RDA 113: Control Theory

Lecture 1:

What is Control?

It is the ability to dictate, modify and regulate the behavior of a system or something so that it behaves in a certain way. (…..)

Where do we need control?

Examples:

- **To maintain the altitude of an aircraft regardless of wind gust, velocity profile, aircraft inclination**
- **To maintain the speed of a car (cruise control) on the highway as close as possible to 60 kilometers per hour regardless of road inclination, wind resistance and road condition. (….CSBI)**

- **To maintain a desired temperature and pressure level in a reactor vessel regardless of mixture concentration, mass flow rate and external conditions.**

Controlled System

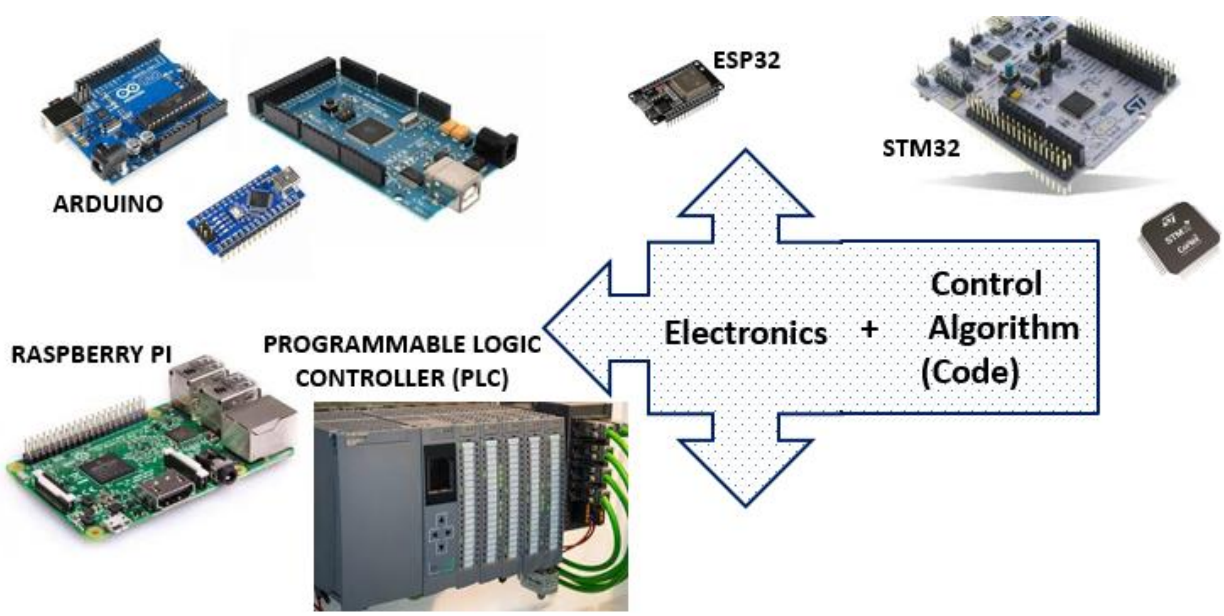
It is a system whose output is controlled to produce a desired result e.g DC Motor of robot car or drone. For a system to be controlled, a controller is required. (…BB)

A controller manipulates the systems inputs via an actuator.

Components of a controller

A controller consists of the following:

- Electronic parts: e.g Arduino, ESP, Raspberry Pi, STM32 , PLC
- Control Algorithm (Code)



Controller Design

For a system to be controlled, the controller must be well designed for the particular task. (…)

A controller for one task might not work for another task or system. In other to effectively design a controller for a system, we need to understand the system. To do this, we need some mathematical tools for predicting the behavior of the system. This process is called System Modelling. And one of the most commonly used ways of predicting system behavior is mathematical modelling. In this course, we will learn; Differential equations, Laplace Transforms etc and their simulations.

There are 3 ways to assess a system's behaviour:

- steady state behaviour,
- transient behaviour or
- both

Steady State Behavior

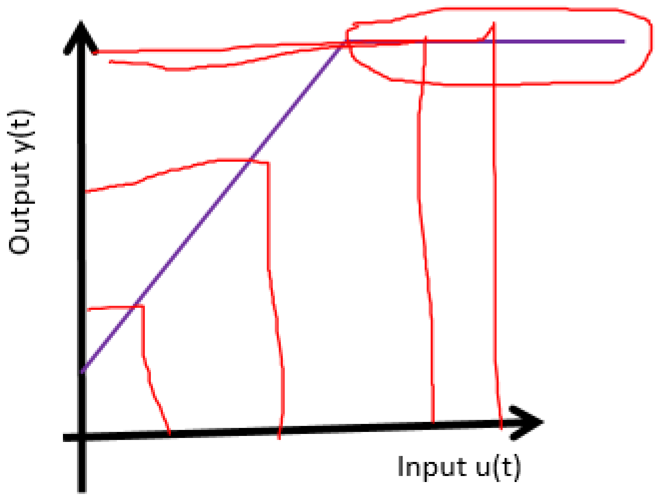
This refers to the condition in which a system has reached a stable and unchanging state in response to its inputs or disturbance. In this state, the system's output or response remains constant. (…graph)

Transient Behavior

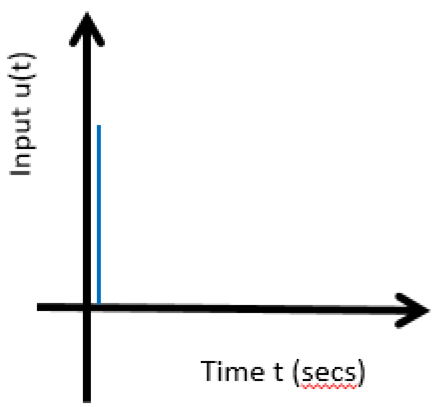
This refers to how the system behaves during transition period i.e when it is adjusting to the changes. Imagine you have a bouncing ball. The period when it is bouncing is its transient behavior.

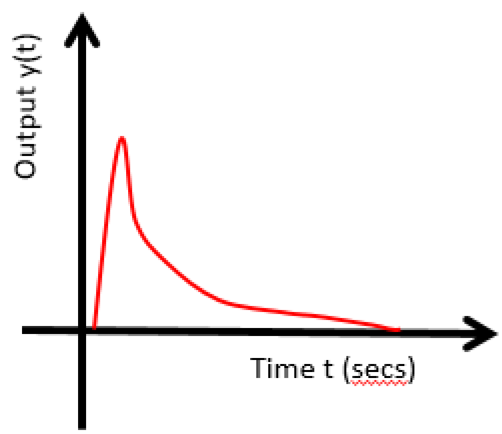
These behaviors can be determined using:

- **Input/output Characteristic:** shows steady state gain and limits.

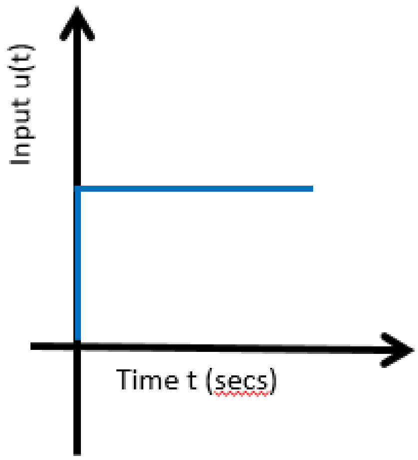


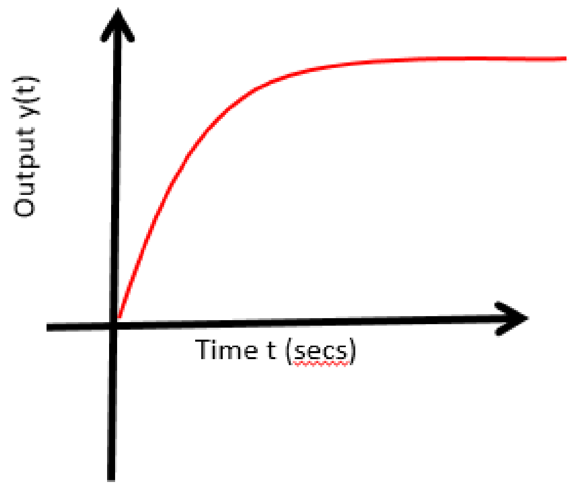
- **Impulse Response:** Shows transient behaviour, i.e, the output in response to an impulse input signal.





- **Step Response:** Shows both transient and steady state behaviour, i.e, the output in response to an step input signal.


Input u(t)
Time t (secs)


Output y(t)
Time t (secs)

FUNDAMENTALS OF CONTROL SYSTEMS

What is a control system?

It is the set of components (mechanical or electrical) designed to regulate, manage and dictate the output of another system to achieve a desired result. OR

A control system is a system, which provides the desired response by controlling the output. The following figure shows the simple block diagram of a control system.

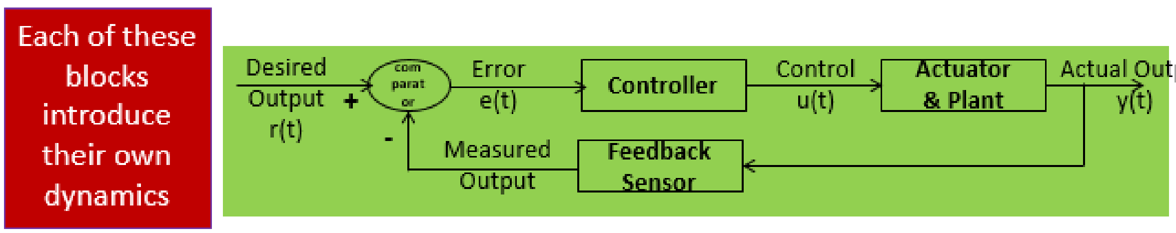
Key terms in Control System

1. **Plant/System:** This is the Process or Machine which has a behaviour that need to be brought under control.
2. **Actuators:** This is a device that drives the operation of the Plant/System.
3. **Controller/Micro-Controller Unit (MCU):** This is the device which helps to regulate the behaviour of the Plant/System. This consists of a processor, memory, I/O devices and power circuitry.
4. **Feedback Sensors:** These are components which can measure or detect certain behaviour in a plant/system (using appropriate transducers), and transmit such information to the MCU.
5. **Controlled Variable y(t) :** This refers to the parameter characterising the behaviour of the plant/system which is to be controlled.

6. **Manipulated Variable/Control Signal u(t) :** This is the input parameter of the plant/system which when varied does influence a change in its behaviour under focus.

7. **Reference or Set Point or Desired Output r(t) :** This is the value which the controller must drive the controlled variable to.

8. **Error Variable e(t) :** This is simply the difference between r(t) and the y(t)



Classification of Control Systems

Based on some parameters, we can classify the control systems into the following ways.

Continuous time and Discrete-time Control Systems

Control Systems can be classified as continuous time control systems and discrete time control systems based on the **type of the signal** used. In **continuous time** control systems, all the signals are continuous in time. But, in **discrete time** control systems, there exists one or more discrete time signals.

SISO and MIMO Control Systems

Control Systems can be classified as SISO control systems and MIMO control systems based on the **number of inputs and outputs** present. **SISO** (Single Input and Single Output) control systems have one input and one output. Whereas, **MIMO** (Multiple Inputs and Multiple

Outputs) control systems have more than one input and more than one output.

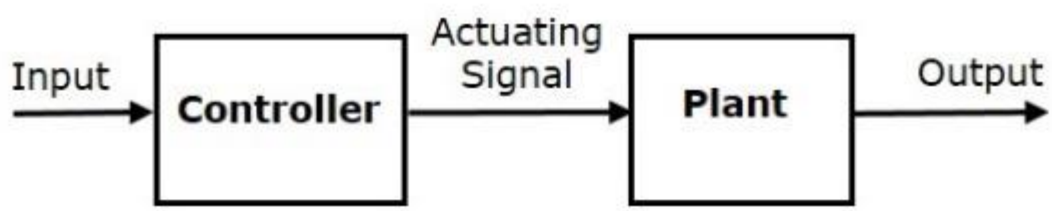
Open Loop and Closed Loop Control Systems

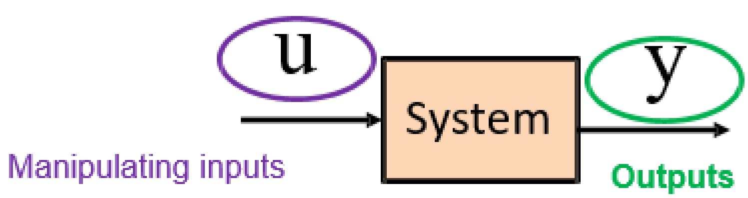
Control Systems can be classified as open loop control systems and closed loop control systems based on the **feedback path**.

Open Loop Control

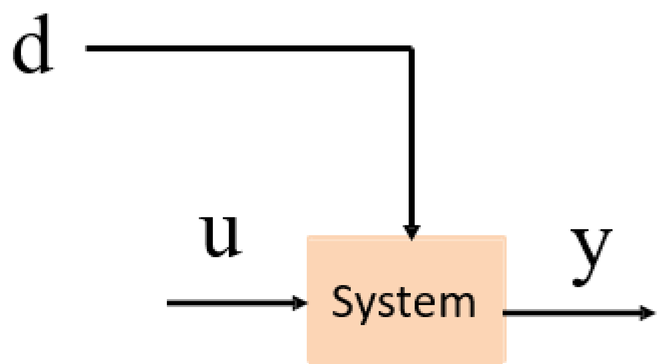
In open loop control systems, output is not fed-back to the input. So, the control action is independent of the desired output.

The following figure shows the block diagram of the open loop control system.





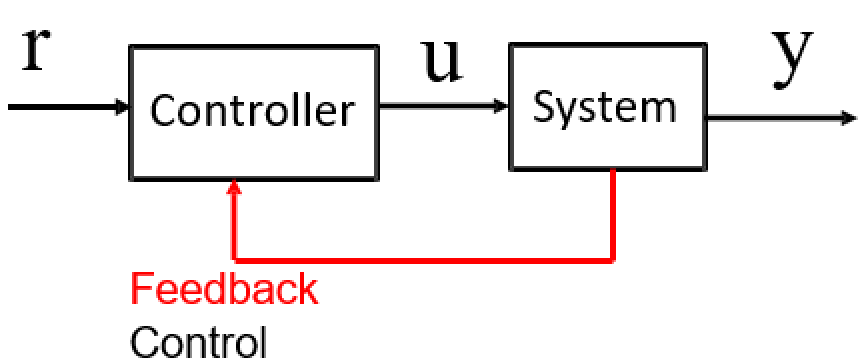
Open Loop systems do not account for external disturbances



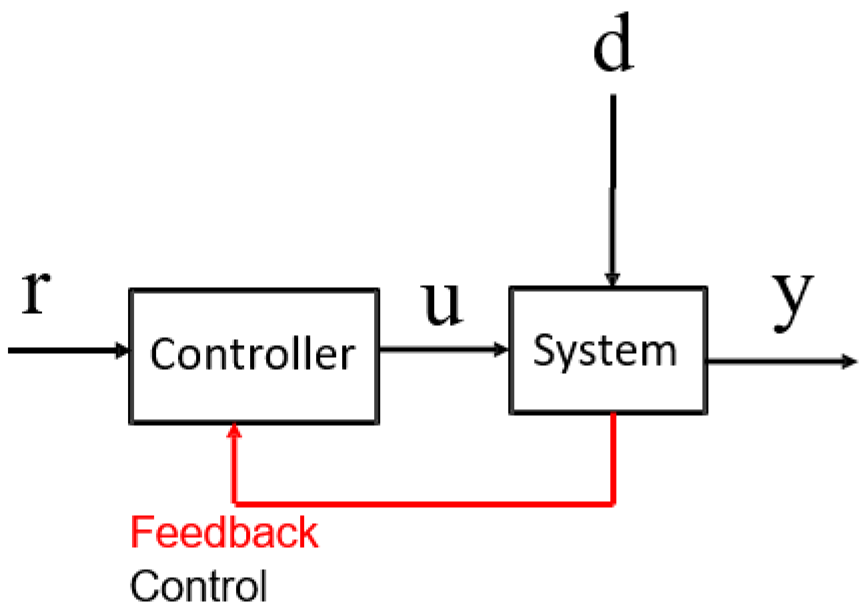
In the diagram, d is disturbance. Since Open loop systems do not account for unknown external disturbance, the system would misbehave in the presence of the disturbance. For example, ambient temperature (surrounding temperature), wind speed and material composition can be disturbance to a system. This makes Open loop controls UNCERTAIN.

Close Loop Control

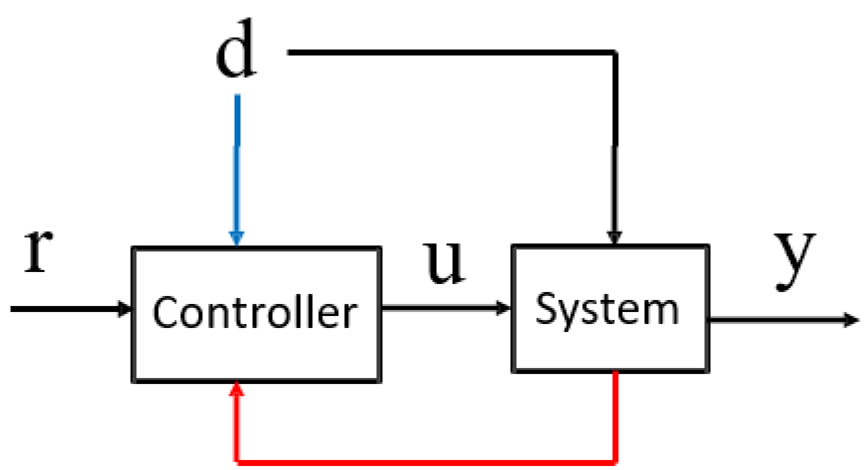
In closed loop control systems, output is fed back to the input. So, the control action is dependent on the desired output. This is shown below.



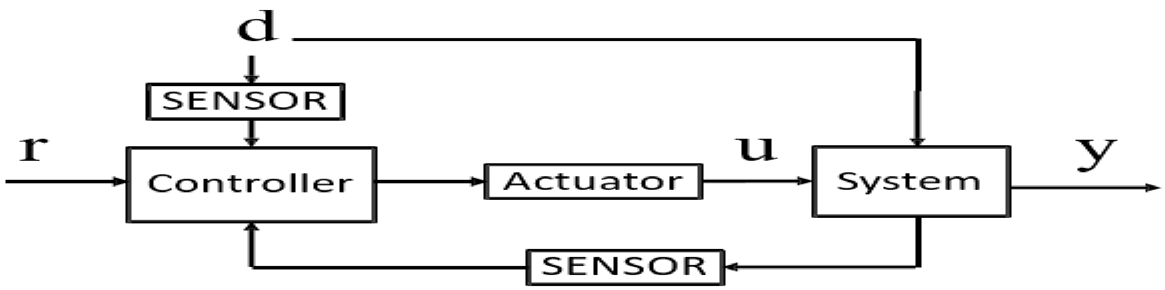
When there is an external disturbance, the system readjusts to the disturbance as shown below.



If we can estimate or anticipate the disturbance, we can use it in planning of our controller by feeding or telling our controller about the disturbance in advance and this is called: **Feedforward** control as shown below.



We should also note that both our Feedback and Feedforward loops need sensors as shown in the block.



Defining a Control Problem

Control Problem: Maintaining a comfortable temperature in a house using a thermostat.

For every control problem, we have to define two things (in most cases). They are, the components and the Process. Note that the process describes the activity.

For the control problem we have above,

Components:

Thermostat (Thermometer): The control device that measures the current temperature and allows you to set a desired temperature.

Heating or Cooling System: The furnace, air conditioner, or heater that regulates the temperature in the room.

Process:

Measurement: The thermostat measures the temperature in the room at every instance.

Comparison: It compares the current temperature of the room to the temperature we set as the target.

Decision: If the current temperature is lower than the set temperature, the thermostat sends a signal to the heating system to turn on. If it's higher, it signals the cooling system to turn on.

Control Action: The heating or cooling system adjusts its operation to increase or decrease the room's temperature accordingly.

Feedback Loop: The thermostat continues to monitor the temperature and adjusts the heating or cooling system as needed to maintain the desired temperature.

In a more broad approach, we need to understand the following about the problem;

- What are we trying to control?
- What are the manipulated variables?
- What are the possible disturbance? Should we neglect or take it into account?

For our above problem, the controlled variable (the variable we are trying to control) is the temperature while the manipulated variable (the thing that affects the temperature) is the heating or cooling system.

Assignment

Solve the following control problems: and for each case state the controlled and manipulated variable.

1. Water Tank Level Control
2. Self-balancing robot

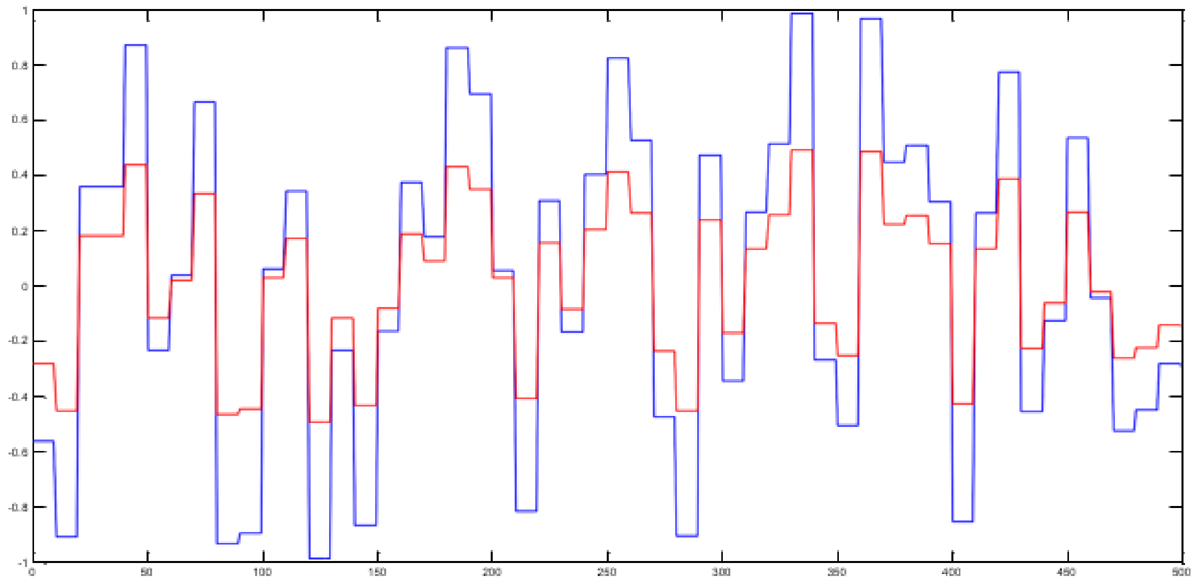
Mathematical Modelling of System Dynamics

Systems Dynamics /Behaviour

This refers to how a given input to a system yields or produces a given output. Imagine a scenario where you are told to stand up, you obey the order and stand. But do you know the process that led to you standing? The input is the command given to you. Your brain gets this impulse and process it then sends a response to target parts. This is the same Idea in system dynamics. We are referring to how the system behaves.

Therefore, in system dynamics, we are trying to understand the relationship between input and output.

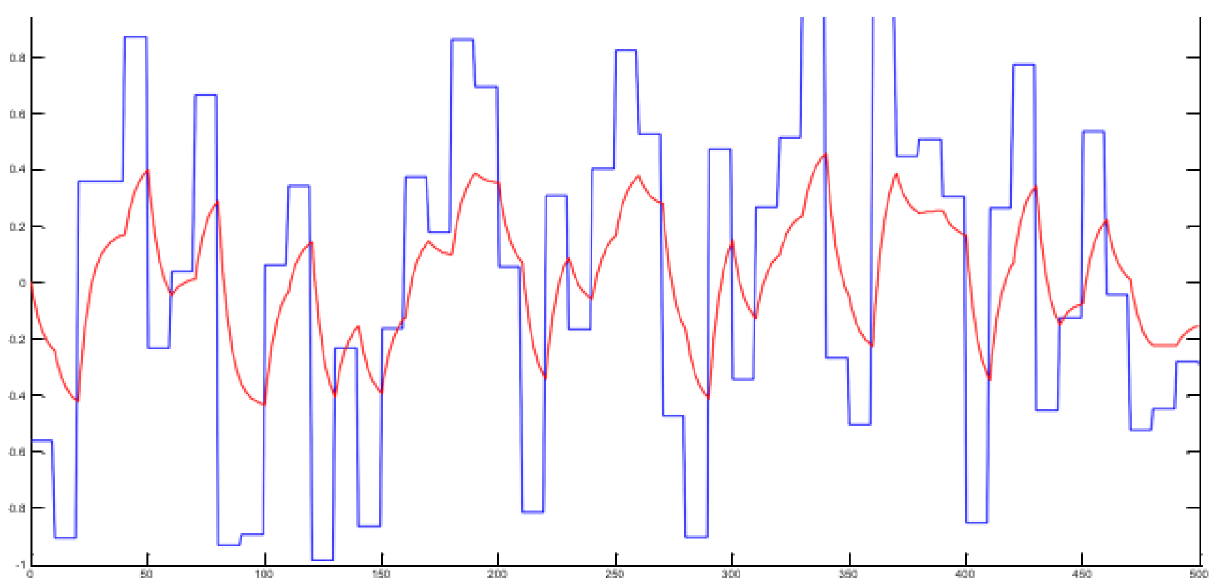
Consider a system where the input/output relationship is output = 0.5 * input. This means that when input is 10, the output is 10 * 0.5 which gives 5 and so on. But this would be the case in ideal an condition. But in the real world, our output might be affected by external factors example delay. Let's use the graph to explain further.



Ideal output to a given input

N.B: blue is input and red is output.

But in reality, we have this



Real output by system

We see that the output does not follow the input as we thought. It takes time for it to get to the desired value. Every system has an inertia and so does not respond immediately.

The inertia we talked about can vary in deferent systems e.g in a mechanics, the inertia is the mass of the object. In electrical system, the inertia might be resistance.

Now we would try to understand the dynamics of a simple mechanical system. This is a system where an objects falls freely under the influence of gravity. Ensure you listen to the explanations and ask question.

A Free Falling Object

Force of GRAVITY → **Object** → VELOCITY

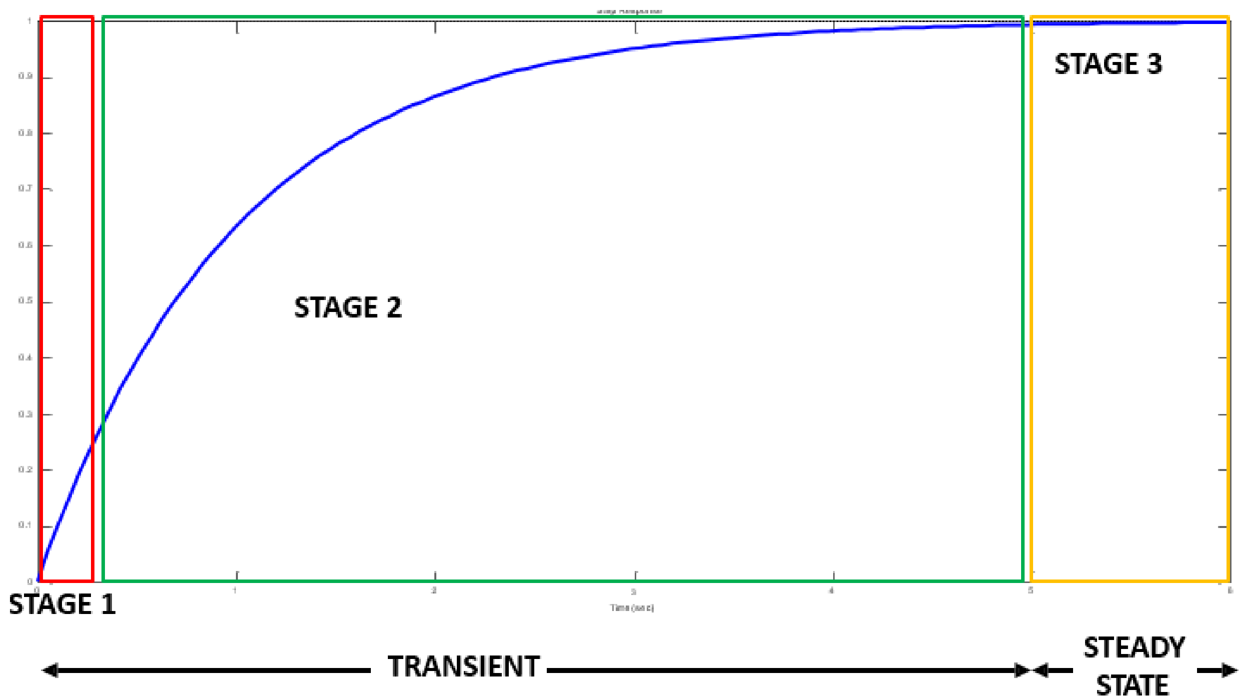
$$\textit{Force of Gravity} - \textit{Drag Force} = \textit{Resultant Force}$$

$$F_G - b\,v = ma$$

$$F_G - b\,v = m\frac{dv}{dt}$$

Where;

F_G = Force of Gravity	dv = infinitesimal change in v	dt = infinitesimal change in t
M = mass of object	v = velocity of object	$\frac{dv}{dt}$ = time rate of change of v
t = time	a = acceleration	b = drag coefficient



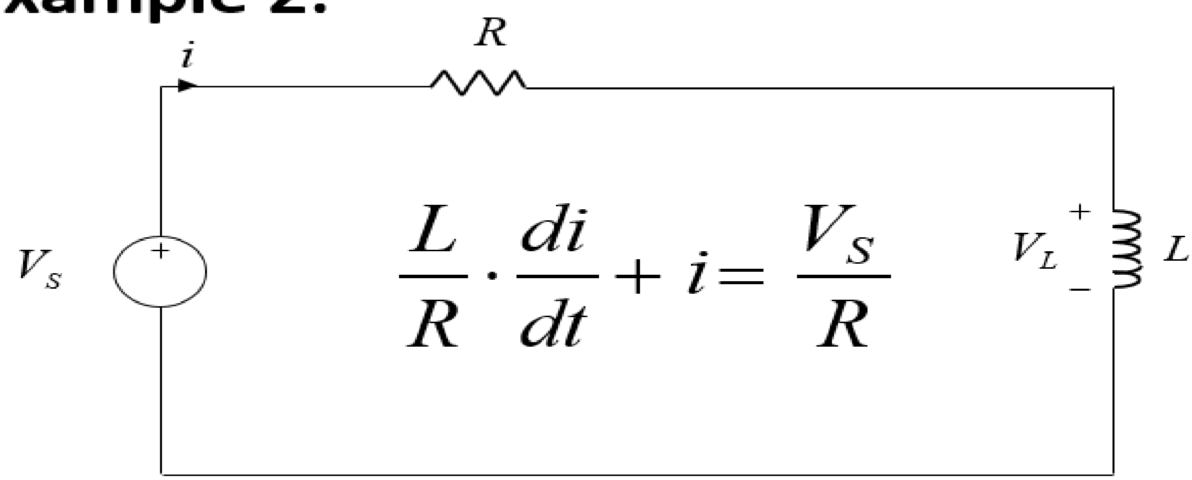
Based on our analysis, the graph above shows the response of our mechanical system; showing the transient state and steady state.

Dynamics of Electrical Circuit System

A control engineer must understand the dynamics of the electrical system before he starts designing a controller of the system and that means understanding the Current and Voltage relationships.

Dynamic System – RL Circuit

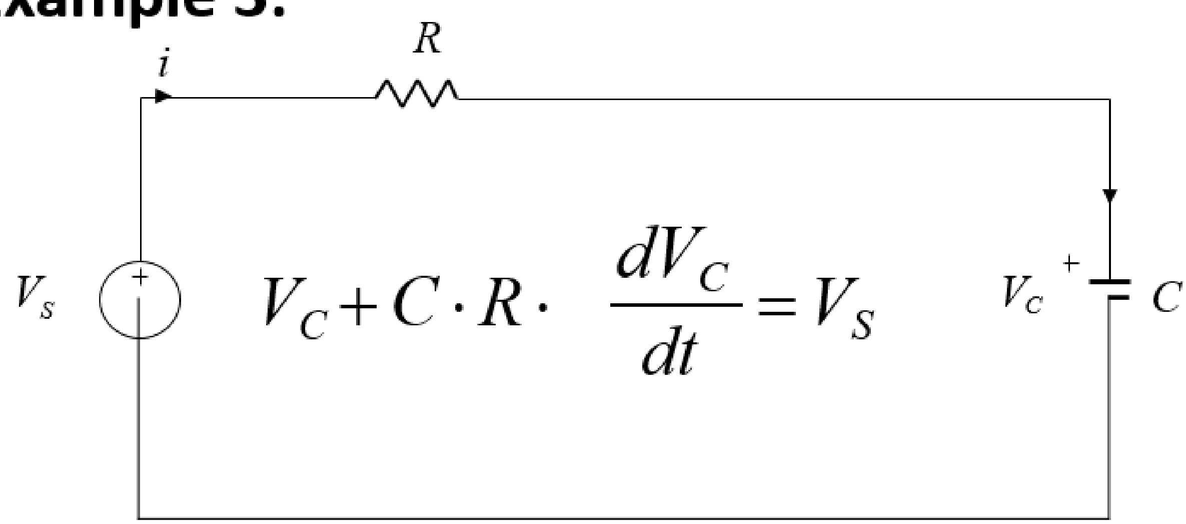
Example 2:



$$\frac{L}{R}\cdot\frac{di}{dt}+i=\frac{V_S}{R}$$

Dynamic System – RC Circuit

Example 3:



$$V_C+C\cdot R\cdot\frac{dV_C}{dt}=V_S$$

After the analysis of the systems above, the summary is shown above.

The examples considered so far are:

First order differential equations

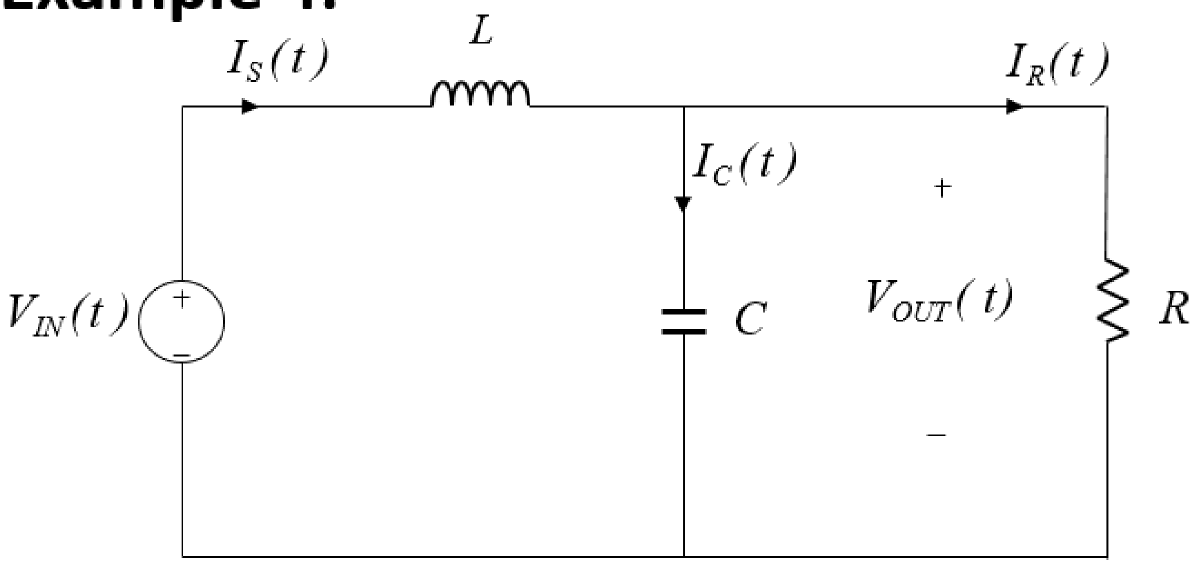
$$\boxed{C \cdot R \cdot \frac{dV_C}{dt} + V_C = V_S}$$

$$\boxed{\frac{L}{R} \cdot \frac{di}{dt} + i = \frac{V_S}{R}}$$

$$\boxed{\frac{m}{b} \cdot \frac{dv}{dt} + v = \frac{F}{b}}$$

Dynamic System – RLC Circuit

Example 4:

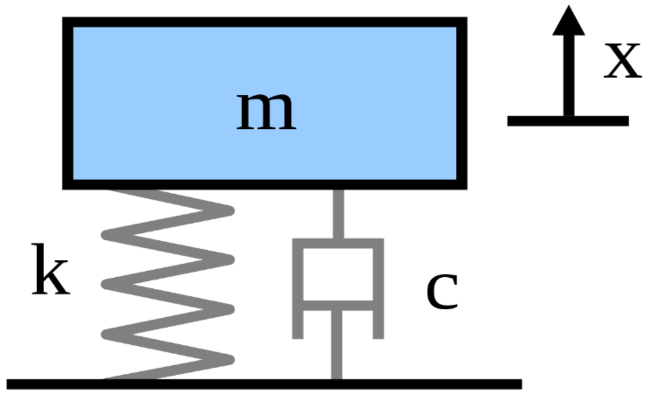


$$\boxed{L \cdot C \cdot \frac{d^2 V_{OUT}(t)}{dt^2} + \frac{L}{R} \cdot \frac{dV_{OUT}(t)}{dt} + V_{OUT}(t) = V_{IN}(t)}$$

The above system is a second order system.

Assignment

Develop a mathematical model of a mass and spring system shown below.


m
x
k
c

S-Domain: Laplace Transform

Solving control problems whose mathematical model are differential equations as learned earlier, most times become problematic. Hence the need for a better, “easier” way to solve. One of this ways is to transform the equation from time domain to S-domain. This can be achieved using Laplace Transforms.

Mathematically, the Laplace transform of a function f(t) is defined as:

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Where S is a complex frequency domain.

So it means that we can easily convert differential equations in to algebraic equations. Example

$$\boldsymbol{\frac{d^2v}{dt^2}+\frac{dv}{dt}+v(t)} \longrightarrow \boldsymbol{s^2V+sV+V}$$

$$\boldsymbol{\frac{d}{dt}} \longrightarrow \boldsymbol{s}$$

$$\boldsymbol{\frac{d^2}{dt^2}} \longrightarrow \boldsymbol{s^2}$$

$$\boldsymbol{v(t)} \longrightarrow \boldsymbol{V}$$

Properties of Laplace Transform

There are features that make Laplace Transforms very useful. They include:

1. **Linearity**

   In simple terms, the linearity property of the Laplace transform means that when you apply the Laplace transform to a linear combination of functions (e.g., sums or scaled versions of functions), the result is equal to the linear combination of the individual Laplace transforms of those functions. Mathematically,

$$L[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

2. **Differentiation**

   Mathematically,

$$Laplace\left(\frac{dv(t)}{dt}\right) = s \cdot V(s) - v(t=0)$$

   With all initial conditions set to zero.

3. **Integration**

$$Laplace \int v(t)dt = \frac{1}{s} \cdot V(s)$$

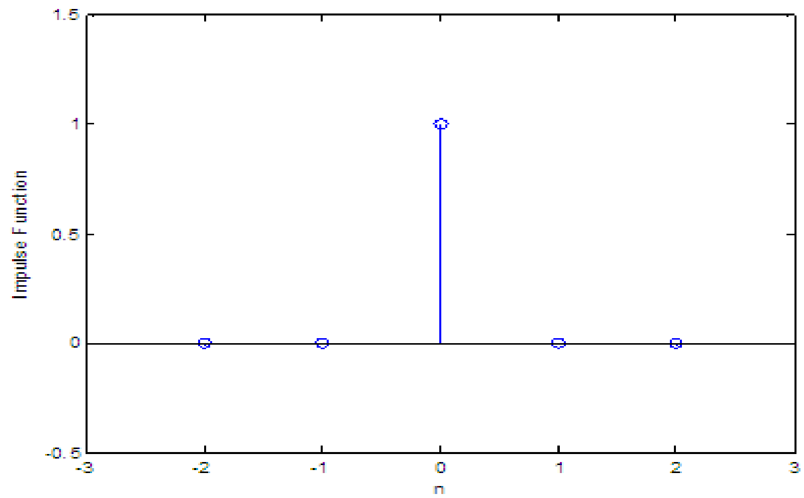
4. **Time Delay**

$$Laplace[x(t-\tau)] = e^{-s \cdot \tau} \cdot V(s)$$

Types of Functions/Input signals

1. Dirac Delta or Impulse

The Dirac delta function $\delta(x-\sigma)$, also called the impulse function, is usually defined as a function which is zero everywhere except at $x=\sigma$, where it has a spike such that it's length is infinite and the area under the line or spike is 1.



Example

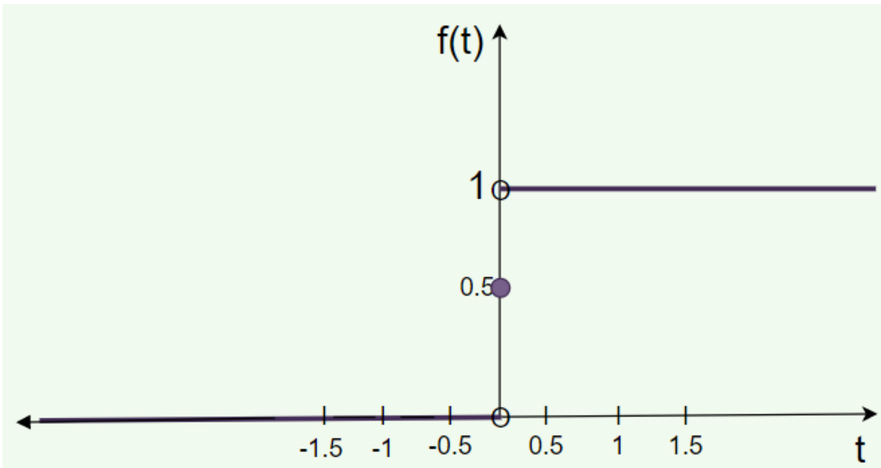
- Dirac of t i.e $\delta(t)$ will mean that it has a value of 0 everywhere except at t = 0. Observe the plot I will draw.
- Dirac of t-3 i.e $\delta(t-3)$ will mean that it has a value of 0 everywhere except at t = 3. Observe the plot I will draw.
- Dirac of t+5 i.e $\delta(t+5)$ will mean that it has a value of 0 everywhere except at t = -5. Observe the plot I will draw.

We use the Dirac Delta function when we want to get the impulse response of our system.

N.B $\delta(t-\sigma)$ in S-domain is $e^{-\sigma s}$

2. Step Function/ Heaviside Function step function

The step function u(t- σ) is a function that is equal to 0 for all values of t less than 0.
At t = σ, it jumps to a different value, typically 1, and remains at that value for all t greater than or equal to σ.



Example

- Step of t i.e u(t) will mean that it has a value of 0 everywhere except at t >= 0 where the value will be 1. Observe the plot I will draw.
- Step of t-3 i.e u(t-3) will mean that it has a value of 0 everywhere except at t> = 3. Observe the plot I will draw.
- Step of t+5 i.e u(t+5) will mean that it has a value of 0 everywhere except at t >= -5. Observe the plot I will draw.

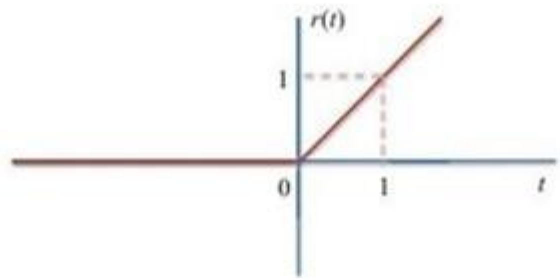
We use Step function when we want to give input to our controller.

N.B the Step function u(t- σ) in S-domain is 1/s * $e^{-\sigma s}$

3. Ramp Function

The ramp function r(t- σ), is a mathematical function used to describe linear growth or increase over time. The ramp function is equal to 0 for all values of t less than 0.

For t greater than or equal to 0, it increases linearly with a slope of 1



Example

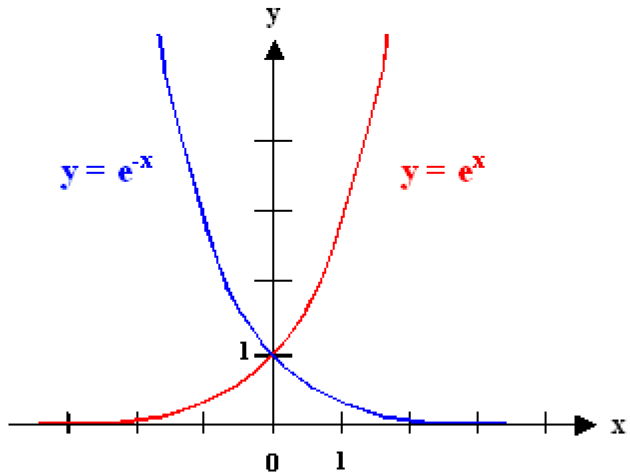
- Ramp of t i.e r(t) will mean that it starts from 0. Observe the plot I will draw.
- Ramp of t-3 i.e r(t-3) will mean that it starts at t = 3. Observe the plot I will draw.
- Ramp of t+5 i.e r(t+5) will mean that it starts at t = -5. Observe the plot I will draw.

We can use Ramp functions to create zig-zig functions.

N.B the S-domain of the Ramp function is $1/s^2$

4. Exponential function

An exponential function is a mathematical function in which a variable appears as an exponent. It is typically written in the form of $f(t) = e^{-at}$ or $f(t) = e^{at}$



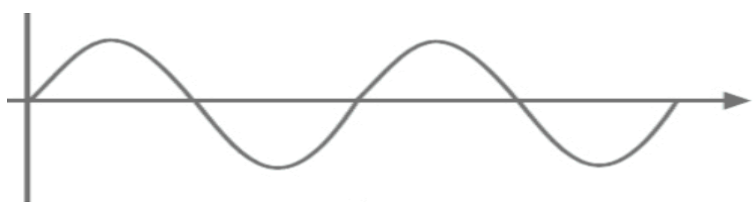
In S-domain, we have;

$\frac{1}{s+a}$ and $\frac{1}{s-a}$

Usually, we stick with the first because it has bounds that means it ends.

5. Sinusoidal Function

The term sinusoidal is used to describe a curve, referred to as a sine wave or a sinusoid, that exhibits smooth, periodic oscillation. It is of the form **sinwt** or **coswt**



In S-domain, it is

$$\frac{\omega}{s^2+\omega^2} \quad or \quad \frac{s}{s^2+\omega^2}$$

N.B first is for sine and the second is for cosine

We don’t have to always derive or cram the Laplace transforms of functions we can always search on the internet or refer to a Laplace Transform table as shown here.

f(t)	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > 0$
t^n, $n = 1,2,\ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2 + b^2}$, $s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}$, $s > 0$
$e^{at}t^n$, $n = 1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$

Another Useful Mathematical Tool

Final Value Theory

Final value theory, in simple terms, is a concept in control theory that helps predict where a system will settle after a disturbance or change. Or we can say that it helps us to know where steady state will occur.

Mathematically,

$$\lim_{t \to \infty} x(t) = x_{SS} = \lim_{s \to 0} \left(s \cdot X(s) \right)$$

Pay attention to the explanation as we find the final value of some of the functions we studied earlier.

Transforming Differential Equations

Observe how we transform the following equations because it would take us into transfer function of a system.

1. $$\frac{dx(t)}{dt} = u(t)$$

2. $$\frac{dx(t)}{dt} + x(t) = u(t)$$

3. $$2 \cdot \frac{dx(t)}{dt} + x(t) = u(t)$$

4. $$\frac{d^2 x(t)}{dt^2} = u(t)$$

5. $$\frac{d^2 x(t)}{dt^2} + 2 \cdot \frac{dx(t)}{dt} + x(t) = 5 \cdot u(t)$$

Transfer Function

This is nothing complex. It is what we have been solving earlier. Transfer function of a system is simply the ratio of the output of the system Y(s) to the input U(s). We should note that the transfer function is represented by G(s) and is in S-domain.

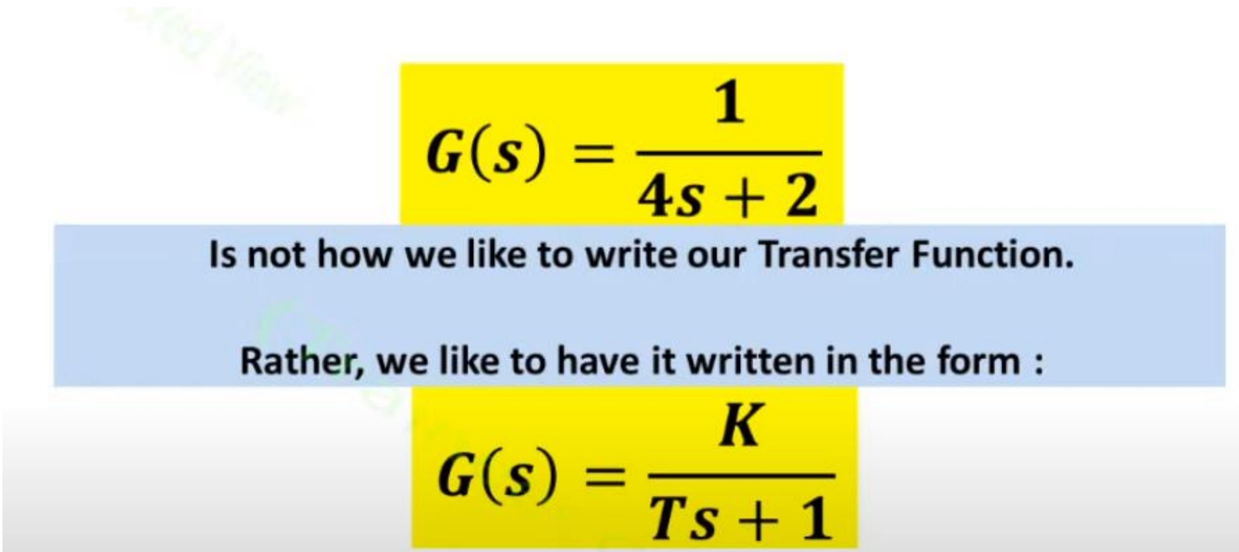
From now on, when we talk about the behavior of a system, we are referring to its Transfer function.

Poles and Zeros

The poles of a system's transfer function is the value(s) of S which makes the transfer functions undefined or infinity. While zeros are the values of S which makes the equation zero.

Pay attention to the explanations

Quick note



Where K is the open loop gain and T is the time constant. As a control Engineer, we want to see your K and your T. Because, with your K, we will know what the gain at Steady state will be and with your T, we will know how the system gets to the steady state.

Pay attentions to examples given.

$$G(s) = \frac{1}{4s + 2} \longrightarrow G(s) = \frac{0.5}{2s + 1}$$

How? Listen…………

Exercise

Determine the Open Loop gain and Time constants of the following First order equations and hence their poles and zeros.

a) $G(s) = \dfrac{1}{6s + 10}$

b) $G(s) = \dfrac{12}{6s + 5}$

c) $G(s) = \dfrac{1}{5s + 20}$

d) $G(s) = \dfrac{1}{6.3\, s + 0.5}$

e) $G(s) = \dfrac{1}{6.3\ + 0.5s}$

f) $G(s) = \dfrac{1}{10\ + s}$

Inverse Laplace Transform

Inverse Laplace can be obtained from Inverse tables obtained online. But most times, another mathematical technique is needed and this technique is called Partial Fractions.

But we can always seek external help (online) when we need that. For this course not to get boring and confusing, we are going to skip that part and proceed to simulations but remember you have to seek help !!!.

Step Response

The step response of a system is the output Y(s) of a system when you give it a unit step function 1/s (in s-domain) as an input. Mathematically,

$$Y(s) = G(s) \cdot \frac{1}{s}$$

Y(s) is the step response in s-domain,

G(s) is transfer function

1/s is the unit step

N.B pay attention to the examples given.

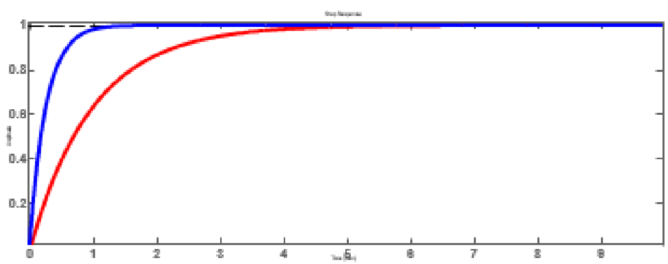
Quick Note:

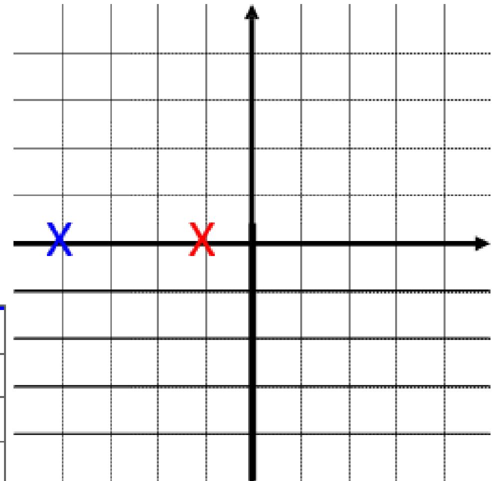
When your system's pole is further away from the Y-axis (on the negative), it means your system will have a smaller response time and hence stable.

Example is shown below.

First-Order System

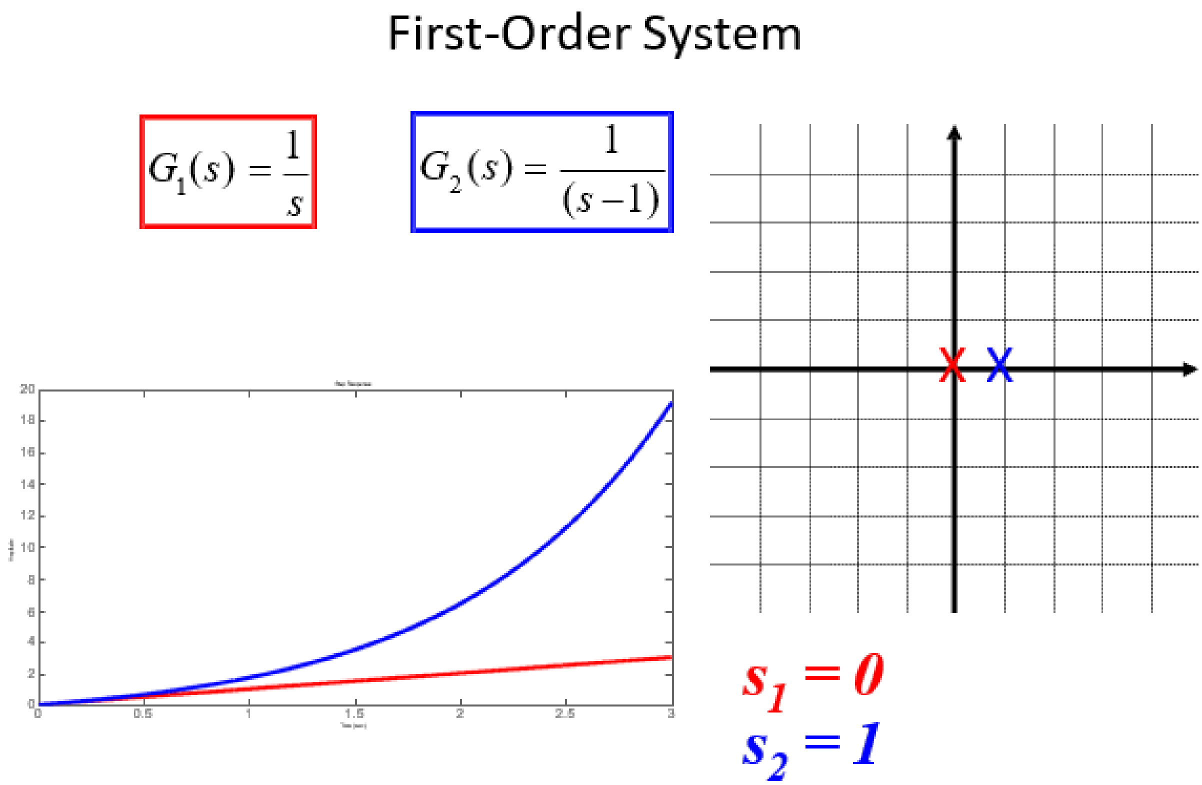
$$\boxed{G_1(s) = \frac{1}{s+1}} \qquad \boxed{G_2(s) = \frac{4}{(s+4)}}$$





$s_1 = -1$

$s_2 = -4$



The system is unstable here because it has a pole on the positive X-axis.

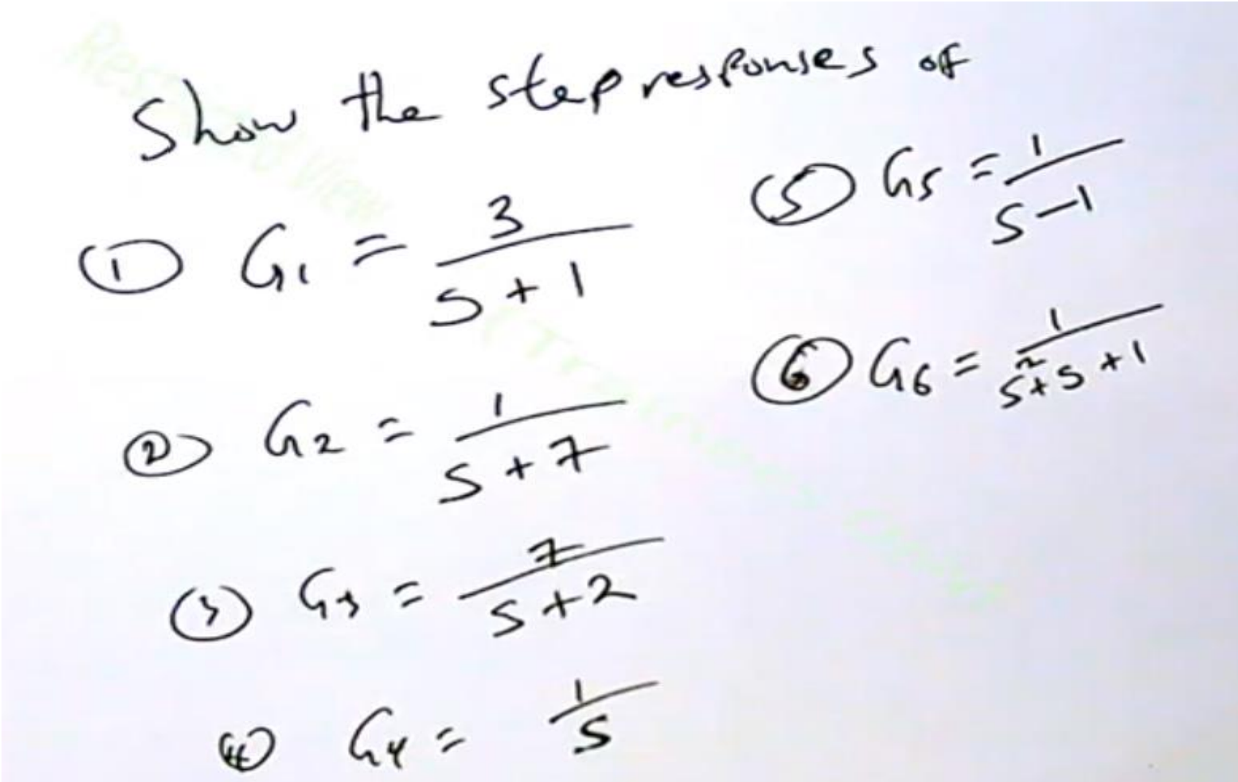
Control Simulations Using python Programming

As control engineers, we need ways to be able to visualize or system. So we can easily make adjustments where needed. One way we can do this is using Python programming since it can make our work easier.

Packages to Install

- **Python Software**
- **Numpy**
- **Scipy**
- **Matplotlib**
- **Sympy**
- **Control**

The systems we will simulate are shown.



Simulation!!!!!

Simulation!!!!!

Simulation!!!!!

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