Linear Quadratic Regulators (LQR)

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Introduction

1. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost function.
2. A linear quadratic problem is a problem where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function.

A typical optical problem is given by;

$$\begin{aligned} \underset{x}{Minimize} \quad & f_0(x) \\ s.t \quad f_i(x) \quad & \leq \quad b_i, \qquad i = 1, 2, ..., m \end{aligned}$$

In control problems, we optimize our trajectories to minimize the cost function.

The control optimization problem is written as,

$$\underset{u_0 \ldots N}{Minimize} \quad \sum_{i=0}^{N-1} g(x_i, u_i) + E(x_N)$$
$$s.t \quad x_{k+1} = f(x_k, u_k)$$
$$x - \bar{x} \quad = \quad 0$$

1. The function f is a linear function of x and u and the function g a quadratic function of x and u. This makes our problem a linear quadratic problem.
2. The function g is convex which makes the problem simple to solve.

Linear Quadratic Regulator Problem

The control optimization problem for finite-horizon, discrete time linear quadratic regulator is given;

$$\underset{u_{0\ldots N}}{Minimize} \quad \sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i) + x_N^T E x_N$$
$$s.t \quad x_{k+1} = Ax_k + Bx_k$$
$$x - \bar{x} = 0$$

Variable	Explanation
x	State variable at each time step
$\bar{x}$	Start state
N	Time horizon
u	Regulator/ Action
g	Cost function
E	Final cost
Q	Cost (weight) on variable x
R	Cost (weight) on variable u

The cost matrices Q and R must be positive definite.
Thus,

$$\forall, x^T Qx > 0$$
$$\forall, u^T Ru > 0$$

Infinite-horizon, continuous-time LQR

For a continuous time linear system given by,

$$\dot{x} = Ax + Bu$$

The cost can be defined as,

$$J = \int_0^{\infty} (x^T Qx + u^T Ru + 2x^T Nu)dt$$

The feedback control law that minimizes the value of the cost J is given by:

$$u = -kx \tag{1}$$

k which is known as the gain matrix is defined as,

$$k = R^{-1}(B^T P + N^T) \tag{2}$$

P is found by solving the continuous time algebraic Riccati equation.

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0 \tag{3}$$

The optimization problem for the Infinite-horizon, continuous-time LQR is given as,

$$\begin{aligned} Minimize \quad & J \\ s.t. \quad \dot{x} &= Ax + Bu \end{aligned}$$

Procedure for LQR

1. Given A and B
2. Choose the weights Q and R
3. Solve the algebraic Riccati equation.
4. Compute k
5. Chose the k solution that yields a stable system

Python output of LQR

```
In [4]: control.lqr?

Signature: control.lqr(*args, **keywords)
Docstring:
lqr(A, B, Q, R[, N])

Linear quadratic regulator design

The lqr() function computes the optimal state feedback controller
that minimizes the quadratic cost

.. math:: J = \int_0^\infty (x' Q x + u' R u + 2 x' N u) dt

The function can be called with either 3, 4, or 5 arguments:

* ``lqr(sys, Q, R)``
* ``lqr(sys, Q, R, N)``
* ``lqr(A, B, Q, R)``
* ``lqr(A, B, Q, R, N)``

where `sys` is an `LTI` object, and `A`, `B`, `Q`, `R`, and `N` are
2d arrays or matrices of appropriate dimension.

Parameters
----------
A, B: 2-d array
    Dynamics and input matrices
sys: LTI (StateSpace or TransferFunction)
    Linear I/O system
Q, R: 2-d array
    State and input weight matrices
```

Application of LQR

1. Adam Coates, Pieter Abbeel and Andrew Y. Ng in their paper, "Apprenticeship learning for Helicopter control", present an algorithm (Reinforcement and Dynamic programming) to model and build high performance control systems for the helicopter.
2. Reshmi and Priya designed LQR based controller to control the depth of Autonomous underwater vehicle (AUV). The (AVU) is a programmable robotic submarine which can drift, drive or glide without real time control, without any human, depending on the system and it can be able to swim at different depth.

Conclusion

1. The goal of LQR is to stabilize a system

1. Thank you

References

1. Marin Vlastelica Pogančić, Optimal Control: LQR, (May -2019) from: https://towardsdatascience.com/optimal-control-lqr-417b41e10d0d, Last Acess Date[26/11/2019: 12:00 pm]
2. Linear–quadratic regulator, Wikipedia, (last edited : 5/11/2019) from: https://en.wikipedia.org/wiki/LinearLast Access[26/11/2019: 06:00 am]