Stats 6545 Project 1

Emmeline Williams

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1.a)

The probability density function for Binomial$(10, \frac{1}{3})$ is

$$f(k) = \binom{10}{i}(\frac{1}{3})^i(\frac{2}{3})^{10-i}$$

The possible number of successes here are $k = 0, 1, ..., 9, 10$. For each, I calculate the CDF, which is

$$F(k) = \sum_{i=0}^{k} \binom{10}{i}(\frac{1}{3})^i(\frac{2}{3})^{10-i}$$

For each sample from the uniform distribution, I check which interval $[F(i), F(i+1))$ it falls into and sample $k = i$.

Plotting a histogram over 1000 samples shows that the outcomes do in fact follow a Binomial$(10, \frac{1}{3})$ distribution as expected.

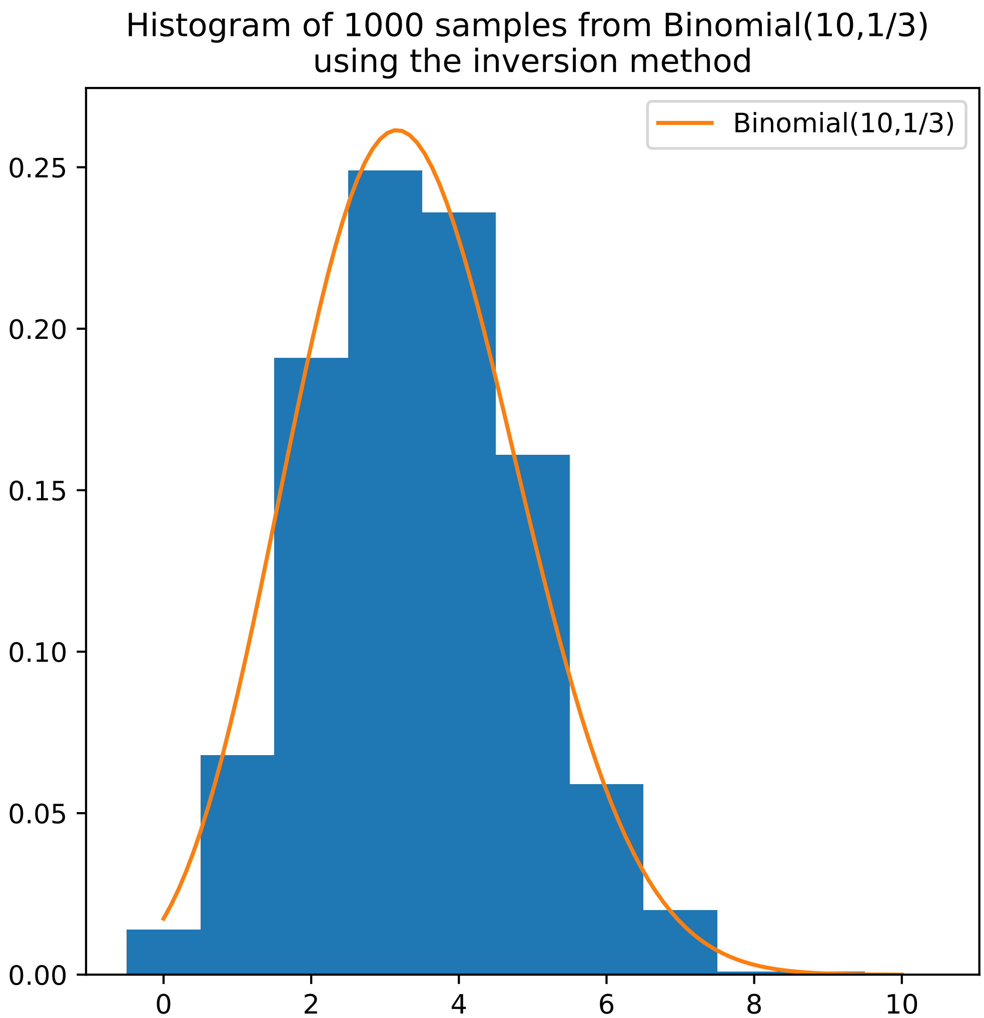


Figure 1: (1.a)

1.b)

The Binomial$(10, \frac{1}{3})$ distribution equals the sum of 10 independent Bernoulli$(\frac{1}{3})$ distributions. This can be shown by comparing the moment generating functions of each distribution. The Bernoulli PDF is $f(k) = \begin{cases} 1/3 \text{ if } k = 1 \\ 2/3 \text{ if } k = 0 \end{cases}$ and the

MGF is

$$E_{Ber}(e^{tk}) = \frac{1}{3}e^{t(1)} + \frac{2}{3}e^{t(0)} = \frac{1}{3}e^{t} + \frac{2}{3}.$$

The Binomial MGF is

$$E_{Bi}(e^{tx}) = \sum_{k=0}^{10} \binom{10}{k} (\frac{1}{3})^k (\frac{2}{3})^{10-k} e^{tk} = \sum_{k=0}^{10} \binom{10}{k} (\frac{e^t}{3})^i (\frac{2}{3})^{10-k} = (\frac{1}{3}e^t + \frac{2}{3})^{10} = (E_{Ber}(e^{tk}))^{10}$$

by the binomial expansion.

I use inversion to sample a Bernoulli($\frac{1}{3}$) distribution from a uniform distribution based on whether the uniform sample is greater than or less than 1/3. I take 10 of these samples and take their sum as a sample from the Bernoulli distribution.

Plotting the histogram again with 1000 samples shows that this joint inversion and transformation method does result in the expected Bernoulli distribution.

Both methods seem similarly accurate.

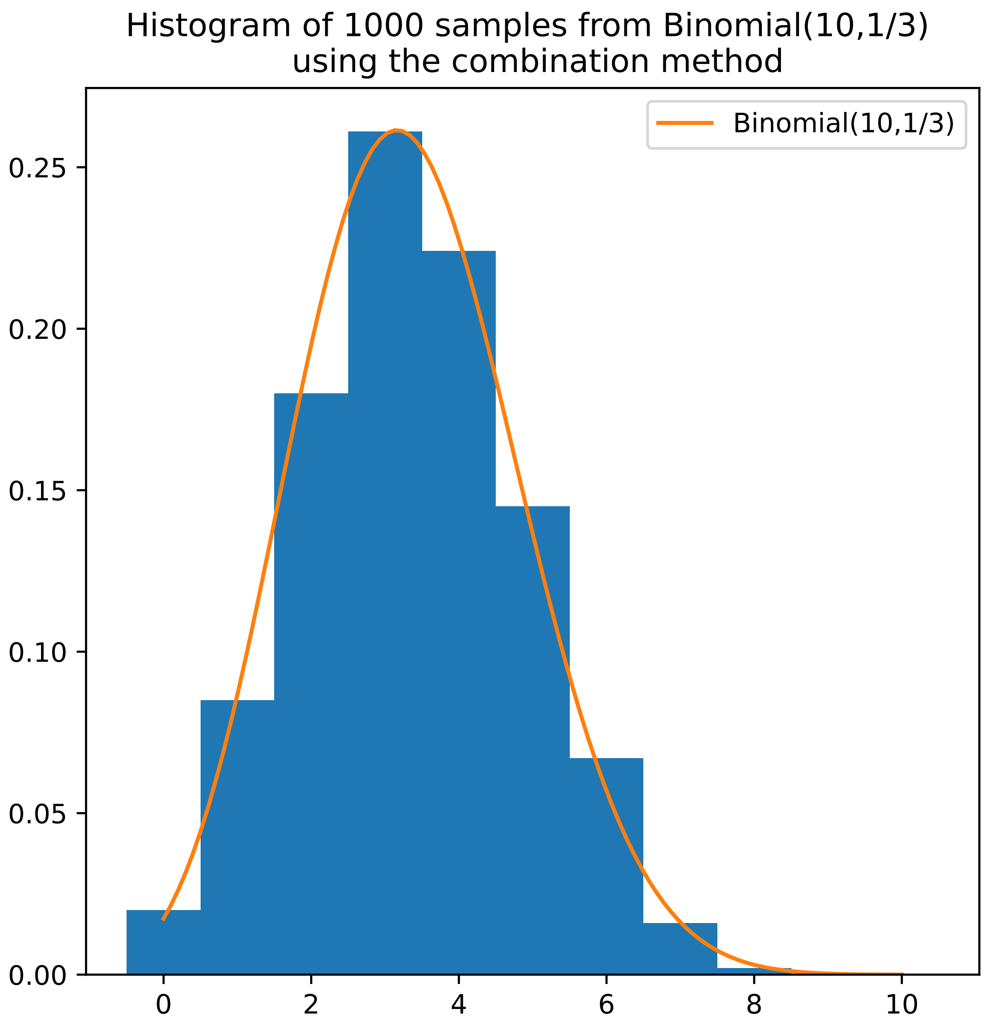


Figure 2: (1.b)

1.c)

The Central Limit Theorem says that for n iid variables taken from a distribution with expectation μ and variance σ^2, in the limit of large n, the sample mean converges to μ and the sample variance converges to $n\sigma^2$. It approximates a normal distribution $\mathrm{N}(\mu, \sigma^2)$.

Using 100 samples from the first method, the sample mean is 3.28, giving an estimate of the expectation of the Binomial distribution as $\mu = 3.28$. The sample variance is 2.00 giving an estimate $\sigma^2 = 2.00/100 = 0.02$ and a standard error $\sigma = \sqrt{\sigma^2} = 0.14$.

Using the normal approximation $\mathrm{N}(\mu, \sigma^2)$ to the Binomial, a 95% confidence bound on the expectation is

$$(\mu \mp z_{0.025}\sigma) = (3.28 \mp 1.96 * 0.14) = (3.00, 3.56).$$

A 99% confidence bound is

$$(\mu \mp z_{0.005}\sigma) = (3.28 \mp 2.576 * 0.14) = (2.91, 3.65).$$

Both intervals contain the true expectation, $np = 10/3 = 3.33$. The true variance is $np(1-p) = 2.22$.

2)

The Poisson distribution is $f(k, \lambda) = e^{-\lambda} * \frac{\lambda^k}{k!}$. We know that if X_i are Exponential(1) iid random variables, taking the sum $S_n = \sum_{i=1}^{n} X_i$ gives a probability $P(S_n \leq t \leq S_{n+1}) = e^{-\lambda} * \frac{\lambda^k}{k!}$, which is precisely the Poisson probability. Exponential(1) has PDF $f(x) = \begin{cases} e^{-x} & \text{if } 0 \leq x \\ 0 & \text{if } x < 0 \end{cases}$. If u is drawn from Uniform$[0, 1]$, then $x = -\ln(1-u)$ follows Exponential(1) using the inversion method. Taking repeated samples until their sum is greater than $t = 1$ gives a sample from the Poisson(t) distribution.

A histogram with 1000 samples shows that the generated samples do follow the Poisson(t) distribution. As before, I apply the central limit theorem to obtain confidence bounds on the mean.

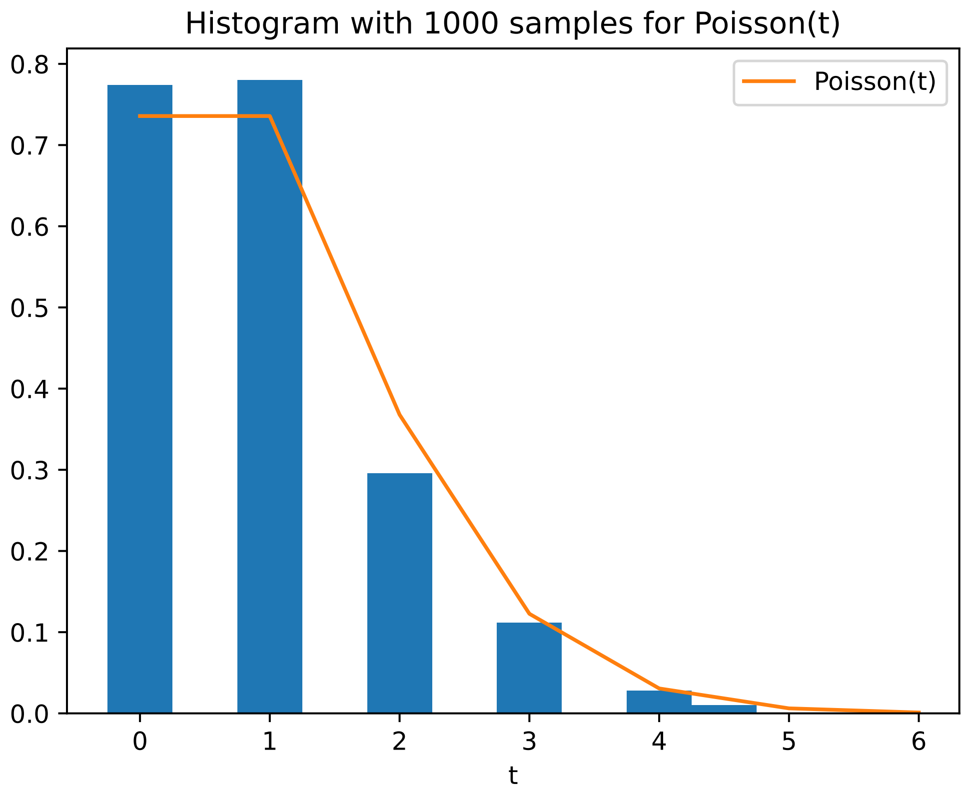


Figure 3: (2)

For $n = 10$ samples, the mean is 1.1, the standard error is 0.2769, and the 95% confidence bounds are $(0.5573, 1.6427)$.

For $n = 100$ samples, the mean is 0.81, the standard error is 0.095, and the 95% confidence bounds are $(0.6238, 0.9962)$.

For $n = 1000$ samples, the mean is 0.977, the standard error is 0.0316, and the 95% confidence bounds are $(0.915, 1.039)$.

For $n = 10000$ samples, the mean is 1.0022, the standard error is 0.01, and the 95% confidence bounds are $(0.9826, 1.0218)$.

As the number of samples increases, the mean approaches the true mean $t = 1$ of the Poisson distribution as expected from the central limit theorem.

3.a)

$$\pi(x) = \frac{0.2}{\sqrt{2\pi \times 0.5}} \exp(\frac{-(x-1)^2}{2 \times 0.5}) + \frac{0.8}{\sqrt{2\pi \times 0.1}} \exp(\frac{-(x-2)^2}{2 * 0.1}) \quad (1)$$

$$\approx 0.113 e^{(-(x-1)^2)} + 1.009 e^{(-5(x-2)^2)} \quad (2)$$

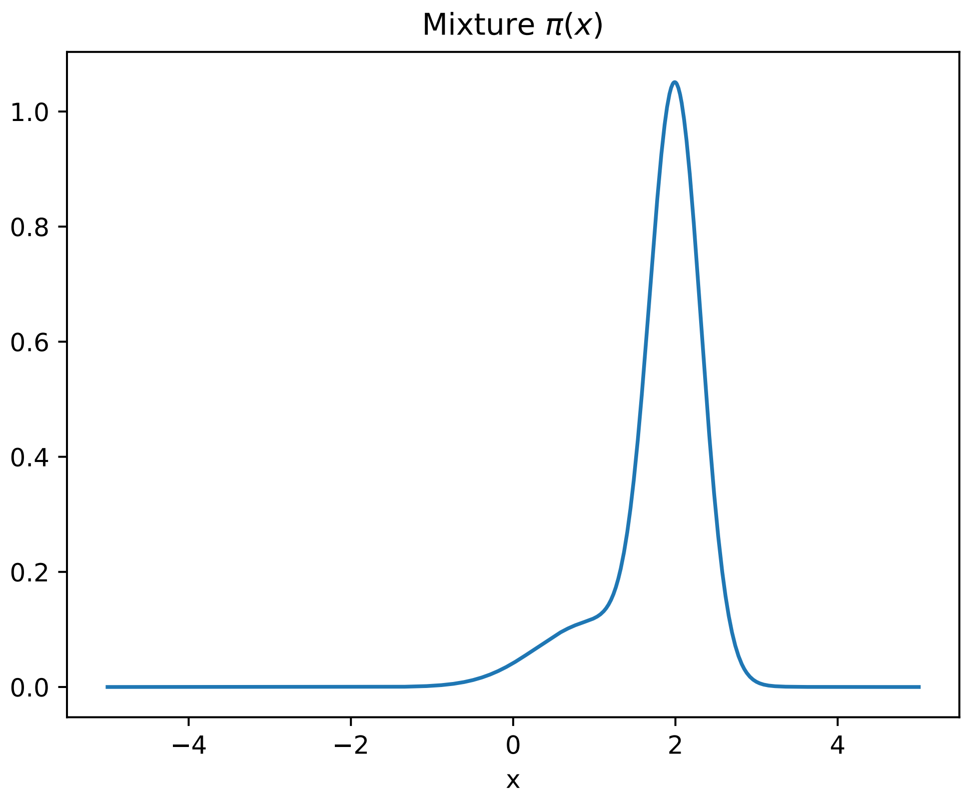


Figure 4: (3.a)

The second component of the mixture, $N_2(2, 0.1)$, contributes much more to the density than the first component $N_1(1, 0.5)$, and so I choose to center my proposal normal distribution q around $x = 2 = \mu$. The variance needs to be chosen to include the higher density to the left of $x = 2$.

The first distribution $N_1(1, 0.5)$ has three standard deviations in $(\mu_1 \mp 3\sigma_1) = (1 \mp 3\sqrt{0.1}) = (-1.12, 3.12)$. For the proposal to include this region, I want to have $\mu - 3\sigma = 2 - 3\sigma \approx -1.12$, or $\sigma = \sigma^2 = 1$. That gives my proposal as $N(2, 1)$.

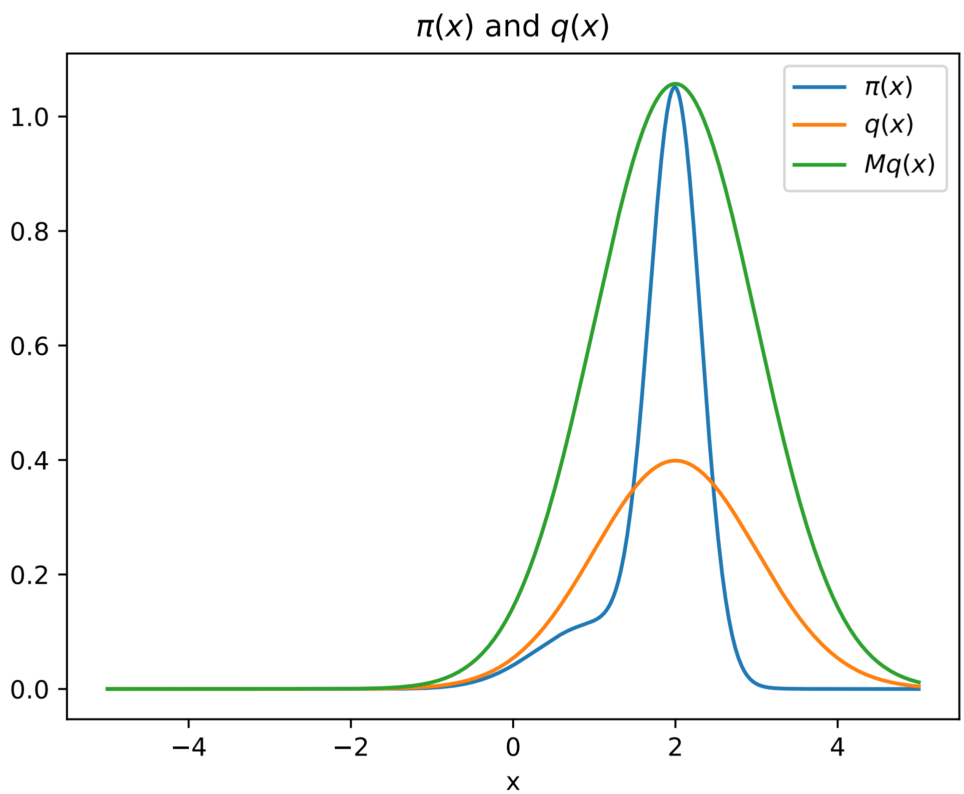


Figure 5: (3a)

I want to bound $\pi(x) \leq M * q(x)$. To ensure this, I need $\pi(2) \leq M * q(2) \implies 1.051 \leq M/\sqrt{2\pi} \implies M \geq 2.63$. To be safe, I take $M = 2.65$.

I verify the correctness of the sampler by comparing a histogram to the mixture $\pi(x)$.

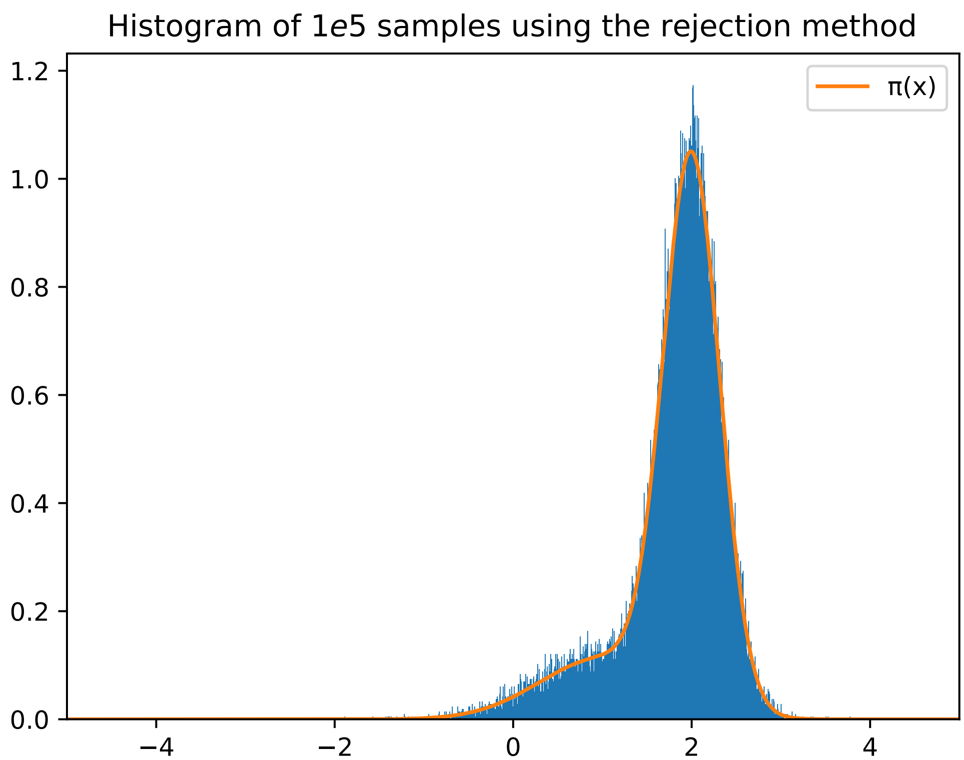


Figure 6: (3a)

The acceptance rate is $0.377 = 1/M$ as expected.

3.b)

The proposal here is $q(x) = \lambda e^{-\lambda x}$ on $x \geq 0$, the density of Y. The density of X is $\pi(x) = \lambda e^{-\lambda(x-a)}$ on $x \geq a > 0$. On $x \geq a$, we get $\pi(x) = \lambda e^{-\lambda x} e^{\lambda a} = q(x)e^{\lambda a}$.

The bound $\pi(x) \leq Mq(x)$ gives $0 \leq Mq(x) \implies 0 \leq M$ on $x < a$. On $x \geq a$ it gives

$$q(x)e^{-\lambda a} \leq Mq(x) \implies e^{-\lambda a} \leq M.$$

I take $M = e^{\lambda a} \geq 0$.

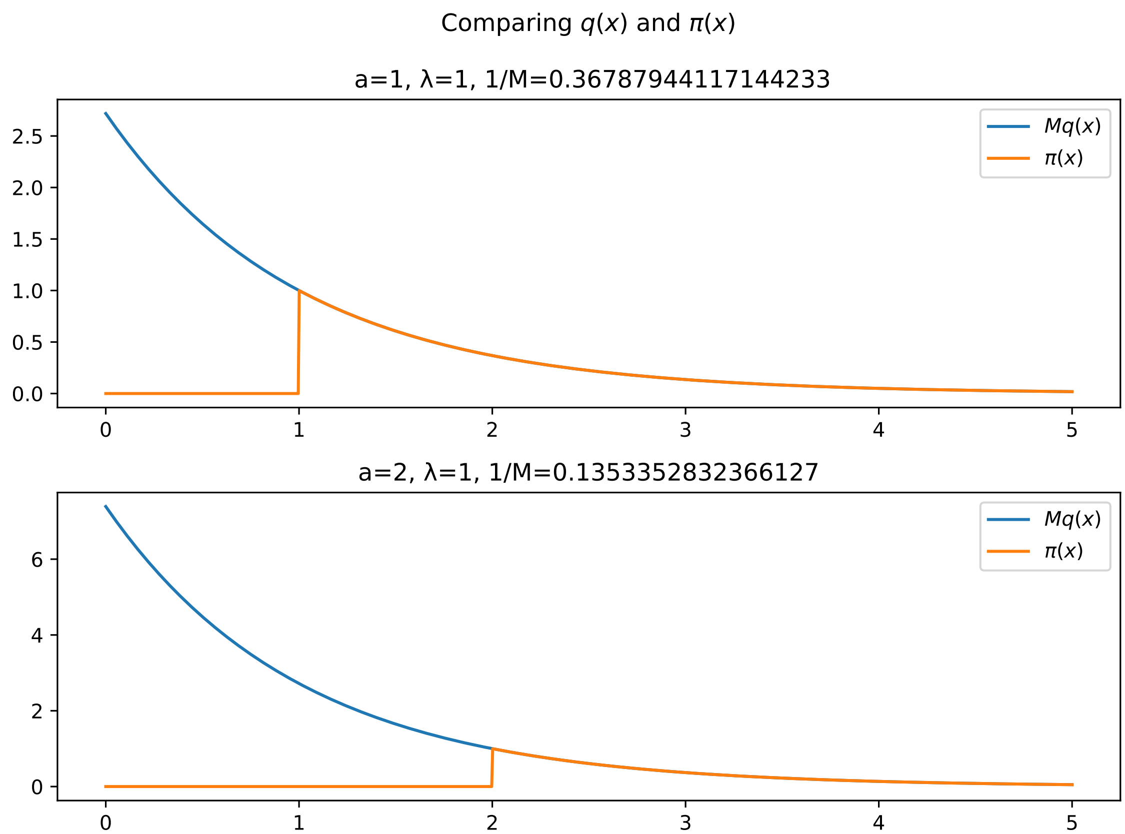


Figure 7: (3b)

This choice of M gives acceptance criteria as

$$\frac{\pi(x)}{Mq(x)} = \begin{cases} 0 \text{ if } 0 < x < a \\ 1 \text{ if } x \geq a \end{cases}$$

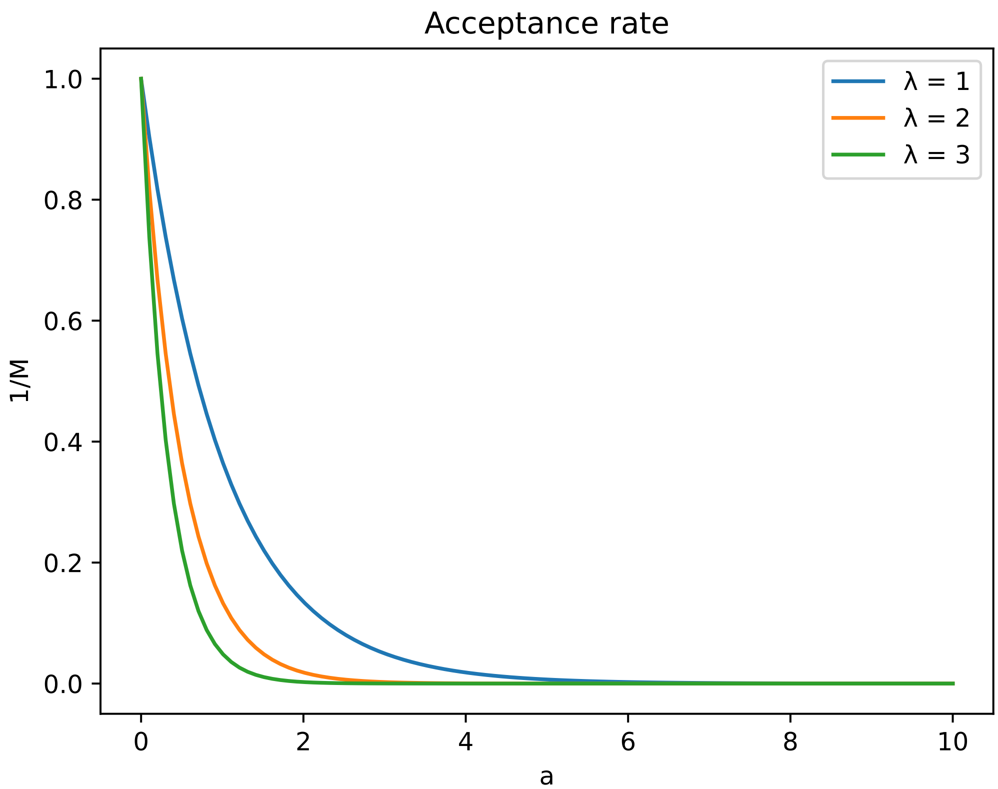
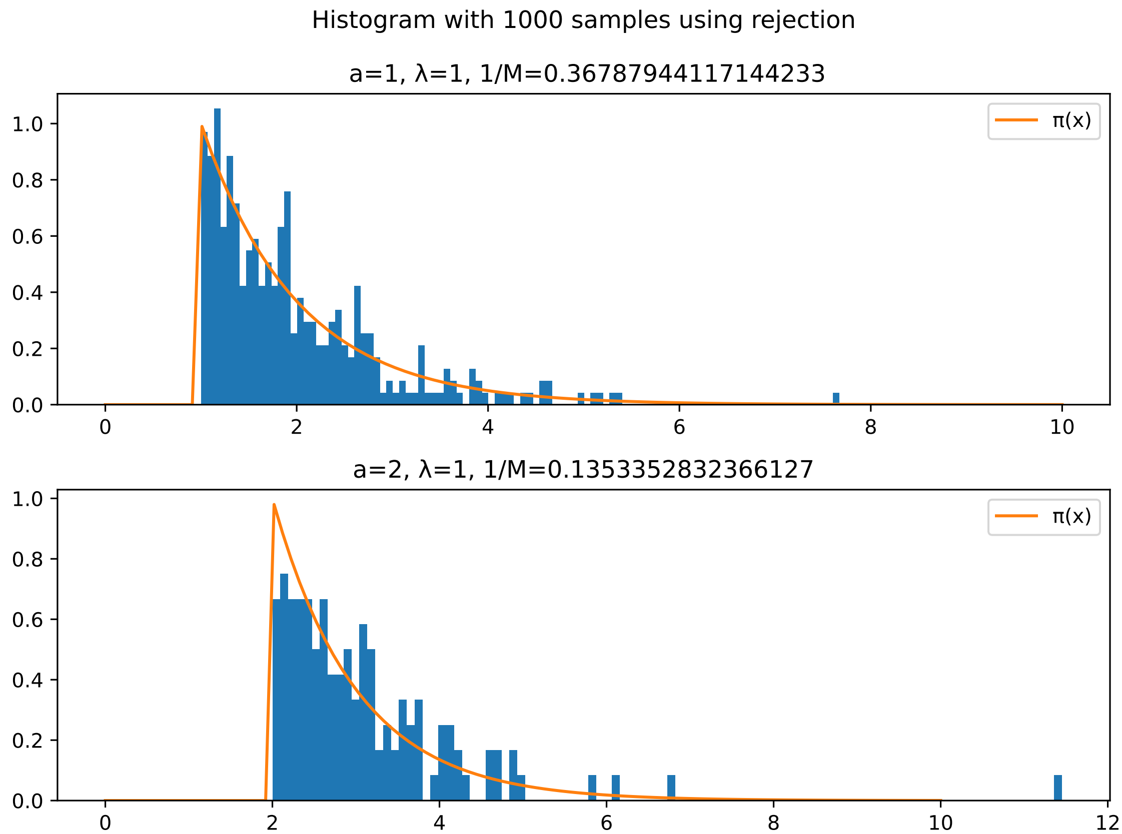


Figure 8: (3b)

The acceptance rate is $1/M = e^{-\lambda a}$ which decreases exponentially as a increases. This means the efficiency of the sampler decreases significantly as a increases. The target $\pi(x)$ shifts further with a while the proposal $q(x)$ is independent of a, so there is a larger region $0 < x < a$ where the proposal will be much greater than $\pi(x) = 0$.


Histogram with 1000 samples using rejection
a=1, λ=1, 1/M=0.36787944117144233
π(x)
a=2, λ=1, 1/M=0.1353352832366127
π(x)
Figure 9: (3b)