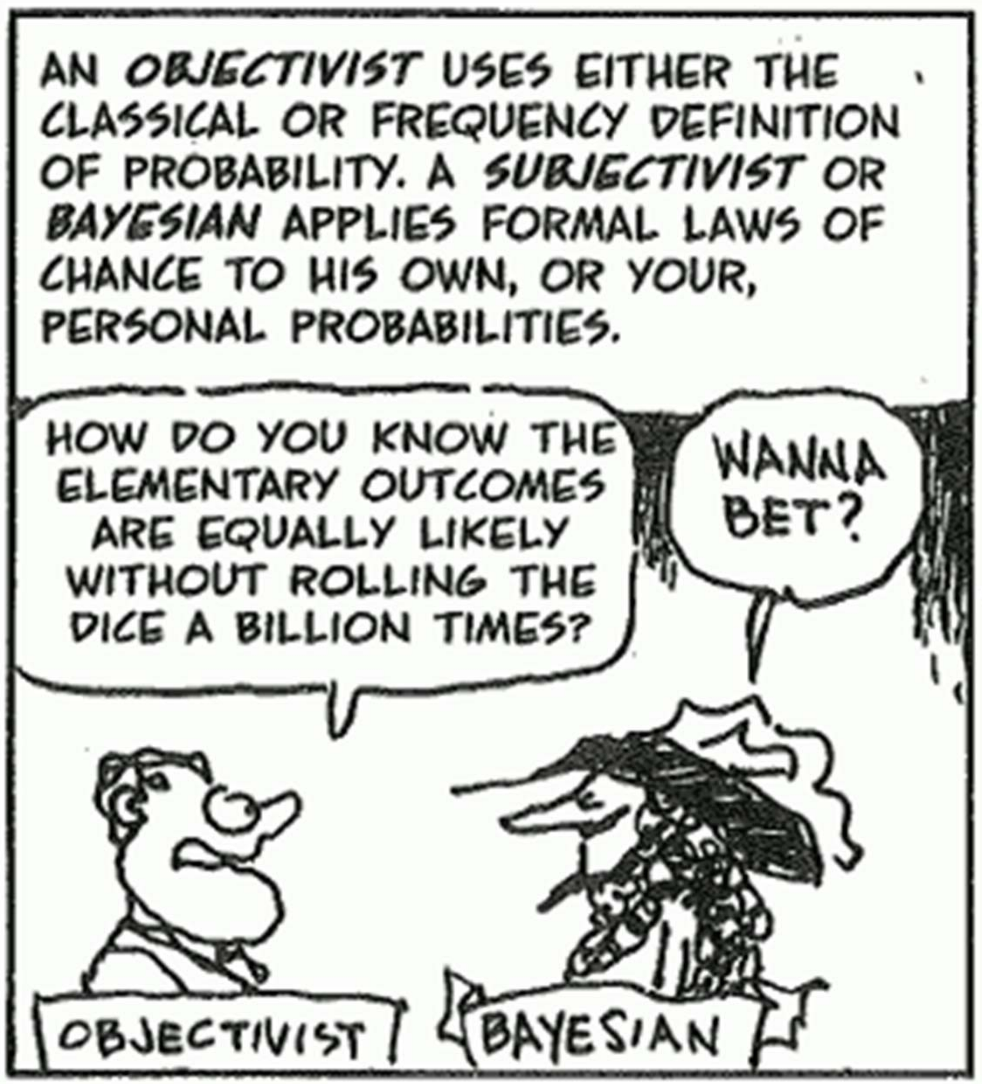
UNIVERSITY of WASHINGTON

Data Science UW
Methods for Data Analysis

Bayesian models, Part 2

Steve Elston


AN OBJECTIVIST USES EITHER THE CLASSICAL OR FREQUENCY DEFINITION OF PROBABILITY. A SUBJECTIVIST OR BAYESIAN APPLIES FORMAL LAWS OF CHANCE TO HIS OWN, OR YOUR, PERSONAL PROBABILITIES.
HOW DO YOU KNOW THE ELEMENTARY OUTCOMES ARE EQUALLY LIKELY WITHOUT ROLLING THE DICE A BILLION TIMES?
WANNA BET?
OBJECTIVIST
BAYESIAN

Review

- Bayesian Statistics
  - Bayesian Inference
  - MCMC distributions

Bayesian Model Summary

- Bayesian view of the world includes updating/changing beliefs new observations
- Bayesian view takes prior beliefs into account
- Based on Bayes theorem

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

- Can use simplified formulation with no P(B)

$$P(A|B) \propto P(B|A)P(A)$$

Posterior Distribution

The Likelihood

Prior Distribution

Bayes Model Summary

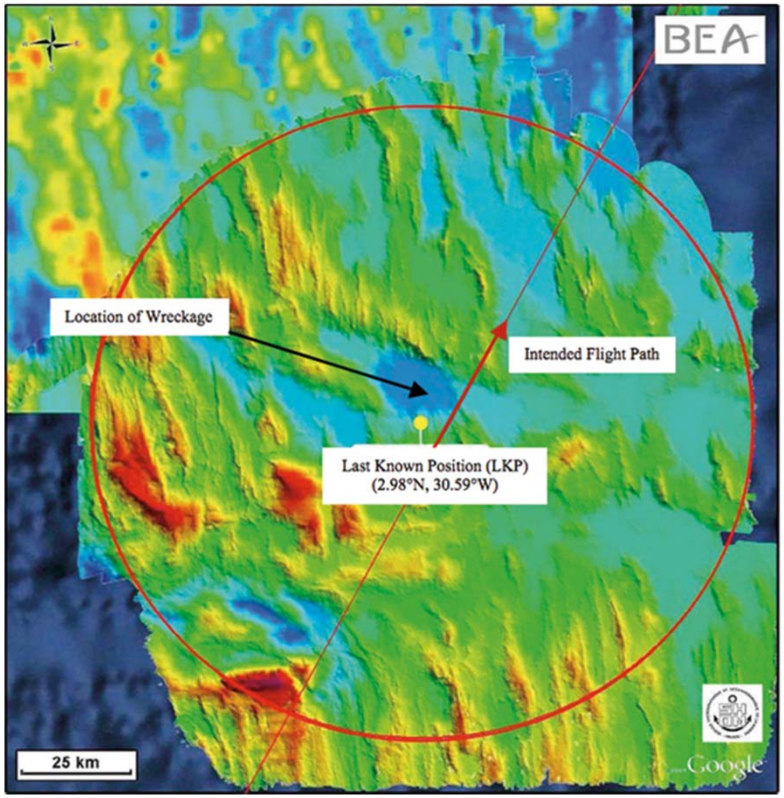
> Use MCMC models to scale Bayesian analysis
  – Metropolis-Hastings Algorithm
  – Gibbs sampling for better convergence

Frequentist	Bayesian
Goal is a point estimate and confidence interval	Goal is posterior distribution
Start from observations	Start from prior distribution
Re-compute model given new observations	Update belief (posterior) given new observations
Examples: Mean estimate, t-test, ANOVA	Examples: posterior distribution of mean, overlap in highest density interval (HDI)

Reading assignment: Bayesian Inference Successes

$$P(parameters|data) \propto P(data|parameters)P(parameters)$$

- Bayesian inference used to successfully to find lost planes. E.g. Air France 447
- https://www.informs.org/ORMS-Today/Public-Articles/August-Volume-38-Number-4/In-Search-of-Air-France-Flight-447

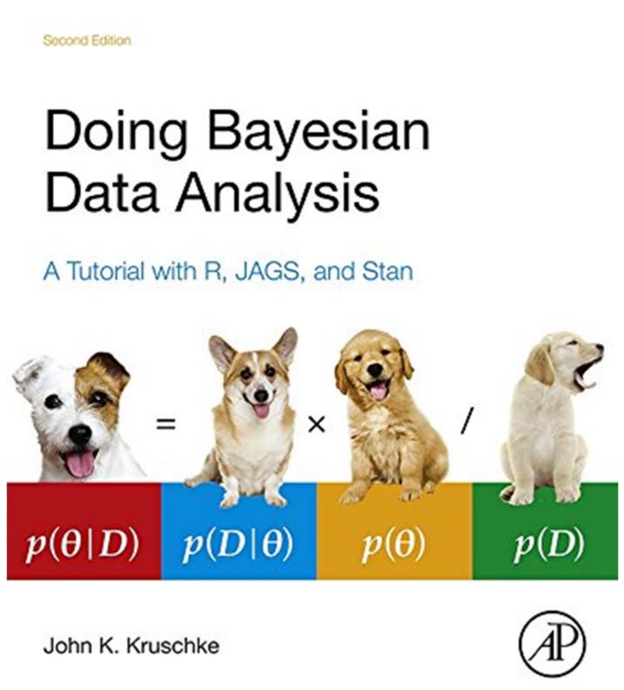


Topics

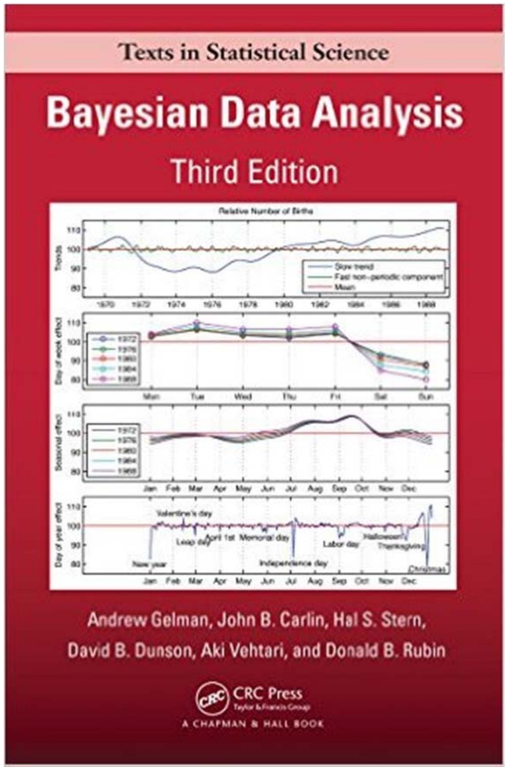
- Bayesian Statistics
  - Multi-level (Hierarchical) models)
  - Bayes factor
  - Bayes hypothesis testing
  - MCMC diagnostics
- Naive Bayes

References

Bayesian modeling is a deep and wide subject



Introductory, but deep text



Seminal book

Multi-level or Hierarchical Bayes Model

Simple Bayes models have all coefficients at same level

$$P(parameters|data) \propto P(data|parameters)P(parameters)$$

- Recall the Beta distribution used as prior for Bernoulli likelihood

  $P(\theta| a, b) \quad = \kappa\, \theta^{(a-1)} (1 - \theta)^{(b-1)}$

- But what if θ is not from a single population?

Multi-level or Hierarchical Bayes Model

How to model real-world hierarchies?

- Sub-populations may behave differently
- How to we partition the our model to account for sub-populations?
- Multi-level or hierarchical models accommodate this structure

Multi-level or Hierarchical Bayes Model

Examples

> Distinguish effect of individual player vs. team
> Performance of students vs. performance of school
> Product sales vs. store sale effect
> Species population vs. habitat

Multi-level or Hierarchical Bayes Model

Can use multi-level models to apply adjustments

> Individual player performance for team performance
> Individual students performance for school performance
> Sales for store effect
> Species population for habitat changes

Multi-level or Hierarchical Bayes Model

Extending Bayesian model

> Bayes rule becomes

$$P(\theta, \omega | D) \propto P(D | \theta, \omega)\, p(\theta, \omega)$$

$$\propto P(D | \theta)\, p(\theta | \omega)\, p(\omega)$$

where

θ = parameters for each sub-group

ω = parameter for population

Multi-level or Hierarchical Bayes Model

Bayes rule for multi-level models

> Hierarchy of priors

$$P(\theta, \omega | D) \propto P(D | \theta)\, p(\theta | \omega)\, p(\omega)$$

Prior Distribution of ω

Posterior Distribution

Prior Distribution of θ given ω

The Likelihood

Multi-level or Hierarchical Bayes Model

Extending Bayesian model

> Bayes rule becomes

$$P(\theta, \omega | D) = P(D | \theta, \omega)\, p(\theta, \omega)$$
$$= P(D | \theta)\, p(\theta | \omega)\, p(\omega)$$

> Example: for beta prior, posterior probability is now:

$$\text{Prior of } \omega = \text{Beta}(A_\omega, B_\omega)$$

$$P(\theta, \omega | D) = \text{Bernoulli}(\theta)\, \text{Beta}(\omega\,(K-2) + 1, (1 - \omega)(K - 2) + 1)$$

Joint Prior

Multi-level or Hierarchical Bayes Model

Extending Bayesian model

> With Bayes rule:

$P(\theta, \omega | D) = P(D | \theta)\, p(\theta | \omega)\, p(\omega)$

> Example: for beta prior, the joint posterior probability is now:

$p_j \sim \text{Beta}(y_j + K\eta,\ n_j - y_j + K(1 - \eta))$

where

$\eta = a / (a+b)$

$K = a + b$

n_j = sample size

y_j = number of hits for player j

Multi-level or Hierarchical Bayes Model

The posterior is proportional to the product of individual probabilities

$$P(\theta, \omega | D) \propto \prod^{N}_{j=1} p_j$$

To simplify computation in example we reparameterize

$\theta_1 = \log[\, \eta / (1 - \eta) \,]$

$\theta_2 = \log(K)$

Bayesian Model Selection

How do we find the best model

> Different likelihood distributions
> Different prior distributions
> Compare hierarchies of models

Compare Performance of Bayesian Models

Bayes Factor – identify the most likely model

> For a single model:

$P[\Theta_1, \Theta_2, \ldots m|D] \propto P[\Theta_1, \Theta_2, \ldots, m]\, P[D|\Theta_1, \Theta_2, \ldots m]$

> Compare (hierarchy) of two models as a ratio:

$$\frac{p(m=1|D)}{p(m=2|D)} \propto \frac{p(D|m=1)}{p(D|m=2)} \quad \frac{p(m=1)}{p(m=2)}$$

> Reduces to

$$\frac{p(m=1|D)}{p(m=2|D)} = \frac{p(D|m=1)}{p(D|m=2)} = Bayes\ Factor$$

Hypothesis Testing with Bayes Models

Use HCI to perform hypothesis tests

- Analogous to bootstrap resample hypothesis test
- Test conditions for **posterior** distribution
  - If HCI overlap; accept Null Hypothesis
  - If no HCI overlap reject Null Hypothesis
- HCI is different from Confidence Interval
  - HCI is for interval with greatest probability mass
  - Difference with CI is greatest for asymmetric prior
- Tests can be one-sided or two-sided

Diagnostics for MCMC

Multiple ways to look at convergence

- Summary statistics
  - Mean, median, se, time series se, quantiles
  - Plot cumulative mean and quantiles
  - Plot trace of each chain
  - Plot posterior distribution
- Plots based on convergence of multiple chains
  - Gelman-Rudin plot of chain convergence
  - Compares shrinkage of between chain and within chain variance
  - Should converge to 1.0

Diagnostics for MCMC

Detect convergence issues with autocorrelation

- High autocorrelation inhibits convergence
- High rejection rate inhibits convergence
- Use ACF
- Effective Sample Size

$$ESS = N \,/\, \left(1 + 2 \sum_{K} ACF(k)\right)$$

Introduction to Naïve Bayes

Naïve Bayes is a remarkably good and flexible classifier

- Widely used classifier
  - Document classification
  - SPAM detection
  - Image classification
- Scales well
  - Does not require a prior
  - Computation linear in number of parameter/features
  - Requires minimal data
  - Simple regularization

Introduction to Naïve Bayes

Simplify the conditional probability calculation

> Starting with Bayes Theorem: $P(A|B) = P(B|A)\frac{P(A)}{P(B)}$

> The probability of class C_k is the joint distribution:

$$p(C_k, x_1, x_2, \ldots, x_n) = p(x_1, x_2, \ldots, x_n, C_k)$$
$$= p(x_1| x_2, \ldots, x_n, C_k)\, p(x_2, \ldots, x_n, C_k)$$
$$= p(x_1| x_2, \ldots, x_n, C_k)\, p(x_2| x_3, \ldots, x_n, C_k)\, p(x_3, \ldots, x_n, C_k)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= p(x_1| x_2, \ldots, x_n, C_k)\, p(x_2| x_3, \ldots, x_n, C_k) \ldots p(C_k)$$

> **But if $\{x_1, x_2, \ldots, x_n\}$ are independent:**

$$p(x_i| x_{i+1}, \ldots, x_n, C_k) = p(x_i, | C_k)$$

Introduction to Naïve Bayes

Simplify the conditional probability calculation

> With $\{x_1, x_2, \ldots., x_n\}$ independent:
>
> $$p(x_i | x_{i+1}, \ldots., x_n, C_k) = p(x_i, | C_k)$$

> The probability of class C_k is the joint distribution:

$$p(C_k | x_1, x_2, \ldots., x_n) \propto p(C_k) \prod^{N}_{j=1} p(x_j | C_k)$$

> And the most likely class y_{hat} is:
>
> $$y_{hat} = \text{argmax}_k \left[p(C_k) \prod^{N}_{j=1} p(x_j | C_k) \right]$$

No Prior

Naïve Bayes Classifiers

Different distributions lead to different classifiers

- Difference Naïve Bayes models are not the same!
- Normal naïve Bayes classifier
- Multinomial naïve Bayes classifier

$$\mathrm{Log}(p(C_k \mid x)) \propto \log[\, p(C_k) \prod^{N}_{j=1} p_{kj}^{\,Xi} \,]$$

$$= \log(\, p(C_k)\,) + \sum^{N}_{j=1} xi \log(\, p_{kj} \,)$$

- Bernoulli naïve Bayes classifier

$$p(x \mid C_k) = \prod^{N}_{j=1} p_{kj}^{\,Xi} (1 - p_{kj})^{(1 - Xi)}$$

Naïve Bayes Document Classification

Use ‘bag of words’ model

> Want the probability of topic C in document D given set of words in topic $\{\mathbf{w_1, w_2, \ldots, w_n}\}$:

$$p(D \mid C) = \prod_{j=1}^{N} p(w_j \mid C)$$

> Spam classification:

$$p(S+ \mid D) \propto p(S+) \prod_{j=1}^{N} p(w_j \mid S+)$$

> Test the hypothesis text is spam:

$$\ln(p(S+ \mid D) / p(S- \mid D)) =$$

$$\ln(p(S) / p(S-)) + \sum_{j=1}^{N} \ln(p(w_j \mid S+)/ p(w_j \mid S-)) > 0$$

Naïve Bayes Pitfalls

A few points to watch out for

- > Multiplication of small probabilities can lead to floating point underflow
  - – Compute with ln(p)
- > If no samples get probability = 0
  - – Product of probabilities = 0
  - – Use Laplace smoother to ensure all p > 0
- > Collinear features can be a problem
  - – Do not exhibit independence
- > Regularization is minor issue
  - – Uninformative feature tends to uniform distribution

Final Projects

Only one week to go!

- This project gives you a chance to demonstrate your knowledge of the topics covered in the course
- You must create your report independently
  - Collaboration with others on the analysis is okay
- Report must contain:
  - Introduction and summary with clearly stated conclusions
  - Support your conclusions based on exploration of data and model results
  - See Florence Nightingale report for example

Final Projects, Continued

- Steps which you must show
  - Exploration of data from several views using graphics and summary statistics as appropriate
    - Demonstrate your understanding of the data relationships and properties
  - Comparison of several models
    - Compare difference classes of models and/or features as required
- R Code must in a professional style
  - Well structured
  - Clean comments
- **Due Monday August 29**
- **NO EXTENSIONS!** University policy