$$\begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

$$A - \lambda I \Rightarrow \begin{pmatrix} (4-\lambda) & 8 & -1 & -2 \\ -2 & (-9-\lambda) & -2 & -4 \\ 0 & 10 & (5-\lambda) & -10 \\ -1 & -13 & -14 & -13-\lambda \end{pmatrix}$$

$$(4-\lambda)\begin{bmatrix} (-9-\lambda) & -2 & -4 \\ 10 & (5-\lambda) & -10 \\ -13 & -14 & (-13-\lambda) \end{bmatrix} - 8\begin{pmatrix} -2 & -2 & -4 \\ 0 & (5-\lambda) & -10 \\ -1 & -14 & (-13-\lambda) \end{pmatrix} - \begin{vmatrix} -2 & (-9-\lambda) & -2 \\ 0 & 10 & (5-\lambda) \\ -1 & -13 & -14 \end{vmatrix}$$

$$+ 2\begin{pmatrix} -2 & (-9-\lambda) & -2 \\ 0 & 10 & (5-\lambda) \\ -1 & -13 & -14 \end{pmatrix}$$

Gaussian elimination

$$\begin{pmatrix} (-9-\lambda) & -2 & -4 \\ 10 & (5-\lambda) & -10 \\ -13 & -14 & (-13-\lambda) \end{pmatrix}$$

$$R_2 = R_2 - \left(\frac{10}{-9-\lambda} R_1\right) \Rightarrow \bullet\ 10 - \left(\frac{10}{-9-\lambda} \cdot (-9-\lambda)\right) = 0 \quad \bullet\ R_2(2) = (5-\lambda) - \left(\frac{10}{-9-\lambda} \cdot -2\right)$$

$$= (5-\lambda + \frac{20}{9+\lambda}$$

let $D_1 = \begin{pmatrix} -9-\lambda & -2 & -4 \\ 10 & (5-\lambda) & -10 \\ -13 & -14 & (-13-\lambda) \end{pmatrix}$

$$D_1 = (-9-\lambda)\begin{vmatrix} 5-\lambda & -10 \\ -14 & (-13-\lambda) \end{vmatrix} - (-2)\begin{vmatrix} 10 & -10 \\ -13 & (-13-\lambda) \end{vmatrix} - 4\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix}$$

$$= (-9-\lambda)\Big[(5-\lambda)(-13-\lambda) - (140)\Big] + 2\Big[(-130) - 130 - 10\lambda\Big] - 4\Big[-140 - (65 + 13\lambda\Big]$$

$$= (-9-\lambda)\left[-65 + 8\lambda + \lambda^2 - 140\right] + 2\left[-260 - 10\lambda\right] - 4\left[-75 + 13\lambda\right]$$

$$= (-9-\lambda)\left[\lambda^2 + 8\lambda - 205\right] + 2\left[-260 - 10\lambda\right] - 4\left[-75 - 13\lambda\right]$$

$$= -\lambda^3 - 17\lambda^2 + 133\lambda + 1845 - 520 - 20\lambda + 300 + 52\lambda$$

$$= -\lambda^3 - 17\lambda^2 + 165\lambda + 1625$$

let $D_2 = \begin{vmatrix} -2 & -2 & -4 \\ 0 & (5-\lambda) & -10 \\ -1 & -14 & (-13-\lambda \end{vmatrix}$

$$D_2 = -2\begin{vmatrix} (5-\lambda) & -10 \\ -14 & (-13-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 0 & -10 \\ -1 & (-13-\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & (5-\lambda) \\ -1 & -14 \end{vmatrix}$$

$$= -2\Big[(5-\lambda)(-13-\lambda) - (140)\Big] + 2\Big[0 - (+10)\Big] - 4\Big[0 - (-5+\lambda)\Big]$$

$$= -2\left[\lambda^2 + 8\lambda - 205\right] + -20 - 20 + 4\lambda$$

$$= -2\lambda^2 - 16\lambda + 410 + 40 + 4\lambda \Rightarrow -2\lambda^2 - 12\lambda + 370$$

let $\Delta_3 = \begin{vmatrix} -2 & -9-\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda \end{vmatrix}$

$$\Delta_3 = -2\begin{vmatrix} 10 & -10 \\ -13 & -13-\lambda \end{vmatrix} - (-9-\lambda)\begin{vmatrix} 0 & -10 \\ -1 & -13-\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & 10 \\ -1 & -13 \end{vmatrix}$$

$$= -2\left[(-130-10\lambda)-130\right]\ 9+\lambda\left[0-10\right] - 4\left[0-(-10)\right]$$

$$= -2(-260-10\lambda) - 90 - 10\lambda - 40$$

$$= +520 + 20\lambda - 90 - 10\lambda - 40$$

$$= 10\lambda + 390$$

let $\Delta_4 = \begin{vmatrix} -2 & (-9-\lambda) & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$

$$\Delta_4 = -2\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix} - (-9-\lambda)\begin{vmatrix} 0 & 5-\lambda \\ -1 & -14 \end{vmatrix} - 2\begin{vmatrix} 0 & 10 \\ -1 & -13 \end{vmatrix}$$

$$= -2\left[-140-(-65+\lambda)\right] + (9+\lambda)\left[0-(5+\lambda)\right] - 2\left[0-(-10)\right]$$

$$= \cancel{-2(-78-13\lambda) - 45 + 9\lambda + 5\lambda + \lambda^2 - 20}$$

$$= \cancel{150 + 26\lambda}$$

$$= -2(-75-13\lambda) + 45 - 9\lambda + 5\lambda - \lambda^2 - 20$$

$$= +150 + 26\lambda + 45 - 9\lambda + 5\lambda - \lambda^2 - 20$$

$$= -\lambda^2 + 22\lambda + 175$$

So,

$$\det(A-\lambda I) = 4-\lambda . D_1 - 8\Delta_2 - \Delta_3 + 2\Delta_4$$

$$\det(A-\lambda I) = (4-\lambda)(-\lambda^3 - 17\lambda^2 + 165\lambda + 1625) - 8(-2\lambda^2 - 12\lambda + 370)$$
$$- (10\lambda + 390) + 2(-\lambda^2 + 22\lambda + 175)$$

$$= -4\lambda^3 - 68\lambda^2 + 660\lambda + 6500 + \lambda^4 + 17\lambda^3 - 165\lambda^2 - 1625\lambda + 16\lambda^2$$
$$+ 96\lambda - 2960 - 10\lambda - 390 - 2\lambda^2 + 44\lambda + 350$$

$$= +\lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$$

Newton - Raphson method.

$$\lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)}$$

$f(\lambda_n) = \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$

$f'(\lambda_n) = 4\lambda^3 + 39\lambda^2 - 438\lambda - 835$

$f(0) = 3500 \quad f(1) = 2460 \quad f(2) = 1074 \quad f(3) = -544$

It changes signs between 2 & 3 so the root is between those values.

At $\lambda_0 = 2.5 \qquad \lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)}$

$$\lambda_1 = 2.5 - \frac{285.9375}{-1623.75} = 2.7$$

$$\lambda_2 = 2.7 - \frac{41.9869}{-1654.558} = 2.674$$

$$\lambda_3 = 2.674 - \frac{0.983}{-1650.87} = 2.674$$

So, one of the root is $\boxed{\lambda_1 = 2.674}$

$$(A - \lambda I) = \begin{pmatrix} 4-2.674 & 8 & -1 & -2 \\ -2 & -9-2.674 & -2 & -4 \\ 0 & 10 & 5-2.674 & -10 \\ -1 & -13 & -14 & -13-2.674 \end{pmatrix}$$

$$= \begin{pmatrix} 1.326 & 8 & -1 & -2 \\ -2 & -11.674 & -2 & -4 \\ 0 & 10 & 2.326 & -10 \\ -1 & -13 & -14 & -15.674 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Gaussian elimination.

- $R_1 = R_1 / 1.326 \Rightarrow (1, 6.034, -0.754, -1.509)$
- $R_2 = R_2 + 2R_1 \Rightarrow (0, 0.394 \quad -3.508 \quad -7.018)$
- $R_4 = R_4 + R_1 \Rightarrow (0, -6.966 \quad -14.754 \quad -17.183)$
- Normalise ; $R_2 = R_2 / 0.394 \Rightarrow (0\ 1, -8.904 \quad -17.812)$
- R_1 $\begin{pmatrix} \cancel{1} & \cancel{6.034} & \cancel{-0.754} & \cancel{-1.509} \\ \cancel{0} & \cancel{0.0394} & \cancel{-3.508} & \\ \cancel{0} & & & \\ 0 & & & \end{pmatrix}$

$$\begin{pmatrix} 1 & 6.034 & -0.754 & -1.509 \\ 0 & 1 & -8.904 & -17.812 \\ 0 & 10 & 2.326 & -10 \\ 0 & 1 & -8.904 & -17.812 \end{pmatrix}$$

- $R_1 = R_1 - 6.034R_2 \Rightarrow (1 \ 0 \ 52.977 \ 106.977)$
- $R_3 = R_3 - 10R_2 \Rightarrow (0 \ 0 \ 91.366 \ 168.120)$
- $R_4 = R_4 - 6.966R_2 \Rightarrow (0 \ 0 \ -76.808 \ -141.366)$
- Normalise : $R_3 = R_3 / 91.366 \Rightarrow (0 \ 0 \ 1 \ 1.840)$
- $R_1 = R_1 - 52.977R_3 \Rightarrow (1, 0, 0, 8.494)$
- $R_2 = R_2 + 8.904R_3 \Rightarrow (0, 1, 0, -1.428)$
- $R_4 = R_4 + 76.808R_3 \Rightarrow (0 \ 0 \ 0 \ 0)$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} -8.494 \\ 1.428 \\ -1.840 \\ 1 \end{pmatrix}$$ The eigen vector is $$\begin{pmatrix} -8.494 \\ 1.428 \\ -1.840 \\ 1 \end{pmatrix}$$

$\lambda_2 = 11.054.$

$$A = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

Eigenvector fo $\lambda_2 \Rightarrow (A - \lambda I)x = 0.$

$$A - \lambda I = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - 11.054 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -7.054 & 8 & -1 & -2 \\ -2 & -20.054 & -2 & -4 \\ 0 & 10 & -6.054 & -10 \\ -1 & -13 & -14 & -24.054 \end{pmatrix}$$

$$\text{Let } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -7.054 & 8 & -1 & -2 \\ -2 & -20.054 & -2 & -4 \\ 0 & 10 & -6.054 & -10 \\ -1 & -13 & -14 & -24.054 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

* Swap R_1 and R_4

$$\begin{pmatrix} -1 & -13 & -14 & -24.054 \\ -2 & -20.054 & -2 & -4 \\ 0 & 10 & -6.054 & -10 \\ -7.054 & 8 & -1 & -2 \end{pmatrix}$$

$R2 = R_2 - 2 * R_1$ and $R_4 = R_4 - 7.054 * R_1$.

$$\begin{pmatrix} 1 & -13 & -14 & -24.054 \\ 0 & -20.054 - 2(-13) & -2-2(-14) & -4-2(24.054) \\ 0 & 10 & -6.054 & -10 \\ 0 & 8 - 7.054(-13) & -7.054(-14) & -2-7.054(-24.054) \end{pmatrix}$$

$$\begin{pmatrix} -1 & -13 & -14 & -24.054 \\ 0 & 5.946 & 26 & 44.108 \\ 0 & 10 & -6.054 & -10 \\ 0 & 99.702 & 97.716 & 169.576 \end{pmatrix}$$

R_2 normalisatem $R_2 = R_2 / 5.946$.

$R_3 = R_3 - 10 \times R_2$

$R_4 = R_4 - 99.702 \times R_2$.

Normalisatem of R_3.

$$\begin{pmatrix} -1 & -13 & -14 & -24.054 \\ 0 & 1 & 4.3725 & 7.4187 \\ 0 & 0 & 1 & 1.6913 \\ 0 & 0 & -337.52 & -570.61 \end{pmatrix}$$

$R_4 = R_4 - (-337.52) R_3$.

$$\begin{pmatrix} -1 & -13 & -14 & -24054 \\ 0 & 1 & 4.3725 & 7.4187 \\ 0 & 0 & 1 & 1.6913 \\ 0 & 0 & 0 & -570.61 - (-337.52)(1.6913) \end{pmatrix}$$

$$-570.61 + 337.52 \times 1.6913 = -570.61 + 570.78$$
$$= 0.17.$$

$$\Rightarrow \begin{pmatrix} -1 & -13 & -14 & -24.054 \\ 0 & 1 & 4.3725 & 7.4187 \\ 0 & 0 & 1 & 1.6913 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-x_1 - 13x_2 - 14x_3 - 24.054x_3 = 0$$
$$x_2 + 4.3725x_3 + 7.4187x_4 = 0$$
$$x_3 + 1.6913x_4 = 0$$

$$x_4 = y.$$

$$x_2 + 4.3725(-1.6913y) + 7.4187y = 0$$
$$x_2 - 7.3992y + 7.4187y = 0$$
$$x_2 + 0.0192y = 0$$
$$x_2 = -0.0192y.$$

x_2 into equation (1)

$$-x_1 - 13(-0.0192y) - 14(-1.6913y) - 24.054y = 0$$

$$-x_1 + 0.2496y + 25.675 2y - 24.054y = 0$$

$$-x_1 + 25.9272y - 24.054y = 0.$$

$$-x_1 - 0.1262y = 0$$

$$x_1 = -0.1262y.$$

$$x = \begin{pmatrix} -0.1262y \\ -0.0192y \\ -1.6913y \\ y \end{pmatrix}$$

$$x = \begin{pmatrix} -0.1262 \\ -0.0192 \\ -1.6913 \\ 1 \end{pmatrix}$$

$\lambda_3 = -5.6$

$$A - \lambda_3 I = \begin{pmatrix} 9.6 & 8 & -1 & -2 \\ -2 & -3.396 & -2 & -4 \\ 0 & 10 & 10.6 & -10 \\ -1 & -13 & -14 & -7.396 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 9.6 & 8 & -1 & -2 & 0 \\ -2 & -3.3 & -2 & -4 & 0 \\ 0 & 10 & 10.604 & -10 & 0 \\ -1 & -13 & -14 & -7.3 & 0 \end{array}\right) \times (0.104) \underset{R_1}{\sim} \left(\begin{array}{cccc|c} 1 & 0.837 & -0.1 & -0.2 & 0 \\ -2 & -3.3 & -2 & -4 & 0 \\ 0 & 10 & 10.6 & -10 & 0 \\ -1 & -13 & -14 & -7.0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0.837 & -0.1 & -0.2 & 0 \\ 0 & -1.7 & -2.2 & -4.4 & 0 \\ 0 & 10 & 10.6 & -10 & 0 \\ -1 & -13 & -14 & -7 & 0 \end{array}\right) \times 1 \; \underset{R_4}{(-1)} \quad \left(\begin{array}{cccc|c} 1 & 0.8 & -0.1 & -0.208 & 0 \\ 0 & -1.7 & -2.2 & -4.416 & 0 \\ 0 & 10 & 10.6 & -10 & 0 \\ 0 & -12.167 & -14.1 & -7.6 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0.837 & -0.1 & -0.2 & 0 \\ 0 & 1 & 1.2 & 2.5 & 0 \\ 0 & 0 & -2.1 & -35.5 & 0 \\ 0 & 0 & 1.4 & 23.4 & 0 \end{array}\right) \times 0.463 \; R_3 \quad \begin{pmatrix} 1 & 0.8 & -0.1 & -0.208 \\ 0 & 1 & 1.2 & 2.5 \\ 0 & 0 & -2.1 & -35.5 \\ 0 & -12.1 & -14.1 & -7.6 \end{pmatrix} \times 12.167$$

$$\left(\begin{array}{cccc|c} 1 & 0.83 & -0.1 & -0.2 & 0 \\ 0 & 1 & 1.2 & 2.5 & 0 \\ 0 & 0 & -2.1 & -35.5 & 0 \\ 0 & 0 & 1.4 & 23.4 & 0 \end{array}\right) \times 0.46 \; R_3 \quad \left(\begin{array}{cccc|c} 1 & 0.83 & -0.1 & -0.2 & 0 \\ 0 & 1 & 1.27 & 2.5 & 0 \\ 0 & 0 & 1 & 16.4 & 0 \\ 0 & 0 & 1.4 & 23.4 & 0 \end{array}\right) \times -1.4$$

$$\left(\begin{array}{cccc|c} 1 & 0.8 & -0.1 & -0.2 & 0 \\ 0 & 1 & 1.2 & 2.5 & 0 \\ 0 & 0 & 1 & 16.44 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \times (-1.2) \; R_2 \quad \left(\begin{array}{cccc|c} 1 & 0.8 & -0.104 & -0.2 & 0 \\ 0 & 1 & 0 & -18 & 0 \\ 0 & 0 & 1 & 16.4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \times 0.1$$

$$\left(\begin{array}{cccc|c} 1 & 0.8 & 0 & 1.5 & 0 \\ 0 & 1 & 0 & -18.4 & 0 \\ 0 & 0 & 1 & 16.44 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \times 0.8 \underset{R_1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 16.8 & 0 \\ 0 & 1 & 0 & -18.4 & 0 \\ 0 & 0 & 1 & 16.4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = -16.446\, x_4$

$x_2 = 18.439\, x_4$

$x_1 = -16.884\, x_4$

$x_4 = x_4$

$$V_3 = \begin{pmatrix} -16.86 \\ 18.43 \\ -16.44 \\ 1 \end{pmatrix}$$

$f(-20) = -11,400$ $f(-21) = -1456$ $f(-22) = 11,706$

the root is between ~~21 & 22~~ -21 & -22

At $\lambda_1 = -21$

$$\lambda_2 = -21 - \frac{-1456}{-11,482} = -21.12$$

$$\lambda_3 = -21.12 - \frac{-54.88}{-11,870.44} = -21.1246$$

Eigen Value = -21.1246

$(A - \lambda I)v = 0$ $\lambda = -21.1246$

$A - (-21.1246)I = A + 21.1246 \Rightarrow$

$$\begin{pmatrix} 4+21.1246 & 8 & -1 & -2 \\ -2 & -9+21.1246 & -2 & -4 \\ 0 & 10 & 5+21.1246 & -10 \\ -1 & -13 & -14 & -13+21.1246 \end{pmatrix}$$

$$= \begin{pmatrix} 25.1246 & 8 & -1 & -2 \\ -2 & 12.1246 & -2 & -4 \\ 0 & 10 & 26.1246 & -10 \\ -1 & -13 & -14 & 8.1246 \end{pmatrix}$$

$$\begin{pmatrix} 25.1246 & 8 & -1 & -2 \\ -2 & 12.1246 & -2 & -4 \\ 0 & 10 & 26.1246 & -10 \\ -1 & -13 & -14 & 8.1246 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

$Row_1 / 25.1246$. Normalising it.

$Row_1 \rightarrow (1 \quad 0.3184 \quad -0.0398 \quad -0.796)$

Eliminate v_1 from Row_2 and 4

* $Row\,2 + 2 \times Row\,1$:

$Row_2 : (0 \quad 12.7614 \quad -2.0796 \quad -4.1592)$

* $Row_4 + Row_1$:

$Row\,4 : (0 \quad -12.6816 \quad -14.0398 \quad 8.0450)$

Pivot $Row\,2$:

$Row\,2 / 12.7614$.

$Row\,2 \rightarrow (0 \quad 1 \quad -0.1630 \quad -0.3260)$

Now eliminate v_2 from Row 1, 3 and 4

* $Row_1 - 0.3184 \times Row_2$.

$R_1 \to (1 \quad 0 \quad 0.0122 \quad 0.0244)$

* $Row_3 - 10 \times Row_2$:

$R_3 \to (0 \quad 0 \quad 27.7546 \quad -6.7400)$

* $Row_4 + 12.6816 \times Row_2$!

$R_4 \to (0 \quad 0 \quad -16.1816 \quad 3.8818)$

→ Pivot Row 3:

$Row_3 / 27.7546$.

$R_3 \to (0 \quad 0 \quad 1 \quad -0.2429)$

→ Eliminate V_3 from $R_{1,2}$ and 4:

$R_1 - 0.0122 \times R_3$

$R_1 \to (1 \quad 0 \quad 0 \quad 0.0271)$

$R_2 \to (0 \quad 1 \quad 0 \quad -0.3655)$

$R_4 \to (0 \quad 0 \quad 0 \quad 0)$

$$\begin{cases} V_1 + 0.0271 V_4 = 0 \\ V_2 - 0.3655 V_4 = 0 \\ V_3 - 0.2429 V_4 = 0 \end{cases}$$

let $V_4 = 1$

$V_1 = -0.0271$, $V_2 = 0.3655$ $V_3 = 0.2429$

$$V = \begin{pmatrix} -0.0271 \\ 0.3655 \\ 0.2429 \\ 1 \end{pmatrix}$$