PRECISE CALCULATION OF HAUSDORFF DIMENSION OF APOLLONIAN GASKET

Article *in* Fractals · March 2018

DOI: 10.1142/S0218348X18500500

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PRECISE CALCULATION OF HAUSDORFF DIMENSION OF APOLLONIAN GASKET

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ABSTRACT. A transfer operator method is proposed to calculate d_H, the Hausdorff dimension of the Apollonian gasket. Compared with previous operator-based methods, we make two improvements in this paper. We adopt an infinite set of contractive Möbius transformations (rather than a finite set of parabolic ones) to generate the Apollonian gasket. We also apply an efficient finite matrix approximation of an infinite sum of infinite-dimensional operators. By using this method, a high precision estimate of d_H is obtained:

$$d_H = 1.305\,686\,728\,049\,877\,184\,645\,986\,206\,851\,0\ldots.$$

Keywords: Apollonian gasket, Hausdorff dimension, transfer operator

1. Introduction: Description of Method

The Apollonian gasket is one of the most beautiful fractals with an intriguing and rich mathematical structure.[1,2] This fractal may be constructed in a manner similar to that for the Sierpinski gasket (see Figure 1). For three distinct points z_1, z_2 and z_3 in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, there exist uniquely three mutually tangent circles C_1, C_2 and C_3 that touch with each other at the given three points. The arcs between z_1, z_2 and z_3 form a curvilinear triangle; denote it by Δ_0. Starting from Δ_0, find within it a circle C_4 that is tangent to its three sides. Remove the interior of C_4 from Δ_0; the remaining part consists of three smaller curvilinear triangles Δ_1, Δ_2 and Δ_3. Iterate this procedure on each smaller curvilinear triangle and the limit set is the Apollonian gasket. Besides its geometry meaning, the Apollonian gasket can also be

explained as a random walk on the light cone in Minkowski space.[3,4] In recent years, there has been a revival of interest in number theoretic aspects of Apollonian packing of circles.[5–8]

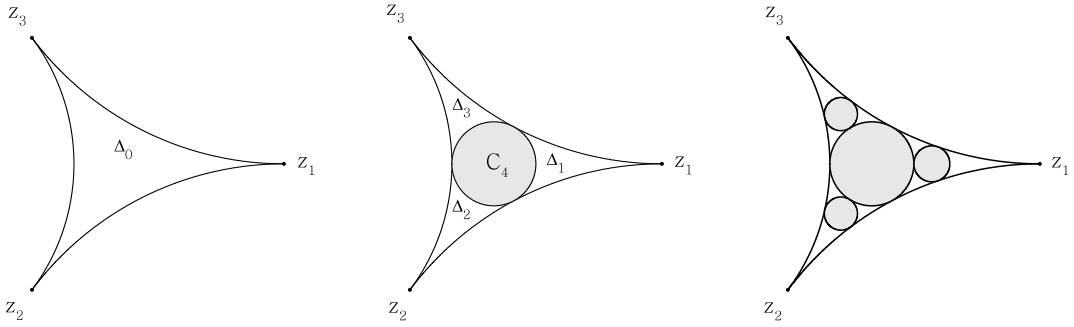

FIGURE 1. Construction of Apollonian gasket

A long unsolved problem regarding the Apollonian gasket is the analytical determination of its Hausdorff dimension(d_H). High precision numerical estimation of d_H remains a challenge. The best result is $d_H \approx 1.305686729(10)$, which was obtained by Thomas and Dhar more than 20 years ago.[4] In this paper we propose a method that can yield more than 30 reliable decimal digits of d_H within an hour CPU time on a personal computer. The improvement can only be partly attributed to the increasing of computing power. A better treatment of an infinite-dimensional operator is more crucial. The method can be briefly described as follows.

The Apollonian gasket can be regarded as a fixed set of an iterated function system(IFS) $\{f_j : \Delta_0 \to \Delta_j | j = 1, 2, 3\}$. Each f_j, which we shall call a generator, is a Möbius transformation. The initial curvilinear triangle Δ_0 can be arbitrarily chosen, but for simplicity, we set the vertexes $(z_1, z_2, z_3) = (\infty, 0, i)$. With a slight abuse of notation, we write the Möbius transformation associated with a 2×2 matrix A by $A(z)$,

i.e.,

$$A(z) = \frac{a_{1,1}z + a_{1,2}}{a_{2,1}z + a_{2,2}}.$$

Then we have $f_j(z) = A_j(z)$, where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix}. \tag{1}$$

The three matrices are connected by a 3-fold rotation of Δ_0. Letting

$$R_0 = -\begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix},$$

we have $R_0(z) : (z_1, z_2, z_3) \to (z_2, z_3, z_1)$, $R_0^3 = I$ and $A_2 = R_0 A_1 R_0^2$, $A_3 = R_0^2 A_1 R_0$.

Compared with those of the Sierpinski gasket, the generators of Apollonian gasket are not uniform contractions. Specifically, at the vertexes of Δ_0, we have $A_j(z_j) = z_j$ and $A_j'(z_j) = 1$, $j = 1, 2, 3$. In other words, all $A_j(z)$'s are parabolic Möbius transformations. Most difficulties in numerical calculation of d_H can be attributed to this fact. To get rid of parabolicity, we consider an alternative but infinite set of generators $\{A_1^k R^i | k \in \mathbb{N}^+, i = 1, 2\}$, where $R = A_1^{-1} R_0 A_1$. All generators in this set are contraction mappings and they produce the same fixed set as $\{A_1, A_2, A_3\}$.

We now consider the transfer operator associated with the IFS. Let $f = f(z)$ be a conformal map; define a transfer operator $\mathcal{L}_{f,q}$ by

$$\mathcal{L}_{f,q}(z_2, z_1) = |f'(z_1)|^q \delta(z_2 - f(z_1))$$

where q is a real parameter and $\delta(z)$ denotes the Dirac delta-function. $\mathcal{L}_{f,q}$ is known as the Ruelle-Perron-Frobenius operator in the literature, and it provides a powerful tool for the study of statistical properties of chaotic dynamic systems.[9] In particular, similar to our computation, a transfer operator associated with a series of special Möbius

transformations (the Ruelle-Mayer operator) has been used to analyze the average-case complexity of the Gaussian algorithm and the Euclidean algorithm for lattice reduction.[10,11]

It can be readily verified that $\mathcal{L}_{f,q}\mathcal{L}_{g,q} = \mathcal{L}_{f\circ g,q}$ for two arbitrary conformal maps f and g, and therefore $\mathcal{L}_{A,q}\mathcal{L}_{B,q} = \mathcal{L}_{AB,q}$ for any two 2×2 matrices A and B. Let

$$\mathcal{L}_q = \sum_{k=1}^{\infty}\sum_{i=1}^{2} \mathcal{L}_{A_1^k R^i,q}$$

and denote its leading eigenvalue by $\lambda_0 = \lambda_0(q)$, then d_H is determined by $\lambda_0(d_H) = 1$. To numerically determine the leading eigenvalue of $\mathcal{L}_q$, we consider its dual action on a function space. Let $\Phi = \Phi(z)$ be a function on the complex plane,

$$\Phi\mathcal{L}_q = \sum_{k=1}^{\infty}\sum_{i=1}^{2} \Phi\mathcal{L}_{A_1^k R^i,q}$$

where

$$(\Phi\mathcal{L}_{A,q})(z) = \int_{\mathbb{C}} \Phi(\tilde{z})\mathcal{L}_{A,q}(\tilde{z},z)d\tilde{z} = |A'(z)|^q \Phi(A(z)).$$

In the space spanned by $\{\Phi_{m,n}(z) \equiv z^{*m}z^n | m,n\in\mathbb{N}\}$, $\mathcal{L}_{A,q}$ is represented by an infinite dimensional matrix

$$\Phi_{m,n}\mathcal{L}_{A,q} = \sum_{r,s=0}^{\infty} \mathcal{L}_{A,q}(m,n;r,s)\Phi_{r,s}$$

where $\mathcal{L}_{A,q}(m,n;r,s) = \mathcal{M}^*_{A,q}(m,r)\mathcal{M}_{A,q}(n,s)$ and $\mathcal{M}_{A,q}(n,s)$ is given by the expansion

$$\frac{(a_{1,1}z + a_{1,2})^n}{(a_{2,1}z + a_{2,2})^{n+q}} = \sum_s \mathcal{M}_{A,q}(n,s)z^s$$

($\det A = 1$ is assumed).

To deal with the sum of infinite operators, for sufficiently large k, we expand

$$\mathcal{L}_{A_1^k R,q} = \sum_{l=0}^{\infty} k^{-(l+2q)}\mathcal{F}_q^{(l)} \quad \text{and} \quad \mathcal{L}_{A_1^k R^2,q} = \sum_{l=0}^{\infty} k^{-(l+2q)}\hat{\mathcal{F}}_q^{(l)}.$$

The explicit form of $\mathcal{F}_q^{(l)}$ and $\hat{\mathcal{F}}_q^{(l)}$ will be given in Section 3. Therefore

$$\mathcal{L}_{A,q} = \sum_{k=1}^{k_0-1} \sum_{i=1}^{2} \mathcal{L}_{A_1^k R^i,q} + \sum_{l=0}^{\infty} \zeta(l+2q, k_0)(\mathcal{F}_q^{(l)} + \hat{\mathcal{F}}_q^{(l)})$$

where $\zeta(s,k)$ denotes the Hurwitz zeta function

$$\zeta(s,k) = \sum_{j=0}^{\infty} (j+k)^{-s}.$$

Finally, we set an upper bound to the summation over l and approximate $\mathcal{L}_{A,q}$ by a finite-dimensional matrix. In this method, high precision estimation of d_H can be obtained at modest cutoff parameters.

From the viewpoint of nonlinear dynamics, our method can be regarded as an example of how to efficiently extract the statistical properties of a dynamic system with nonuniform hyperbolicity. We organize the paper as follows. In the next section, we explain the relationship of our method with previous works, especially the transfer operator approaches of McMullen,[12] Thomas and Dhar,[4] and one we used to calculate the generalized Lyapunov exponent of 2×2 positive matrices.[13] The numerical method and certain results are described in detail in Section 3. Finally, we present some open questions arising from the method in Section 4.

2. Related Work

Hirst obtained the first rigorous bounds of d_H:[14]

$$1 < d_H < \frac{\log 3}{\log(1 + 2/\sqrt{3})} = 1.4311\ldots.$$

The result was later refined by Boyd, who proved that[15]

$$1.300197 < d_H < 1.314534.$$

Numerical methods to calculate d_H can be classified into two categories: by circle count[16–18] and by spectral radius of a transfer operator.[4,12] We will focus on the second category.

The transfer operator used by McMullen is essentially equivalent to $\mathcal{L}'_q = \sum_{j=1}^{3} \mathcal{L}_{A_j,q}$. If $A_1(z), A_2(z)$ and $A_3(z)$ are contraction mappings, the leading eigenvalue of $\mathcal{L}'_q$ can be effectively computed by cycle (periodic orbit) expansion.[9,19] But this is not valid for the Apollonian gasket. Parabolic generators cause serious difficulty in analytical as well as numerical procedures for estimating d_H. To circumvent this difficulty, Mauldin and Urbański proposed an infinite set of contractive generators.[20] Their technique can be formulated as a factorization of spectral determinant.[21] Due to the symmetry of R_0, if only the leading eigenvalue is concerned, we can restrict $\mathcal{L}'_q$ to the space that is invariant under the action of R_0. In this invariant subspace $\mathcal{L}_{A_2,q}$ and $\mathcal{L}_{A_3,q}$ can be identified with $\mathcal{L}_{R_0A_1,q}$ and $\mathcal{L}_{R_0^2A_1,q}$, respectively. Then consider the spectral determinant:

$$\begin{aligned}
&\det(1 - z\mathcal{L}_{A_1,q} - z\textstyle\sum_{i=1}^{2} \mathcal{L}_{R_0^iA_1,q}) \\
&= \det(1 - z\mathcal{L}_{A_1,q}) \det(1 - z(1 - z\mathcal{L}_{A_1,q})^{-1} \textstyle\sum_{i=1}^{2} \mathcal{L}_{R_0^iA_1,q}) \\
&= \det(1 - z\mathcal{L}_{A_1,q}) \det(1 - \textstyle\sum_{k=0}^{\infty} z^{k+1}\mathcal{L}_{A_1,q}^k \sum_{i=1}^{2} \mathcal{L}_{R_0^iA_1,q}) \\
&= \det(1 - z\mathcal{L}_{A_1,q}) \det(1 - \textstyle\sum_{k=0}^{\infty} z^{k+1} \sum_{i=1}^{2} \mathcal{L}_{A_1^kR_0^iA_1,q})
\end{aligned}$$

The last equality implies that $\{A_1^kR_0^iA_1 = A_1^{k+1}R^i | k \in \mathbb{N}, i = 1, 2\}$ can be used as an alternative set of generators. One step further,

$$\begin{aligned}
&\det(1 + z\textstyle\sum_{i=1}^{2} \mathcal{L}_{R_0^iA_1,q}) \det(1 - \textstyle\sum_{k=0}^{\infty} z^{k+1} \sum_{i=1}^{2} \mathcal{L}_{A_1^kR_0^iA_1,q}) \\
&= \det(1 - \textstyle\sum_{k=1}^{\infty} z^{k+1}(\sum_{i=1}^{2} \mathcal{L}_{A_1^kR_0^iA_1,q} + \sum_{i,j=1}^{2} \mathcal{L}_{R_0^iA_1^kR_0^jA_1,q})),
\end{aligned}$$

we arrive at the generator set proposed by Mauldin and Urbański:

$$\{A_1^k R_0^i A_1, R_0^j A_1^k R_0^i A_1 | k \in \mathbb{N}^+, i, j = 1, 2\}.$$

Thomas and Dhar proposed a transfer operator from a different viewpoint.[4] Considering the initial curvilinear triangle Δ_0, let a, b, c, d be the curvatures of C_1, C_2, C_3 and C_4, respectively. It is well known that the curvature vector $X = X(\Delta_0) = (a, b, c, d)^T$ satisfies $X^T M X = 0$ with

$$M = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

Since the square matrix M is congruent to $\mathrm{diag}(1, -1, -1, -1)$, X can be regarded as a light-like vector in Minkowski space. Moreover, the curvature vectors of the three smaller curvilinear triangles in the next generation of Apollonian packing are given by $X(\Delta_j) = B_j X(\Delta_0)$, $j = 1, 2, 3$, where

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 2 & 2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 2 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & -1 & 2 \end{pmatrix}.$$

The B_j's preserve the quadratic form $X^T M X$, i.e., $B_j^T M B_j = M$, hence can be regarded as elements in the Lorentzian group. Thomas and Dhar defined an operator $\hat{\mathcal{L}}$, which acts on a real function $f(X)$ of curvature vectors via

$$(\hat{\mathcal{L}} f)(X) = \sum_{j=1}^{3} f(B_j X).$$

In addition, they required $f(X)$'s to be homogeneous functions of degree $-q$, i.e., $f(\lambda X) = \lambda^{-q} f(X)$ for all nonzero real λ. Under this restriction, $\hat{\mathcal{L}}$ is denoted by $\hat{\mathcal{L}}_q$. They then computed d_H under the condition that the leading eigenvalue of $\hat{\mathcal{L}}_{d_H}$ should be 1.

The relation between $\hat{\mathcal{L}}_q$ and $\mathcal{L}'_q$ can be readily derived from the well-known isomorphism of the Lorentz group with the Möbius group. To be concrete, we parameterize light-like vectors by

$$X = \begin{pmatrix} 2 & i & -i & 0 \\ 0 & i & -i & 2 \\ 0 & -i & i & 0 \\ 2 & 2+i & 2-i & 2 \end{pmatrix} \begin{pmatrix} z_1 z_1^* \\ z_1 z_2^* \\ z_2 z_1^* \\ z_2 z_2^* \end{pmatrix} \equiv X(Z),$$

where $z_1, z_2 \in \mathbb{C}$ and $Z \equiv (z_1, z_2)^T$. Then straightforward calculation shows that

$$B_j X(Z) = X(A_j Z), j = 1, 2, 3$$

where A_1, A_2, A_3 are the three 2×2 matrices given in formula (1). Consider the following base of homogeneous functions:

$$\{\Psi_{m,n}(X(Z)) = \frac{1}{|z_2|^{2q}} \left(\frac{z_1^*}{z_2^*}\right)^m \left(\frac{z_1}{z_2}\right)^n | m, n \in \mathbb{N}\}.$$

For a 2×2 matrix A,

$$\Psi_{m,n}(X(AZ)) = \frac{(a_{1,1}^* z_1^* + a_{1,2}^* z_2^*)^m (a_{1,1} z_1 + a_{1,2} z_2)^n}{(a_{2,1}^* z_1^* + a_{22}^* z_2^*)^{m+q} (a_{2,1} z_1 + a_{22} z_2)^{n+q}}.$$

Letting $z = z_1/z_2$, we have

$$\begin{aligned} \Psi_{m,n}(X(AZ)) &= \tfrac{1}{|z_2|^{2q}} \tfrac{(a_{1,1}^* z^* + a_{1,2}^*)^m (a_{1,1} z + a_{1,2})^n}{(a_{2,1}^* z^* + a_{22}^*)^{m+q} (a_{2,1} z + a_{22})^{n+q}} \\ &= \tfrac{1}{|z_2|^{2q}} \textstyle\sum_{r,s=0}^{\infty} \mathcal{L}_{A,q}(m, n; r, s) z^{*r} z^s \\ &= \textstyle\sum_{r,s=0}^{\infty} \mathcal{L}_{A,q}(m, n; r, s) \Psi_{r,s}(X(Z)). \end{aligned}$$

Therefore $\hat{\mathcal{L}}_q = \mathcal{L}'_q$ in this representation. In their numerical computation, Thomas and Dhar used a different representation, in which they approximated $\hat{\mathcal{L}}_q$ by a finite dimensional sparse matrix. However, since parabolic generators were used, the result converged rather slowly with increasing matrix size (for details, see Table 1 in Thomas and Dhar[4]).

The transfer operator $\mathcal{M}_{A,q}$ has been used to compute the generalized Lyapunov exponent of a product of random matrices.[13,22] Suppose $\{M_\sigma | \sigma \in \Lambda\}$ is a finite set of 2×2 real matrices. Let $S = \sigma_1\sigma_2\cdots\sigma_n, \sigma_i \in \Lambda$, be an arbitrary sequence and $M_S \equiv M_{\sigma_1}M_{\sigma_2}\cdots M_{\sigma_n}$ be a product of n matrices. To characterize how M_S grows with increasing sequence length $|S|$, a generalized Lyapunov exponent $\tau(q)$ may be defined as

$$\tau(q) = \lim_{n\to\infty} \frac{1}{n} \log(\sum_{|S|=n} ||M_S||^{-q})$$

where $||\cdot||$ is an arbitrary matrix norm. If all $A_\sigma(z)$'s map the unit disk to its interior, $\tau(q)$ can be calculated from the transfer operator

$$\mathcal{M}_q = \sum_{\sigma\in\Lambda} \mathcal{M}_{A_\sigma,q}.$$

Specifically, $\tau(q) = \ln \lambda_q(0)$ where $\lambda_q(0)$ is the leading eigenvalue of $\mathcal{M}_q$. The reason to require $|A(z)| < 1$ for all $z \leq 1$ lies in the fact that, if it is true, the elements of $\mathcal{M}_{A,q}$ will be exponentially small, i.e.,

$$|\mathcal{M}_{A,q}(n,s)| < C|\theta|^{n+s}$$

for $\theta \in (0,1)$.[13] Thus $\mathcal{M}_{A,q}$'s can be efficiently approximated by a finite matrix when numerically calculating the leading eigenvalue of $\mathcal{M}_q$. In this paper we generalize the transfer operator to the case of complex matrices M_σ (note that $\mathcal{L}_{A,q} = \mathcal{M}^*_{A,q} \otimes \mathcal{M}_{A,q}$).

If one of M_S has identical eigenvalues (i.e., $M_S(z)$ is parabolic), there may exist a critical q_c above which $\tau(q)$ is identically zero. A well-known example is the Farey spin chain model,[23,24] in which

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and q_c is found to be 2. In another example, which is related to the count of Fibonacci representations of non-negative integers,[25]

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $q_c = 2.47875\ldots$ is determined by $2\zeta(q_c) - \zeta(q_c - 1) = 0$. Similar to the Apollonian gasket, parabolicity in the two examples can be removed by adopting an infinite set of generators. We have applied our method to the two examples. Compared with the exact values, the errors of the calculated q_c's are less than 10^{-32}.

3. Method and Result

It can be readily shown that all generators $\{A_1^k R^i(z) | i = 1, 2, k \in \mathbb{N}^+\}$ are contraction mappings in a domain that contains the right half-plane.[20] Hence we should map the right half-plane by a Möbius transformation $M(z)$ to the interior of the unit circle. For a given $M(z)$, we make the substitutions $A_1 \to MA_1M^{-1}$, $R \to MRM^{-1}$ and then evaluate the matrix elements of $\mathcal{L}_q$. The choice of M has some effect on the efficiency of our method. Since a Möbius transformation is specified at three points, for convenience, we require $M(z_l) = -1$, $M(z_r) = 1$ and $M(\infty) = z_\infty$. To facilitate numerical computation, we choose z_l and z_r from the line of symmetry, i.e., $\mathrm{Im}(z_l) = \mathrm{Im}(z_r) = 1/2$, and require z_∞ to be a real number. It follows that the new initial curvilinear triangle is always symmetric with respect to the real axis, and hence A_1 is

TABLE 1. Two representations of G and R used in our computation.

No.	(z_l, z_r, z_∞)	R	G
I	$(\frac{-1+i}{2}, \frac{-9+i}{2}, \frac{1}{2})$	$\frac{1}{12}\begin{pmatrix} -6+8i & 11i \\ 4i & -6-8i \end{pmatrix}$	$\frac{1}{12}\begin{pmatrix} 2-2i & -1-5i \\ 4-4i & -2-10i \end{pmatrix}$
II	$(\frac{-1+i}{2}, \frac{-10+i}{2}, \frac{2}{3})$	$\frac{1}{180}\begin{pmatrix} -90+111i & 121i \\ 99i & -90-111i \end{pmatrix}$	$\frac{1}{45}\begin{pmatrix} 12-6i & -8-26i \\ 18-9i & -12-39i \end{pmatrix}$

a real matrix and $R^2 = R^*$. Two representations we use in numerical computation are listed in Table 1.

Noticing that $A_1^k = I + k(A_1 - I)$, we have $A_1^k R = R + k(A_1 - I)R \equiv R + kG$. For sufficiently large k, expanding

$$\begin{aligned}\frac{((kg_{1,1}+r_{1,1})z + kg_{1,2} + r_{1,2})^n}{((kg_{2,1}+r_{2,1})z + kg_{2,2} + r_{2,2})^{n+q}} &= \frac{(g_{1,2}+g_{1,1}z+r_{1,2}\frac{1}{k}+r_{1,1}\frac{z}{k})^n}{k^q(g_{2,2}+g_{2,1}z+r_{2,2}\frac{1}{k}+r_{2,1}\frac{z}{k})^{n+q}} \\ &= \sum_{s,l=0}^{\infty} \frac{z^s}{k^{q+l}} F_q(n,s,l),\end{aligned}$$

we have

$$\begin{aligned}\mathcal{L}_{A_1^k R,q}(m,n;r,s) &= \sum_{l_1,l_2=0}^{\infty} k^{-(l_1+l_2+2q)} F_q^*(m,r,l_1) F_q(n,s,l_2) \\ &= \sum_{l=0}^{\infty} k^{-(l+2q)} \sum_{l'=0}^{l} F_q^*(m,r,l-l') F_q(n,s,l').\end{aligned}$$

Thus

$$\mathcal{F}_q^{(l)}(m,n;r,s) = \sum_{l'=0}^{l} F_q^*(m,r,l-l') F_q(n,s,l').$$

From the reflection symmetry, we have

$$\mathcal{L}_{A_1^k R^2,q}(m,n;r,s) = \mathcal{L}_{A_1^k R,q}(n,m;s,r) = \mathcal{L}^*_{A_1^k R,q}(m,n;r,s)$$

and

$$\mathcal{L}_q(m,n;r,s) = 2\mathrm{Re}[\sum_{k=1}^{k_0-1} \mathcal{L}_{A_1^k R,q}(m,n;r,s) + \sum_{l=0}^{\infty} \zeta(l+2q,k_0)\hat{\mathcal{F}}_q^{(l)}(m,n;r,s)].$$

By setting two upper bounds N_c and L_c, one for m, n, r, s and the other for l, we approximate $\mathcal{L}_q$ by a real square matrix of order $(N_c+1)^2$. Taking the symmetry $\mathcal{L}_q(m,n;r,s) = \mathcal{L}_q(n,m;s,r)$ into consideration, the matrix order can be further reduced to $(N_c+1)(N_c+2)/2$. We then calculate its leading eigenvalue $\lambda_0(q; N_c, L_c, k_0)$ by the power method and expect it to converge quickly to $\lambda_0(q)$ when $N_c, L_c \to \infty$ for a given k_0. The calculation is implemented in a Fortran program at extended (128-bit) precision.

The convergence of power iteration is demonstrated in Figure 2, where

$$\varepsilon_n = \left| \frac{\langle 0|\mathcal{L}_q^n|0\rangle}{\langle 0|\mathcal{L}_q^{n-1}|0\rangle} - \frac{\langle 0|\mathcal{L}_q^{n+1}|0\rangle}{\langle 0|\mathcal{L}_q^n|0\rangle} \right|$$

are plotted at $q = 1.25$ and 1.35 (the initial vector $|0\rangle$ is set to $\Phi_{0,0}$). The exponential convergence rate indicates that there is a finite gap between the leading and next-to-leading eigenvalues of $\mathcal{L}_q$.

The test results of the convergences of $\lambda_0(q; N_c, L_c, k_0)$ with respect to cutoff parameters are summarized in Tables 2-4, from which we can reasonably assume that, when $q \sim 1.3$ and the representation I is adopted, the difference between $\lambda_0(q; N_c, L_c, 100)$ and $\lambda_0(q)$ can be less than 10^{-32} if the cutoff parameters are set to $(N_c, L_c) = (50, 16)$. Finally, starting from $q_1 = 1.30$ and $q_2 = 1.31$, we numerically solve the equation $\lambda_0(q; 50, 16, 100) = 1$ by the secant method. The result is listed in Table 5, from which we conclude that

$$d_H = 1.305\,686\,728\,049\,877\,184\,645\,986\,206\,851\,0\ldots.$$

TABLE 2. The convergence of $\lambda_0(1.25; N_c, 20, 100)$ in two representations.

N_c	Representation I	Representation II
20	1.10715659811323221714726154986444	1.10715659811335690873966838429508
25	1.10715659811323194939540134972834	1.10715659811323179105990871489484
30	1.10715659811323194877161509543907	1.10715659811323194835657900327526
35	1.10715659811323194877194177602755	1.10715659811323194877409799217705
40	1.10715659811323194877194236791574	1.10715659811323194877194064276988
45	1.10715659811323194877194236749029	1.10715659811323194877194235147117
50	1.10715659811323194877194236748952	1.10715659811323194877194236754802
55	1.10715659811323194877194236748952	1.10715659811323194877194236748955
60	1.10715659811323194877194236748952	1.10715659811323194877194236748952

TABLE 3. The convergence of $\lambda_0(1.35; N_c, 20, 100)$ in two representations.

N_c	Representation I	Representation II
20	0.92489050890072982281591329193136	0.92489050890079857461387627030495
25	0.92489050890072959502796402628310	0.92489050890072950231364349085568
30	0.92489050890072959471224591193980	0.92489050890072959452111359936473
35	0.92489050890072959471247908140078	0.92489050890072959471358059328314
40	0.92489050890072959471247933421092	0.92489050890072959471247816276157
45	0.92489050890072959471247933393508	0.92489050890072959471247932711270
50	0.92489050890072959471247933393481	0.92489050890072959471247933396356
55	0.92489050890072959471247933393481	0.92489050890072959471247933393480
60	0.92489050890072959471247933393481	0.92489050890072959471247933393481

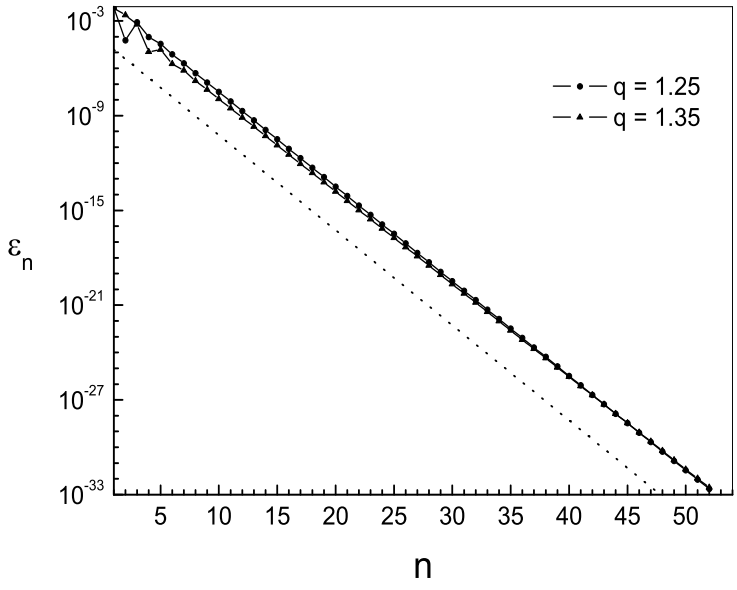

FIGURE 2. The convergence of power iteration. The result is calculated in representation I with $(N_c, L_c, k_0) = (50, 16, 100)$. For reference, also plotted is a power decay proportional to 4^{-n} (dotted line).

4. Open Questions

In this paper we have applied a transfer operator method to calculate the Hausdorff dimension of the Apollonian gasket. The method works well in computation. Nevertheless, it also gives rise to some theoretical questions. For example, the function space that the transfer operator acts on has not been explicitly addressed. We have not specified the domain of $\Phi_{m,n}(z)$. The span of $\{\Phi_{m,n}(z)|m,n \in \mathbb{N}\}$ is dense in the Hilbert space of square-integrable functions on the unit disk. However, since $\{\Phi_{m,n}(z)|m,n \in \mathbb{N}\}$ is an overcomplete basis of this space, it seems that more (analytic or algebraic) structures should be added to the desired function space. Another related question concerns the properties of the transfer operators $\mathcal{L}_q$ and $\mathcal{L}'_q$. Given the space they act on, can their

TABLE 4. The convergence of $\lambda_0(q; 60, L_c, 100)$ with respect to L_c. As $N_c = 60$ is sufficiently large, the listed data do not depend on the choice of representation I or II.

L_c	$\lambda_0(1.25; 60, L_c, 100)$	$\lambda_0(1.35; 60, L_c, 100)$
0	1.10718795145785699244797581365823	0.92490429253100699442137347774716
2	1.10715660257550600489781821805272	0.92489051124704233841106198831713
4	1.10715659811376944785573025619042	0.92489050890104147555960395033494
6	1.10715659811323201333009896832257	0.92489050890072963429770935322239
8	1.10715659811323194878642241501579	0.92489050890072959472115079713602
10	1.10715659811323194877194897146940	0.92489050890072959471248309518991
12	1.10715659811323194877194237036301	0.92489050890072959471247933552761
14	1.10715659811323194877194236749047	0.92489050890072959471247933393533
16	1.10715659811323194877194236748952	0.92489050890072959471247933393481

full spectra be determined and used to obtain a deeper understanding of the Apollonian gasket? Finally, higher dimensional generalization of the method to various sphere packings is also worth studying.[26–28]

TABLE 5. Convergence of d_H calculated by the secant method.

iteration	q	$\lambda_0(q; 50, 16, 100)$
1	1.30000000000000000000000000000000	1.01025172690474995538446832172419
2	1.31000000000000000000000000000000	0.99232164756874177600814135695325
3	1.30571761380004709128799140886178	0.99994472156496530174188253930065
4	1.30568678379980069804409034733407	0.99999990021650256123471447441947
5	1.30568672814834731134627807640386	0.99999999982375412009960628397841
6	1.30568672805005110384068886864491	0.99999999999968871227718607186508
7	1.30568672804987749182426639645206	0.99999999999999945019968898995182
8	1.30568672804987718518852836828354	0.99999999999999999902893574080975
9	1.30568672804987718464694445167470	0.99999999999999999999828489403044
10	1.30568672804987718464598789931516	0.99999999999999999999999697075816
11	1.30568672804987718464598620984029	0.99999999999999999999999999464971
12	1.30568672804987718464598620685632	0.99999999999999999999999999999055
13	1.30568672804987718464598620685105	0.99999999999999999999999999999998
14	1.30568672804987718464598620685104	1.00000000000000000000000000000000

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