x-1) + x$, and split the sum:

$$\mathbb{E}[X^2] = \sum_{x=0}^{\infty} (x(x-1) + x) \frac{\lambda^x e^\lambda}{x!} = \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^\lambda}{x!} + \sum_{x=0}^{\infty} x \frac{\lambda^x e^\lambda}{x!}$$

For the first term, notice that when $x = 0$ or $x = 1$, the term is zero.

$$\sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^\lambda}{x!} = \sum_{x=2}^{\infty} \frac{\lambda^x e^\lambda}{(x-2)!}$$

Change the index $y = x - 2$:

$$= \sum_{y=0}^{\infty} \frac{\lambda^{y+2} e^\lambda}{y!} = \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^\lambda}{y!} = \lambda^2 e^{2\lambda}$$

For the second term, we can start the sum from $x = 1$:

$$\sum_{x=1}^{\infty} x \frac{\lambda^x e^\lambda}{x!} = \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^\lambda}{(x-1)!} = \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^\lambda}{y!} = \lambda e^{2\lambda}$$

Combining the two terms, we get:

$$\mathbb{E}[X^2] = (\lambda^2 + \lambda) e^{2\lambda}$$