# Bicubic interpolation in Height Map Terrain context

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**Abstract**

The bicubic interpolation method produces a suitable degree 3 polynomial surface satisfying first (position) and second (tangent) order constraints. The standard use case is the design of a smooth surface, continuously wrapping a sample of points. The method may apply to different contexts (grid sampling, free 3D vertices) with different types and dimensions of inputs and therefore lead to different formulations but using the same resolution process.

## Problem statement

In the context of a Raster Digital Elevation Model (DEM), the input set of data is height value sampled for each vertex of a regular grid on the earth-ground.

The trivial terrain modelling surface is a set of triangles connecting straightforward the set of points (two triangles connecting each 4-tuple of adjacent vertices, two possible solutions). But this solution is usually not satisfactory because of sampling artefacts (especially discontinuity of C0 solution). The poor modelling might be balanced by an improved visualisation using C0 normals but "peak effect" would be still visible in some configurations (near view, low density sampling of complex terrain). Moreover, this modelling method produces a static representation whereas, to comply with performance issue, it has to be adaptative. Indeed, the level of accuracy has to be correlated both with the distance to the observer and with height distortion (in the regions where the terrain is nearly plane, the sampling doesn't need high intensity).

A better solution is to considerer a complex surface function (also called interpolation surface), with continuity attribute (at least C1), underlying the input sampling (i.e. every single point of the input sampling is on that surface) and to build the modelling surface from a set of triangle connecting another sampling of this formal function. This other sampling is set to be more precise in the neighbourhood of the observer (lower away from it) and might be desynchronized with the original sampling. In the bicubic interpolation case, the complex surface is a piecewise degree 3 polynomial surface (see chapter "Inputs and constraints" for the reason of dergree 3), with a single piece for every 4-tuple and continuity quality along the piece borders.

## Function definition

To be able to determine easily the coefficients, both parameters of the function vary inside [0,1] interval. The polynomial function $\varphi$ is therefore defined by:

$$\varphi : [0,1] \times [0,1] \to \mathbb{R} \qquad \varphi(u,v) = \sum_{i,j=0}^{3} \alpha_{i,j} u^i v^j \qquad \text{with } \alpha_{i,j} \in \mathbb{R}$$

Note that the the polynomial surface is defined with scalar coefficients (i.e. in $\mathbb{R}$) since the input points are defined as parameteric point with an altitude (z-component) resulting from sampling. In the situation where the input points would be free vertices, the polynomial surface will be defined with coeeficients in $\mathbb{R}^3$.

The real numbers $\alpha_{i,j}$ are the 16 coefficients to determine The degree of the polynomial function is determined by the number of constraints to satisfy. In the current context, there are 4 scalar contraints for the positions, 8 scalar constraints for the tangents (4 in each plane direction). A total of 12, which is more than 9 but less than 16, the number of degrees of freedom respectively of a degree 2 and degree 3 polynomial surface. Therefore, the degree of the polynomial has to be at least 3. To be able to find a solution for a degree 3 polynomial it is necessary to add 4 scalar constraints. It might be the constraints on "mixed tangent variation" (the way a tangent in one direction varies along the other direction). The geometrical interpretation of this mathematical quantity is not obvious, let's simply say that it is corresponding to a twist factor. Of course, this quantity is not part of the input set of data but might be estimated out of it (see below).

The polynomial function has the trivial derivatives:

$$\begin{aligned}
\frac{\partial\varphi}{\partial u}(u,v) &= \sum_{i=1,j=0}^{3} i.\alpha_{i,j}u^{i-1}v^j \\
\frac{\partial\varphi}{\partial v}(u,v) &= \sum_{i=0,j=1}^{3} j.\alpha_{i,j}u^{i}v^{j-1} \\
\frac{\partial^2\varphi}{\partial u\partial v}(u,v) &= \sum_{i,j=1}^{3} i.j.\alpha_{i,j}u^{i-1}v^{j-1}
\end{aligned}$$

**Inputs, constraints and resolution**

The input set of data is made a matrix of height values, named $h_{r,s}$, $r$ and $s$ being the integer index within the matrix or grid. Considering a single piece of the global solution, we have 4 heights values, named $z_0$ to $z_3$ and associated to 4 corners of the surface $(u,v)$-parametrized respectively $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ and equal to $h_{r,s}$, $h_{r+1,s}$, $h_{r,s+1}$ and $h_{r+1,s+1}$.

The tangent constraints, named $z'_{u0}$ to $z'_{u3}$ (with respect to the first paramater/direction) and $z'_{v0}$ to $z'_{v3}$ (with respect to the second paramater/direction), are estimated from $h_{r,s}$ values ($z'_{u0}$ by $\frac{h_{r+1,s}-h_{r-1,s}}{2}$ and so forth).

In the same way, the second order constraints, named $z''_{uv0}$ to $z''_{uv3}$, are estimated from $h_{r,s}$ values ($z''_{uv0}$ by $\frac{h_{r+1,s+1}-h_{r+1,s-1}-h_{r-1,s+1}+h_{r-1,s-1}}{4}$ and so forth).

The above-mentioned estimations are a little bit different for bordering pieces (for instance considering surface for $h_{0,0}$, $h_{1,0}$, $h_{0,1}$ and $h_{1,1}$, since $h_{0,-1}$ does not exist).

The position constraints to the surface (i.e. height at 4 corners) leads to the following 4 equations:

$$\varphi(0,0) = \alpha_{0,0} = z_0 \tag{1a}$$
$$\varphi(1,0) = \alpha_{0,0} + \alpha_{1,0} + \alpha_{2,0} + \alpha_{3,0} = z_1 \tag{1b}$$
$$\varphi(0,1) = \alpha_{0,0} + \alpha_{0,1} + \alpha_{0,2} + \alpha_{0,3} = z_2 \tag{1c}$$
$$\varphi(1,1) = \sum_{i,j=0}^{3} \alpha_{i,j} = z_3 \tag{1d}$$

The tangency constraints, along the first parameter, to the surface (i.e. partial derivative with respect to the first parameter at 4 corners) leads to the following 4 equations:

$$\frac{\partial\varphi}{\partial u}(0,0) = \alpha_{1,0} = z'_{u0} \tag{2a}$$
$$\frac{\partial\varphi}{\partial u}(1,0) = \alpha_{1,0} + 2\alpha_{2,0} + 3\alpha_{3,0} = z'_{u1} \tag{2b}$$
$$\frac{\partial\varphi}{\partial u}(0,1) = \alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3} = z'_{u2} \tag{2c}$$
$$\frac{\partial\varphi}{\partial u}(1,1) = \sum_{i=1,j=0}^{3} i.\alpha_{i,j} = z'_{u3} \tag{2d}$$

The tangency constraints, along the second parameter, to the surface (i.e.partial derivative with respect to the second parameter at 4 corners) leads to the following 4 equations:

$$\frac{\partial\varphi}{\partial v}(0,0) = \alpha_{0,1} = z'_{v0} \tag{3a}$$
$$\frac{\partial\varphi}{\partial v}(1,0) = \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} = z'_{v1} \tag{3b}$$
$$\frac{\partial\varphi}{\partial v}(0,1) = \alpha_{0,1} + 2\alpha_{0,2} + 3\alpha_{0,3} = z'_{v2} \tag{3c}$$
$$\frac{\partial\varphi}{\partial v}(1,1) = \sum_{i=0,j=1}^{3} j.\alpha_{i,j} = z'_{v3} \tag{3d}$$

The tangency constraints, along the combined parameters, to the surface (i.e. mixed second order partial derivative at 4 corners) leads to the following 4 equations:

$$\frac{\partial^2\varphi}{\partial u\partial v}(0,0) = \alpha_{1,1} = z''_{uv0} \tag{4a}$$
$$\frac{\partial^2\varphi}{\partial u\partial v}(1,0) = \alpha_{1,1} + 2\alpha_{2,1} + 3\alpha_{3,1} = z''_{uv1} \tag{4b}$$
$$\frac{\partial^2\varphi}{\partial u\partial v}(0,1) = \alpha_{1,1} + 2\alpha_{1,2} + 3\alpha_{1,3} = z''_{uv2} \tag{4c}$$
$$\frac{\partial^2\varphi}{\partial u\partial v}(1,1) = \sum_{i=1,j=1}^{3} i.j.\alpha_{i,j} = z''_{uv3} \tag{4d}$$

Equations (1a), (2a), (3a) and (4a) provide straightforward 4 coefficients values.

$$\boxed{\begin{array}{ll} \alpha_{0,0} = z_0 & \alpha_{1,0} = z'_{u0} \\ \alpha_{0,1} = z'_{v0} & \alpha_{1,1} = z''_{uv0} \end{array}}$$

Combining equations (1c) and (3c) gives the following solutions:

$$\boxed{\begin{array}{l} \alpha_{0,2} = 3(z_2 - z_0) - 2z'_{v0} - z'_{v2} \\ \alpha_{0,3} = z'_{v0} + z'_{v2} - 2(z_2 - z_0) \end{array}}$$

Combining equations (1b) and (2b) gives the following solutions:

$$\boxed{\begin{array}{l} \alpha_{2,0} = 3(z_1 - z_0) - 2z'_{u0} - z'_{u1} \\ \alpha_{3,0} = z'_{u0} + z'_{u1} - 2(z_1 - z_0) \end{array}}$$

Combining equations (3b) and (4b) gives the following solutions:

$$\boxed{\begin{array}{l}\alpha_{2,1} = 3(z'_{v_1} - z'_{v_0}) - 2z''_{uv_1} - z''_{uv_1} \\ \alpha_{3,1} = z''_{uv_0} + z''_{uv_1} - 2(z'_{v_1} - z'_{v_0})\end{array}}$$

Combining equations (2c) and (4c) gives the following solutions:

$$\boxed{\begin{array}{l}\alpha_{1,2} = 3(z'_{u_2} - z'_{u_0}) - 2z''_{uv_0} - z''_{uv_2} \\ \alpha_{1,3} = z''_{uv_0} + z''_{uv_2} - 2(z'_{u_2} - z'_{u_0})\end{array}}$$

Eventually, the last 4 coefficients yet to be determined — $\alpha_{2,2}, \alpha_{2,3}, \alpha_{3,2}\ and\ \alpha_{3,3}$ — lead to the following linear system (equations (1d), (2d), (3d) and (4d)):

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & A \\ 2 & 2 & 3 & 3 & B \\ 2 & 3 & 2 & 3 & C \\ 4 & 6 & 6 & 9 & D \end{array}\right]$$

$$A = z_3 - \sum_{i,j=0}^{3} \alpha_{i,j} \qquad B = z'u_3 - \sum_{i=1,j=0}^{3} i.\alpha_{i,j}$$

$$C = z'v_3 - \sum_{i=0,j=1}^{3} j.\alpha_{i,j} \qquad D = z''uv_3 - \sum_{i=1,j=1}^{3} i.j.\alpha_{i,j}$$

with (i,j) $\notin$ {(2,2), (2,3), (3,2), (3,3)}

Using Gaussian elimination algorithm, we have the intermediate and final states:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & A \\ 0 & 1 & 0 & 1 & C-2A \\ 0 & 0 & 1 & 1 & B-2A \\ 0 & 0 & 2 & 3 & D-2C \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9A-3B-3C+D \\ 0 & 1 & 0 & 0 & -6A+2B+3C-D \\ 0 & 0 & 1 & 0 & -6A+3B+2C-D \\ 0 & 0 & 0 & 1 & 4A-2B-2C+D \end{array}\right]$$

Which can be written:

$$\boxed{\begin{array}{l}\alpha_{2,2} = 9A - 3B - 3C + D \\ \alpha_{2,3} = -6A + 2B + 3C - D \\ \alpha_{3,2} = -6A + 3B + 2C - D \\ \alpha_{3,3} = 4A - 2B - 2C + D\end{array}}$$

**Continuity between patches**

The way the inputs $z'_u$, $z'_v$ and $z''_{uv}$ are calculated (i.e. symetrically in both direction) enforce a full continuity (i.e. $\mathcal{C}^\infty$) between two adjacent bicubic surfaces within the grid. This is proved by verifying that ($r$ and $s$ identifying the patch inside the grid):

$$
\begin{aligned}
\varphi_{r,s}(1,v) &= \varphi_{r+1,s}(0,v) && \mathcal{C}^0\\
\frac{\partial \varphi_{r,s}}{\partial u}(1,v) &= \frac{\partial \varphi_{r+1,s}}{\partial u}(0,v) && \mathcal{C}^1\\
\frac{\partial \varphi_{r,s}}{\partial v}(1,v) &= \frac{\partial \varphi_{r+1,s}}{\partial v}(0,v) && \mathcal{C}^1\\
\frac{\partial^2 \varphi_{r,s}}{\partial u^2}(1,v) &= \frac{\partial^2 \varphi_{r+1,s}}{\partial u^2}(0,v) && \mathcal{C}^2\\
\frac{\partial^2 \varphi_{r,s}}{\partial v^2}(1,v) &= \frac{\partial^2 \varphi_{r+1,s}}{\partial v^2}(0,v) && \mathcal{C}^2\\
\frac{\partial^2 \varphi_{r,s}}{\partial u \partial v}(1,v) &= \frac{\partial^2 \varphi_{r+1,s}}{\partial u \partial v}(0,v) && \mathcal{C}^2\\
&...
\end{aligned}
$$

and the same system replacing $..._{r,s}(1,v) = ..._{r+1,s}(0,v)$ with $..._{r,s}(u,1) = ..._{r,s+1}(u,0)$.