Homework 6

Emmet Allen

February 20, 2021

Question 1

See C++ File.

Question 2

See C++ File.

Question 3

See C++ File.

Question 4

See C++ File.

Question 5

5.a) $5n^2 + 2n^2 + 3n = \Theta(n^3)$

Proof by Cases.

We need to find the Upper Bounds of the function defined by $O(g)$ and Lower Bounds of the function defined by $\Omega(g)$ in order to prove $\Theta(g)$

Case 1: Upper Bound $O(g)$

First we will solve for the Upper Bound of the function, defined by $O(g)$ where $f(n) \leq O(g)$ and c_2 is a constant. We will also define $f(n)$ as the function and $g(n)$ as the bounding function.

$$f(n) = 5n^2 + 2n^2 + 3n$$
$$g(n) = n^3$$

$$5n^3 + 2n^2 + 3n \leq c_2 \cdot n3$$
$$5n^3 + 2n^2 + 3n \leq 5n^3 + 2n^3 + 3n^3 \text{ when } n \leq 1$$
$$5n^3 + 2n^3 + 3n^3 = 10n^3 \leq c_2 \cdot n^3 \; 10n^3 \leq 11n^3$$

Because we know $c2$ is a constant, and $c_2 \cdot n^3 \geq 10n^3$ we can have $c_2 = 11$

Case 2: Lower Bound $\Omega(g)$

Secondly, we will solve for the Lower Bound of the function, defined by $\Omega(g)$ where $f(n) \geq g(n)$ and c_1 is a constant, using the same $f(n)$ function and $g(n)$ bounding function.

$$f(n) = 5n^2 + 2n^2 + 3n$$
$$g(n) = n^3$$

$$5n^3 + 2n^2 + 3n \geq c_1 \cdot n^3$$
$$5n^3 + 2n^2 + 3n \geq 5n^3 + 2n^2 \text{ when } n \geq 0$$
$$5n^3 + 2n^2 \geq 5n^3$$
$$5n^3 \geq c_1 \cdot n^3$$
$$5n^3 \geq 4 \cdot n^3$$

Because we know c_1 is a constant, and $c_1 \cdot n^3 \leq 5n^3$ we can have $c_1 = 4$

With the Lower Bounds being defined as $\Omega(g) = 4g(n) = 4 \cdot n^3$ and the Upper Bounds being defined as $O(g) = 11g(n) = 11 \cdot n^3$ for all $n \geq 1$ we can conclude that $5n^2 + 2n^2 + 3n = \Theta(n^3) = \Theta(g)$. □

5.b) $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

Proof by Cases.

We need to find the Upper Bounds of the function defined by $O(g)$ and Lower Bounds of the function defined by $\Omega(g)$ in order to prove $\Theta(g)$

Case 1: Upper Bound $O(g)$

First we will solve for the Upper Bound of the function, defined by $O(g)$ where $f(n) \le O(g)$ and c_2 is a constant. We will also define $f(n)$ as the function and $g(n)$ as the bounding function.

$$f(n) = \sqrt{7n^2 + 2n - 8}$$
$$g(n) = \sqrt{n}$$

$$\sqrt{7n^2 + 2n - 8} \le c_2$$
$$\sqrt{7n^2 + 2n - 8} \le \sqrt{7n^2 + 2n^2 - 8^2} \text{ when } n \ge 1$$
$$\sqrt{7n^2 + 2n - 8} \le 7n^2 + 2n - 8$$
$$7n^2 + 2n - 8 \le c_2$$
$$7n^2 + 2n - 8 \le 7n + 2n$$
$$7n + 2n \le c_2 \cdot n$$
$$9n \le c_2 \cdot n$$

Because we know c_2 is a constant, and $c_2 \cdot n \ge 9n$ we can have $c_2 = 10$

Case 2: Lower Bound $\Omega(g)$

Secondly, we will solve for the Lower Bound of the function, defined by $\Omega(g)$ where $f(n) \ge g(n)$ and c_1 is a constant, using the same $f(n)$ function and $g(n)$ bounding function.

$$f(n) = \sqrt{7n^2 + 2n - 8}$$
$$g(n) = \sqrt{n}$$

$$\sqrt{7n^2 + 2n - 8} \ge c_1$$
$$(\sqrt{7n^2 + 2n - 8})^2 \ge (\sqrt{7n^2 + 2n - 8})^2$$
$$7n^2 + 2n - 8 \ge 7n^2 + 2n - 8$$
$$7n^2 + 2n \ge 7n^2 + 2n - 8 \text{ when } n \ge 0$$
$$7n^2 + 2n \ge 7n^2$$
$$7n^2 \ge c_1 \cdot n$$

Because we know c_1 is a constant, and $c_1 \cdot n \le 7n^2$ we can have $c_1 = \sqrt{7}$

With the Lower Bounds being defined as $\Omega(g) = \sqrt{7}g(n) = \sqrt{7}(n)$ and the Upper Bounds being defined as $O(g) = 10g(n) = 10n$ for all $n \ge 1$ we can conclude that $\sqrt{7n^2 + 2n - 8} = \Theta(n) = \Theta(g)$. □