Pricing of Eurasian Options Using Monte-Carlo Methods

Emmet Rice

Department of Mathematics & Physics

Queen's University

Belfast

MSc Applied Mathematics & Physics

2019

In Memory of My Father

Acknowledgements

I would like to thank my supervisor Prof. Hugo Van der Hart, as well as Prof. Mauro Paternostro for their continued support, guidance and understanding throughout the year. Thanks to all of lectures and Staff at Queen's University for their instruction and aid, providing a wonderful learning experience. Finally, to my Family, friends and Fiona Donnelly, to whom I am forever grateful for their unwavering support without which, this report and my continued progress would not have been possible.

Abstract

The pricing of Asian options has garnered a lot of interest and research from the scientific and finance sectors in recent decades; however, no analytical solution yet exists for the popular, arithmetically averaged, Asian option variant. In this report, I investigate alternative methods for pricing these options; specifically, via the Monte-Carlo numerical technique. I demonstrate that the Monte-Carlo pricing method is improved by applying the antithetic and control variate variance reduction techniques. Of the pricing methods investigated, combining the two variance reduction techniques and using the discrete analytical solution for geometrically averaged Asian options is shown to provide the most accurate and precise valuation for the pricing of arithmetic Asian options. This reduced the variance for "in-the-money" options on average by a reduction factor of over 2700 when compared to the base Monte-Carlo approximations. For the more realistic case, where the volatility is modelled stochastically, the Black-Sholes valuations under price options which are significantly "out-of-the-money" and overprice "in the money options" by approximately 11% when compared to the Monte-Carlo numerical prices, in concordance with previous research conducted by Hull & White. The pricing of options using Black-Scholes valuations where the underlying asset's implied volatility appears to vary should therefore be avoided. Further investigation into correcting for the demonstrated error in analytical solutions for use as a control variate for the pricing of arithmetic Asian options is suggested.

Contents

Nomenclature

A	Arithmetic average
G	Geometric average
B	Value of the riskless asset
C	Value of a call option
P	Value of a put option
E_Q	Linear expected value operator conditional on all available information under the risk neutral measure Q
K	Agreed strike price
$N(m, sd)$	Normal distribution of mean m and standard deviation sd
S	Asset value
T	Time until option maturity date, as a fraction of fiscal year
τ	Temporal difference between the current time t and the maturity date T.
n	Number of financial trading days until maturity date (253 in a fiscal year)
m	Number of Monte-Carlo path iterations
V	Portfolio Value
W	a standard gaussian Wiener process
σ	the implied volatility of the asset as a percentage of the asset value
μ	the percentage drift rate of the asset
ρ	Correlation coefficient
x'	Corresponding Antithetic variable of arbitrary variable x
Z	a standard gaussian random variable
s	Standard deviation of a sample
β	Parameter for the optimisation of the Control Variate

Introduction

Financial derivatives are integral instruments in modern financial markets; for instance, the European Securities & Markets Authority (ESMA) valued the 2018 European Union (EU) derivatives market at €660 Trillion. A financial derivative is a contract based on at least one underlying asset; be that stocks, commodities, or a plethora of other measurable holdings or variables. Derivatives base their expected future payoff in various ways relating to how the underlying asset price has evolved over time. There are various types and styles of derivatives. One of the main types of these contracts are options.

Appropriately pricing options is critical; particularly because of the size of the options market, and the increased regulation of the market introduced after the 2008 financial crash. The values of financial assets and instruments can vary both greatly and rapidly. Therefore, traders with the most efficient and accurate pricing models have a distinct advantage, utilizing pricing inaccuracies to generate profit. Black and Scholes revolutionised the pricing of options in their 1973 paper. They provided an analytical solution for the pricing of basic options under certain market assumptions. The demand for options quickly increased and with this came the need for more complex "Exotic" option variants.

One such type of Exotic options are Asian options, first introduced into Tokyo Markets in 1987. Asian options are known as path-dependant options, as the pay-off depends on the average price of the asset along the entire path it has followed for the derivative lifetime. This contrasts with basic "Vanilla" options whose pay-off typically only depends on the final asset price; exposing the option to greater risk of market manipulation.

Unlike the Black-Scholes case, no exact closed-form analytic solution is known for Asian options when the average is calculated arithmetically; due to the arithmetic asset distribution density being unknown. However, an analytical solution exists for the less popular geometrically averaged case under the Black-Scholes assumptions. In reality, the assumptions made by the Black-Scholes model are often not reflected in real markets. One such assumption is that that the volatility of the underlying asset is constant; however, real-world market data suggests that this is not the case. Therefore, numerical methods must be used in order to evaluate the expected value of arithmetic Asian options and stochastic volatility options and infer the appropriate pricing.

Monte Carlo simulations were first used to price options by Boyle in 1977. They are now the most common method used to evaluate options; however, they require substantial simulations and computational effort in order to effectively approximate the value. I investigate the application of two variance reduction techniques, the antithetic and control variate methods, in order to improve the Monte-Carlo valuations for Eurasian Fixed Strike Call options; as well as investigating the modelling of volatility as a stochastic variable. The simulated approximated prices are compared to the Black-Scholes economy analytical valuations to appraise the Monte-Carlo numerical method and the volatility assumption. This paper first introduces the core mathematical concepts in Chapter 1 which are required for the analytical and numerical pricing methodologies discussed in Chapters 2 through 4.

Chapter 1 Financial Background

1.1 Options

Options are a legal agreement between parties that one of them has the option to buy an asset from the other at a later date (known as the maturity date) for a prearranged fair price (known as the strike price). The choice of whether to exercise the option or not provides a level of insurance against large price changes, and thus commands a premium price to be paid to the other party to obtain this right.

Options can be held in either long or short positions. A long position involves buying the option and holding it, expecting the value to increase. A short position sells the option for a premium in the expectation that the value will not increase enough to offset this. These positions can be applied to the most frequent option trades known as “calls” or “puts”. A call option provides the holder (in the long position) the right to buy an asset from the other party, known as the writer of the option (the short position), for the agreed strike price, whereas the holder of a put option has the right to sell it for a potentially different strike price.

Thus, options facilitate traders who speculate that the asset value will deviate in such a way that the expected profit to be made for either longing or shorting the option is greater than if similar positions were taken for the underlying asset. Options are also typically used to reduce the risk of said asset to large price fluctuations and hedge portfolios.

It follows that knowing the correct pricing of a long option, the corresponding payoff of the short position is trivially known. For this investigation the options are assumed to be call options in the long position.

1.1.1 European and American Options

Options also depend on what "style" they are. The two most common are European and American options. American options can be used at any time until the maturity date while European options can only be exercised at the maturity date and are the most basic option type. For this reason, European call and put options are often referred to as "Vanilla" options. In both cases, they can only be exercised once per option.

Equation 1: Value of Vanilla European Long Call Option and Long Put Option

$$C(S,T) = Max(S-K,0) = (S-K)^{+}$$

$$P(S,T) = Max(K-S,0) = (K-S)^{+}$$

Where C is the value of the call option, P the value of the put option, S is the asset value and K the agreed strike price

One perceived problem with a Vanilla European option is that it does not account for how the underlying asset price fluctuates, but only the value at the maturity date. This is known as a "path-independent option". This exposes the holder to a lot of risk of large fluctuations at the maturity date (potentially from manipulation) determining the payoff of such options. While this is easiest to model mathematically, more "Exotic styles", such as Asian options, can be used to account for this.

1.2 Black Scholes Model

The Black-Scholes Equation forms the core mathematical model for this investigation. Fischer Black, Myron Scholes and Robert Merton, revolutionised the theory of option pricing and the field quantitative finance and were awarded the Nobel Prize in economics in 1997. Described in their 1973 seminal papers, the model provided an analytical closed form solution for the pricing of vanilla options under certain market conditions and assumptions, described below. Such a market will hereby be referred to as a *Black-Scholes Economy*.

While this model is an imperfect replica of the real world, it is extremely robust under the following assumptions:

1. There are no transactions costs or taxes, i.e. a frictionless market.
2. There are no penalties for short sales
3. Market trading operates continuously
4. The risk-free intertest rate *r* is a known constant, and is identical for all maturities
5. The asset pays no dividends
6. The option is a Vanilla European type (the option payoff at expiry *T* depends solely on the asset price and strike price)
7. There are no arbitrage opportunities
8. The asset price follows continuous Geometric Brownian Motion (GBM)
9. The percentage drift rate μ, which expresses how the logarithmic average asset value trends over time, and the implied logarithmic volatility of the asset σ are constant.

1.3 Black Scholes Option Asset Pricing

The final two assumptions listed above mean that the asset price is mathematically a *stochastic process*, which are characterised by the Geometric Brownian motion *stochastic differential equation.*

Equation 2: Stochastic Differential Equation (SDE)

$$dS_t = S_t \mu \mathrm{dt} + S_t \sigma \mathrm{d}W_t^P$$

Where S_t is the instantaneous value of the underlying asset, μ is the percentage drift rate of the asset which expresses how the logarithmic average asset value trends over time, and the implied volatility of the asset σ are constant. W_t^P is a standard gaussian Wiener process under the real-world probability measure "P".

This equation defines the change in asset price in terms of a percentage of the previous price, which facilitates large "rough" variations in the stock price over time, while preventing the asset price becoming negative; as in real world prices.

The equation reflects how the price varies due to both the deterministic change of the asset from the expected drift term $S_t \mu dt$, as well as the effect of unforeseen information through the stochastic term $S_t \sigma dW_t^P$, where dW_t^P is the time derivative of standard Brownian motion, known as a Wiener process.

Equation 3: Wiener Process

$$dW_t^P = W_{t+dt} - W_t \sim N(0, dt)$$

This process is defined by a specific probability space, where the P subscript signifying that it is with respect to the real-world historic measure of the asset price probability. This is a Gaussian distribution with mean 0 and a standard deviation of $\sqrt{dt}$. On small *dt* time scales this results in the stochastic term dominating the price change dS_t.

Modelling asset movements in this way follows the *Efficient Random Walk Market Theory*, where in such a market, all currently known available information influencing the intrinsic value of an asset are sufficiently approximated in the market; where the price variations appear random due to unforeseen events. The stochastic term models the seemingly random release of this information and how it affects the stock price.

With the aid of *Ito's lemma*, an adaptation of the standard chain rule for stochastic calculus, a solution to the *Stochastic Differential Equation* can be found

1.3.1 Ito's Lemma

Where W_t is a Wiener process and X_t is an Ito drift-diffusion process satisfying the stochastic differential equation:

Equation 4: Diffusion Stochastic Equation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

And if $f(x,t)$ is a twice differential scalar function where $f(x,t) \in C^2(\mathbb{R}^2, \mathbb{R})$, then $f(X_t, t)$ is also an Ito drift diffusion process, which has a differential form of:

Equation 5: Ito's Lemma

$$d(f(X_t,t)) = \frac{\partial f}{\partial t}(X_t,\mathrm{t})\mathrm{dt} + \frac{\partial f}{\partial x}(X_t,\mathrm{t})\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t,\mathrm{t})\mathrm{d}{X_t}^2$$

$$d(f(X_t,t)) = \left(\frac{\partial f}{\partial t}(X_t,\mathrm{t}) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t,\mathrm{t})\right)\mathrm{dt} + \frac{\partial f}{\partial x}(X_t,\mathrm{t})\mathrm{d}X_t$$

Where: $dt^2 = 0$, $dtdX_t = 0$, $dX_t dX_t = dX_t^2 = dt$

As this is a continuously differentiable diffusion process, one limitation of this model is that it cannot account for discontinuities in the asset price, instantaneous large variations caused by the sudden release of very influential information. More advanced models implement a 'jump' term to account for this behaviour. Due to the constraints on this investigation, jump diffusion processes were not implemented.

Equation 6: Jump Diffusion Stochastic Equation

$$dX_t = \mu(X_t,\mathrm{t})\mathrm{dt} + \sigma(X_t,\mathrm{t})\mathrm{d}W_t + \mathrm{d}J_t$$

Where: $\mathrm{d}J_t$ *is the jump term often modelled via Markov Processes.*

1.3.2 Stochastic Differential Equation (SDE) Solution

A solution to the *Diffusive Stochastic Differential Equation* for asset prices can then be found by applying Ito's lemma.

Let $f(S_t) = \ln S_t$ be independent of *t*, and applying Ito's Lemma yields:

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{dt} + \sigma\mathrm{d}W_t^P$$

Where: $f(S_{t0}) = \ln S_0$, S_0 *is the initial stock price at t=0*

Changing to the integral form and exponentiation gives the solution:

Equation 7: Solution to SDE

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t^P}$$

Recalling the *Wiener process'* Gaussian distribution, which can be equated as $N(0, dt) = \sqrt{dt}Z(0,1)$, where Z is a standard normal Gaussian variable, the solution can be written as:

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z_t}$$

Where Z is a random variable which takes a different value for each temporal point along the asset path.

Fortunately, unlike most stochastic differential equations, an explicit closed form analytical solution for this case can be found by utilizing the theory of arbitrage elimination.

1.4 Arbitrage

Arbitrage is one of the most important concepts in financial markets. Arbitrage can be defined as any process which leads to an immediate risk-free profit due to market inefficiencies. On a basic level, if the same item can be bought for a lower price on one market, or an equivalent item or financial instrument potentially on the same market, and sold for a higher price on another, an immediate guaranteed profit can be made. Due to how trading affects supply and demand, which in turn affects the price, arbitrage opportunities are time limited. In today's markets these opportunities are incredibly short and as such are exploited by computer programmes; however, the underlying principle is the same.

Bonds are typically considered safer investments than stocks, and based on the reduced risk, have a lower rate of return. Ideally, and assumed for this report, government issued bonds are risk-free investments with guaranteed yet small return on investment. This risk-free profit is the benchmark for investment opportunities, but by the nature of investments, it requires time. Though interest payments come in discrete instalments, by approximating these as continuous payments, a simple equation can be used to calculate the future value of an investment.

Equation 8: Interest Equation

$$B_t = B_0 e^{rt}$$

Where B_t is the future value of the bond, $B_0$is the initial value, r is the equivalent annual continuous compound interest rate, t is the time period as a fraction of a financial year.

One consequence of this guaranteed return is known as the "Time Value of Money". This is the idea that the value of money now is higher than that same amount in the future, as the amount in the future can be obtained by investing a lower amount now and making up the difference from the risk-free interest. As such, the value of money in the future must be *discounted* to obtain the current value. This discount equation is then trivially the inverse of the interest equation.

Equation 9: Discount Equation

$$B_0 = B_t e^{-rt}$$

Where e^{-rt} is known as the discount factor, and r is risk-free annual continuous compound interest rate

This discount factor for the risk-free no-arbitrage market is the reason the asset price in the SDE derivation is of logarithmic form.

The elimination of arbitrage leads to an important principle for options known as the put-call parity. It states that the value of a portfolio consisting of an asset and a long European put option must be equal to a portfolio of a European call option and a bond investment. Therefore, the effective pricing of a call option enables the valuation of the corresponding put option. The same principle holds for Asian options, and the portfolio holdings are altered depending on the expected payoff function of the option.

Equation 10: Put-Call Parity for Vanilla European Options

$$S_0 + P_0 = C_0 + Ke^{-rt}$$

1.5 Black Scholes Partial Differential Equation (PDE) Solution

The famous Black-Sholes PDE can be derived by utilizing the principle of arbitrage elimination. This is done by constructing a *self-financing portfolio* consisting of a risk-free bond "B" and the underlying stochastic asset.

Equations 11 (a), 11(b): Black-Scholes Asset Model

(a) $$dB_t = rB_t dt$$

(b) $$dS_t = S_t \mu \mathrm{dt} + S_t \sigma \mathrm{d}W_t^P$$

Where equation (a) models the instantaneous change in value of risk-free asset, and equation (b) the instantaneous change in value of underlying Stochastic asset of the Derivative

This portfolio is then traded continuously so that it is always equivalent to the value of the desired Vanilla European option. Logically, the derivative and its underlying asset follow the same stochastic nature which negate each other when comparing the derivative to the self-financing portfolio. This facilitates the creation of a risk-free portfolio through the use of continuous trading. This risk-free portfolio therefore must be equivalent in value to the risk-free asset for no arbitrage opportunity to exist.

Using the portfolio as a proxy, allowed the creation of the Black-Scholes Equation and the theoretically exact initial pricing for a vanilla European option to be found.

Equation 12: Black Scholes PDE

$$\frac{\partial\, V(S,t)}{\partial t} = rV(S,t) - rS\frac{\partial\, V(S,t)}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\, V(S,t)}{\partial t^2}$$

With Boundary Conditions:

$$S \in \,]0,\infty[\,, \qquad t \in [0,T]$$

And initial Conditions for call and put options respectively:

$$C(S,T) = (S-K)^+, \qquad P(S,T) = (K-S)^+$$

Where V is the value of the portfolio, and thus the desired vanilla option derivative value for a call C, or a Put P

An analytical solution for this PDE is found by first transforming the equation and boundary condition into the one-dimensional heat equation, solving it, and then transforming back. Black and Scholes gave the solution for a European option as:

Equation 13: BS European Call Option

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

Equation 14: BS European Put Option

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - \ SN(-d_1)$$

Where N(x) is a cumulative normal distribution $\frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{\frac{-z^2}{2}} dz$

$$d_1(S,t) = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2(S,t) = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_1(S,t) = \ d_2(S,t) + \sigma\sqrt{(T-t)}$$

Notably, this form of the BS PDE is a terminal value backward propagating equation, where the solution at maturity T is known and extrapolated backwards in time. This can easily be transformed to be forward propagating by substituting for a new temporal variable $\tau = T - t$.

1.6 Risk Neutral valuation & Equivalent Martingale Measure

The Black-Sholes Equation can also be derived using what is known as the Equivalent Martingale Measure (EMM), or the risk neutral valuation. For a market to be arbitrage free then the expected pay-off of any derivative in the future, with the discount factor applied, must be equal to its current listed price. A variable which satisfies this condition is known as a martingale, where the subsequent conditional expectation is equal to the previous value. By this definition, a martingale has zero drift as the expected value does not change.

For the Black-Scholes asset model, the ratio of the two assets $\frac{S_t}{B_t}$ forming the self-financing portfolio is required to be a martingale. This martingale asset has the form:

$$d\left(\frac{S_t}{B_t}\right) = \frac{1}{B_t} dS_t + S_t d\left(\frac{1}{B_t}\right)$$

Substituting for dS_t and dB_t from the BS model Equation 11 yields a new stochastic equation:

Equation 15: Martingale Stochastic Asset

$$d\left(\frac{S_t}{B_t}\right) = (\mu - \mathrm{r})\left(\frac{S_t}{B_t}\right)\mathrm{dt} + \sigma\left(\frac{S_t}{B_t}\right)\mathrm{d}W_t^P$$

For this new stochastic asset to be a martingale, the drift term $(\mu - \mathrm{r})$ must be zero.

This requires both assets, regardless of associated risk, to have the same expected growth as the interest of the risk-free asset, requiring the drift of the stochastic asset to be equal to the interest rate of the risk-free asset $\mu = r$, due to the time value of money and the associated discount factor. Such a market is said to exist in a risk neutral world. For such a world, the expected value is subject to the risk neutral probability measure Q which is also known as the equivalent martingale measure. These measures asses the likelihood of the underlying asset taking certain values if the economy behaved as if it were risk-neutral and arbitrage free.

Equation 16: Expected Value of Call Option

$$C_0 = \mathrm{E}_Q(e^{-rT} C_T)$$

Where C is the value of the call option, E_Q *is the linear expected value operator conditional on all available information under the risk neutral measure Q.*

In reality, investors are typically risk-averse, which means that under the real world measure P, $\frac{S_t}{B_t}$ is not a true martingale; however, for no arbitrage opportunities to exist under the real world probabilities, they must also not exist under the risk-neutral world. This is the basis for pricing derivatives in such a hypothetical risk neutral world. Once the risk-neutral measure Q and how to represent the asset in a risk-neutral world is known, from Equation 16, the derivative price can be found.

1.6.1 Girsanov's Theorem

Reviewing the Black-Scholes asset model in Equations 11, Girsanov's Theorem predicts that a risk-neutral probability measure Q must exist, and the BS asset under such a measure is represented as:

Equation 17: Stochastic Asset under risk-neutral measure

$$dS_t = S_t \mathrm{rdt} + S_t \sigma \mathrm{d}W_t^Q$$

As investors are risk adverse, they expect greater returns for taking on the risk associated with the stochastic assets from the Geometric Brownian Motion. The level of such risk is encapsulated and quantified in the volatility term σ as a decimal percentage of the asset. The true volatility of the asset cannot be directly measured or known, and so different markets asses what they believe the true volatility to be, known as *implied volatility.* Markets therefore price in a risk premium in order to compensate for the adversity to risk, known as the market price of risk. It should be noted that this is only the market premium associated with the stochastic geometric Brownian motion, and markets may price according to additional risks.

Equation 18: Market Price of Risk

$$\frac{\mu - r}{\sigma}$$

Girsanov's theorem states that for stochastic Brownian motion W_t^p defined on a specific proability space (Ω, F, P), there exists and equivaelent probability measure Q on the sample space Ω such that:

Equation 19: Girsanov's Theorem

$$dW_t^Q = dW_t^p - vdt$$

Where v is a reasonable function

Substituting Equation 19 into Equation 15 yields:

$$d\left(\frac{S_t}{B_t}\right) = (\mu - \mathrm{r})\left(\frac{S_t}{B_t}\right)\mathrm{dt} + \sigma\left(\frac{S_t}{B_t}\right)\mathrm{d}W_t^Q + v\sigma\left(\frac{S_t}{B_t}\right)dt$$

For this to be a martingale with zero drift requires:

$$v = -\frac{\mu - r}{\sigma}$$

Which is the negative of the market price of risk. This is supported by substituting for v into Equation 19, and rearranging to give:

$$\sigma dW_t^P = \sigma dW_t^Q - (\mu - \mathrm{r})dt$$

And substituting into the BS stochastic asset in Equation 17 to give:

$$dS_t = S_t\big(\mu - (\mu - \mathrm{r})\big)dt + S_t\sigma \mathrm{d}W_t^Q$$

Which is just a new stochastic differential equation, identical in form to Equation 17 for a stochastic asset under the risk neutral measure, with solution:

Equation 20: Risk-Neutral SDE Solution

$$S_t = S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)\mathrm{t}+\sigma\sqrt{t}z_t(0,1)}$$

1.6.2 EMM Call Option Solution

From Equation 20, the call option formula from Equation 1, and the discount factor, then:

$$C_t = e^{-r\tau}\mathrm{E}(S_t - K)^+$$

$$C_t = e^{-r\tau}\mathrm{E}(S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}z_t(0,1)} - K)^+$$

Recalling the density of a gaussian random variable Z (0,1) as $\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}$:

$$C_t = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \max\left[(S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}} - K, 0)\right] e^{\frac{-z^2}{2}} dz$$

Due to the max function in the integrand, the integrand is only non-zero if and only if:

$$S_t e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}z} \geq K$$

Which corresponds to a lower limit z value of:

$$z \geq -\left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$

Which is denoted d_2

$$C_t = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} \left(S_\tau e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}} - K, 0\right) e^{\frac{-z^2}{2}} dz$$

Separating the integral:

$$C_t = \frac{e^{-r\tau}}{\sqrt{2\pi}} S_0 e^{r\tau} \int_{-d_2}^{\infty} e^{\sigma\sqrt{\tau}-\left(\frac{1}{2}\sigma^2\right)\tau-\frac{z^2}{2}} dz - \frac{e^{-r\tau}}{\sqrt{2\pi}} K \int_{-d_2}^{\infty} e^{\frac{-z^2}{2}} dz$$

Using a change of variable $j = z - \sigma\sqrt{\tau}$ in the first integral terms gives:

$$C_t = \frac{S_t}{\sqrt{2\pi}}\int_{-d_2-\sigma\sqrt{\tau}}^{\infty} e^{\frac{-j^2}{2}} dj - \frac{e^{-r\tau}}{\sqrt{2\pi}} K \int_{-d_2}^{\infty} e^{\frac{-z^2}{2}} dz$$

The two integrals are two separate cumulative normal distributions, which can be written as:

Equation 21: BS European Call Option from EMM

$$C_t = SN(d_1) - Ke^{-r\tau}N(d_2)$$

Where:

$$d_1(S,\tau) = d_2(S,\tau) + \sigma\sqrt{\tau} = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

This is the same solution as quoted in the BS PDE section. The price of the option is the expected value when t = 0, hence $\tau = T$.

1.6.3 Equivalence to Black-Scholes PDE

The EMM derived solution in Equation 21 is the same as that given by the Black-Sholes PDE Equation 13. This is verified by the Feynman – Kac theorem, as well as using the following martingale method.

For a general derivative of value $V(S,t)$ under the risk neutral measure, $\frac{V_t}{B_t}$ is required to be a martingale. Applying Ito's lemma for $V(S,t)$ and substituting for risk neutral dS_t Equation X yields:

$$d(V(S_t,t)) = \left(\frac{\partial V}{\partial t}(S_t,\mathrm{t}) + \mathrm{r}S_t\frac{\partial V}{\partial S}(S_t,\mathrm{t}) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}(S_t,\mathrm{t}) - rV(S_t,\mathrm{t})\right)\mathrm{dt} + \sigma\mathrm{S}\frac{\partial V}{\partial S}(S_t,\mathrm{t})dW_t^Q$$

By dividing across by B_t gives the Martingale differential equation for $\frac{V_t}{B_t}$; however, for this to be a martingale requires the drift term (the pre-factor to dt term) to be equal to zero:

$$\frac{\partial V}{\partial t}(S_t,\mathrm{t}) + \mathrm{r}S_t\frac{\partial V}{\partial S}(S_t,\mathrm{t}) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}(S_t,\mathrm{t}) - rV(S_t,\mathrm{t}) = 0$$

Which is identical to the Black-Scholes PDE from Equation 12.

Chapter 2 Asian Options

Asian options are a type of "Exotic" option that were first introduced into the Asian market, hence their naming, in 1987 by the Tokyo branch of Banker's Trust. These options were introduced in order to discourage and reduce the effects of market manipulation of the underlying asset price at the maturity date. From the theory of supply and demand, large and sudden buying and selling of the underlying asset would affect the market value of said asset. As can be seen from the payoff formulae for vanilla European options, where only the maturity price is considered, such manipulation dramatically affects the profitability of the options.

Asian options offer price protection as they depend on the average price of the underlying asset over a specified window of the options lifetime; therefore, the value of Asian options are dependent on the path the stock price evolved along. In this investigation we will focus on Asian option monitoring over the complete lifetime of the option. The majority of Asian options are of the European type, known as Eurasian, as early exercise American style Asian options may be exercised before the averaging process has weighed against the price manipulation; which would negate the price protection. Only the pricing of Eurasian options are investigated in this report.

An additional benefit of the averaging process is the inherent reduction of the associated volatility.

Equation 22: $Avergaing\ and\ Volaitlity\ Reduction$

$$\mathrm{Var}(\mathrm{X_i}) = \sigma^2$$

$$Var(\bar{\bar{X}}_i) = Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Where X_i are independent identically distributed values.

With lower volatility, the expected payoff and thus initial price are lower compared to Vanilla European options. This makes Asian options attractive as a cheaper alternative for those interested in reducing risk via hedging.

2.1 Asian Option Types

Typically, there are two main classes of Asian options, *floating strike and fixed strike* Asian options. Floating strike Asian options replace the strike price of a typical European option with an average price of the underlying asset; whereas, fixed strike options replace the maturity asset price with the average price. The following are the payoff formulae for options of each type:

Table 1: Standard Eurasian Payoff Formulae

	Fixed Strike	*Floating Strike*
Call	$(\bar{\bar{S}}_T - K)^+$	$(S_T - \bar{\bar{S}}_T)^+$
Put	$(K - \bar{\bar{S}}_T)^+$	$(\bar{\bar{S}}_T - S_T)^+$

Where: $\bar{\bar{S}}_T$ *is the average value of the stock price since the option was taken out.*

Asian options are further sub-divided based on how the average is calculated, either arithmetically or geometrically; and whether the average is sampled discretely or continuously.

Table 2: Asian Options Averaging Equations

	Continuous	*Discrete*
Arithmetic	$A_T = \frac{1}{T}\int_0^T S_t\, dt$	$A_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$
Geometric	$G_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t\, dt\right)$	$G_T = \left[\prod_{i=1}^{n} S_{t_i}\right]^{\frac{1}{n}}$

Where S_{t_i} is the Asset price at discrete times t_i for i=1,…,n and in the limit as $n \to \infty$ the discrete cases converge to the continuous cases. A_T and G_T represent the arithmetic and geometric averages and can be substituted for the asset average $\bar{\bar{S}}_T$ in the option formula.

Arithmetic averaging is typically the more common form used in everyday life and is also the preferred type for typical financial markets. For practical reasons this also is true for the discrete form. While this is the case, the Geometric averaging has distinct advantages. In order to calculate return on investment for portfolio performance, geometric averaging is often considered a more reliable measure as it better accounts for the compounding investment of assets as well as it being more difficult to recover from negative returns; thus, carries higher importance for using geometric averaging for highly volatile assets. Notably, the geometric average is always equal to or less than that of the arithmetic average, which among other proofs, is given by Jensen's Inequality. This states that any concave function of an arithmetic mean is greater than or equal to the arithmetic mean of the function's values. As the logarithm function is a concave function, this inequality applies to the geometric average.

The true distinction between geometric and arithmetic averaging is that under the BS model, only a closed form analytical solution for geometrically averaged Asian options exist. The arithmetically averaged case is analytically intractable as the sum of log-normal densities, which are how stochastic assets are distributed due to being partially modelled by Wiener processes, have no known explicit representation. For the geometric case; the product of these log-normal distributions is itself a known log-normal, which can therefore be solved.

2.2 Analytic Solutions

As before, the geometric Asian options can either be solved via the Black-Scholes PDE method, or by utilizing the theory of equivalent martingales. As Asian options have a further dependency on the running asset average, when compared to the BS model for vanilla options, this additional state variable results in the subsequent PDE being two dimensional. This complicates the solution and requires a transformation of variables in order to reduce the dimensionality to a solvable first order PDE. For consistency, the martingale method derivation will again be followed for this investigation.

2.2.1 Martingale Continuous Gematric Asian Option

Equation 23: EMM Geometric Asian Call Option Price

$$C_G = E\left[e^{rt}\left(\bar{\bar{S}}_T - K\right)^+\right]$$

As Asian options depend on a running average, this separates the valuation into two distinct temporal sections. The first is from the initial time $t_0$of the option creation to the current valuation time *t,* where the path the option has taken is already determined. Therefore, the running geometric average G_t in the range *(0,t)* is known. The second term is the average over the range *(t,T)* which is dependent on currently unknown future information, and as such is seen to be stochastic in nature. As the average at the current time can only be calculated once the asset value has been determined, the average calculation must lag slightly behind the instantaneously known asset value, which is denoted as the time frame t',where $t' > t$.

From the continuous geometric averaging formula:

$$G_T = \exp\left(\frac{1}{T}\int_0^t ln\, S_{t'}\, dt' + \frac{1}{T}\int_t^T ln\, S_{t'}\, dt'\right)$$

Substituting for the deterministic $\mathrm{G_t}$ value:

$$\ln G_T = \left(\frac{1}{T}\ln G_t + \frac{1}{T}\int_t^T ln\, S_{t'}\, dt'\right)$$

Under the risk neutral valuation, $\mathrm{S}_{t'} = \mathrm{S}_t e^{\left(r-\frac{1}{2}\sigma^2\right)(t'-t)+\sigma W^Q_{t'-t}}$ which is log normally distributed, is substituted giving;

$$\ln G_T = \frac{1}{T}\ln G_t + \frac{1}{T}\int_t^T [\ln S_t + \left(r-\frac{1}{2}\sigma^2\right)(t'-t) + \sigma W^Q_{t'-t}]dt'$$

$$\ln G_T = \frac{1}{T}\ln G_t + \frac{1}{T}\left((T\ln S_t - t\ln S_t) + \left(r-\frac{1}{2}\sigma^2\right)\left(\frac{T^2}{2} - tT + t^2 - \frac{t^2}{2}\right) + \frac{\sigma}{T}\int_t^T W^Q_u dt'\right)$$

$$\ln G_T = \frac{1}{T}\ln G_t + \left(1-\frac{t}{T}\right)\ln S_t + \frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)\frac{(T-t)^2}{T} + \frac{\sigma}{T}\int_t^T W^Q_{t'-t}\,dt'$$

Changing the limits of integration to $\int_{t-t}^{T-t} W^Q_{t'-t}\,d(t'-t)$ and by rearranging, and substituting for $\tau = T - t$, $u = t' - t$ gives:

$$\ln G_T = \frac{1}{T}\ln G_t + \left(1-\frac{t}{T}\right)\ln S_t + \frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)\frac{\tau^2}{T} + \frac{\sigma}{T}\int_0^\tau W^Q_u\,du$$

Rearranging and using laws of logarithms gives:

Equation 24: *Continous Geometric Average Integral Form*

$$G_T = S_t\left(\frac{G_t}{S_t}\right)^{\frac{t}{T}} exp\left[\frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)\frac{\tau^2}{T} + \frac{\sigma}{T}\int_0^\tau W^Q_u\,du\right]$$

The only unknown at this point is the stochastic integral term. The analytical solution requires this be given in terms of a standard gaussian random variable Z (0,1). This entails calculating the correlation of the integrated Wiener process and a standard Wiener process; as well as the variance of the integral process. The formulation of these is given in the appendix, which gives the values as:

$$Var(I) = \frac{T^3}{3}, \qquad Std(I) = \frac{T^{\frac{3}{2}}}{\sqrt{3}} \qquad \boldsymbol{Corr}\left(W^Q_t, \boldsymbol{I}\right) = \frac{\sqrt{3}}{2}$$

Where $I = \int_0^\tau W^Q_u\,du$, *and* $\boldsymbol{\sigma}_{W^Q_t} = \sqrt{T}$

Therefore, the stochastic integral for the continuous geometric average can be written in terms of a new standard gaussian random variable Z'.

$$I = \int_0^{\tau} W_u^Q \, du = \frac{T^{\frac{3}{2}}}{\sqrt{3}} Z'$$

Where $(Z', Z) = (0,1,\frac{\sqrt{3}}{2})$

Substituting this into Equation 24, and rearranging gives:

$$G_T = S_t \left(\frac{G_t}{S_t}\right)^{\frac{t}{T}} exp\left[\frac{1}{2}\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)\frac{\tau^2}{T} + \sigma\frac{\tau^{\frac{3}{2}}}{\sqrt{3}T}Z'\right]$$

This resembles the equation for a stochastic asset and can be written in terms of a new dividend paying Stochastic asset. This requires transforming $\frac{1}{2}\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)\frac{\tau^2}{T}$ to $\left(\mathrm{r} - \mathrm{q} - \frac{1}{2}\sigma'^2\right)\tau$, by using the following substitutions:

$$a = \frac{\tau}{T} = 1 - \frac{t}{T} \qquad Y = S_t\left(\frac{G_t}{S_t}\right)^{1-a} \qquad \sigma(\mathrm{a})' = \frac{\sigma a}{\sqrt{3}}$$

Which Gives:

$$G_t = Yexp\left[\frac{1}{2}\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)a\tau + \sigma\frac{a\sqrt{\tau}}{\sqrt{3}}Z'\right] = Yexp\left[\left(\mathrm{r} - \mathrm{q} - \frac{1}{2}\sigma'^2\right)\tau + \sigma'\sqrt{\tau}Z'\right]$$

Where: $q(a) = r - \frac{1}{2}\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)a - \frac{1}{6}\sigma^2\mathrm{a}^2$

The initial value of the Geometric average at the initial time $\mathrm{t} = 0$, $\tau = \mathrm{T}$, $\mathrm{a} = 1$, is then:

Equation 25: EMM Continuous Geometric Average Price

$$G_T = S_0 exp\left[\left(\mathrm{r} - q_0 - \frac{1}{2}{\sigma_0}^2\right)T + \sigma_0\sqrt{T}Z'\right]$$

Where: $q_0(1) = \frac{1}{2}\left(\mathrm{r} + \frac{1}{2}{\sigma_0}^2\right)$, $\qquad \sigma_0 = \sigma(0)' = \frac{\sigma}{\sqrt{3}}$ $\qquad Y = S_0$

In this form, this is effectively identical to that of a divided paying asset. Thus, it follows that the same martingale derivation applies, and subsequently, the same form of the solution for a call option of this hypothetical asset Y as for a vanilla European call option.

$$C_G = E[e^{rt}(G_T - K)^+]$$

Equation 26: *Continous Geometric Asian Call Option Analytic Price*

$$C_{G0} = S_0 e^{-q_0 T} N(d_1) - K e^{-rT} N(d_2)$$

Where:

$$d_1(S_0, T) = d_2(S_0, T) + \sigma_0\sqrt{T} = \frac{ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T}{o\sqrt{\frac{T}{3}}}$$

Where all values are fully deterministic, therefore and closed analytical solution

Where $(Z', Z) = (0,1,\frac{\sqrt{3}}{2})$ *for the Normal distribution N*

2.2.2 Martingale Discrete Geometric Asian Option

In real markets, stock prices and option values are not calculated truly continuously but are instead measured at discrete time intervals to the buyer's discretion. The times t_i are consecutively and uniformly spread, with temporal spacing Δ such that $t_i = i\Delta$. Therefore, the continuous differential stochastic asset in Equation 19 can be written a discrete form.

Equation 27: Discrete Stochastic Asset Motion

$$\Delta S = S_{t+!} - S_t = S_t \mathrm{r}\Delta t + S_t \sigma\sqrt{\Delta t} Z(0,1)$$

Which is equivalent to:

$$S_{t+!} = S_t exp\left[\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} Z\right]$$

With corresponding Discrete Geometric Asian Option:

$$G_T = \left[\prod_{i=1}^{n} S_{t_i}\right]^{\frac{1}{n}}$$

Where the underlying stochastic asset follows the black-Scholes model in the risk neutral world and is log normally distributed. $t_i > 0$ *denote the discrete asset valuation times, with initial time* $t = 0$ *and maturity date* $t_n = T$.

$$G_T = S_0\left[\prod_{i=1}^{n} \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\frac{t_i}{n} + \frac{\sigma}{n}W_{t_i}^{Q}\right)\right]^{\frac{1}{n}}$$

Where: $W_{t_i}^{Q}$ *is a Geometric Brownian motion Wiener process under the risk neutral measure Q at discrete time steps.*

$$\ln\frac{G_T}{S_0} = \frac{1}{n}\sum_{i=1}^{n}\left(r - \frac{1}{2}\sigma^2\right)t_i + \frac{\sigma}{n}\sum_{i=1}^{n}\left(W_{t_i}^{Q}\right)$$

This is a new stochastic asset, normally distributed $N(\mu_n, {\sigma_n}^2)$ with drift and variance given by:

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n}\left(r - \frac{1}{2}\sigma^2\right)t_i = \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{n}\sum_{i=1}^{n}(t_i) = \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{n}\sum_{i=1}^{n}(i\Delta)$$

$$\mu_n = \left(r - \frac{1}{2}\sigma^2\right)\bar{T}_n$$

Where $\bar{\mathrm{T}}_\mathrm{n} = \frac{\Delta}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{i})$ which is a basic geometric series, therefore:

$$\bar{T}_n = \frac{1}{2}(n+1)\Delta$$

Variance is calculated as in the same way as the continuous case in the appendix as:

$$\sigma_n{}^2 = Var\left(\frac{\sigma}{n}\sum_{i=1}^{n}(W_{t_i}^Q)\right) = E\left(\frac{\sigma}{n}\sum_{i=1}^{n}(W_{t_i}^Q)\right)^2 - \left(E\left(\frac{\sigma}{n}\sum_{i=1}^{n}(W_{t_i}^Q)\right)\right)^2 = E\left(\frac{\sigma}{n}\sum_{i=1}^{n}(W_{t_i}^Q)\right)^2$$

$$= \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n}E\left(W_{t_i}^Q W_{t_l}^Q\right) = \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n}\min(t_i, t_l) = \frac{\Delta\sigma^2}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n}\min(i, l) = \frac{\Delta\sigma^2}{n^2}\sum_{L=1}^{n}L^2$$

Substituting for the known series sum $\sum_{L=1}^{n} L^2 = \frac{n(n+1)(2n+1)}{6}$ and defining the new variable:

$$\hat{T}_n = \frac{\Delta}{n^2}\sum_{L=1}^{n}L^2 = \Delta\frac{(n+1)(2n+1)}{6n}$$

Then:

$$\sigma_n{}^2 = \sigma^2\hat{T}_n$$

Where $W_{t_j}^Q$ is a Wiener process which is separate to $W_{t_i}^Q$; however, as i =l =L time increments fully overlap, the variables are logically fully dependant. This follows as:

$$Corr\left(W_{t_j}^Q, W_{t_i}^Q\right) = \frac{\min(t_i, t_j)}{\sqrt{t_i}\sqrt{t_j}} = \frac{t_L}{t_L} = 1$$

Therefore as $\ln\frac{G_T}{S_0}$ follows the distribution $N(\mu_n, \sigma_n{}^2)$, when standardised yields:

$$\ln\frac{G_T}{S_0} = \mu_n + \sigma\sqrt{\hat{T}_n}\,Z(0,1)$$

Which follows that:

Equation 27: EMM Discrete Geometric Average Price

$$G_T = S_0 exp\left[\left(r - \frac{1}{2}\sigma^2\right)\bar{T}_n + \sigma\sqrt{\hat{T}_n}\,Z(0,1)\right]$$

Notably the Gaussian random variable in this case is the same as that of the underlying asset. The derivation of the discrete geometric Asian call option from this point follows the EMM procedure and gives:

Equation 28: Continous Geometric Asian Call Option Analytic Price

$$C_{Gn} = S_0 e^{-q_n T} N(d_3) - K e^{-rT} N(d_4)$$

Where: $T = n\Delta$

$$d_3(S_0, T) = d_4(S_0, T) + \sigma\sqrt{\hat{T}_n} = \frac{\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)\bar{T}_n}{\sigma\sqrt{\hat{T}_n}}$$

$$q_n = r - \left(\mathrm{r} - \frac{1}{2}\sigma^2\right)\frac{\bar{T}_n}{T} - \frac{1}{2}\sigma^2\frac{\hat{T}_n}{T}$$

2.2.3 Discrete and Continuous equivalence

In limit of $n \to \infty, \Delta \to 0$ the discrete geometric retrieves the continuous case.

$$\bar{T}_n = \frac{1}{2}(n+1)\Delta = \frac{n\Delta}{2} + \frac{\Delta}{2} = \frac{T}{2} + \frac{\Delta}{2}$$

$$\hat{T}_n = \Delta\left(\frac{(n+1)(2n+1)}{6n}\right) = \Delta\left(\frac{n}{3} + \frac{1}{2} + \frac{1}{6n}\right)$$

$$\lim_{n\to\infty,\Delta\to 0} \bar{T}_n = \frac{T}{2} \qquad \lim_{n\to\infty,\Delta\to 0} \hat{T}_n = \frac{n\Delta}{3} = \frac{T}{3}$$

$$\lim_{n\to\infty,\Delta\to 0} q_n = r - \frac{1}{2}\left(\mathrm{r} - \frac{1}{2}\sigma^2\right) - \frac{1}{6}\sigma^2 = \frac{1}{2}\left(\mathrm{r} + \frac{o^2}{6}\right) = q_0$$

$$\lim_{n\to\infty,\Delta\to 0} d_3(S_0, T) = d_4(S_0, T) + \sigma\sqrt{\frac{T}{3}} = \frac{log\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T}{o\sqrt{\frac{T}{3}}} = d_1(S_0, T)$$

Which trivially gives Equation 28 equal to Equation 26.

Chapter 3 Monte Carlo Pricing Options

3.1 Monte Carlo Simulation

Derivatives and various other mathematical problems can be also be solved via numerical techniques rather than analytical methods. These are often used in cases where no analytical solution is available, or otherwise inefficient to calculate. One of the most wides spread and powerful numerical techniques used for analysing complex problems is the Monte-carol method. While it is often not be the most efficient method for solving such problems, it is relatively simple to implement and has a broad range of applications. First used in the pricing of derivatives by Boyle (1977), the method was initially developed in 1946 by Ulam and von Neumann. It was first envisioned as a thought experiment to analyse the game “Solitaire”; however, the method was integral to the Manhattan Project.

The basis of the method is the process of random sampling with application of the strong law of large numbers and the central limit theorem. The method can be used to estimate the expected value of a random variable. In Monte Carlo Simulation method, the previous states of the variable are known and so there is path memory. With this property and the ability to solve analytically intractable problems, this technique lends itself exceptionally well to the pricing of Asian options.

The simulation procedure is summarized below. The previous Black-Scholes economy assumptions that volatility and risk-free interest rate are constant, and that the option only depends on the stock price and maturity date hold:

1) As this is a numerical method, time must be discretised into equal segments $\Delta t = \frac{T}{n}$ and the underlying stochastic asset S_t, using the discrete form given in Equation 27 rather than the continuous differential given in Equation 19. A random number generator (RNG) is then used to generate values of Z.
2) The parameters are entered into the equation and looped continuously "n" times to generate a random path for the risk-neutral stock originating from S_0 iterating until $S_T = S_{t+n\Delta t}$.
3) The payoff of the desired option is then calculated, discounted, and stored for the path
4) Previous steps are repeated for a substantially large number of paths.
5) Calculate the sample mean for all discounted option values, which were calculated based on each sample path in step 3. This sample mean is the discounted expected payoff of the asset, hence the approximate price of the option.

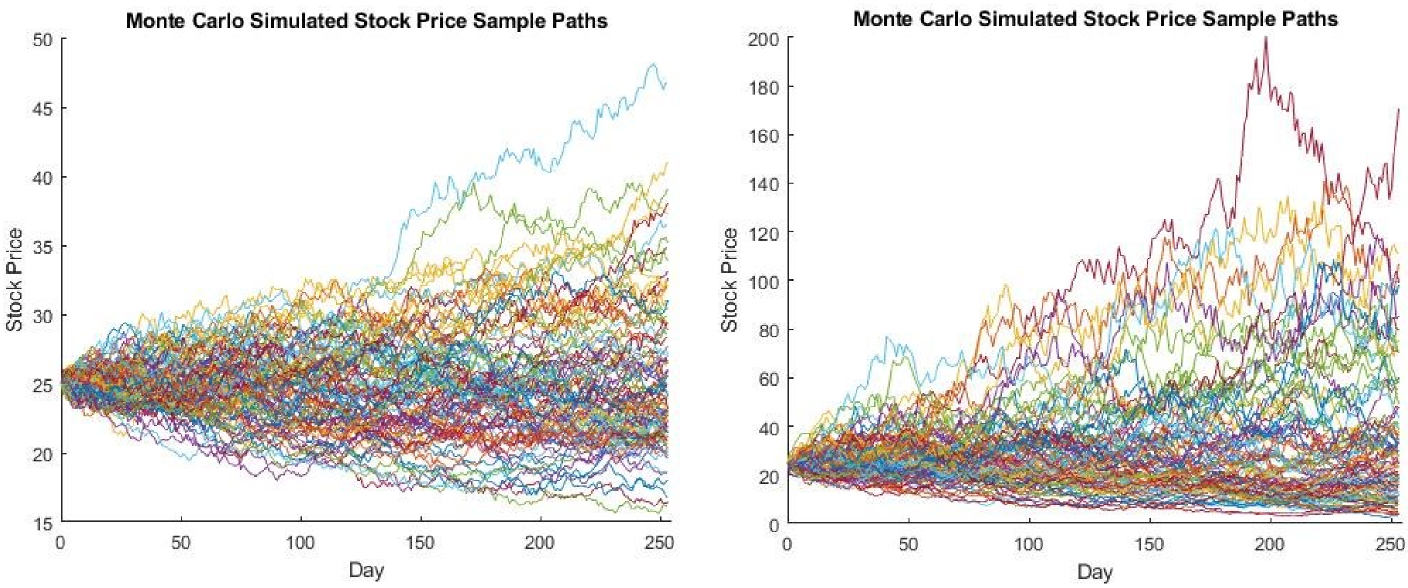


Figure 1 (a) left (b) right: Simulated asset price paths under Geometric Brownian Motion for low volatility ($\sigma = 0.2$) *and high volatility* ($\sigma = 0.8$) *respectively. (S_0 = 25, K = 20, r = 0.05, n = 253, m = 100)*

Figure 1 shows the stock paths simulated for the alternative form of Equation 27. From the comparison of the two volatility cases, as the stock price cannot be negative, the average asset path is a higher value in the high volatility model; as expected for risk compensation.

For a European call option with expected price $\hat{C}$, is given from the Monte-carlo method for "m" path iterations as:

From the strong law of large numbers:

$\hat{C} = \frac{1}{m}\sum_{j=1}^{m} C_j$ $\quad\quad$ $\lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} C_j = E(C) = C$ $\quad\quad$ $\lim_{m\to\infty} \hat{C} = C$

Where $\hat{C}$ is the approximate price, C_j is the call option price from iteration I, and C is the true exact price of the option.

This shows that the estimator is unbiased as it converges to the correct mean value. For an estimator to be valid, the variance must decrease as the sample size increases, which can be seen as:

$$Var(\widehat{C}) = Var\left(\frac{1}{m}\sum\nolimits_{j=1}^{m} C_j\right) = \frac{1}{m^2}Var\left(\sum\nolimits_{j=1}^{m} C_j\right) = \lim_{m\to\infty} \frac{{\sigma_C}^2}{m}$$

Where: ${\sigma_C}^2$ is the true population variance Var(C)

As the number of iterations "m" is finite however, sample variance exists in the model.

$$\hat{s}_c{}^2 = \frac{1}{m-1}\sum\nolimits_{j=1}^{m} (C_i - \hat{C})^2$$

Where $\hat{s}^2$ is the unbiased sample variance and converges to the true population variance $\lim_{m\to\infty} \hat{s}_c{}^2 = {\sigma_C}^2$

Equation 29: Central Limit Theorem

$$\frac{\hat{C} - C}{\sigma_C/\sqrt{n}} = \lim_{m\to\infty} \frac{\hat{C} - C}{\hat{s}_C/\sqrt{n}} \xrightarrow{d} N(0,1)$$

And from the central limit theorem, a 95% confidence interval for the Monte-Carlo approximation is found.

$$1 - 0.05 = P\left(z_{\frac{0.05}{2}} \le \frac{\hat{C} - C}{\frac{\hat{s}_c}{\sqrt{n}}} \le z_{1-\frac{0.05}{2}} \right)$$

$$= P\left(\hat{C} - z_{1-\frac{0.05}{2}}\frac{\hat{s}_c}{\sqrt{n}} \le C \le \hat{C} + z_{1-\frac{0.05}{2}}\frac{\hat{s}_c}{\sqrt{n}}\right)$$

$$\hat{C} - \frac{1.96\widehat{s_c}}{\sqrt{m}} < C < \hat{C} + \frac{1.96\widehat{s_c}}{\sqrt{m}}$$

$$\hat{C} - \frac{1.96\hat{s}}{\sqrt{m}} < C < \hat{C} + \frac{1.96\hat{s}}{\sqrt{m}}$$

Equation 30: Variance of Monte Carlo Simulation

$$Var(\hat{C}) = \sigma_{\hat{C}}{}^2 = \frac{\hat{s}^2}{m}$$

This shows that the standard deviation of the MC approximation $\sigma_{\hat{C}}$ is equal to $\frac{\hat{s}}{\sqrt{m}}$, which means that the reduction of error is an inverse square relationship with the number of iterations, requiring substantially more iterations and computational effort for an equivalent improvement in accuracy.

As the Black-Scholes model is known to give the theoretically exact price for a risk-neutral call option C, the validity of the Monte-Carlo model implementation can be affirmed, along with showing that the error decreases as expected. In stock trading, typically stocks are listed to the accuracy of the “Tick Rate” which is typically $0.01. This is the value used to determine the amount of iterations required, as the Monte-Carlo simulation is deemed sufficient when the difference between $\hat{C} - C$ is equal to 0.01.

Once the European option is within the tick rate, the same method and formula for the MC variance can be used by substituting the call option calculation with either the arithmetic or geometric Asian option pricing formula

It should be noted that for this investigation, the model was implemented in MATLAB version R20183b, as seen in the CODEX, and the pseudo RNG function built into MATLAB deemed sufficiently suitable for depth of the investigation. A modification is made when calculating the geometric average, as when the stock price is null, the natural log is negative infinity which would dominate the average. Therefore, before the geometric average is taken, all stocks are increased in value by 1 and the resultant average then subtracted by 1.

3.2 Monte-Carlo Variance Reduction Techniques

The accuracy of the Monte-Carlo method is determined by the computationally undesirable inverse square ratio:$\frac{\hat{s}}{\sqrt{m}}$. This is due the inherent error cancelation from the averaging process. In order to reduce the number of trials "m", and thus computational effort, needed for the approximation to be within tick rate accuracy, two variance reduction techniques were investigated.

3.2.1 Antithetic Variates Method

The first approach improves the efficiency of the simulation by sampling the input variables, in this case the asset values, more strategically to further reduce the error cancelation. As the random variables $Z_{t_i}(0,1)$ are drawn from a standard gaussian distribution, the corresponding negative values $-Z_{t_i}(0,1)$ are also an equally probable valid draw from the distribution, and are known as the antithetic counter sample. These values can then be used to generate an antithetic counter stock price $S_{ti}^*(-Z_{t_i})$, and subsequent antithetic option value $C_j^*(S_{ti}^*, T)$ for an equally probable antithetic world.

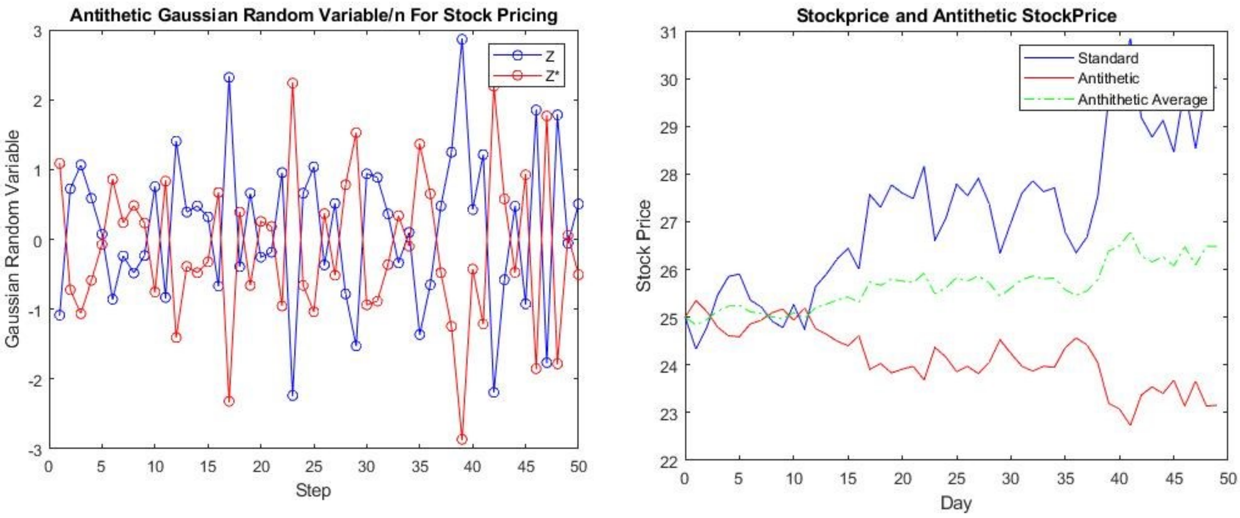


Figure 2: (a) left (b) right: (a) Shows how the Simulated Gaussian random variables Z and antithetic values Z are symmetric. (b) Shows the corresponding asset paths and their average. (S_0 = 25, K = 20, σ= 0.2, r = 0.05, n = 253, m = 100)*

From the formula for the call option, it would then be expected that the risk neutral world option price C_j and the antithetic risk neutral world option price C_j^* to be negatively correlated. Thus, when C_j overshoots the real-world true value, C_j^* would undershoot it, balancing the two depending on the degree of correlation. Averaging the Estimates of the two then utilises this correlation to form what is known as an antithetic estimate $\bar{C}_{AV}$.

$$\bar{C}_{AV} = \frac{1}{2}\left[\frac{1}{m}\sum\nolimits_{j=1}^{m} C_j + \frac{1}{m}\sum\nolimits_{j=1}^{m} C_j^*\right] = \frac{\hat{C} + \widehat{C^*}}{2}$$

Where: $\bar{C}_{AV}$ is the antithetic option price, C_j the originally simulated option price for a specific path iteration j. $C_j{}^*$ is the corresponding antithetic option price, $\hat{C} + \widehat{C^*}$ are the average option prices respectively.

With variance:

$$Var(\bar{C}_{AV}) = \frac{1}{2}\left[Var\left(\frac{1}{m}\sum\nolimits_{j=1}^{m} C_j\right) + Var\left(\frac{1}{m}\sum\nolimits_{j=1}^{m} C_j^*\right)\right] = \frac{1}{2m^2}\left[\sum\nolimits_{j=1}^{m} Var\left(C_j + C_j^*\right)\right]$$

$$= \frac{1}{2m^2}\left[m{\sigma_C}^2 + m{\sigma_C}^2 + 2\sum\nolimits_{j=1}^{m} Cov\left(C_j, C_j^*\right)\right] = \frac{1}{2m}\left(2{\sigma_C}^2 + 2Cov(C, C^*)\right)$$

Substituting for the unbiased sample variance $\hat{s}^2$and the correlation $\rho_{C,C^*} = \frac{Cov(C,C^*)}{\sigma_C{}^2}$ where $-1 \leq \rho_{C,C^*} \leq 1$, and as they are effectively drawn from the same distribution $\sigma_C{}^2 = \sigma_{C*}{}^2$

$$Var(\bar{C}_{AV}) = \frac{\hat{s}_C{}^2}{m}\left(1 + \rho_{C,C^*}\right) \xrightarrow{\lim\limits_{m\to\infty}} \frac{\sigma_{\hat{C}}{}^2}{m}\left(1 + \rho_{C,C^*}\right)$$

Where the sample variance is now given by:

$$\hat{s}_C{}^2 = \frac{1}{m-1}\sum\nolimits_{j=1}^{m}(C_{AV} - \hat{C})^2$$

Where: $C_{AV} = \frac{C_j + C_j^*}{2}$

Therefore, the variance will be reduced when the covariance is negative $Cov(C, C^*) < 0$, and in such a case where the negative correlation is large, showing it to be more effective than if the number of iterations had been doubled. While the sample of antithetic random variables are perfectly negatively correlated, the degree of the negative correlation is reduced when applied in a function such as the calculation of the stock price and the subsequent option calculation. This can be seen visually when comparing the reduction in symmetry between the antithetic values in Figure 2(a) and (b).

3.2.2 Control Variate Method

When conducting numerical analysis, it is beneficial to compare the results against analytical results, as utilised previously to support the validity of the European Monte-Carlo model using the Black Scholes formula. The control variate technique exploits the information available in such closed form results to improve numerical estimates. This method can be applied when there are two similar highly correlated problems, and when the analytic solution is known for one of them. This lends itself to the estimation of analytically intractable arithmetic Asian options. This is due to only differing in nature from the previously solved geometric Asian options in their averaging process of the underlying asset thus $C_A(S_t) \approx C_G(S_t)$. As either averaging technique is trivially calculated in the simulation, information encoded in the errors between the numerical and analytic geometric options can be used to correct the corresponding simulated arithmetic option value (Kemma and Vosrt 1990).

$C_{Arith} = E(C_A)$ is the true Arithmetic Asian option price, $C_{Geo} = E(C_G)$ is the true geometric Asian option price which can be calculated analytically. C_A and C_G are the prices calculated for a single path, as well as, $\widehat{C_A} = \frac{1}{m}\sum_{j=1}^{m} C_{A_j}$ and $\widehat{C_G} = \frac{1}{m}\sum_{j=1}^{m} C_{G_j}$ are the estimated prices calculated from multiple monte Carlo simulation paths. A new control variate estimator for the arithmetic option, ${C_{Arith}}^{CV}$, is given by:

$$ {C_{Arith}}^{CV} = \frac{1}{m}\sum_{j=1}^{m} C_{A_j} - (C_{G_j} - C_{Geo}) + C_{Geo} = \frac{1}{m}\sum_{j=1}^{m} (C_{A_j} - C_{G_j}) + C_{Geo} $$

$$ {C_{Arith}}^{CV} = \widehat{C_A} - \widehat{C_G} + C_{Geo} $$

And from the strong law of large numbers, this estimator is unbiased as $\lim_{m\to\infty}$, $\widehat{C_A} = E(C_A)$, $\widehat{C_G} = E(C_G)$, therefore:

$$ E\left({C_{Arith}}^{CV}\right) = \widehat{C_A} - \widehat{C_G} + C_{Geo} = C_{Arith} $$

And following similar steps as in the antithetic variance derivation, the variance of ${C_{Arith}}^{CV}$ is given by:

$$ Var({C_{Arith}}^{CV}) = Var\left(\frac{1}{m}\sum_{j=1}^{m} C_{A_j}\right) + Var\left(\frac{1}{m}\sum_{j=1}^{m} C_{G_j}\right) - 2\frac{1}{m^2}\sum_{j=1}^{m} Cov\left(C_{A_j}, C_{G_j}\right) $$

$$ Var({C_{Arith}}^{CV}) = \frac{1}{m}\left(Var(C_A) + Var(C_G) - 2Cov(C_A, C_G)\right) = \frac{1}{m} Var(C_A - C_G) $$

Therefore the variance will be reduced if $\widehat{C_A}$ and $\widehat{C_G}$ are sufficiently close in value to satisfy: $Var(C_A - C_G) < Var\left(\widehat{C_A}\right)$. As seen earlier that $Var(\widehat{C}) = \frac{{\sigma_C}^2}{m} \xrightarrow{\lim_{m\to\infty}} \frac{{\hat{s}_C}^2}{m}$, this is equivalent to $Var({C_{Arith}}^{CV})$ being lower if and only if $\frac{Cov(C_A, C_G)}{Var(C_G)} > \frac{1}{2}$.

3.2.2.1 Optimised Control Variate

From this inequality, the reduction in variance due to the control variate can be maximised by optimising the control variate. This is known as the parametrised control variate estimator, or the regression estimator, and is denoted β where $\beta \in \mathbb{R}$.

Equation 31: Parametrised Control Variate

$$C_{Arith}(\beta)^{CV} = \widehat{C_A} - \beta\,(\widehat{C_G} - C_{Geo})$$

Notably, when $\beta = 0$ the original MC estimator $\widehat{C_A}$ is obtained, and the original control variate technique when $\beta = 1$. The minimal associated variance of $C_{Arith}(\beta)^{CV}$ is found by differentiating with respect to β, and setting the derivative to zero in order to minimise it. The optimal value of β is found to be:

Equation 32: Variance of Parametrised Control Variate

$$Var(C_{Arith}(\beta)^{CV}) = \frac{1}{m}\left(Var(C_A) + \beta^2 Var(C_G) - 2\beta Cov(C_A, C_G)\right)$$

$$0 = \frac{1}{m}\left(2\beta_{opt} Var(C_G) - 2Cov(C_A, C_G)\right)$$

$$\beta_{opt} = \frac{Cov(C_A, C_G)}{Var(C_G)}$$

Which when substituted back in for Equation 32 gives:

Equation 33: Variance of Optimised Control Variate

$$Var(C_{Arith}(\beta_{opt})^{CV}) = \frac{Var(C_A)}{m}\left(1 - {\rho_{C_A,C_G}}^2\right) = \frac{{\sigma_{C_A}}^2}{m}\left(1 - {\rho_{C_A,C_G}}^2\right)$$

$$\xrightarrow{\lim_{m\to\infty}} \frac{{\hat{s}_{C_A}}^2}{m}\left(1 - {\rho_{C_A,C_G}}^2\right) = Var(\hat{C}_A)\left(1 - {\rho_{C_A,C_G}}^2\right)$$

Where: ρ_{C_A,C_G} *is the* $Corr(C_A, C_G) = \frac{Cov(C_A,C_G)}{\sigma_{C_A}\sigma_{C_G}}$

The variance of the MC will be reduced via the control variate method if and only if $\rho_{C_A,C_G}{}^2 < 1$.

Notably, due to the squared term, the correlation between the geometric and arithmetic simulated Asian options can be negative for a variance reduction. This also results in the variance reduction sharply increasing as $|\rho_{C_A,C_G}|$ tends towards 1, and vice a versa. Similar to the antithetic method, the variance depends heavy upon a correlation relationship, though now between the simulated geometric and arithmetic options.

As before, these values must be calculated from the simulation and as such, the sample covariance and sample variances must be used.

$$\widehat{\beta_{opt}} = \frac{Sample\ Cov(C_A, C_G)}{Sample\ Var(C_G)}$$

$$Sample\ Cov(C_A, C_G) = \frac{1}{m-1}\sum_{j=1}^{m}(C_{A_j} - \widehat{C_A})(C_{G_j} - \widehat{C_G})$$

$$Sample\ Var(C_G) = \frac{1}{m-1}\sum_{j=1}^{m}(C_{G_j} - \widehat{C_G})^2$$

Where $\widehat{\beta_{opt}}$ converges to β_{opt} for large iterations "m" due to the strong law of large numbers.

3.2.2.2 Variance reduction of Optimal Control Variate

A bias exists in the estimation of $\widehat{\beta_{opt}}$ from the "m" simulations due to the interdependency of $\widehat{\beta_{opt}}$ on C_A, C_G and thus Equation 33 cannot be used to calculate the variance directly. A Theorem published by Levenberg, Moeller and Welch (1982) allows the calculation of the variance in such a case.

$$Var(C_{Arith}(\widehat{\beta_{opt}})^{CV}) \approx \frac{m-2}{m-3}(1 - \rho_{C_A,C_G}{}^2)Var(\widehat{C_A}) = \frac{m-2}{m-3}(1 - \rho_{C_A,C_G}{}^2)\frac{\sigma_{C_A}{}^2}{m}$$

This shows that the control variate method has lower variance than the basic MC when:

$$Var(C_{Arith}(\widehat{\beta_{opt}})^{CV}) < Var(\widehat{C_A})$$

$$\frac{m-2}{m-2-1}\left(1-\rho_{C_A,C_G}{}^2\right) < 1 \rightarrow (m-2)\left(1-\rho_{C_A,C_G}{}^2\right) < (m-2)-1$$

Equation 34: Condition for Optimal Control Variate Variance Reduction

$$\rho_{C_A,C_G}{}^2 > \frac{1}{m-2}$$

Notably, as m is very large, $\lim_{m\to\infty} \frac{m-2}{m-3} = 1$ and so $\rho_{C_A,C_G}{}^2$ is expected to dominate the variance reduction. The variance of C_G, while theoretically known, must be calculated numerically due to integral terms which arise in the function.

Importantly, when the variance is calculated this way, the choice of control variate does not affect the variance.

Chapter 4 Stochastic Volatility

Unlike in the Black Scholes model where the volatility is assumed to be constant, in reality the implied volatilities for options vary. This variation in the volatility can be viewed to be the result of the arrival of information influencing the trading of the asset greatly due to fluctuating demand. This behaviour suggests that the inherent volatility, or at least the *implied volatility* traders infer, is also a stochastic process. As the volatility of an asset cannot be directly measured, it is difficult to estimate how volatile the volatility is.

In order to improve the model and further mimic reality, the instantaneous risk-neutral world volatility used was simulated as a stochastic variable in the Monte-Carlo method. The method implemented is similar to that used by Hull and White (1987), which was one of the first pricing models to implement stochastic volatilities.

Hull and White assumed that volatility was not a directly traded asset and that the instantaneous volatility and, by extension, the drift of the volatility and the associated volatility Wiener process were not directly correlated with the asset price and the Wiener process of the asset. This means that information may flow from the volatility to the asset price but not the other way around.

Since the Monte-Carlo method relies on multiple simulations and, as previously mentioned, it is often not the most efficient method, correlating the two Wiener processes could significantly increase the computational time. Hull and White concluded that for a large number of simulation iterations, assuming the Wiener processes for the instantaneous volatility and the stock were not correlated, the MC method could be used effectively to derive option prices under these assumptions.

For a stochastic risk-neutral asset under all other previous assumptions with instantaneous volatility σ and associated instantaneous variance $V = \sigma^2$, the stochastic diffusive asset with stochastic diffusive volatility was modelled as:

$$dS = rS_t\mathrm{dt} + V^{0.5}S_t\,\mathrm{d}W_t^Q$$

$$dV = \mu_V V\mathrm{dt} + \xi_V V_t \mathrm{d}W_t^v$$

Where: $\mu_V$is the instantaneous drift of the volatility variance, ξ_V is the standard deviation of the variance assumed to be constant, $\mathrm{d}W_t^Q$ *and $dW_t^v$are Wiener processes with correlation $\rho_{\mathrm{dW,dW}}$, assumed to be 0*

The true process the stochastic volatility follows is likely very complex, but it must not take on negative values and the process must tend to zero as the variance of the asset σ^2tends to zero. As the drift of the variance μ_V only affects the magnitude of the stock changes, it does not change the expected value of said stock and thus, does not interfere with the martingale calculation. This, along with the assumption that the volatility itself is not a tradeable asset in the model, allows for the substitution of a relatively arbitrary function $\widetilde{\mu_V}$.

$$dV = \widetilde{\mu_V}\, V\mathrm{dt} + \xi_V V\mathrm{d}$$

As the real-world volatility is normally inferred to be mean-reverting once perturbed by information, the risk-neutral volatility and by extension $\widetilde{\mu_V}$, is typically modelled to be mean reverting as well.

$$\widetilde{\mu_V} = \lambda(\hat{\sigma} - \sigma_t)$$

Where: λ is a constant determining the speed of mean reversion, and $\hat{\sigma}$ is the volatility the asset tends to revert to, $\sigma_t = \sqrt{V_t}$

Following the same process for deriving the discrete asset price Monte-Carlo formula, the variance was modelled as:

$$\Delta V = \widetilde{\mu_V}\, V_t \Delta \mathrm{t} + \ \xi_V V_t Z_{V_t} \sqrt{\Delta t}$$

$$V_{t+\Delta t} = V_t e^{\left(\widetilde{\mu_V} - \frac{1}{2}\xi_V{}^2\right)\mathrm{t} + \sigma\sqrt{\Delta t} Z_{V_t}(0,1)}$$

$$S_{t+\Delta t} = S_t e^{\left(\mathrm{r} - \frac{1}{2}\mathrm{V}\right)\mathrm{t} + \sigma\sqrt{\Delta t} Z_t(0,1)}$$

Where the volatility variance of the asset V_t is always positive so long as $2\lambda\hat{\sigma} > \xi_V{}^2$

This implementation is known as the Basic Heston model for pricing financial assets with stochastic volatility.

Chapter 5 Results

The investigation aims to determine the most appropriate and accurate pricing for the analytically intractable arithmetically averaged Asian Options.

In this section, the base Monte Carlo price approximation for Vanilla European options are reviewed in comparison to the Black-Scholes analytical prices, to first evaluate the accuracy of the base model. I investigate the effect of increasing the number of iterations on the variance and the error of the numerical model. I also examine how many iterations are required for the error between the analytical and numerical valuations to fall within an uncertainty of the typical market tick rate of 0.01. I then compare these with the results given by the Antithetic variance reduction technique.

The effect that varying the gap between the initial stock price and strike price has for the vanilla options and the more complex Asian option pricing is then analysed and compared to those given from the corresponding analytical solutions. The discrepancy between the analytical geometric prices informs the choice of the control variate for the reduction technique.

5.1 European Call Option Results

The following are the results for a vanilla European call option using the base Monte Carlo method discussed in section 4.1 and the Antithetic Monte Carlo method in Section 4.2.1 compared to the theoretical price given from the Black-Scholes Pricing method discussed in Chapter 2. The number of Monte Carlo iterations ‘m’ were doubled until the Base Monte Carlo approximation was within the tick rate of 0.01 of the theoretical prices.

Table 3

Valuation of Non-Dividend Paying Vanilla European Option Using Base Monte Carlo and Antithetic Monte Carlo Methods. $S_0 = 25, K = 20, r = 0.05, \sigma = 0.2, n = 253$. *Corresponding Black Scholes Price* $C_0 = 6.1472$

Number of Path Iterations 'm'	*Base MC European Call Option Price*	*95% Confidence Error*	*Comparative Pricing Error to Analytical*	*Variance*	*Antithetic MC European Call Option Price*		*95% Confidence Error*	*Comparative Pricing Error to Analytical*	*Variance*
100	6.844728	0.94206	0.697528	0.231017	6.146607	0.222563	0.000593	0.012894	
200	5.765492	0.632448	0.381708	0.104121	6.055321	0.129115	0.091879	0.004340	
400	6.274700	0.489233	0.127500	0.062304	6.169588	0.121642	0.022388	0.003852	
800	6.178555	0.339315	0.031355	0.029971	6.132806	0.072751	0.014394	0.001378	
1600	6.080969	0.233373	0.066231	0.014177	6.166311	0.059817	0.019111	0.000931	
3200	5.928921	0.161391	0.218279	0.006780	6.127109	0.037996	0.020091	0.000376	
6400	6.171011	0.116577	0.023811	0.003538	6.134653	0.027024	0.012547	0.000190	
12800	6.193424	0.084302	0.046224	0.001850	6.164862	0.02022	0.017662	0.000106	
25600	6.170854	0.059376	0.023654	0.000918	6.162977	0.014502	0.015777	5.47E-05	

Table 3 shows that simulated pricing becomes more accurate the higher the number of iterations, as seen from the reduced error and confidence intervals; in the relationship predicted by Equation 30, where the factor of variance reduction is directly proportional to the factor the number of iterations has increased. This trend is seen for both the base and antithetic Monte Carlo cases and is visualised in appendix Figure 1.

Notably, the theoretical price given from Black-Scholes falls within the 95% confidence interval in all but two cases being 3200 iterations for the base method and 25600 iterations for the Antithetic case. This discrepancy is likely due to the inherent random sampling of the process.

As expected, increasing the number of iterations also decreases the confidence error. While the option price estimated in previous iterations may be closer to the theoretical value, they have a larger margin of error. Thus, the accuracy of the Monte Carlo method increases with the number of iterations. This property, along with the discrepancy between the Black-

Scholes and the base Monte Carlo pricing falling within the tick rate, supports the validity of this method for the pricing of options.

The results also demonstrate that the Antithetic method is more accurate than the base method, as in the majority of cases the pricing error compared to the analytical is smaller. The confidence error range and the variance are decreased in all cases giving a consistently more precise estimation of the option prices. For these simulation values, the percentage variance reduction via the Antithetic process is consistently above 93% for these simulation cases, and averages a significant 94.45% reduction, with a corresponding average reduction factor larger than 8 over the base Monte Carlo. This variance reduction is equivalent to increasing the number of iterations by the same factor. See appendix Figure 2.

The Antithetic method does not significantly increase the computational effort required, ignoring the averaging procedure the work is approximately equivalent to doubling the iterations. From the Appendix Figure 1, the antithetic method converges towards the analytical solution faster than when the iterations are doubled for the Base approximation. This comes with a reduced calculation time for the valuation to fall within the uncertainty.

5.2 Constant Volatility Asian Option Price Results

The following are the results for fixed call Asian options using the base Monte Carlo method, the Antithetic Monte Carlo and the control variate reduction technique for the pricing of arithmetic Asian options. These are compared to both the discrete and continuous analytical prices derived in section 3.2. The initial stock price was varied, and the strike price held constant to investigate the effects on both the simulation accuracy. This is effectively equivalent to varying the strike price and a fixed initial price for the payoff of Asian options as seen from the payoff formula in Table 1 of Chapter 2. The Tables Containing the Results for the 1,000-path iteration case can be found in Appendix B.2.

Table 4

Valuation of Non-Dividend Geometric Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$

S_0	*Analytical Discrete Geometric Asian Call Option*	*Analytical Continuous Geometric Asian Call Option*	*MC Geometric Asian Call Option*	*95% Confidence Error*	*Comparative Pricing Error Discrete Analytical*	*Comparative Pricing Error Continuous Analytical*	*Antithetic MC Geometric Asian Call Option*	*95% Confidence Error*	*Comparative Pricing Error Discrete Analytical*	*Comparative Pricing Error Continuous Analytical*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000002	0.000002	0.000003	0.000002	0.000001	0.000001	0.000003	0.000002	0.000000	0.000000
7	0.000518	0.000500	0.000510	0.000035	0.000009	0.000009	0.000524	0.000025	0.000006	0.000024
8	0.016541	0.016249	0.016495	0.000231	0.000046	0.000246	0.016481	0.000161	0.000060	0.000232
9	0.145321	0.144145	0.144992	0.000757	0.000328	0.000848	0.144999	0.000497	0.000322	0.000854
10	0.556543	0.554682	0.555769	0.001512	0.000774	0.001088	0.555724	0.000743	0.000820	0.001042
11	1.276680	1.274950	1.276579	0.002161	0.000101	0.001630	1.276719	0.000559	0.000038	0.001769
12	2.172749	2.171332	2.173670	0.002577	0.000922	0.002338	2.173766	0.000362	0.001018	0.002435
13	3.128725	3.127410	3.129767	0.002854	0.001042	0.002358	3.130110	0.000273	0.001385	0.002700
14	4.098312	4.096954	4.100946	0.003085	0.002635	0.003992	4.099663	0.000260	0.001351	0.002709
15	5.070149	5.068705	5.071506	0.003306	0.001357	0.002801	5.071334	0.000272	0.001185	0.002628
16	6.042275	6.040738	6.043981	0.003531	0.001706	0.003244	6.043648	0.000290	0.001373	0.002911
17	7.014433	7.012799	7.015018	0.003750	0.000585	0.002219	7.015592	0.000308	0.001159	0.002793
18	7.986593	7.984863	7.987254	0.003976	0.000661	0.002391	7.988026	0.000327	0.001433	0.003163
19	8.958753	8.956927	8.958928	0.004188	0.000174	0.002000	8.959607	0.000344	0.000853	0.002680
20	9.930914	9.928992	9.929586	0.004410	0.001328	0.000594	9.931778	0.000361	0.000864	0.002786
21	10.903074	10.901056	10.905496	0.004633	0.002422	0.004440	10.903969	0.000381	0.000895	0.002913
22	11.875235	11.873120	11.876030	0.004855	0.000795	0.002910	11.876056	0.000399	0.000822	0.002936
23	12.847395	12.845184	12.844119	0.005072	0.003276	0.001066	12.847966	0.000415	0.000571	0.002782
24	13.819555	13.817249	13.822286	0.005296	0.002730	0.005037	13.820109	0.000434	0.000553	0.002860
25	14.791716	14.789313	14.790052	0.005508	0.001664	0.000738	14.791769	0.000451	0.000053	0.002456

Table 5

Valuation of Non-Dividend Paying Vanilla European Option and Non-Dividend Paying Arithmetic Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$

S_0	*Analytical BS European Call Option*	*Antithetic MC European Call Option*	*95% Confidence Error*	*Base MC Arithmetic Asian Call Option*	*95% Confidence Error*	*Antithetic MC Arithmetic Asian Call Option*	*95% Confidence Error*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000002	0.000002	0.000002	0.000000	0.000000	0.000000	0.000000
5	0.000240	0.000249	0.000022	0.000000	0.000000	0.000000	0.000000
6	0.005444	0.005413	0.000116	0.000006	0.000004	0.000006	0.000003
7	0.044145	0.044325	0.000364	0.000807	0.000046	0.000823	0.000033
8	0.185942	0.185642	0.000774	0.019850	0.000262	0.019831	0.000183
9	0.509122	0.509517	0.001219	0.156321	0.000805	0.156340	0.000526
10	1.045058	1.043703	0.001438	0.575738	0.001562	0.575687	0.000766
11	1.766295	1.765622	0.001360	1.304406	0.002207	1.304563	0.000582
12	2.616904	2.616638	0.001204	2.208488	0.002616	2.208603	0.000392
13	3.544027	3.543887	0.001060	3.169803	0.002890	3.170129	0.000318
14	4.511061	4.510759	0.000966	4.144846	0.003120	4.143556	0.000315
15	5.497014	5.497032	0.000930	5.118852	0.003343	5.118666	0.000334
16	6.491302	6.491821	0.000933	6.094771	0.003571	6.094409	0.000357
17	7.489063	7.488728	0.000962	7.069082	0.003792	7.069668	0.000379
18	8.488209	8.489410	0.001012	8.044740	0.004021	8.045510	0.000403
19	9.487890	9.487427	0.001055	9.019711	0.004235	9.020411	0.000423
20	10.487772	10.487845	0.001109	9.993655	0.004460	9.995867	0.000445
21	11.487730	11.487537	0.001168	10.972968	0.004685	10.971445	0.000469
22	12.487714	12.487796	0.001223	11.946789	0.004909	11.946828	0.000491
23	13.487709	13.486906	0.001269	12.918169	0.005129	12.922055	0.000511
24	14.487707	14.488505	0.001332	13.899884	0.005356	13.897681	0.000536
25	15.487706	15.486823	0.001384	14.870841	0.005570	14.872585	0.000557

Table 6

Valuation of Non-Dividend Paying Arithmetic Asian Option Using Control Variate and Combined Antithetic Control Variate Monte Carlo Methods for varying initial asset price S_0 . Variables quoted to 6 decimal places. Where $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$

S₀	*Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric*	*95% Confidence Error*	*Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric*	*95% Confidence Error*	*Antithetic Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric*	*95% Confidence Error*	*Antithetic Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric*	*95% Confidence Error*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000005	0.000001	0.000005	0.000001	0.000005	0.000001	0.000005	0.000001
7	0.000818	0.000010	0.000795	0.000010	0.000816	0.000007	0.000792	0.000007
8	0.019902	0.000023	0.019571	0.000023	0.019900	0.000016	0.019568	0.000016
9	0.156670	0.000032	0.155421	0.000032	0.156681	0.000022	0.155436	0.000022
10	0.576538	0.000040	0.574615	0.000040	0.576532	0.000029	0.574614	0.000029
11	1.304509	0.000050	1.302742	0.000050	1.304523	0.000038	1.302725	0.000038
12	2.207552	0.000061	2.206114	0.000061	2.207508	0.000048	2.205984	0.000048
13	3.168748	0.000071	3.167417	0.000071	3.168543	0.000055	3.167036	0.000055
14	4.142182	0.000079	4.140810	0.000079	4.141944	0.000059	4.140326	0.000059
15	5.117480	0.000085	5.116021	0.000085	5.117239	0.000063	5.115501	0.000063
16	6.093046	0.000092	6.091491	0.000092	6.092752	0.000068	6.090896	0.000068
17	7.068491	0.000097	7.066840	0.000097	7.068269	0.000072	7.066297	0.000072
18	8.044072	0.000104	8.042323	0.000104	8.043777	0.000077	8.041685	0.000077
19	9.019535	0.000109	9.017689	0.000109	9.019379	0.000081	9.017173	0.000081
20	9.994997	0.000115	9.993054	0.000115	9.994822	0.000086	9.992499	0.000086
21	10.970520	0.000121	10.968479	0.000121	10.970362	0.000090	10.967920	0.000090
22	11.945985	0.000127	11.943848	0.000127	11.945834	0.000095	11.943276	0.000095
23	12.921480	0.000133	12.919246	0.000133	12.921365	0.000099	12.918692	0.000099
24	13.897123	0.000140	13.894791	0.000140	13.897010	0.000104	13.894218	0.000104
25	14.872524	0.000145	14.870095	0.000145	14.872521	0.000108	14.869610	0.000108

In all cases, for the European option and both forms of Asian options, the price of the options are positive and increase as the initial stock price increases. This follows logically, due to the lower limit of stock prices being zero, and when the initial price is higher, simulation paths of the stock price are likely to reach higher values. As the value of the initial price increases, and therefore the option price increases, the error between the theoretical price and the range of the confidence interval also grows. This creates a larger probabilistically significant range of values that the option value can take, and thus increasing the variance and pricing error.

As expected, the European options are always estimated to be more expensive than either Asian option types. The Arithmetic Asian options are also valued to be more expensive than the geometrically averaged counter parts. Therefore, the confidence ranges are also larger for the more expensive option type, as seen in the results. See appendix Figure 3.

The valuation of the Arithmetic Asian option via the control variate method depends heavily on the choice of the appropriate geometric analytical solution. For the m=1000 iteration case, the difference between the two errors is not significant; however, for m = 1,000,000 iterations the discrepancy becomes substantial. From this point, all analysis in this section will only refer to the more accurate m=1,000,000 values. The discrete solution is comparatively more than twice as accurate a predictor of the simulated values on average than the continuous case initial stock prices. This increased accuracy is unsurprising given the discrete computational nature of the simulation, as it is measured in steps of one day intervals. When the Antithetic method is used to price the geometric options, the disparity between the two analytical cases increases.

It is reasonable then to conclude that using the discrete analytical solution values in the control variate method, when used in combination with the antithetic method, will provide the most accurate valuation of arithmetic Asian options out of the methods investigated for the Black-Scholes economy. This can be inferred visually from appendix Figure 4.

For all geometric Asian valuations using the base Monte Carlo and the antithetic case for all initial stock prices, the error between both analytical cases and the simulated values are one magnitude less than the typical market tick rate of 0.01. This gives further support to the validity of the simulation valuations.

Of note, following previous arguments, the error between both analytical solutions would be expected to increase as the difference between the initial stock and strike price increases. In the more accurate m=1,000,000 sample set this appears to occur for both analytical solutions when compared to the antithetic simulation, initiated near the strike price. However, the error plateaus and begins to decrease for the discrete case. This is visualised in appendix Figure 5.

The simulated Geometric Asian values tend to overshoot the analytical values in both cases. The control variate method corrects for this disparity, comparatively reducing the antithetic Asian option price. As the error is smaller for the discrete solution, when used to correct the simulated antithetic prices in the control method, the antithetic prices are greater than when the continuous solution is used. See Appendix Figure 6.

The continuous analytical solution prices the geometric Asian options more accurately than the discrete case when the initial stock price is below a value of 6. The absolute differences between the simulated and analytical prices are more than 8 orders of magnitude smaller than the tick rate, and the price of the option itself in this region is negligible as it is also below the market tick rate of 0.01, this does not alter the conclusion that the discrete solution is most appropriate.

Given the improvements offered by the antithetic method, it was also utilized in combination with the control variate method. This increased accuracy, using the antithetic control variate, results in the arithmetic price being larger with a lower error range. Therefore, the most accurate valuation of arithmetic Asian option pricing comes from the antithetic control variate utilising the analytical discrete geometric Asian option pricing.

5.2.1 Constant Volatility Asian Option Variance Reduction

Table 7

Variance of Non-Dividend Paying Vanilla European Options, and Non-Dividend Paying Arithmetic and Geometric Asian Options Using Monte Carlo, Antithetic Monte Carlo, and Control Variate Methods for Varying Initial Asset Price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000$. *Variables quoted to 6 decimal places.*

S_0	*Variance of Base MC European Call Option*	*Variance of Antithetic MC European Call Option*	*Variance of Base MC Geometric Asian Call Option*	*Variance of Antithetic MC Geometric Asian Call Option*	*Variance of Base MC Arithmetic Asian Call Option*	*Variance of Antithetic MC Arithmetic Asian Call Option*	*Variance of Control Variate MC Arithmetic Asian Call Option*	*Variance of Antithetic Control Variate MC Arithmetic Asian Call Option*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000000	0.000001	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7	0.000081	0.000029	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
8	0.000479	0.000198	0.000017	0.000008	0.000023	0.000010	0.000000	0.000000
9	0.001077	0.000401	0.000142	0.000068	0.000162	0.000077	0.000000	0.000000
10	0.001994	0.000553	0.000547	0.000141	0.000585	0.000150	0.000000	0.000000
11	0.003814	0.000526	0.001268	0.000094	0.001326	0.000102	0.000001	0.000000
12	0.004786	0.000284	0.001670	0.000024	0.001712	0.000028	0.000001	0.000001
13	0.006421	0.000285	0.002066	0.000015	0.002115	0.000022	0.000001	0.000001
14	0.007136	0.000272	0.002286	0.000016	0.002340	0.000024	0.000001	0.000001
15	0.009178	0.000271	0.002832	0.000020	0.002904	0.000031	0.000002	0.000001
16	0.010385	0.000210	0.003249	0.000020	0.003325	0.000030	0.000002	0.000001
17	0.011073	0.000187	0.003490	0.000019	0.003556	0.000029	0.000002	0.000001
18	0.013579	0.000271	0.004197	0.000030	0.004302	0.000045	0.000003	0.000002
19	0.014634	0.000302	0.004591	0.000033	0.004685	0.000049	0.000003	0.000002
20	0.016473	0.000335	0.005144	0.000033	0.005248	0.000050	0.000004	0.000002
21	0.016642	0.000281	0.005203	0.000035	0.005341	0.000053	0.000004	0.000002
22	0.020611	0.000415	0.006598	0.000046	0.006761	0.000071	0.000005	0.000003
23	0.020209	0.000373	0.006270	0.000038	0.006395	0.000059	0.000004	0.000003
24	0.024026	0.000502	0.007165	0.000046	0.007332	0.000074	0.000006	0.000003
25	0.026793	0.000573	0.008505	0.000066	0.008749	0.000100	0.000007	0.000003

Table 8

Variance of Non-Dividend Paying Vanilla European Options, and Non-Dividend Paying Arithmetic and Geometric Asian Options Using Monte Carlo, Antithetic Monte Carlo and Control Variate Methods for Varying Initial Asset Price S_0 . Where $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$.

S_0	*Variance Base MC European Call Option*	*Variance Antithetic MC European Call Option*	*Variance Base MC Geometric Asian Call Option*	*Variance Antithetic MC Geometric Asian Call Option*	*Variance Base MC Arithmetic Asian Call Option*	*Variance Antithetic MC Arithmetic Asian Call Option*	*Variance Control Variate MC Arithmetic Asian Call Option*	*Variance of Antithetic Control Variate MC Arithmetic Asian Call Option*
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
3	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
4	5.37E-12	1.37E-12	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
5	2.43E-10	1.27E-10	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
6	7.03E-09	3.50E-09	1.61E-12	6.55E-13	4.04E-12	0.00E+00	3.43E-13	1.68E-13
7	7.05E-08	3.45E-08	3.16E-10	1.67E-10	5.49E-10	0.00E+00	2.42E-11	1.17E-11
8	3.46E-07	1.56E-07	1.39E-08	6.78E-09	1.79E-08	0.00E+00	1.32E-10	6.45E-11
9	1.03E-06	3.87E-07	1.49E-07	6.42E-08	1.69E-07	7.21E-08	2.60E-10	1.28E-10
10	2.16E-06	5.38E-07	5.95E-07	1.44E-07	6.35E-07	1.53E-07	4.11E-10	2.24E-10
11	3.56E-06	4.82E-07	1.22E-06	8.14E-08	1.27E-06	8.83E-08	6.53E-10	3.74E-10
12	5.03E-06	3.77E-07	1.73E-06	3.41E-08	1.78E-06	4.01E-08	9.84E-10	6.09E-10
13	6.43E-06	2.92E-07	2.12E-06	1.94E-08	2.17E-06	2.63E-08	1.32E-09	7.94E-10
14	7.75E-06	2.43E-07	2.48E-06	1.75E-08	2.53E-06	2.59E-08	1.61E-09	9.03E-10
15	9.08E-06	2.25E-07	2.84E-06	1.93E-08	2.91E-06	2.90E-08	1.89E-09	1.04E-09
16	1.04E-05	2.27E-07	3.25E-06	2.19E-08	3.32E-06	3.31E-08	2.18E-09	1.19E-09
17	1.18E-05	2.41E-07	3.66E-06	2.47E-08	3.74E-06	3.74E-08	2.45E-09	1.35E-09
18	1.33E-05	2.67E-07	4.12E-06	2.79E-08	4.21E-06	4.23E-08	2.80E-09	1.53E-09
19	1.47E-05	2.90E-07	4.57E-06	3.07E-08	4.67E-06	4.66E-08	3.11E-09	1.71E-09
20	1.63E-05	3.20E-07	5.06E-06	3.40E-08	5.18E-06	5.16E-08	3.45E-09	1.91E-09
21	1.80E-05	3.55E-07	5.59E-06	3.77E-08	5.71E-06	5.73E-08	3.84E-09	2.10E-09
22	1.98E-05	3.89E-07	6.13E-06	4.14E-08	6.27E-06	6.29E-08	4.23E-09	2.33E-09
23	2.15E-05	4.19E-07	6.70E-06	4.48E-08	6.85E-06	6.80E-08	4.59E-09	2.54E-09
24	2.36E-05	4.62E-07	7.30E-06	4.90E-08	7.47E-06	7.47E-08	5.09E-09	2.81E-09
25	2.54E-05	4.99E-07	7.90E-06	5.30E-08	8.08E-06	8.08E-08	5.50E-09	3.05E-09

Table 9

Variance Reduction Factor of Non-Dividend Paying Vanilla European Options, and Non-Dividend Paying Arithmetic and Geometric Asian Options Using Monte Carlo, Antithetic Monte Carlo and Control Variate Methods for Varying Initial Asset Price S_0 when the number of iterations was increased by a factor of 1000. Where $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m_1\ = 1{,}000,\ m_2\ = 1{,}000{,}000$.

Average Variance Reduction Factor Base MC European Call Option	*Average Variance Reduction Factor Antithetic MC European Call Option*	*Average Variance Reduction Factor Base MC Geometric Asian Call Option*	*Average Variance Reduction Factor Antithetic MC Geometric Asian Call Option*	*Average Variance Reduction Factor Base MC Arithmetic Asian Call Option*	*Average Variance Reduction Factor Antithetic MC Arithmetic Asian Call Option*	*Average Variance Reduction Factor Control Variate MC Arithmetic Asian Call Option*	*Average Variance Reduction Factor Antithetic Control Variate MC Arithmetic Asian Call Option*
1021.186414	1005.678346	961.694940	933.871010	975.610675	946.477878	997.405765	981.398919

Table 12 shows the average variance reduction factor for all options and pricing methodologies is relatively similar when the number of iterative paths is increased, and within the reduction predicted by Equation 30.

While similar, the average variance reduction factor from increasing the number of iterations is lower for both the geometric and arithmetic Asian options, as well as for all antithetic cases. This is likely due to the reduction of variance from applying the averaging process in the antithetic method lowering the subsequent impact of increasing the number of iterations, as some of the associated error cancelation has likely already occurred.

As the initial stock price varies further from the strike price, the variance reduction increases in all, except for the control variate cases, before plateauing. I theorise that this due to the selection of the maximum value in the option payoff functions varying the most when near the strike price region. Logically, when the initial price is below the strike price, which is known as the option being “out-of-the-money”, the zero limit dominates, and when above the strike, known as “in-the-money”, the option value dominates creating an inflection point. This inflection occurs around the region where the initial and strike price are equal, known as the option being “at-the-money”. This is further supported by the spike in variance of the all the pure antithetic methods in this region. Past this region, the variance begins to increase relatively linearly as the initial stock price increases, as expected from the larger probabilistic upper limit for the stock price. These trends are visualised in Appendix B Figure 8, for the antithetic arithmetic Asian Option. This seemingly linear increase explains why the variance reduction begins to plateau, as well as the analytical pricing error trend discussed in the previous section.

While the antithetic method reduces the variance when compared to the base Monte Carlo method, the averaging process compounds the relative degree of the variance spike at the strike price. The variance reduction factor is larger for both the geometric and arithmetic Asian option valuations when compared to that of the European options. When the options are "in-the-money" and out of the strike price region, applying the antithetic method for European, geometric and arithmetic Asian options gave an average reduction factor of approximately 50, 150, and 100 respectively. This is comparable to variance reduction from increasing the number of iterations by the same amount as the quoted antithetic reduction factors. Applying the antithetic method, on average, provides a threefold and twofold effective variance reduction over the European case for the geometric and arithmetic Asian options respectively. This is likely due to the path dependence of these options, providing more instances for variation cancelation of the antithetic method to apply.

The most variance reduction comes from the control variate method. When compared to the variance of the arithmetic option from the base and antithetic Monte-Carlo methods, the variance reduction factor for the "in-the-money" cases are approximately 1500 and 15 respectively. Using the control variate method offers approximately a 1400%. increase in variance reduction over the antithetic method when the option is "in-the-money". In all control variate cases the factor begins to decay after the strike price region. The reduction in variance is typically dominated by the correlation between the geometric and arithmetic prices as discussed in section 4.2.2.2. The correlation decreases slightly as the variance increases for the base and antithetic methods with the initial stock price.

When the antithetic prices are used in the control variate method, the variance compared to the standard control variate is approximately reduced by a further factor of 2. This is substantially less than the antithetic reduction over the base Monte Carlo approximation, even as the sample antithetic variance used in the calculation improves by a factor of 100. This is again due to the covariance between the antithetic option prices decreasing compared to that of the base Monte Carlo method.

5.3 Varying Constant Volatility Asian Option Results

Table 10

Valuation of Non-Dividend Geometric Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for Varying Volatility σ *. Where* $S_0 = 25, K = 20, r = 0.05,\ n = 253\ (days), m = 1{,}000{,}000$

Vol	*Analytical Discrete Geometric Asian Call Option*	*Analytical Continuous Geometric Asian Call Option*	*MC Geometric Asian Call Option*	*95% Confidence Error*	*Comparative Pricing Error Discrete Analytical*	*Comparative Pricing Error Continuous Analytical*	*Antithetic MC Geometric Asian Call Option*	*95% Confidence Error*	*Comparative Pricing Error Discrete Analytical*	*Comparative Pricing Error Continuous Analytical*
0.2	5.294866	5.292188	5.294225	0.005454	0.000641	0.002038	5.294822	0.000605	0.000045	0.854113
0.3	5.319004	5.315106	5.320592	0.007843	0.001588	0.005486	5.319030	0.001730	0.000026	1.300159
0.4	5.437743	5.432202	5.433506	0.010030	0.004238	0.001303	5.440306	0.003185	0.002563	1.812866
0.5	5.614233	5.606997	5.614914	0.012131	0.000681	0.007917	5.615400	0.004770	0.001167	2.343683
0.6	5.817650	5.808772	5.818340	0.014258	0.000689	0.009568	5.827799	0.006448	0.010149	2.888340
0.7	6.028844	6.018404	6.034617	0.016380	0.005773	0.016213	6.030243	0.008162	0.001398	3.422132
0.8	6.236165	6.224249	6.229836	0.018569	0.006330	0.005586	6.244356	0.009931	0.008191	0.020107

Table 11

Valuation of Non-Dividend Paying Vanilla European Option and Non-Dividend Paying Arithmetic Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for Varying Volatility σ*. Where* $S_0 = 25,\ K = 20, r = 0.05, n = 253\ (days), m = 1{,}000{,}000$

Vol	*Analytical BS European Call Option*	*Antithetic MC European Call Option*	*95% Confidence Error*	*Base MC Arithmetic Asian Call Option*	*95% Confidence Error*	*Antithetic MC Arithmetic Asian Call Option*	*95% Confidence Error*
0.2	6.147209	6.146301	0.002247	5.372263	0.005529	5.372814	0.000681
0.3	6.615521	6.615265	0.005040	5.477354	0.008059	5.475846	0.001871
0.4	7.244102	7.245068	0.008292	5.688401	0.010489	5.695518	0.003444
0.5	7.948130	7.950679	0.011915	5.988897	0.012949	5.989674	0.005218
0.6	8.687882	8.697112	0.015990	6.333116	0.015570	6.342876	0.007173
0.7	9.442948	9.440536	0.020548	6.712505	0.018331	6.707584	0.009270

Table 12

Valuation of Non-Dividend Paying Arithmetic Asian Option Using Control Variate and Combined Antithetic Control Variate Monte Carlo Methods for Varying Volatility σ . Variables Quoted to 6 Decimal Places. Where $S_0 = 25, K = 20, r = 0.05, n = 253\ (days), m = 1{,}000{,}000$

Vol	*Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric*	*95% Confidence Error*	*Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric*	*95% Confidence Error*	*Antithetic Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric*	*95% Confidence Error*	*Antithetic Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric*	*95% Confidence Error*
0.2	5.372913	0.000138	5.370198	0.000138	5.372864	0.000109	5.369887	0.000109
0.3	5.475724	0.000289	5.471721	0.000289	5.475819	0.000221	5.471634	0.000221
0.4	5.692827	0.000503	5.687039	0.000503	5.692764	0.000374	5.686808	0.000374
0.5	5.988171	0.000795	5.980462	0.000795	5.988405	0.000585	5.980541	0.000585
0.6	6.332366	0.001191	6.322699	0.001191	6.331669	0.000874	6.321866	0.000874
0.7	6.706072	0.001682	6.694437	0.001682	6.706010	0.001236	6.694259	0.001236
0.8	7.094834	0.002308	7.081220	0.002308	7.095584	0.001694	7.081862	0.001694

Table 13

Variance of Non-Dividend Paying Vanilla European Options, and Non-Dividend Paying Arithmetic and Geometric Asian Options Using Monte Carlo, Antithetic Monte Carlo and Control Variate Methods for Varying Volatility σ. Where $S_0 = 25, K = 20, r = 0.05,, n = 253\ (days), m = 1{,}000{,}000$.

Vol	*Variance Base MC European Call Option*	*Variance Antithetic MC European Call Option*	*Variance Base MC Geometric Asian Call Option*	*Variance Antithetic MC Geometric Asian Call Option*	*Variance Base MC Arithmetic Asian Call Option*	*Variance Antithetic MC Arithmetic Asian Call Option*	*Variance Control Variate MC Arithmetic Asian Call Option*	*Variance of Antithetic Control Variate MC Arithmetic Asian Call Option*
0.2	2.29E-05	1.31E-06	7.74E-06	9.52E-08	7.96E-06	1.21E-07	4.94E-09	3.12E-09
0.3	4.79E-05	6.61E-06	1.60E-05	7.80E-07	1.69E-05	9.11E-07	2.17E-08	1.27E-08
0.4	8.40E-05	1.79E-05	2.62E-05	2.64E-06	2.86E-05	3.09E-06	6.58E-08	3.64E-08
0.5	1.36E-04	3.70E-05	3.83E-05	5.92E-06	4.36E-05	7.09E-06	1.65E-07	8.91E-08
0.6	2.09E-04	6.66E-05	5.29E-05	1.08E-05	6.31E-05	1.34E-05	3.69E-07	1.99E-07
0.7	3.09E-04	1.10E-04	6.98E-05	1.73E-05	8.75E-05	2.24E-05	7.37E-07	3.97E-07
0.8	4.45E-04	1.72E-04	8.98E-05	2.57E-05	1.19E-04	3.48E-05	1.39E-06	7.47E-07

Aligning with the principles set out, the results in Tables 13 through 15 show that when the volatility of the underlying asset is increased, every option price also increases. With this higher volatility it is natural that the confidence errors and the variance also increase. While not strictly consistent, the error between the analytical geometric Asian options and the simulation values tends to increase as the volatility increases. Further analysis is required to verify this trend due to the nature of simulating high volatility scenarios. The degree of variance reduction also decreases for all reduction methods implemented and does so exponentially.

5.4 Stochastic volatility

Here the option valuations from the Monte Carlo method are simulated for the mean reverting stochastic volatility case using the Basic Heston model and compared to the analytical valuations in order to assess the constant volatility assumption made in a Black-Scholes economy. The asset price and the underlying volatility are assumed to be uncorrelated. The prices are sampled daily assuming that there are 253 trading days in a year, and all options have a maturity date of one trading year. The rate of mean reversion $\lambda = 10$, is the same as that used by Hull & White. The volatility of the volatility $\xi_V = 0.3$, as in their 1987 paper Hull & White found that the best estimates for ξ_V to be in the region of 10% to 40% of the implied volatility. In order to compare to the non-stochastic case previously analysed, the initial implied volatility is the same as before $\sigma_0 = 0.2$ and the run average volatility is also $\hat{\sigma} = 0.2$. There should be no expected drift of the volatility. For real markets these stochastic volatility parameters are measured from the historical stock prices and their implied volatilises. All other parameters are identical to the non-stochastic case.

Table 14

Valuation of Non-Dividend Geometric Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 and Modelling Stochastic Volatility Via the Basic Heston Model. Where $K = 20, r = 0.05$, $\xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10$, $n = 253\ (days), m = 1{,}000{,}000$

S_0	*Analytical Discrete Geometric Asian Call Option*	*Analytical Continuous Geometric Asian Call Option*	*MC Geometric Asian Call Option*	*95% Confidence Error*	*Comparative Pricing Error Discrete Analytical*	*Comparative Pricing Error Continuous Analytical*	*Antithetic MC Geometric Asian Call Option*	*95% Confidence Error*	*Comparative Pricing Error Discrete Analytical*	*Comparative Pricing Error Continuous Analytical*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000002	0.000002	0.000014	0.000006	0.000002	0.000002	0.000014	0.000004	0.000011	0.000011
7	0.000518	0.000500	0.000890	0.000054	0.000152	0.000134	0.000881	0.000038	0.000363	0.000381
8	0.016541	0.016249	0.015615	0.000241	0.001952	0.002244	0.015567	0.000168	0.000974	0.000681
9	0.145321	0.144145	0.124121	0.000722	0.002235	0.001059	0.124444	0.000482	0.020877	0.019701
10	0.556543	0.554682	0.486112	0.001457	0.008169	0.010030	0.486328	0.000783	0.070215	0.068354
11	1.276680	1.274950	1.132043	0.002169	0.016391	0.018122	1.131488	0.000951	0.145193	0.143462
12	2.172749	2.171332	1.930030	0.002753	0.040112	0.038695	1.930810	0.001378	0.241939	0.240522
13	3.128725	3.127410	2.778815	0.003291	0.012537	0.011222	2.779341	0.001933	0.349383	0.348068
14	4.098312	4.096954	3.640665	0.003834	0.033062	0.031705	3.639790	0.002518	0.458522	0.457165
15	5.070149	5.068705	4.505490	0.004391	0.057900	0.059343	4.505606	0.003103	0.564543	0.563100
16	6.042275	6.040738	5.369478	0.004967	0.027016	0.028554	5.371153	0.003692	0.671123	0.669585
17	7.014433	7.012799	6.237518	0.005547	0.007531	0.009165	6.237166	0.004279	0.777267	0.775633
18	7.986593	7.984863	7.100614	0.006146	0.075101	0.076831	7.100631	0.004875	0.885962	0.884232
19	8.958753	8.956927	7.966134	0.006759	0.073637	0.075463	7.961892	0.005476	0.996862	0.995035
20	9.930914	9.928992	8.833243	0.007355	0.107271	0.105349	8.830430	0.006059	1.100484	1.098562
21	10.903074	10.901056	9.692522	0.007968	0.020502	0.018483	9.691675	0.006658	1.211399	1.209381
22	11.875235	11.873120	10.561319	0.008577	0.044316	0.042201	10.559523	0.007242	1.315711	1.313597
23	12.847395	12.845184	11.425018	0.009191	0.043653	0.041443	11.425247	0.007833	1.422148	1.419938
24	13.819555	13.817249	12.275687	0.009831	0.086485	0.088792	12.277726	0.008457	1.541829	1.539522
25	14.791716	14.789313	13.146685	0.010433	0.003254	0.000852	13.150514	0.009027	1.641202	1.638799

Table 15

Valuation of Non-Dividend Paying Vanilla European Option and Non-Dividend Paying Arithmetic Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 and Modelling Stochastic Volatility Via the Basic Heston Model. Where $K = 20, r = 0.05,\ \xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10\ ,\ n = 253\ (days), m = 1{,}000{,}000$

S_0	*Analytical BS European Call Option*	*Antithetic MC European Call Option*	*95% Confidence Error*	*Base MC Arithmetic Asian Call Option*	*95% Confidence Error*	*Antithetic MC Arithmetic Asian Call Option*	*95% Confidence Error*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.001411	0.000579	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.004291	0.000930	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.046108	0.073160	0.000292	0.000571	0.000146	0.000286
4	0.000002	0.018592	0.003588	0.000001	0.000002	0.000010	0.000013
5	0.000240	0.030299	0.003344	0.000042	0.000057	0.000027	0.000029
6	0.005444	0.059598	0.009766	0.000077	0.000040	0.000105	0.000043
7	0.044145	0.112987	0.005414	0.001673	0.000100	0.001657	0.000072
8	0.185942	0.243410	0.005208	0.019876	0.000291	0.019806	0.000204
9	0.509122	0.538052	0.030821	0.136389	0.000788	0.136862	0.000545
10	1.045058	1.010423	0.008823	0.507645	0.001526	0.507835	0.000822
11	1.766295	1.650563	0.004711	1.162105	0.002232	1.161569	0.000987
12	2.616904	2.421002	0.014795	1.967877	0.002811	1.968711	0.001413
13	3.544027	3.302097	0.123878	2.822727	0.003347	2.823472	0.002026
14	4.511061	4.091127	0.032735	3.689257	0.003904	3.688256	0.002562
15	5.497014	4.946548	0.009637	4.558037	0.004453	4.558088	0.003143
16	6.491302	5.824827	0.015857	5.426036	0.005029	5.427737	0.003733
17	7.489063	6.688621	0.009372	6.297639	0.005610	6.297359	0.004321
18	8.488209	7.575206	0.014224	7.164453	0.006212	7.164515	0.004919
19	9.487890	8.448637	0.013989	8.034092	0.006832	8.029807	0.005522
20	10.487772	9.334674	0.018197	8.904945	0.007441	8.901887	0.006111
21	11.487730	10.214130	0.013852	9.768051	0.008046	9.767373	0.006717
22	12.487714	11.094588	0.014377	10.640716	0.008660	10.638844	0.007295
23	13.487709	11.986826	0.019095	11.508059	0.009277	11.508318	0.007888
24	14.487707	12.892616	0.054195	12.362261	0.009919	12.364563	0.008517
25	15.487706	13.760510	0.031489	13.237171	0.010523	13.241391	0.009091

Table 16

Valuation of Non-Dividend Paying Arithmetic Asian Option Using Control Variate and Combined Antithetic Control Variate Monte Carlo Methods for varying initial asset price S_0 and Modelling Stochastic Volatility Via the Basic Heston Model. Variables quoted to 6 decimal places. Where $K = 20, r = 0.05,\ \xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10\ ,\ n = 253\ (days), m = 1{,}000{,}000$

S_0	Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric	95% Confidence Error	Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric	95% Confidence Error	Antithetic Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric	95% Confidence Error	Antithetic Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric	95% Confidence Error
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000053	0.000038	0.000053	0.000038	0.000082	0.000042	0.000082	0.000042
7	0.001140	0.000064	0.001114	0.000064	0.001130	0.000046	0.001104	0.000046
8	0.020974	0.000057	0.020627	0.000057	0.020961	0.000041	0.020614	0.000041
9	0.159343	0.000101	0.158069	0.000101	0.159423	0.000162	0.158152	0.000162
10	0.581149	0.000125	0.579206	0.000125	0.581113	0.000087	0.579170	0.000087
11	1.310659	0.000137	1.308882	0.000137	1.311389	0.000110	1.309603	0.000110
12	2.215215	0.000175	2.213772	0.000175	2.215470	0.000146	2.214025	0.000146
13	3.177904	0.000204	3.176570	0.000204	3.177984	0.000509	3.176650	0.000509
14	4.152959	0.000391	4.151584	0.000391	4.152312	0.000263	4.150938	0.000263
15	5.129557	0.000279	5.128096	0.000279	5.128546	0.000208	5.127088	0.000208
16	6.106424	0.000248	6.104868	0.000248	6.105179	0.000206	6.103627	0.000206
17	7.082800	0.000224	7.081148	0.000224	7.081249	0.000207	7.079601	0.000207
18	8.059362	0.000240	8.057615	0.000240	8.057522	0.000223	8.055778	0.000223
19	9.036395	0.000306	9.034551	0.000306	9.034096	0.000250	9.032256	0.000250
20	10.012964	0.000502	10.011024	0.000502	10.010163	0.000322	10.008227	0.000322
21	10.989524	0.000324	10.987488	0.000324	10.986924	0.000432	10.984892	0.000432
22	11.966231	0.000341	11.964098	0.000341	11.963099	0.000291	11.960971	0.000291
23	12.942587	0.000366	12.940357	0.000366	12.939433	0.000303	12.937209	0.000303
24	13.918901	0.000358	13.916576	0.000358	13.915836	0.000385	13.913516	0.000385
25	14.895492	0.000353	14.893070	0.000353	14.892348	0.000443	14.889931	0.000443

Table 17

Variance of Non-Dividend Paying Vanilla European Options, and Non-Dividend Paying Arithmetic and Geometric Asian Options Using Monte Carlo, Antithetic Monte Carlo and Control Variate Methods for Varying Initial Asset Price S_0 and Modelling Stochastic Volatility Via the Basic Heston Model. Where $K = 20, r = 0.05,\ \xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10\ ,\ n = 253\ (days), m = 1{,}000{,}000$.

S_0	*Variance Base MC European Call Option*	*Variance Antithetic MC European Call Option*	*Variance Base MC Geometric Asian Call Option*	*Variance Antithetic MC Geometric Asian Call Option*	*Variance Base MC Arithmetic Asian Call Option*	*Variance Antithetic MC Arithmetic Asian Call Option*	*Variance Control Variate MC Arithmetic Asian Call Option*	*Variance of Antithetic Control Variate MC Arithmetic Asian Call Option*
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
1	2.19E-08	8.72E-08	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
2	1.38E-07	2.25E-07	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
3	5.57E-03	1.39E-03	0.00E+00	0.00E+00	8.50E-08	2.12E-08	0.00E+00	0.00E+00
4	4.46E-06	3.35E-06	0.00E+00	0.00E+00	6.06E-13	4.57E-11	0.00E+00	0.00E+00
5	4.69E-06	2.91E-06	0.00E+00	0.00E+00	8.55E-10	2.18E-10	0.00E+00	0.00E+00
6	1.52E-06	2.48E-05	9.26E-12	5.26E-12	4.09E-10	4.81E-10	3.73E-10	4.61E-10
7	7.51E-06	7.63E-06	7.60E-10	3.76E-10	2.61E-09	1.34E-09	1.05E-09	5.54E-10
8	6.38E-06	7.06E-06	1.51E-08	7.38E-09	2.21E-08	1.08E-08	8.42E-10	4.45E-10
9	2.32E-05	2.47E-04	1.36E-07	6.04E-08	1.62E-07	7.74E-08	2.64E-09	6.85E-09
10	6.17E-05	2.03E-05	5.53E-07	1.60E-07	6.06E-07	1.76E-07	4.04E-09	1.98E-09
11	1.37E-05	5.78E-06	1.23E-06	2.35E-07	1.30E-06	2.54E-07	4.86E-09	3.13E-09
12	3.30E-05	5.70E-05	1.97E-06	4.94E-07	2.06E-06	5.20E-07	8.01E-09	5.52E-09
13	1.79E-04	3.99E-03	2.82E-06	9.72E-07	2.92E-06	1.07E-06	1.09E-08	6.76E-08
14	1.10E-03	2.79E-04	3.83E-06	1.65E-06	3.97E-06	1.71E-06	3.99E-08	1.80E-08
15	6.61E-05	2.42E-05	5.02E-06	2.51E-06	5.16E-06	2.57E-06	2.02E-08	1.13E-08
16	2.33E-04	6.55E-05	6.42E-06	3.55E-06	6.58E-06	3.63E-06	1.61E-08	1.10E-08
17	4.97E-05	2.29E-05	8.01E-06	4.77E-06	8.19E-06	4.86E-06	1.31E-08	1.11E-08
18	7.83E-05	5.27E-05	9.83E-06	6.19E-06	1.00E-05	6.30E-06	1.50E-08	1.29E-08
19	1.44E-04	5.09E-05	1.19E-05	7.80E-06	1.22E-05	7.94E-06	2.44E-08	1.63E-08
20	2.40E-04	8.62E-05	1.41E-05	9.56E-06	1.44E-05	9.72E-06	6.55E-08	2.70E-08
21	1.27E-04	4.99E-05	1.65E-05	1.15E-05	1.69E-05	1.17E-05	2.73E-08	4.85E-08
22	1.33E-04	5.38E-05	1.92E-05	1.37E-05	1.95E-05	1.39E-05	3.03E-08	2.20E-08
23	1.56E-04	9.49E-05	2.20E-05	1.60E-05	2.24E-05	1.62E-05	3.48E-08	2.39E-08
24	1.19E-04	7.65E-04	2.52E-05	1.86E-05	2.56E-05	1.89E-05	3.34E-08	3.85E-08
25	2.26E-04	2.58E-04	2.83E-05	2.12E-05	2.88E-05	2.15E-05	3.24E-08	5.11E-08

The previous trend from the non-stochastic volatility cases hold; Asian options are estimated to be cheaper than the European options and increasing the stock price affects the options and their errors. The degree of reduction from the variance reduction techniques is also consistent with the non-stochastic case.

Unsurprisingly, the variance and confidence intervals of the options also increase when modelling with stochastic volatility; however only the European options confidence errors increase by values more than the market tick rate. As the absolute difference between values is small it is beneficial to use the percentage change compared to the constant volatility model for analysis. The tables depicting the percentage change are found in the Appendix B, Tables 4 through 6

Viewed as a percentage change, the confidence intervals are increased substantially. In all cases the Black-Scholes based analytical solutions and constant volatility cases drastically under-price both European and Asian options when the option is significantly “out-of-the-money”. The over pricing reduces exponentially as the initial stock price approaches just below the strike price. When the options are the “in-the-money” region, all analytical and constant volatility values comparatively over price the options. This over pricing seemingly levels to an over pricing in the region 11% for the range of initial prices investigated for all options. These trends concur with what Hull & White found in their paper. Notably, the geometrically averaged options over pricing approaches this apparent plateau faster and is likely explained by the advantages that are associated when averaging geometrically.

This over pricing, compared to the analytical case, is eliminated, due to how the control variate method operates. Thus, the antithetic option valuation is overpriced for all initial stock prices. Applying the control variate method here is flawed as the analytical solution used is not modelling strictly the same asset for the control variate.

Chapter 6 Review & Conclusion

6.1 Conclusion

The investigation shows that the Monte Carlo method can be used to evaluate both European and Asian options to within the typical market tick rate of the Black-Scholes economy analytical prices. Increasing the number of iteration paths substantially decreased the error between the simulated price approximations and the analytical values, as well as the variance and confidence intervals. The computational effort required for Monte Carlo simulations, while less demanding with technological improvements, is still a limiting factor.

In retrospect, given the large variance reduction and accuracy improvements from the Antithetic and Control variate techniques and the associated reduced computation time, which is not documented here, implementing a measure of how efficient the ???? are compared to the workload would have been beneficial to evaluate the reduction methods.

In this report, the expected theoretical pricing of European options, geometric Asian options and arithmetic Asian options relative to each other are reproduced in the simulated results. When the antithetic variance reduction method was implemented, the accuracy of the valuation was significantly increased and reduced the variance on average by two orders of magnitude.

The control variate method for the pricing of arithmetic Asian options was preferable to the base Monte Carlo and antithetic method as the variance reduction provided was in the region of 1400% more effective for “in-the-money” options over the antithetic case. Using the discrete solution for the control variate gave more accurate pricing for the arithmetic Asian option, as the simulated values were closer to the discrete form of the geometric analytical solution. When the control variate was used in conjunction with the antiemetic method, the variance for the “in-the-money” arithmetic Asian options were reduced on average by more than a factor of 2700.

For assets with stochastic volatility, assuming the stock price and volatility were uncorrelated, the Black-Scholes economy models consistently under-priced the “out-off the-money” options and overpriced the “in-the-money” options. These findings coincide with those found by Hull; & White. The pricing of real stochastically volatile assets by the Black-Scholes analytical based solutions therefore should be avoided.

Using the discrete analytical solution for the geometric Asian option as a control variate in combination with the antithetic prices in the control variate reduction technique is shown to give the most accurate and precise valuation for the pricing of arithmetic Asian options out of the pricing techniques investigated.

6.2 Further Investigation

Knowing that the analytical solutions over price the asset by approximately 11% in the higher initial stock price cases, it may be beneficial to first reduce the analytical geometric control variate valuation by 11% when investigating “in-the-money” options in order to utilise the variance reduction the method offers. This is suggested cautiously and further analysis into the seeming convergence of the pricing bias and control variate method is required.

In real markets the correlation between the asset volatility and price is negative and is known as the leverage effect. When the value of a company’s stock declines, the debt of the company becomes larger relative to the equity, therefore the company is more leveraged and seen as a riskier investment. The Basic Heston model used in this investigation can be improved by implementing this leverage effect, as well as stochastic jumps as investigated by Veraart & Veraart and published in the Annals of Finance in 2012.

In their paper “Volatility is Rough” published in 2014 by Gatheral, Jaisson and Rosenbaum, showed their Rough Fractional Stochastic Volatility (RFSV) model to replicate financial market data remarkably well. Implementing and further investigating their process should be considered for analysis of the pricing of arithmetic Asian options to investigate the impact and potential improvements.

Bibliography

Black, F. and Scholes, M. (1973), The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81, 637-654*

Boyle, P. (1977), Options: A Monte Carlo Approach*, Journal of Financial Economics,4, 323-338*

Heston, S. (1993), A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies, 6, 327-343*

Hull, J. and White, A. (1987), The Pricing of Options on Assets with Stochastic Volatility, *Journal of Finance, 42, 281-300*

Hull, J. and White, A. (1993), Efficient Procedures for Valuing European and American Path-Dependant Options, *Journal of Derivatives, 1, 21-31*

Kemma, A. and Vorst, A. (1993), A Pricing Method for Options Based on Average Asset Values, *Journal of Banking and Finance, 14, 113-129*

Veraart, A. and Veraart, L. (2012), Stochastic Volatility and Stochastic Leverage, *Annals of Finance, Springer, Volume 8, 2, 205:233*

Gatheral, J. Jaisson, T. and Rosenbaum, M. (2011), Variance Reduction with Control Variate Pricing Asian Options in a Geometric Lévy Model*, IAENG International Journal of Applied Mathematics, 41:4*

Boughamoura, W. and Trabelsi, F. (2014), Volatility is Rough, *Quantitative Finance, Volume 18, 6, 933:949*

Zhong, Y. and Deng, G. (2019), Geometric Asian Options Pricing Under the Double Heston Stochastic Volatility Model with Stochastic Interest Rate*, Complexity Journal, Wiley Hindawi, Volume 2019, Article 4316272*

Buchen, P, (2012), An Introduction to Exotic Option Pricing, *Chapman and Hall*

Kwok, Y. (Original 1998, Revised 2008), Mathematical Models of Financial Derivatives, *Second Edition, Springer Finance*

Joshi, M.S. (2003), The Concepts and Practice of Mathematical Finance, *First Edition*, *Cambridge University Press*

Zhang, H. (2009), Pricing Asian Options Using Monte Carlo Methods, *Uppsala University*

Mraovic, A. and Zhang, Q. (2014), Valuation of Asian Options with Lévy Approximation, *Lund University*

Galda, G. (2008), Variance Reduction for Asian Options, *Halmstad University*

Ramstrom, A. (2017), Pricing of European and Asian Options with Monte Carlo Simulations, *Umea University*

Fouque, J.P. and Han, C.H. (2003), Pricing Asian Options with Stochastic Volatility, *Quantitative Finance, 353-362*

Lidebrandt, T. (2007), Variance Reduction Three Approaches to Control Variates, *Stockholm University*

Chung, S. and Wong, H. (2014), Analytical Pricing og Discrete Arithmetic Asian Options with Mean Reversion and Jumps*, Journal of Banking and Finance*

Vajargah, B. Salimipor, A. and Salahshour, S. (2015), Variance Analysis of Control Variate Technique and Application in Asian Option Pricing*, International Journal Industrial Mathematic, Volume 8*

Nieslsen, L. (2001), Pricing Asian Options

ESMA, (2018), ESMA Annual Statistical Report EU Derivatives Markets, *ESMA 50-165-670*

Appendix A – Supplementary Derivations

A.1 Variance of Wiener Process Integral

W_u^Q is a Wiener process and has the basic properties:

$$E[W_a] = 0 \qquad\qquad E[W_a W_b] = \min(a, b)$$

$$Var\left(\int_0^\tau W_u^Q\, du\right) = E\left(\int_0^\tau W_u^Q\, du\right)^2 - \left(E\left(\int_0^\tau W_u^Q\, du\right)\right)^2$$

$$= E\left(\int_0^\tau W_u^Q\, du\right)^2 - \left(\int_0^\tau E\left(W_u^Q\right) du\right)^2 = E\left(\int_0^\tau W_u^Q\, du\right)^2$$

$$E\left(\iint_0^\tau W_u^Q W_{u'}^Q\, du du'\right) = \iint_0^\tau E(W_u^Q W_{u'}^Q) du du' = \iint_0^\tau \min(\mathrm{u}, \mathrm{u}')\, du du'$$

$$= 2\int_0^\tau \int_u^\tau \min(\mathrm{u}, \mathrm{u}')\, du du' = 2\int_0^\tau \int_u^\tau u du'\, du = 2\int_0^\tau u(\tau - u) du$$

$$= 2\left(\left[\frac{u^2\tau}{2} - \frac{u^3}{3}\right]_0^\tau\right) = \tau^3 - \frac{\tau^3}{3} = \frac{\tau^3}{3}$$

As the integral lower limit is 0, $\tau = T$, hence: $\frac{T^{\frac{3}{2}}}{\sqrt{3}}$

$$Var\left(\int_0^\tau W_u^Q\, du\right) = \frac{T^3}{3}, \qquad Std\left(\int_0^\tau W_u^Q\, du\right) = \frac{T^{\frac{3}{2}}}{\sqrt{3}}$$

A.2 Correlation of Wiener Process and Wiener Integral

$$\boldsymbol{Corr}\left(\boldsymbol{W_t^Q}, \boldsymbol{I}\right) = \frac{\boldsymbol{Cov}\left(\boldsymbol{W_t^Q}, \boldsymbol{I}\right)}{\boldsymbol{\sigma_{W_t^Q}} \boldsymbol{\sigma_I}}$$

Where $I = \int_0^\tau W_u^Q \, du$, *and* $\boldsymbol{\sigma}_{W_t^Q} = \sqrt{T}$

Using Ito's Isometry and the principles of Wiener process, it follows:

$$\boldsymbol{Cov}\left(W_t^Q, \boldsymbol{I}\right) = \boldsymbol{E}\left(W_t^Q \int_0^\tau W_u^Q \, du\right) = \int_0^\tau \boldsymbol{E}(W_t^Q W_u^Q) \, du = \int_0^\tau min(\mathrm{t}, \mathrm{u}) \, du$$

Where $t > \mathrm{u}$ *as* $u = t' - \mathrm{t}$ *and* $t' > \mathrm{t}$

$$\boldsymbol{Cov}\left(W_t^Q, \boldsymbol{I}\right) = \int_0^\tau u \, du = \frac{\tau^2}{2} = \frac{T^2}{2}$$

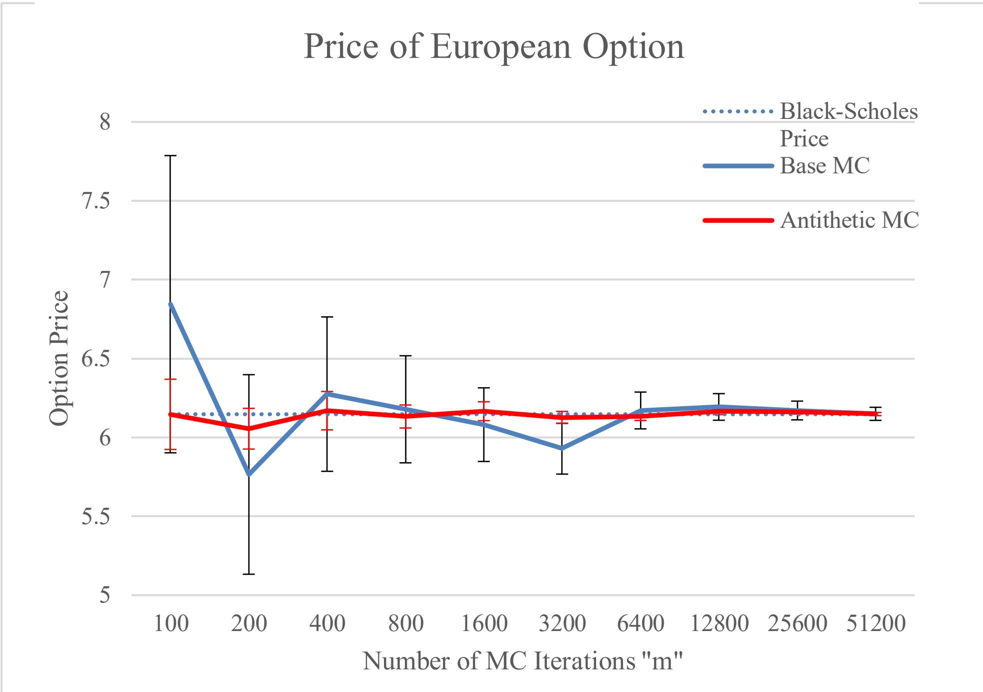
Therefore

$$\boldsymbol{Corr}\left(W_t^Q, \boldsymbol{I}\right) = \frac{1}{2}\left(\frac{\sqrt{3}T^2}{T^{\frac{3}{2}}\sqrt{T}}\right) = \frac{1}{2}\sqrt{3}$$

Appendix B - Results

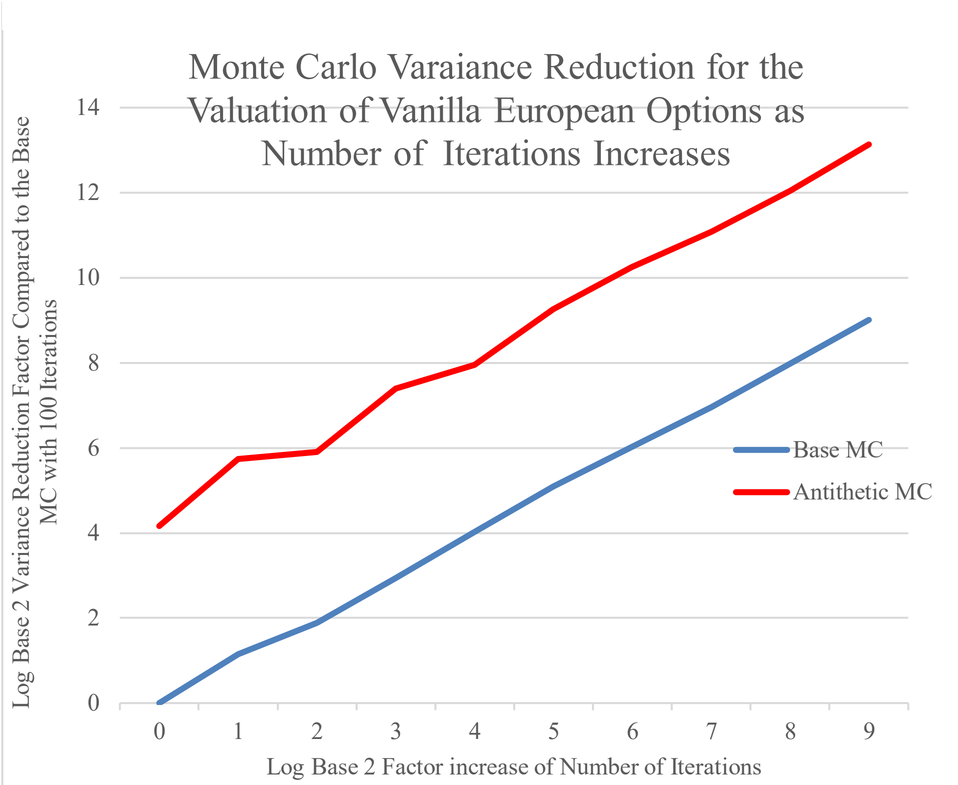
B.1 European option

Appendix Fig 1:



Valuation of Non-Dividend Paying Vanilla European Option Using Base Monte Carlo and Antithetic Monte Carlo Methods. $S_0 = 25, K = 20, r = 0.05, \sigma = 0.2, n = 253$. *The antithetic Method Converges to the Black Scholes Price Faster than the Base Method, Which Corresponds to Faster Computation Time to Fall Within the Desired Margin of Error, as the Computational Load of the Antithetic Method is Comparable to Doubling the Number of Iterations. The Black-Scholes Price Falls Within the 95% Error Bars for the Majority of Cases, Note that the Antithetic Method's Greatly Reduced Error Bars.*

Appendix Fig 2



The Relative Variance Reduction Factor Compared to the Base Monte Carlo Method for Non-Dividend European Call Options With 100 Path Iterations. $S_0 = 25, K = 20, r = 0.05, \sigma = 0.2, n = 253$. *Note The Log Scale is to Base 2. The Variance Reduction of the Monte Carlo Method is Directly Proportional to the Increase in Iterations, the Antithetic Method Provides a Substantial Improvement.*

B.2 Constant Volatility Asian Options 1,000 Iteration Results

Appendix Table 1

Valuation of both Non-Dividend Geometric Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1000.$

S_0	Analytical Discrete Geometric Asian Call Option	Analytical Continuous Geometric Asian Call Option	Base MC Geometric Asian Call Option	95% Confidence Error	Comparative Pricing Error Discrete Analytical	Comparative Pricing Error Continuous Analytical	Antithetic MC Geometric Asian Call Option	95% Confidence Error	Comparative Pricing Error Discrete Analytical	Comparative Pricing Error Continuous Analytical
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000002	0.000002	0.000000	0.000000	0.000002	0.000002	0.000000	0.000000	0.000002	0.000002
7	0.000518	0.000500	0.000393	0.000544	0.000152	0.000134	0.000196	0.000272	0.000322	0.000304
8	0.016541	0.016249	0.017764	0.008186	0.001952	0.002244	0.017971	0.005394	0.001430	0.001722
9	0.145321	0.144145	0.144736	0.023354	0.002235	0.001059	0.148657	0.016145	0.003337	0.004513
10	0.556543	0.554682	0.539478	0.045860	0.008169	0.010030	0.552297	0.023245	0.004246	0.002385
11	1.276680	1.274950	1.252819	0.069792	0.016391	0.018122	1.287553	0.018984	0.010872	0.012603
12	2.172749	2.171332	2.201127	0.080093	0.040112	0.038695	2.163957	0.009606	0.008792	0.007375
13	3.128725	3.127410	3.210179	0.089097	0.012537	0.011222	3.126244	0.007708	0.002481	0.001166
14	4.098312	4.096954	4.131719	0.093714	0.033062	0.031705	4.091550	0.007905	0.006762	0.005405
15	5.070149	5.068705	5.148321	0.104311	0.057900	0.059343	5.069777	0.008762	0.000372	0.001071
16	6.042275	6.040738	6.090179	0.111718	0.027016	0.028554	6.044240	0.008702	0.001964	0.003502
17	7.014433	7.012799	6.998816	0.115787	0.007531	0.009165	7.010554	0.008556	0.003879	0.002245
18	7.986593	7.984863	8.078755	0.126975	0.075101	0.076831	7.988863	0.010703	0.002270	0.004000
19	8.958753	8.956927	8.924499	0.132797	0.073637	0.075463	8.961635	0.011208	0.002882	0.004708
20	9.930914	9.928992	9.875561	0.140576	0.107271	0.105349	9.934311	0.011285	0.003397	0.005319
21	10.903074	10.901056	10.793701	0.141373	0.020502	0.018483	10.895641	0.011624	0.007433	0.005415
22	11.875235	11.873120	11.976313	0.159203	0.044316	0.042201	11.887143	0.013290	0.011908	0.014023
23	12.847395	12.845184	12.821870	0.155205	0.043653	0.041443	12.839612	0.012145	0.007783	0.005572
24	13.819555	13.817249	13.891257	0.165905	0.086485	0.088792	13.816138	0.013238	0.003417	0.001111
25	14.791716	14.789313	14.869474	0.180756	0.003254	0.000852	14.799811	0.015928	0.008095	0.010497

Appendix Table 2

Valuation of Non-Dividend Paying Vanilla European Option and Non-Dividend Paying Arithmetic Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1000$.

S₀	*Analytical BS European Call Option*	*Antithetic MC European Call Option*	*95% Confidence Error*	*Base MC Arithmetic Asian Call Option*	*95% Confidence Error*	*Antithetic MC Arithmetic Asian Call Option*	*95% Confidence Error*
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000002	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000240	0.000113	0.000221	0.000000	0.000000	0.000000	0.000000
6	0.005444	0.001686	0.001801	0.000000	0.000000	0.000000	0.000000
7	0.044145	0.039921	0.010544	0.000814	0.001017	0.000407	0.000508
8	0.185942	0.202687	0.027610	0.021356	0.009316	0.021773	0.006185
9	0.509122	0.527945	0.039225	0.156105	0.024937	0.159997	0.017169
10	1.045058	1.048352	0.046108	0.559026	0.047399	0.572464	0.024041
11	1.766295	1.793980	0.044960	1.281963	0.071362	1.317373	0.019762
12	2.616904	2.587381	0.033027	2.236020	0.081107	2.198617	0.010343
13	3.544027	3.541114	0.033105	3.250246	0.090140	3.165640	0.009165
14	4.511061	4.487662	0.032311	4.172916	0.094821	4.132243	0.009622
15	5.497014	5.493575	0.032283	5.195073	0.105620	5.115875	0.010898
16	6.491302	6.490287	0.028392	6.140879	0.113011	6.094236	0.010690
17	7.489063	7.467951	0.026768	7.049709	0.116880	7.061636	0.010466
18	8.488209	8.495237	0.032275	8.137694	0.128549	8.047069	0.013170
19	9.487890	9.487320	0.034039	8.983496	0.134157	9.020978	0.013657
20	10.487772	10.497582	0.035887	9.939646	0.141993	9.999100	0.013814
21	11.487730	11.455506	0.032858	10.859764	0.143239	10.961564	0.014329
22	12.487714	12.504394	0.039928	12.049653	0.161167	11.959751	0.016486
23	13.487709	13.460698	0.037860	12.894340	0.156734	12.912317	0.015031
24	14.487707	14.495281	0.043922	13.970202	0.167827	13.895045	0.016821
25	15.487706	15.495419	0.046905	14.951927	0.183332	14.880763	0.019620

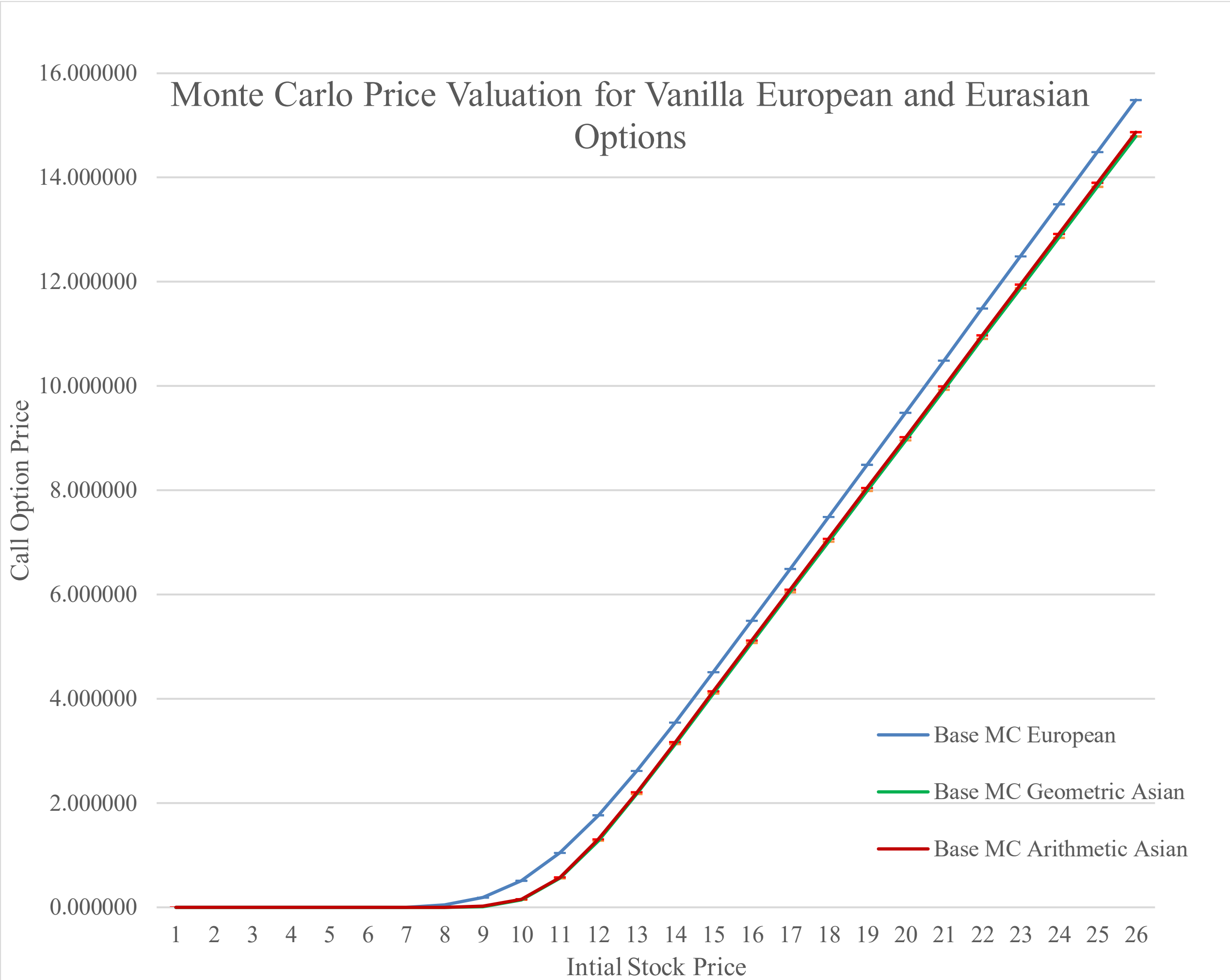
Appendix Table 3

Valuation of Non-Dividend Paying Arithmetic Asian Option Using Control Variate and Combined Antithetic Control Variate Monte Carlo Methods for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1000$..

S_0	Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric	95% Confidence Error	Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric	95% Confidence Error	Antithetic Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric	95% Confidence Error	Antithetic Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric	95% Confidence Error
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
7	0.001045	0.000201	0.001011	0.000201	0.000997	0.000101	0.000964	0.000101
8	0.019970	0.000804	0.019638	0.000804	0.020140	0.000536	0.019806	0.000536
9	0.156728	0.000952	0.155474	0.000952	0.156451	0.000681	0.155202	0.000681
10	0.576659	0.001272	0.574735	0.001272	0.576852	0.000945	0.574928	0.000945
11	1.306354	0.001689	1.304585	0.001689	1.306080	0.001310	1.304282	0.001310
12	2.207289	0.001766	2.205855	0.001766	2.207978	0.001543	2.206469	0.001543
13	3.167863	0.002182	3.166533	0.002182	3.168542	0.001638	3.167004	0.001638
14	4.139124	0.002400	4.137752	0.002400	4.140335	0.001768	4.138711	0.001768
15	5.115953	0.003057	5.114492	0.003057	5.116329	0.002121	5.114568	0.002121
16	6.092435	0.002782	6.090880	0.002782	6.091868	0.002058	6.090014	0.002058
17	7.065469	0.002775	7.063820	0.002775	7.066278	0.002163	7.064323	0.002163
18	8.044419	0.003257	8.042668	0.003257	8.044326	0.002465	8.042235	0.002465
19	9.018091	0.003308	9.016247	0.003308	9.017523	0.002429	9.015333	0.002429
20	9.995537	0.003744	9.993596	0.003744	9.995033	0.002895	9.992732	0.002895
21	10.970540	0.003855	10.968496	0.003855	10.970547	0.002824	10.968108	0.002824
22	11.947365	0.004409	11.945226	0.004409	11.945262	0.003207	11.942690	0.003207
23	12.920108	0.004029	12.917876	0.004029	12.921737	0.003143	12.919061	0.003143
24	13.897697	0.004644	13.895365	0.004644	13.899300	0.003347	13.896428	0.003347
25	14.873092	0.005160	14.870656	0.005160	14.870955	0.003536	14.868044	0.003536

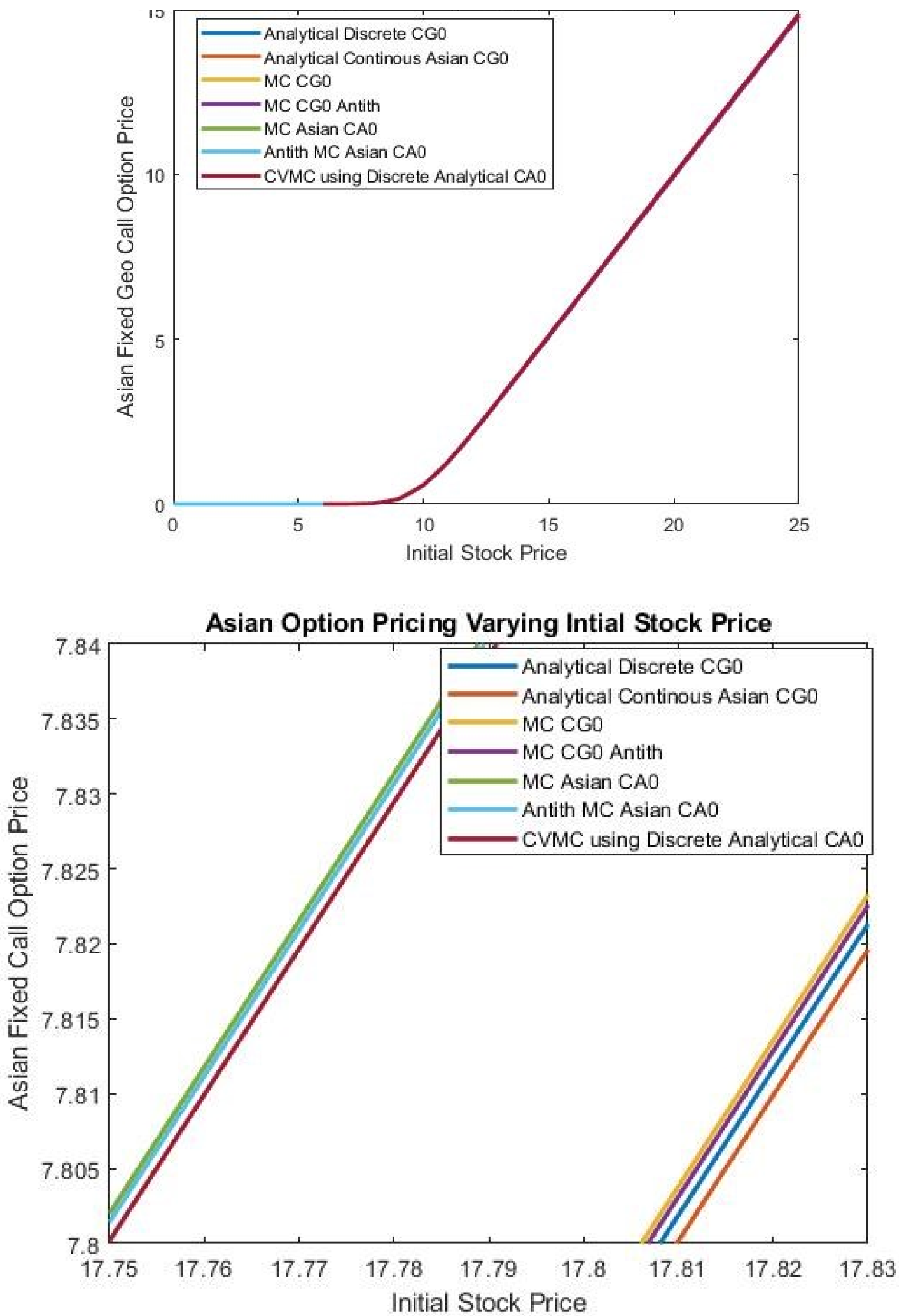
B.3 Graphical Representations of Results for Constant Volatility Asian Options for 1,000,000 Iterations

Appendix Fig 3



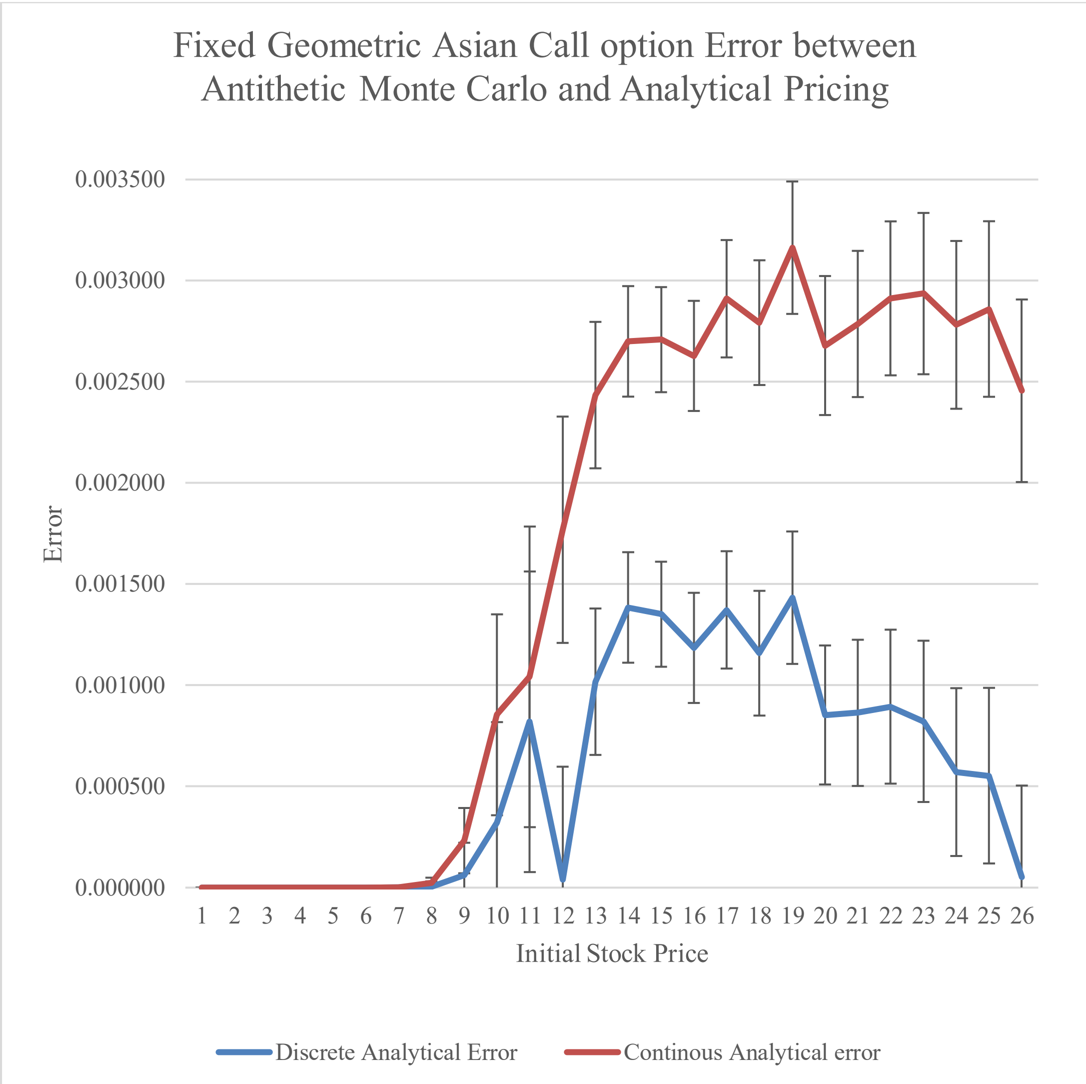
Valuation of Non-Dividend Paying Vanilla European Option, Non-Dividend Paying Geometric and Arithmetic Asian Option Using the Base Monte-Carlo Method for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$*. Error bars for 95% confidence intervals. The Geometric Asian Options are a Lower Price than the Arithmetic Options.*

Appendix Fig 4(a) Top, (b) Bottom



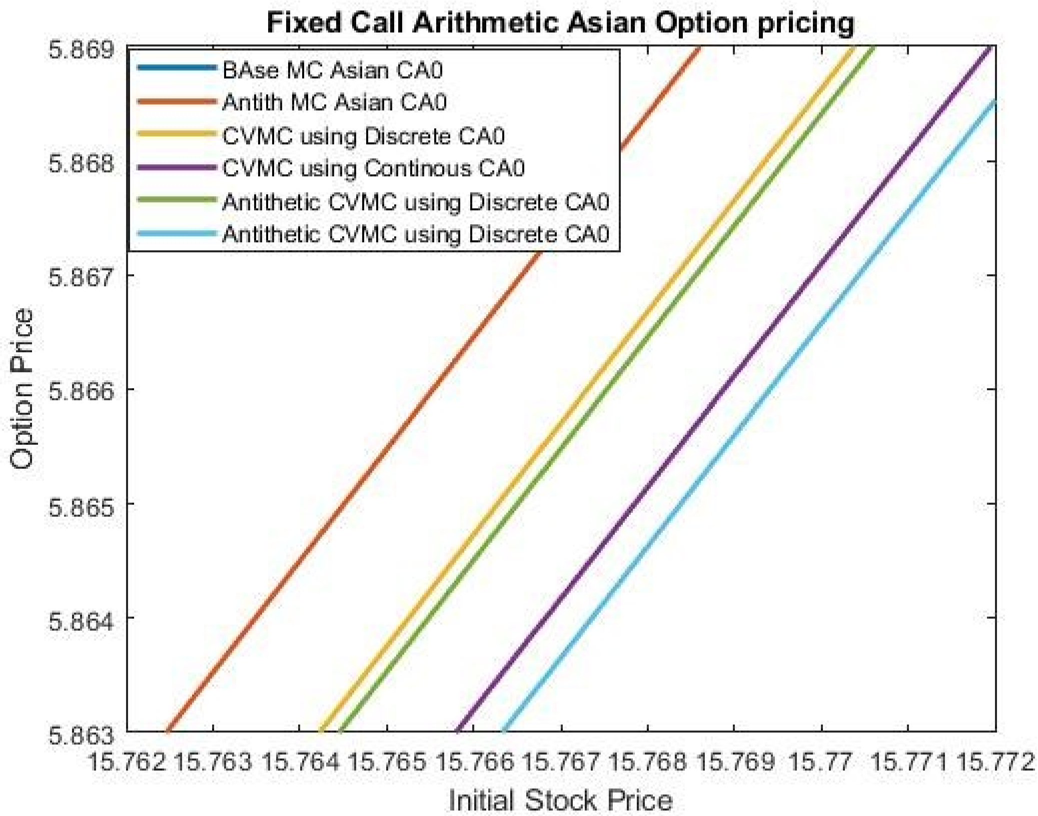
Valuation of Non-Dividend Paying Asian Options for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$*. CG0 is the Price for Geometric Asian Call Options, CA0 is the Price for Arithmetic Asian Call Options, CVMC is the Control Variate Monte-Carlo Method price. 4(b) Shows a Sample of the Valuation Graphing reflects the trends of the full population set investigated. The Discrete Geometric Analytical Valuation Better Fits the Simulations. The Antithetic Pricing Is Better than the Base Monte-Carlo for Approximating the Analytical Prices.*

Appendix Fig 5



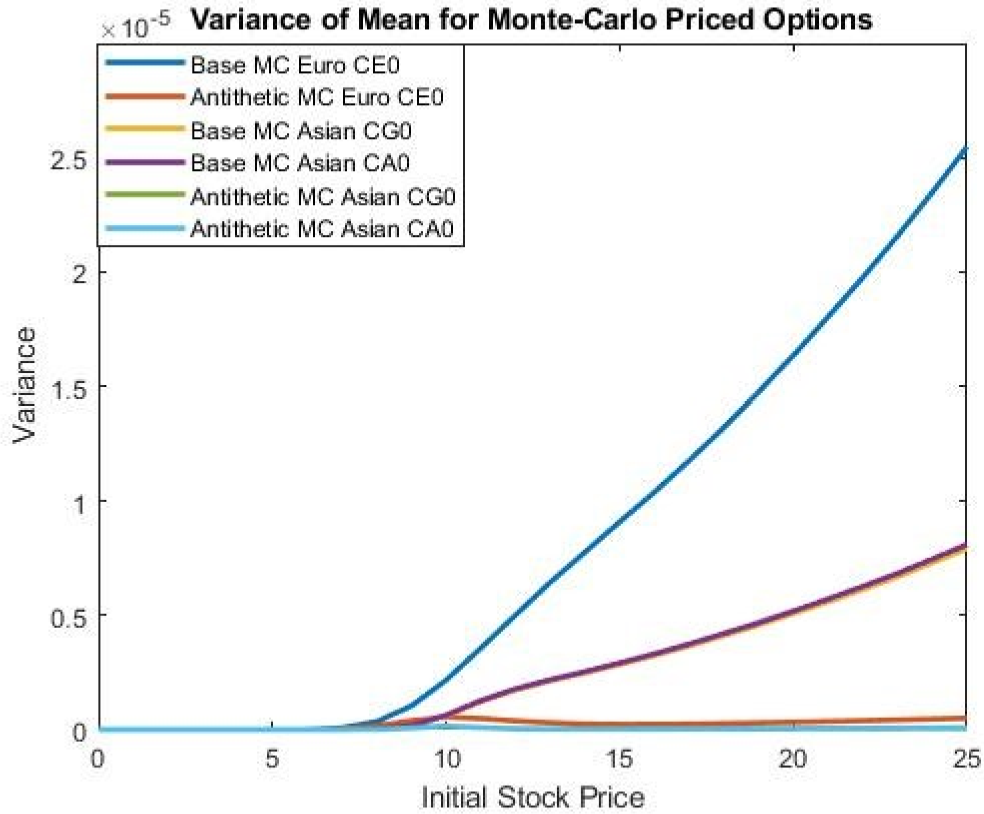
Valuation of Non-Dividend Paying Asian Options for varying initial asset price S_0 . *Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$. *The Discrete Geometric Analytical Valuation Better Fits the Simulations. The Pricing Error Between the Analytical and Numerical Results Appear to Plateau and Decrease for "in-the-money" Options.*

Appendix Fig 6



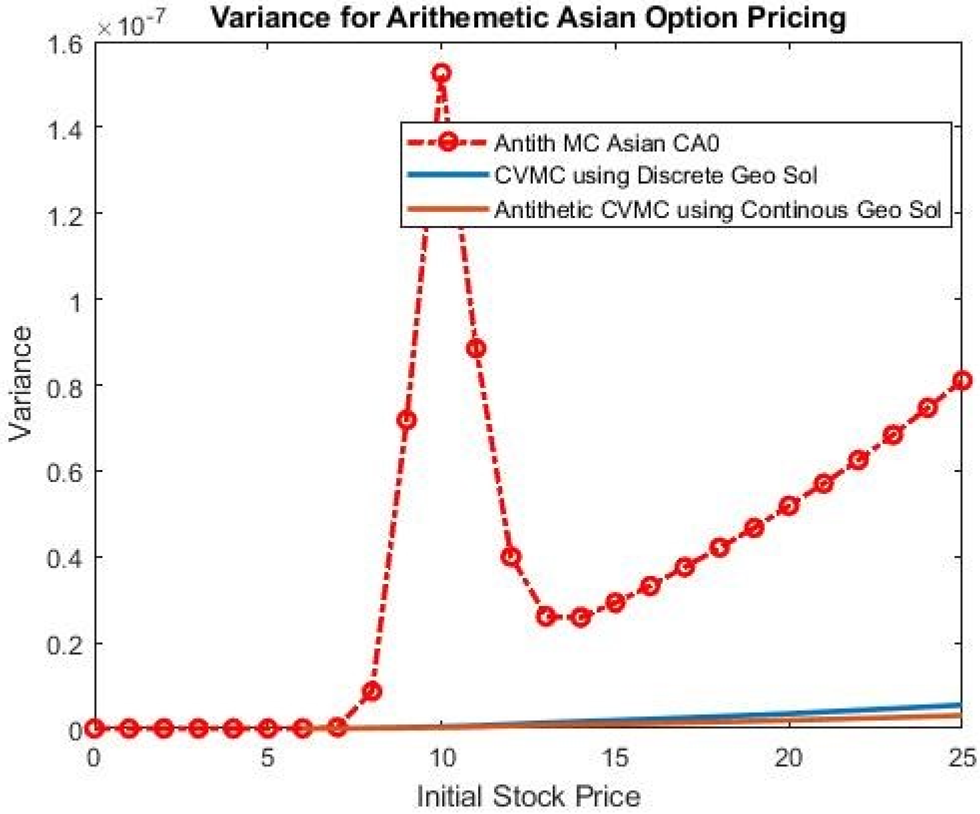
Valuation of Non-Dividend Paying Arithmetic Asian Options for varying initial asset price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$*.CG0 is the Price for Geometric Asian Call Options, CA0 is the Price for Arithmetic Asian Call Options, CVMC is the Control Variate Monte-Carlo Method price.*

Appendix Fig 7



Variance of Non-Dividend Paying Vanilla European Options, and Non-Dividend Paying Arithmetic and Geometric Asian Options Using Monte Carlo, Antithetic Monte Carlo and Control Variate Methods for Varying Initial Asset Price S_0 . Where $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$*. The Base Asian Options Have reduced Variance Compared to the European Option, Which Also Carries Over for the Antithetic Method Where the Variance is Drastically Reduced for Both Cases.*

Appendix Fig 8



Variance of Non-Dividend Paying Arithmetic Asian Options Using Antithetic Monte Carlo and Control Variate Methods for Varying Initial Asset Price S_0 *. Where* $K = 20, r = 0.05, \sigma = 0.2,\ n = 253\ (days), m = 1{,}000{,}000$*. The Control Variate Method Greatly Reduces the Variance When Compared to the Antithetic Method and is Further Improved When the Methods are Combined. There is a Clear Spike in the Variance of the Antithetic Method in the Region of the Strike Price. The Variance Increases Linearly When Substantially Into the "in-the-money" Case.*

Appendix Table 4

Percentage Change in the Relative Valuation of Non-Dividend Geometric Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 between Modelling Stochastic Volatility Via the Basic Heston Model and For the Constant Volatility Case. Where $K = 20, r = 0.05,\ \xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10\ ,\ n = 253\ (days), m = 1{,}000{,}000$

S_0	Analytical Discrete Geometric Asian Call Option	Analytical Continuous Geometric Asian Call Option	MC Geometric Asian Call Option	95% Confidence Error	Comparative Pricing Error Discrete Analytical	Comparative Pricing Error Continuous Analytical	Antithetic MC Geometric Asian Call Option	95% Confidence Error	Comparative Pricing Error Discrete Analytical	Comparative Pricing Error Continuous Analytical
0	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
4	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
5	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
6	0.00%	0.00%	416.98%	139.46%	2135.72%	1695.31%	437.78%	183.48%	3863.19%	2587.29%
7	0.00%	0.00%	74.59%	55.10%	4217.88%	4025.16%	68.16%	50.10%	6118.56%	1495.44%
8	0.00%	0.00%	-5.33%	4.45%	1895.58%	157.08%	-5.54%	4.35%	1513.71%	193.28%
9	0.00%	0.00%	-14.39%	-4.66%	6357.70%	2262.08%	-14.18%	-3.02%	6391.40%	2205.77%
10	0.00%	0.00%	-12.53%	-3.60%	8998.88%	6204.96%	-12.49%	5.42%	8464.95%	6460.90%
11	0.00%	0.00%	-11.32%	0.39%	143156.26%	8669.60%	-11.38%	70.03%	377593.90%	8009.90%
12	0.00%	0.00%	-11.21%	6.82%	26229.54%	10218.80%	-11.18%	280.73%	23667.89%	9779.56%
13	0.00%	0.00%	-11.21%	15.30%	33466.51%	14686.42%	-11.21%	607.57%	25127.15%	12791.20%
14	0.00%	0.00%	-11.22%	24.27%	17271.08%	11330.72%	-11.22%	869.84%	33829.43%	16777.95%
15	0.00%	0.00%	-11.16%	32.83%	41500.21%	20010.49%	-11.16%	1040.14%	47537.88%	21324.24%
16	0.00%	0.00%	-11.16%	40.66%	39342.11%	20594.16%	-11.13%	1173.01%	48782.80%	22903.05%
17	0.00%	0.00%	-11.08%	47.91%	132775.21%	34845.11%	-11.10%	1288.20%	66974.75%	27673.78%
18	0.00%	0.00%	-11.10%	54.56%	133867.18%	36877.65%	-11.11%	1389.58%	61718.43%	27854.30%
19	0.00%	0.00%	-11.08%	61.40%	569243.86%	49429.31%	-11.14%	1493.86%	116698.94%	37034.30%
20	0.00%	0.00%	-11.04%	66.77%	82555.32%	184315.42%	-11.09%	1577.00%	127274.58%	39329.27%
21	0.00%	0.00%	-11.12%	71.98%	49883.72%	27118.10%	-11.12%	1649.54%	135321.70%	41419.09%
22	0.00%	0.00%	-11.07%	76.69%	165151.13%	44986.76%	-11.09%	1716.81%	159975.93%	44636.00%
23	0.00%	0.00%	-11.05%	81.21%	43314.83%	133156.88%	-11.07%	1788.46%	248882.97%	50945.80%
24	0.00%	0.00%	-11.19%	85.62%	56443.12%	30504.48%	-11.16%	1848.87%	278467.81%	53727.50%
25	0.00%	0.00%	-11.11%	89.42%	98745.73%	222330.97%	-11.10%	1900.62%	3078858.90%	66625.47%

Appendix Table 5

Percentage Change in the Relative Valuation of Non-Dividend Paying Vanilla European Option and Non-Dividend Paying Arithmetic Asian Option Using Base Monte Carlo and Antithetic Monte Carlo Methods for varying initial asset price S_0 for Modelling Stochastic Volatility Via the Basic Heston Model and for the Constant Volatility Case. Where $K = 20, r = 0.05$, $\xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10$, $n = 253\ (days), m = 1{,}000{,}000$

S_0	*Analytical BS European Call Option*	*Antithetic MC European Call Option*	*95% Confidence Error*	*Base MC Arithmetic Asian Call Option*	*95% Confidence Error*	*Antithetic MC Arithmetic Asian Call Option*	*95% Confidence Error*
0	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
4	0.00%	785026.72%	156117.12%	0.00%	0.00%	0.00%	0.00%
5	0.00%	12090.40%	15027.50%	0.00%	0.00%	0.00%	0.00%
6	0.00%	1001.05%	8325.55%	1194.51%	906.26%	1718.88%	1561.70%
7	0.00%	154.91%	1386.80%	107.33%	117.91%	101.25%	115.80%
8	0.00%	31.12%	572.87%	0.13%	11.13%	-0.13%	11.23%
9	0.00%	5.60%	2427.51%	-12.75%	-2.02%	-12.46%	3.60%
10	0.00%	-3.19%	513.60%	-11.83%	-2.31%	-11.79%	7.28%
11	0.00%	-6.52%	246.29%	-10.91%	1.15%	-10.96%	69.58%
12	0.00%	-7.48%	1128.72%	-10.89%	7.44%	-10.86%	260.11%
13	0.00%	-6.82%	11590.20%	-10.95%	15.83%	-10.94%	537.69%
14	0.00%	-9.30%	3289.48%	-10.99%	25.13%	-10.99%	712.95%
15	0.00%	-10.01%	936.39%	-10.96%	33.22%	-10.95%	841.54%
16	0.00%	-10.27%	1599.21%	-10.97%	40.83%	-10.94%	946.93%
17	0.00%	-10.68%	874.29%	-10.91%	47.94%	-10.92%	1040.10%
18	0.00%	-10.77%	1305.05%	-10.94%	54.49%	-10.95%	1120.25%
19	0.00%	-10.95%	1225.52%	-10.93%	61.32%	-10.98%	1205.41%
20	0.00%	-11.00%	1541.13%	-10.89%	66.86%	-10.94%	1272.84%
21	0.00%	-11.09%	1086.37%	-10.98%	71.73%	-10.97%	1331.58%
22	0.00%	-11.16%	1075.56%	-10.93%	76.40%	-10.95%	1384.68%
23	0.00%	-11.12%	1404.84%	-10.92%	80.87%	-10.94%	1443.19%
24	0.00%	-11.01%	3968.79%	-11.06%	85.17%	-11.03%	1490.42%
25	0.00%	-11.15%	2175.23%	-10.99%	88.92%	-10.97%	1531.79%

Appendix Table 6

Percentage Change in the Relative Valuation of Non-Dividend Paying Arithmetic Asian Option Using Control Variate and Combined Antithetic Control Variate Monte Carlo Methods for varying initial asset price S_0 for Modelling Stochastic Volatility via the Basic Heston Model and for the Constant Volatility Case. Variables quoted to 6 decimal places. Where $K = 20, r = 0.05,\ \xi_V = 0.3, \sigma_0 = 0.2, \hat{\sigma} = 0.2, \lambda = 10\ ,\ n = 253\ (days), m = 1{,}000{,}000$

S_0	Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric	95% Confidence Error	Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric	95% Confidence Error	Antithetic Control Variate MC Arithmetic Asian Call Option Using Discrete Geometric	95% Confidence Error	Antithetic Control Variate MC Arithmetic Asian Call Option Using Continuous Geometric	95% Confidence Error
0	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
4	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
5	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
6	932.78%	3195.87%	971.93%	3195.87%	1449.11%	5135.21%	1510.06%	5135.21%
7	39.43%	558.63%	40.26%	558.63%	38.59%	586.66%	39.36%	586.66%
8	5.38%	152.40%	5.39%	152.40%	5.33%	162.70%	5.34%	162.70%
9	1.71%	218.22%	1.70%	218.22%	1.75%	632.63%	1.75%	632.63%
10	0.80%	213.47%	0.80%	213.47%	0.79%	196.93%	0.79%	196.93%
11	0.47%	172.74%	0.47%	172.74%	0.53%	189.46%	0.53%	189.46%
12	0.35%	185.28%	0.35%	185.28%	0.36%	200.94%	0.36%	200.94%
13	0.29%	186.43%	0.29%	186.43%	0.30%	822.42%	0.30%	822.42%
14	0.26%	398.48%	0.26%	398.48%	0.25%	346.52%	0.26%	346.52%
15	0.24%	227.46%	0.24%	227.46%	0.22%	230.35%	0.23%	230.35%
16	0.22%	171.31%	0.22%	171.31%	0.20%	203.71%	0.21%	203.71%
17	0.20%	130.72%	0.20%	130.72%	0.18%	186.70%	0.19%	186.70%
18	0.19%	131.68%	0.19%	131.68%	0.17%	190.93%	0.18%	190.93%
19	0.19%	179.65%	0.19%	179.65%	0.16%	208.37%	0.17%	208.37%
20	0.18%	335.71%	0.18%	335.71%	0.15%	276.30%	0.16%	276.30%
21	0.17%	166.56%	0.17%	166.56%	0.15%	380.09%	0.15%	380.09%
22	0.17%	167.75%	0.17%	167.75%	0.14%	207.35%	0.15%	207.35%
23	0.16%	175.42%	0.16%	175.42%	0.14%	206.85%	0.14%	206.85%
24	0.16%	156.42%	0.16%	156.42%	0.14%	270.08%	0.14%	270.08%
25	0.15%	142.65%	0.15%	142.65%	0.13%	309.72%	0.14%	309.72%

Appendix C – MATLAB CODEX

C.1 Black-Scholes

```
function c0 = BlackScholes(S0,K,r,vol,n)

%Calculating The European Call Option Black Scholes Analytical Pirce
    T = n/253;
    d1 = (log(S0/K)+(r+0.5*(vol^2))*T)/(vol*sqrt(T));
    d2 = d1 - (vol*sqrt(T));
    c0 = S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2);
end
```

C.2 Stochastic Volatility Monte-Carlo

```
function [CMC_Mean,CMCArr,CMC_Mean_Antith_Var,CMCAntithArr...

,GT,AT,GTAntith,ATAntith,ST_Mean,ST_MeanArith,CG0Arr,CG0Mean,CA0Arr,CA0Mean,CG0AntithArr,CG0AntithMean,CA0AntithArr,
CA0AntithMean...
          ,CovAntithEuro,CorrAntithEuro...
          ,CovAntithGT,CovAntithAT,GTNeg,ATNeg,CMCNegArr...
```

```
    ,CorrCVBase,CorrCVAnt]  = MonteGTATVolStoch(S0,K,r,vol,n,m,alpha,eta,volhat)

%NOTE dont generally require GT and AT and other matuyrity arrays,
%But useful for debugging to have printed, Code can be altered to
%remove them

%The Stockprice average calculation can also be removed to increase
%code efficency as not required for this investigvation analysis


   T = n/253; %time to maturity in fraction of financial year (253 days)
   dt = T/n; %discretsing time steps

   %setting up arrays

   ZeroArr = zeros(1,n);
   OnesArr = ones(1,n);
   NanArr = nan(1,n);
   CMCArr = zeros(1,m);%size m as m paths
   CMCNegArr = CMCArr;
   CMCAntithArr = CMCArr;


   %ASian Option Set Up

   AT=nan(1,m); %arthimetic mean value at T
   GT=AT; %geometric Mean value at T
   ATNeg = AT;
   GTNeg = AT;
   CG0Arr = AT; %Intial Price arrays
   CA0Arr= AT;
   CG0NegArr= AT;
   CA0NegArr= AT;
```

```
%

        %Arithmetic Variance reduction Asian option values

        GTAntith = AT;  %arrays for the arithmetic variance reduction technique
        ATAntith = AT;
        CG0AntithArr = AT;  %arrays for the arithmetic variance reduction technique
        CA0AntithArr = AT;



        STArr = nan(1,m); %maturity stock price array
        STArithVarArr = STArr;


        vol0 = vol; % setting so that first vol will be the implied

               for j =1:m % running multiple simulations (paths)

                    Z = normrnd(ZeroArr,OnesArr);
                    %faster to calculate all random variables at the same
                    %time
                    ZNeg = (-1).*(Z); %antithetic GRVs
                    Zvol = normrnd(ZeroArr,OnesArr); %Stochastic volatility GRV
                    Stockprices = NanArr;
                    Stockprices(1) = S0; %intial stock price in array
                    %Stockprices2 = Stockprices;
                    StockpricesNeg = Stockprices;
                    StockpricesAv = Stockprices;
                    timestep = ZeroArr;
```

```
 timestep(1) = 0;
 St=S0; %setting intial stock price
 StNeg = S0;


for i = 1:n %Stock path until maturity day

    %Calculating Stochastic volatility

    muvol = alpha*(vol-volhat);
    sigvol = 0.5*(eta^2);


       vola = vol*exp((muvol-sigvol)*dt);
    volexpo = eta*sqrt(dt);

    vol = vola * exp(volexpo*Zvol(i));
    %this is the stochastic volatility

    sig = 0.5*(vol^2); %consatnt value in calculation,


        if i == 1 %this is so starts at intial vol

        vol = vol0;
        %this is the stochastic volatility

        sig = 0.5*(vol^2);
        end
```

```
%St is the discrete form approximation

                        %Splitting equation up to increase computational
                        %efficency and for clarity
                        Sta = St*exp((r-sig)*dt);
                        StaNeg = StNeg*exp((r-sig)*dt);
                        Stbexpo = vol*sqrt(dt);

                        St = Sta * exp(Stbexpo*Z(i));
                        StNeg = StaNeg * exp(Stbexpo*ZNeg(i)); %the stock price if random number was negative

                        Stockprices(i+1)=St;
                        StockpricesNeg(i+1) = StNeg; %negative for arithmetic var reduction technique

                        %Stockprices2(i+1)=St;

                        StockpricesAv(i+1) = (Stockprices(i+1) + StockpricesNeg(i+1))/2;
                        timestep(i+1)=i;


                        if i == n %when at maturity date


                            CMC = exp(-r*T)*max([St-K,0.]);%exponential factor is the discount factor
                            % Calculating european Call option price
                            %Need Neg in order to calculate the covariance
                            CMCNeg = exp(-r*T)*max([StockpricesNeg(end)-K,0.]);%exponential factor is the discount
factor

                            %ASIAN OPTION

                            CMCAntith = (CMC+CMCNeg)/2; %Antithetic Price
```

```
%FIXED CALL ASIAN OPTION Section

%TO DEAL WITH 0 Stockprice Values in Log, add 1
%to all values, then subtract 1 from the final
%geo average

StockpricesPlus = Stockprices + 1;
StockpricesNegPlus = StockpricesNeg + 1;

GT(j) = exp(sum(log(StockpricesPlus))/length(StockpricesPlus)); %geometric average
Geo average final stock price
GT(j) = GT(j) - 1;
%geometric average    Geo average final stock price

GTNeg(j) = exp(sum(log(StockpricesNegPlus))/length(StockpricesNegPlus)); %geometric
average    Geo average final stock price
GTNeg(j) = GTNeg(j) - 1;

AT(j) = sum(Stockprices)/length(Stockprices); %arthimetic average
ATNeg(j) = sum(StockpricesNeg)/length(StockpricesNeg); %arthimetic average

%Arithmetic var reduction asian


GTAntith(j) = (GT(j)+ GTNeg(j))/2; %geometric average
ATAntith(j) = (AT(j)+ ATNeg(j))/2 ; %arthimetic average for arithmetic var reduction


% MAturity Stock price Cen be storred for Floating asian option price
STArr(j) = Stockprices(end); %storing maturity stock price

STArithVarArr(j) = StockpricesAv(end);
```

```
                    %ASIAN OPTION PRICES geo and arith
                    CG0Arr(j) = exp(-r*T)*max([GT(j)-K,0.]);
                    CA0Arr(j) = exp(-r*T)*max([AT(j)-K,0.]);

                    CG0NegArr(j) = exp(-r*T)*max([GTNeg(j)-K,0.]);
                    CA0NegArr(j) = exp(-r*T)*max([ATNeg(j)-K,0.]);

                       %Antithetic method pricing
                      CG0AntithArr(j) = (CG0Arr(j)+CG0NegArr(j))/2;
                      CA0AntithArr(j) = (CA0Arr(j)+CA0NegArr(j))/2;

                end
                %above tells the MC call option pay off at
                %every step

            %     MOVIE comment in and out for debugging

%                 figure(1)
%                 plot (timestep, Stockprices);
%                 title ("Animated: Stock price per day");
%                 xlabel("Day");
%                 ylabel("Stock Price");

            end

        CMCArr(j) = CMC; %creating array of the discounted call option payoffs at maturity (ie end) so can
calculate sample variance
        CMCNegArr(j) = CMCNeg;
        CMCAntithArr(j) = CMCAntith;

        end   %end of monte carlo
    CMC_Mean = mean(CMCArr); %arithemtic mean value of all simultion
```

```
                                  %prices

        CG0Mean = mean(CG0Arr);
        CA0Mean = mean(CA0Arr);


        CG0AntithMean = mean(CG0AntithArr);
        CA0AntithMean = mean(CA0AntithArr);

        CMC_Mean_Antith_Var = mean(CMCAntithArr);

ST_Mean = mean(STArr);
ST_MeanArith = mean(STArithVarArr);

%Correlation and Covariance calculations

CovAntithtemp = cov(CMCArr,CMCNegArr);
CovAntithEuro = CovAntithtemp(1,2); %selecting from covaraince array element (a,b)
%
CorrAntithtemp = corrcoef(CMCArr,CMCNegArr);
CorrAntithEuro = CorrAntithtemp(1,2); %selecting

CovAntithtemp = cov(CG0Arr,CG0NegArr);
CovAntithGT = CovAntithtemp(1,2); %selecting from covaraince array element (a,b)
%
% CorrAntithtemp = corrcoef(CG0Arr,CG0NegArr);
% CorrAntithEuro = CorrAntithtemp(1,2); %selecting

CovAntithtemp = cov(CA0Arr,CA0NegArr);
CovAntithAT = CovAntithtemp(1,2); %selecting from covaraince array element (a,b)
```

```
%for investigating variance reduction when combining control variate with
%anithetic

CorrCVBasetemp = corrcoef(CG0Arr,CA0Arr);
CorrCVBase = CorrCVBasetemp(1,2); %selecting

CorrCVAnttemp = corrcoef(CG0AntithArr,CA0AntithArr);
CorrCVAnt = CorrCVAnttemp(1,2); %selecting

end
```

C.3 Asian Option Valuation for Stochastic Volatility When the Initial Asset Price is Varied

```
%comparing BS and MC for multiple intial stock prices
%CALL OPTION NO DIVIDEND

clear all:
clf(figure(1));
clf(figure(2));
clf(figure(3));


disp('Starting Programme')

global m
m = 1000000;%input('Enter How many monte carlo iterations / paths - ');
```

```
vol = 0.2;%input('Enter Volatility of Underliny Asset - ');
alpha = 10; %stoch vol mean reversion speed
eta = 0.3; %the vol of vol
volhat = vol;


count = 0;

vol = 0.2;%input('Enter Volatility of Underliny Asset - ');
ndash = 253; %input('Enter number of TRADING days till maturity - ');%also number of steps
n=ceil(ndash);
T = n/253; %time in fraction of year
dt = T/n;
%r and mu are equal, check why

r = 0.05;%input('Enter equivaelent annual continous intenerest rate r - ');
K = 10;%input('Enter Strike Price - ');


fprintf('\nStrike Price K:\t %f\nand %d Monte Carlo Path iterations',K,m);
FinS = input('\nUpper Limit of Initial Stock Price (ranging from 0, Recommend 3xK) - ');
FinS1= FinS+1; %Adding 1 as loop iteration i starts at 1 not zero
x=FinS1;
Sarr = (0:FinS);
% disp(size(Sarr))
% Sarr0=zeros(1,FinS1);
% disp(size(Sarr0))

Zeroarr = zeros(1,FinS1);

% disp(Sarr(1))
BSc0=Zeroarr;
```

```
MCc0=Zeroarr;


%need a cell to hold aray of maturityCMC results

MCGTC0 = Zeroarr;
MCATC0 = Zeroarr;
AnaGTDisC0 = Zeroarr;
AnaGTContC0 = Zeroarr;


%Antithmetic Variance reduction method

MCc0Antith = MCc0;
MCGTC0Antith = MCGTC0;
MCATC0Antith = MCATC0;
% C0GTArrayCellAntith = C0GTArrayCell;


%VARAINCE
VarofMeanCMC_Mean = Zeroarr;
VarofMeanCMC_Mean_Antith_Var = VarofMeanCMC_Mean;
VarofMeanFixG0 = VarofMeanCMC_Mean;
VarofMeanFixA0 = VarofMeanCMC_Mean;
VarofMeanFixAntithG0 = VarofMeanCMC_Mean;
VarofMeanFixAntithA0 = VarofMeanCMC_Mean;


%Setting up Control VAriate Variable arrays
```

```
BetaoptHatArithDisGeo = Zeroarr;
BetaoptHatArithContGeo = BetaoptHatArithDisGeo;
BetaoptHatAntArithDisGeo = BetaoptHatArithDisGeo;
BetaoptHatAntArithContGeo = BetaoptHatArithDisGeo;

ControlVarBetaoptDisGeo = Zeroarr;
ConVarArithBetaoptContGeo = ControlVarBetaoptDisGeo;
AntConVarArithBetaoptDisGeo = ControlVarBetaoptDisGeo;
AntConVarArithBetaoptContGeo = ControlVarBetaoptDisGeo;

VarcvBetaoptArithDisGeo = Zeroarr;
VarcvBetaoptArithContGeo = VarcvBetaoptArithDisGeo;
VarAntcvBetaoptArithDIsGeo = VarcvBetaoptArithDisGeo;
VarAntcvBetaoptArithContGeo = VarcvBetaoptArithDisGeo;

%Covariance and correlation arrays
CovAntith= Zeroarr;
CorrAntith= Zeroarr;


%Confidence intervals


ConfofMeanCMC_Mean = CovAntith;
ConfofMeanCMC_Mean_Antith_Var = CovAntith;
ConfofMeanFixG0 = CovAntith;
ConfofMeanFixA0 = CovAntith;
ConfofMeanFixAntithG0 = CovAntith;
ConfofMeanFixAntithA0 = CovAntith;

ConfcvBetaoptArithDisGeo = Zeroarr;
ConfcvBetaoptArithContGeo = VarcvBetaoptArithDisGeo;
ConfAntcvBetaoptArithDIsGeo = VarcvBetaoptArithDisGeo;
```

```
ConfAntcvBetaoptArithContGeo = VarcvBetaoptArithDisGeo;



disp('Running...')

t = cputime;
 for l=1:FinS1 %this is lower case L



   [BSc0(l),MCc0(l),MCc0Antith(l),MCGTC0(l),MCATC0(l),MCGTC0Antith(l), MCATC0Antith(l)...
         ,AnaGTDisC0(l),AnaGTContC0(l)...
         ,ControlVarBetaoptDisGeo(l),VarcvBetaoptArithDisGeo(l),BetaoptHatArithDisGeo(l)...
         ,ConVarArithBetaoptContGeo(l),VarcvBetaoptArithContGeo(l),BetaoptHatArithContGeo(l)...
         ,AntConVarArithBetaoptDisGeo(l),VarAntcvBetaoptArithDIsGeo(l),BetaoptHatAntArithDisGeo(l)...
         ,AntConVarArithBetaoptContGeo(l),VarAntcvBetaoptArithContGeo(l),BetaoptHatAntArithContGeo(l)...
         ,VarofMeanCMC_Mean(l),VarofMeanCMC_Mean_Antith_Var(l),VarofMeanFixG0(l)...
         ,VarofMeanFixA0(l),VarofMeanFixAntithG0(l),VarofMeanFixAntithA0(l)...
         ,CovAntith(l),CorrAntith(l)]  = AsianOption_Loop_FunctionVolStoch(Sarr(l),K,r,vol,n,m,alpha,eta,volhat);


     %Calculating Confidence Intervals

                ConfofMeanCMC_Mean(l) = confidenceVarofMean(VarofMeanCMC_Mean(l),m);
                ConfofMeanCMC_Mean_Antith_Var(l) = confidenceVarofMean(VarofMeanCMC_Mean_Antith_Var(l),m);
                ConfofMeanFixG0(l) = confidenceVarofMean(VarofMeanFixG0(l),m);
                ConfofMeanFixA0(l) = confidenceVarofMean(VarofMeanFixA0(l),m);
                ConfofMeanFixAntithG0(l) = confidenceVarofMean(VarofMeanFixAntithG0(l),m);
                ConfofMeanFixAntithA0(l) = confidenceVarofMean(VarofMeanFixAntithA0(l),m);
```

```
                %Due to how the control variate method works, as well as
                %the approximation of its variance, take the 95% confidence
                %intervals very skeptically for the analytic arithemtic
                %asian option price
                ConfcvBetaoptArithDisGeo(l) = confidenceVarofMean(VarcvBetaoptArithDisGeo(l),m);
                ConfcvBetaoptArithContGeo(l) = confidenceVarofMean(VarcvBetaoptArithContGeo(l),m);
                ConfAntcvBetaoptArithDIsGeo(l) = confidenceVarofMean(VarAntcvBetaoptArithDIsGeo(l),m);
                ConfAntcvBetaoptArithContGeo(l) = confidenceVarofMean(VarAntcvBetaoptArithContGeo(l),m);


     count = count +1;
     fprintf('\n Path Cycles Completed %i',count);
 end




 et = cputime - t;
 fprintf('elapsed time (s):\t %.2f\n',et)



 figure(1)

plot(Sarr,AnaGTDisC0,Sarr,MCGTC0,Sarr,MCGTC0Antith,Sarr,AnaGTContC0,Sarr,MCc0Antith,'c',Sarr,BSc0,Sarr,MCc0,Sarr,MCA
TC0,Sarr,MCATC0Antith,Sarr,ControlVarBetaoptDisGeo,'LineWidth',2)
 title ("Asian Option Pricing Varying Intial Stock Price");
 xlabel("Initial Stock Price");
 ylabel("Asian Fixed Geo Call Option Price");
 legend('Analytical discrete Asian CG0','MC CG0','MC CG0 Antith','Analytical Continous Asian CG0','Antithetc Euro
C0','BS Euro C0','MC Euro C0','MC Asian CA0','Antith MC Asian CA0','CVMC using DisGeo Asian CA0')
```

```
 figure(2)

plot(Sarr,AnaGTDisC0,Sarr,MCGTC0,Sarr,MCGTC0Antith,Sarr,AnaGTContC0,Sarr,MCATC0,Sarr,MCATC0Antith,Sarr,ControlVarBet
aoptDisGeo,'LineWidth',2)
 title ("Comparison Analytical discrete and MC simulation for Asian fixed Call Option Price varying Initial Stock
price");
 xlabel("Initial Stock Price");
 ylabel("Asian Fixed Geo Call Option Price");
 legend


  figure(3)
 plot(Sarr,MCc0Antith,Sarr,BSc0,Sarr,MCc0,'LineWidth',2)
 title ("Comparison Analytical discrete and MC simulation for Asian fixed Call Option Price varying Initial Stock
price");
 xlabel("Initial Stock Price");
 ylabel("Asian Fixed Geo Call Option Price");
 legend

 figure(4)

plot(Sarr,AnaGTDisC0,Sarr,MCATC0,Sarr,ControlVarBetaoptDisGeo,Sarr,ConVarArithBetaoptContGeo,Sarr,AntConVarArithBeta
optDisGeo,Sarr,AntConVarArithBetaoptContGeo,'LineWidth',2)
 title ("Comparison Analytical discrete and MC simulation for Asian fixed Call Option Price varying Initial Stock
price");
 xlabel("Initial Stock Price");
 ylabel("Asian Fixed Geo Call Option Price");
 legend
```

```
%VARAINCE PLOTS




 figure(7)
 plot(Sarr,VarofMeanFixAntithA0,'-
.or',Sarr,VarcvBetaoptArithDisGeo,Sarr,VarcvBetaoptArithContGeo,Sarr,VarAntcvBetaoptArithDIsGeo,Sarr,VarAntcvBetaopt
ArithContGeo,'LineWidth',2)
 title ("Monte Carlo Variance of Mean (Arithmetic Average)");
 xlabel("Initial Stock Price");
 ylabel("Variance");
 legend('Antit MC
A0','BiasVarcvBetaoptArithDisGeo','BiasVarcvBetaoptArithContGeo','BiasVarcvBetaoptAntithArithDisGeo','BiasVarcvBetao
ptAntithArithContGeo');


 figure(8)

plot(Sarr,VarofMeanCMC_Mean,Sarr,VarofMeanCMC_Mean_Antith_Var,Sarr,VarofMeanFixG0,Sarr,VarofMeanFixA0,Sarr,VarofMean
FixAntithG0,Sarr,VarofMeanFixAntithA0,'LineWidth',2)
 title ("Control Variate price (Arithmetic Average)");
 xlabel("Initial Stock Price");
 ylabel("Arithmetic Option Price");

legend('VarofMeanCMC_Mean','VarofMeanCMC_Mean_Antith_Var','VarofMeanFixG0','VarofMeanFixA0','VarofMeanFixAntithG0','
VarofMeanFixAntithA0');
```

```
function [confidenceinterval] = confidenceVarofMean(var,m)

            temp = var * m;
            temp = sqrt(temp);
            confidenceinterval =  1.96*(temp / sqrt(m));
            %returns 95% confidence interval when input is a varaince of
            %the mean
end
```

C.4 Control Variate Reduction

```
function[YcvBetapt,VarYcvBetaopt,BetaoptHat] = Control_Variate_fn(hX,fX,theta)


%hX = C0GTArray the geometric monte carlo price
%fX = C0ATArray the garithmetic monte carlo price
%theta = AnalyGeoPrice

%Note theta can either be continous or discrete - check how similar
%For asian options

%hX is the value of control option each path from MC, its an array
%theta = analytic value of control option, scalar
%fX is the value of arithmetic option each path from MC, array

%assuming Var(h(X)) is not singular (Has inverse)
```

```
%then optimal variance minimzing parameter Beta given by:

m = length ( hX); %ie number of MC paths

%Betaopt = Cov(hX,fX)/Var(hX)
%fXHat = average fX

hXHat = mean(hX);
fXHat = mean(fX);


%Even though know theoretical Var(hX), must use saple from MC due to integrals
% brack = (hX - hXHat).^2;
% VarhX = (sum(brack))/(m-1);  %m-1 as sample

[~,~,StdofhX,VarofhX,] = SampleStdVar(hX,hXHat);
[~,~,StdoffX,VaroffX] = SampleStdVar(fX,fXHat);

% CovhXfX = (sum((hX - hXHat)*transpose((fX-fXHat))))/(m-1);
%splitting for ease
hdash = hX - hXHat;
fdash = fX-fXHat;
hfdash = hdash * transpose(fdash); %transpose for matrix multiplication
CovhXfX = (sum(hfdash))/(m-1);
BetaoptHat = CovhXfX/VarofhX;
%betaopthat must be greater than 1/2 for Var Control to be less than
%Var(FxHat)

%As dont know covariance so estimating it

%and check error mat hes expected

YcvBetapt = fXHat - BetaoptHat*(hXHat - theta); %ie arithmetic from control variable
```

```
%VaCV = VaHat - B(VgeoHat= VgeoAnayltic)

%Taking the variance from the MC is biased as Betaopt is dependant on hX
%and fX. Unbiased if run a pilot sample MC simulation

% rho = corr(hX,fX); %multipliy each aray together and takes the average
%Note as the correlation will be squared, even a negative correlation is
%usable




rho = (CovhXfX)/(StdoffX*StdofhX);

mrat = (m-2)/(m-3);



VarYcvBetaopt = mrat*(1-rho^2)*(VaroffX/m);


%For an unbiased Betaopt run a seperate MC pilot and take betaopt from
%there


end
```

C.5 Geometric Analytical Solutions

```
function [C0fixanalyticalcontPB,C0fixanalyticalDiscretePB]  = ContinousFixedGeoCall(S0,K,r,vol,n)



    T = n/253;

    vol0 = vol/(sqrt(3));
    q0 = 0.5*(r+0.5*(vol0^2));

    d2 = (log(S0/K)+0.5*(r-0.5*(vol^2))*T)/(vol0*sqrt(T));
    d1 = d2 + (vol0*sqrt(T));
    C0fixanalyticalcontPB= S0*exp(-q0*T)*normcdf(d1)-K*exp(-r*T)*normcdf(d2);


    dt = T/n;
    Tn = 0.5*(n+1)*dt;
    TnHat = dt*(((n+1)*(2*n + 1))/(6*n));

    qn = r - ((r -0.5*(vol^2))*Tn/T) - (0.5*(vol^2))*(TnHat/T);

    d4 = (log(S0/K)+(r-0.5*(vol^2))*Tn)/(vol*sqrt(TnHat));
    d3 = d4 + (vol*sqrt(TnHat));
    C0fixanalyticalDiscretePB = S0*exp(-qn*T)*normcdf(d3)-K*exp(-r*T)*normcdf(d4);
```

C.6 Calculation of Sample Standard Deviation and Variance

```
function [StdofMean,VarofMean,SampleStd, SampleVar] = SampleStdVar(x,Mean)

    %where x is an array
    %Sample standard Deviation / Variance with bessel correction
    %(1/N-1)
    %correction (N-1 is degrees of freedom)
    N = length(x);
    %note x must be an array of results (ie the sample)
    SampleVar = sum((x - Mean).^2)/(N-1); %here xbar (Mean) chosen as Expected from BS, could also use CMC_Mean
(both should be similar, ideally within tickrate)
    %this is an unbiased sample variance
    SampleStd = sqrt(SampleVar); %this is biased as sqrt nonlinear

    %Standard error of the mean
    StdofMean = SampleStd / sqrt(N); %this gives proportionality of 1/sqrtN
    VarofMean = (SampleStd^2) / N;
end
```