Homework 7

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Question 1

Refer to C++ File

Question 2

Refer to c++ File

Question 3

3.1 Exercise 8.2.2

B) $n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$

Proof by Cases.
We need to find the Upper Bounds of the function defined by $O(g)$ and Lower Bounds of the function defined by $\Omega(g)$ in order to prove $\Theta(g)$

Proof By Cases

Case 1: Upper Bound $O(g)$

First we will solve for the Upper Bound of the function, defined by $O(g)$ where $f(n) \le O(g)$ and c_2 is a constant. We will also define $f(n)$ as the function and $g(n)$ as the bounding function.

$f(n) = n^3 + 3n^2 + 4$
$g(n) = n^3$

$f(n) \le c \cdot g(n)$
$n^3 + 3n^2 + 4 \le c \cdot n^3$
$n^3 + 3n^2 + 4 \le n^3 + 3n^3 + 4n^3$ for $n \ge 1$
$f(n) \le 8n^3$
$f(n) \le 8 \cdot g(n)$

$c_2 = 8$ when $n \le 1$

Case 2: Lower Bound $\Omega(g)$

Secondly, we will solve for the Lower Bound of the function, defined by $\Omega(g)$ where $f(n) \ge g(n)$ and c_1 is a constant, using the same $f(n)$ function and $g(n)$ bounding function. $f(n) = n^3 + 3n^2 + 4$
$g(n) = n^3$

$f(n) \ge c \cdot g(n)$
$n^3 + 3n^2 + 4 \ge c \cdot n^3$
$n^3 + 3n^2 + 4 \ge n^3 + 3n^2$
$n^3 + 3n^2 \ge n^3$ when $n \ge 1$

$c_1 = 1$ when $n \le 1$

With the Lower Bounds being defined as $\Omega(g) = g(n) = n^3$ and the Upper Bounds being defined as $O(g) = 8g(n) = 8 \cdot n^3$ for all $n \ge 1$ we can conclude that $n^3 + 3n^2 + 4 = \Theta(n^3) = \Theta(g)$. □

3.2 Exercise 8.3.5

A) The sequence of numbers are sorted depending on a comparison value of p.

B) The total number of times the lines are executed depend on the values of the numbers in the sequence, and how they compare to p. So if we have a set of numbers such as $\{0, 1, 2, 3, 4, 5\}$

and the sequence is being compared to $p = 0$, then line $i := i + 1$ will run 5 times and $j := j - 1$ will run 0 times.

C) The swap operation is dependent on the current value of the numbers at a position either a_i or a_j. To minimize this, we would need to have a sequence of numbers that are already ordered to p, such that p transverses through the sequence without performing any of the necessary calculations, enabling the swap function. To maximize this, we would need a sequence of numbers, such that the swapping is performed n - 1 times. This again, also varies between how the sequences of the numbers values and how they relate to p.

D) The lower bound of the algorithm would depend on the necessary amount of increments and decrements of i and j, and how they effect the overall sequence in accordance to p. If the sequence is already ordered in accordance to p, then the algorithm would only one once, without performing any swapping.

E) The upper bound of the algorithm would be $\Theta = (n)$, where the algorithm goes through the sequence once, and depending on the values of the sequence, either i or j will be performed $n - 1$ times. The swapping would only occur once for each of the corresponding a_i and a_j.

Question 4

4.1 Exercise 5.1.1

B) $40^7 + 40^8 + 40^9$

C) $14 \cdot (40^6 + 40^7 + 40^8)$

4.2 Exercise 5.3.2

A) $3 \cdot 2^9$

4.3 Exercise 5.3.3

B) $10 \cdot 26^4 \cdot 9 \cdot 8$

C) $10 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 9 \cdot 8$

4.4 Exercise 5.2.3

A) Show a bijection B^9 between E_{10}.

$B^9 \longrightarrow E_{10}$

Prove one-to-one: If we represent x as all the combinations of B^9, x will have either an even amount of 1's or an odd number of 1's, but not both. So the output will either be $x0$ or $x1$.

Prove onto: If we have a y as all the combination of E_{10}, y must have a co-responding x such that, x is the combination of all B^9 strings. The first 9 bits of of y are defined by z, and if y ends with a 0 there are an even number of 1's in z. If y ends with a 1, there are an odd number of 1's in z. So each combination of z whether it ends with an even or odd number of 1's will always have a corresponding y.

B) $|E_{10}| = 2^9$

Question 5

5.1 Exercise 5.4.2

A) The amount of 7 digit phone numbers with area codes starting with 824 or 825: $2 \cdot 10^4$

B) The amount of 7 digit phone numbers with area codes starting with 824 or 825, with non of the last 4 digits repeating: $2 \cdot (10 \cdot 9 \cdot 8 \cdot 7)$

5.2 Exercise 5.5.3

A) No Restrictions: $2^1 0$

B) Starting with 001: 2^7

C) Starting with 001 or 10: $2^7 + 2^8$

D) First two bits same as last two bits: 2^8

E) String has exactly six 0's : $\binom{10}{6}$

F) Exactly six 0's first bit is 1: $\binom{9}{6}$

G) Exactly one 1 in the first half and three 1's in the second half: $\binom{5}{1} \cdot \binom{5}{3}$

5.3 Exercise 5.5.5

A) 30 Boys and 35 Girls for the choir, choose 10 of each gender to join the chorus $\binom{30}{10} \cdot \binom{35}{10}$

5.4 Exercise 5.5.8

C) A 5 card hand with only Diamond and Heart Suits: $\binom{26}{5}$

D) A 5 card hand with 4 cards of the same rank: $\binom{13}{4}\binom{48}{1}$

E) A "Full House": $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{3}$

F) Five card hands that do not have any 2 cards of the same rank: $\binom{13}{5} \cdot 4^5$

5.5 Exercises 5.6.6

A) 2 Political Parties with 100 members, 44D's 56R's, choose 5 of each party to form a senate with 10 members: $\binom{44}{5} \cdot \binom{56}{5}$

B) Same political parties, now pick a President and Vice President for each. D's: $44 \cdot 43$ R's: $56 \cdot 55$

Question 6

6.1 Exercise 5.7.2

A) $\binom{52}{5} - \binom{39}{5}$

B) $\binom{52}{5} - \binom{13}{5} \cdot 4^5$

6.2 Exercise 5.8.4

A) 20 Different Comic Books to 5 Kids, no Restrictions: 5^{20}

B) 20 Different Comic Books to 5 Kids, divided evenly amongst the kids: $\binom{20}{4} \cdot \binom{16}{4} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}$

Question 7

A) Five Elements to Four Elements: None

B) Five Element to Five Element: 5!

C) Five Elements to Six Elements: 6!

D) Five Elements to Seven Elements: $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$