Determine the optimal weights with the help of Linear Algebra method (Pseudo-inverse)

SOLUTION

$$W = D^T X (X^T X)^{-1}$$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad X^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$X^T . X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$(X^T X)^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 3 & -2 & -2 \\ -2 & 4 & 0 \\ -2 & 0 & 4 \end{bmatrix}}{4(3) - 2(2) + 2(-2)} = \frac{\begin{bmatrix} 3 & -2 & -2 \\ -2 & 4 & 0 \\ -2 & 0 & 4 \end{bmatrix}}{12 - 4 - 4} = \begin{bmatrix} 0.75 & -0.5 & -0.5 \\ -0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$$

$$X (X^T X)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.75 & -0.5 & -0.5 \\ -0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.75 & -0.5 & -0.5 \\ 0.25 & -0.5 & 0.5 \\ 0.25 & 0.5 & -0.5 \\ -0.25 & 0.5 & 0.5 \end{bmatrix}$$

$$W = D^T . X (X^T X)^{-1} = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.75 & -0.5 & -0.5 \\ 0.25 & -0.5 & 0.5 \\ 0.25 & 0.5 & -0.5 \\ -0.25 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0.5 \end{bmatrix}$$

$$\boxed{w_0 = 0.25; \quad w_1 = 0.5; \quad w_2 = 0.5}$$

$$y^* = w_0 x_0 + w_1 x_1 + w_2 x_2$$

$$y^1 = 0.25 \times 1 + 0.5 \times 0 + 0.5 \times 0 = 0.25$$

$$y^2 = 0.25 \times 1 + 0.5 \times 0 + 0.5 \times 1 = 0.75$$

$$y^3 = 0.25 \times 1 + 0.5 \times 1 + 0.5 \times 0 = 0.75$$

$$y^4 = 0.25 \times 1 + 0.5 \times 1 + 0.5 \times 1 = 1.25$$

$$E_{opt} = \frac{1}{2} \sum (D^p - y^p)^2$$

$$E_{opt} = \frac{1}{2} \left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 \right) = \frac{1}{2} \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \right) = \frac{1}{2} \times \frac{4}{16} = \frac{1}{8} = 0.125$$

$$\boxed{E_{opt} = 0.125}$$