1 Preliminaries

We consider a subset of interval graphs with a particular structure. In order to best model the setting of a department schedule, we consider a graph where a node in the graph corresponds to a class, and the corresponding interval(s) in the respresentation correspond to the meeting time(s) of the class, and we assume two properties of these interval graphs:

1. the interval representation is split into separate "days," where any class has no more than one interval per day, and there is a period between each day with no classes, making this a t-union graph;

2. if two classes meet more than once, they overlap in one instance if and only if they overlap in every possible instance.

Whenever we refer to interval graphs in this paper, we assume these properties.

We define an interval clique to be the clique generated by a vertical line drawn through the right endpoint of any interval. We define depth to be the maximum number of intervals that overlap at any time in the interval representation.

We can safely assume that no intervals share endpoints. If a suitable graph does not have this property, then we can shift any identical endpoints by multiples of some arbitrarily small ε.

2 Clique and Depth

We attempt to establish a relationship between the depth and clique size in this particular subset of interval graphs. We start with a lemma.

Lemma 1. *In such a graph, any clique must be an interval clique in the interval representation.*

Proof. Let G be a suitable graph with an interval representation $\mathcal{I}$. By definition, any interval clique in $\mathcal{I}$ must define a clique in G. Consider a day in which every class in $\mathcal{C}$ appears, which must exist because if all classes did not appear together on any day, by the first property, a clique could not exist. Take one class meeting in the clique, which we will call m_1, corresponding to class c_1 in G. Trivially, the clique just containing c_1 can be represented by the interval clique generated by m_1. Then, if c_2 (represented by m_2 on the day in question) is also in the clique, m_1 and m_2 must overlap. We pick the interval clique generated by either m_1 or m_2 that represents this clique of size 2.

Now, let $\ell < k$, and consider a clique of size ℓ that is a subgraph of $\mathcal{C}$. On the day we are considering, all ℓ of these classes meet, and they must meet only once by the first property. By the second property, if these classes meet on the same day and have any overlap at all, which they must as they form a clique, these classes must all overlap. Because we restrict $\mathcal{I}$ to one particular day, we are looking at a 1-interval representation of a clique. The only way to represent this clique is to stack the intervals on top of one another, meaning that there must be one of these ℓ intervals with a an interval clique that crosses all intervals. Therefore, this sub-clique can be represented as an interval clique, and say that meeting m_i defines this clique.

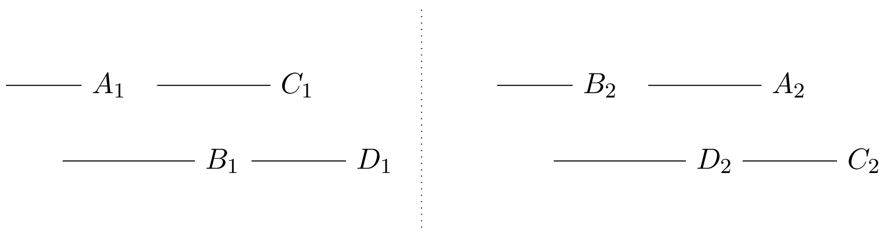
Next, take one more member of this clique, which we will call $c_{\ell+1}$. Since this class has an edge with all classes $c_1, \ldots, c_\ell$, and $m_{\ell+1}$ meets on this day with all other classes, it must overlap all of $m_1, \ldots, m_{\ell+1}$. By the same logic as before, either the interval clique generated by m_i or $m_{\ell+1}$ must generate the clique. We can extend this logic until we find the interval clique that generates the entire clique $\mathcal{C}$.

□

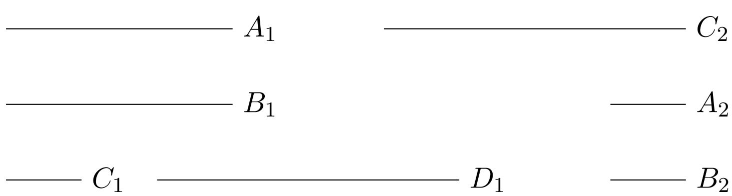
Theorem 2. *These conditions are both sufficient and necessary for the depth and clique size of a graph to be equal.*

Proof. We start by showing that these conditions are sufficient to conclude that the depth and clique size are equal. Let G be a graph with a suitable interval representation $\mathcal{I}$, where ω is the clique size and r is the depth. Trivially, $\omega \geq r$ in any interval graph. We will show that $\omega \not> r$, or in other words, $\omega \not\geq r+1$. By Lemma 1, any clique in G must be represented in $\mathcal{I}$ as an interval clique. If the size of any interval clique cannot reach $r+1$, as it is bounded by the size of the depth, no clique in G can achieve size $r+1$. Thus, $\omega = r$.

We will now prove that these conditions are necessary. At the moment, this must be done by example. First, assume that we have the first property but not the second property. Because we have no constraints on the structures of the representations of the individual days, we can reorder the classes freely to create paths, with depth 2, whose union generates a clique. The following is an example where $t = 2$, $r = 2$, and $\omega = 4$.



Now, assume that we have the second property but not the first property. By using an "overnight" class, we can allow one instance of a class to follow the structural assumption without increasing the depth to the size of the clique. The following is an example where $t = 2$, $r = 3$, and $\omega = 4$.

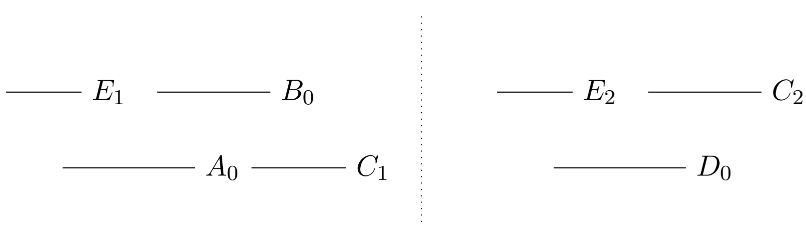


□

3 Coloring Approximations

Theorem 3. *These graphs are not perfect.*

Proof. We proceed with an example. A characteristic of perfect graphs is that they do not have odd holes, meaning that there is no chordless induced odd cycle. Even with the restrictions set by the properties, it is possible to create a 5-cycle



Since this is an odd cycle, it requires 3 colors for a clique size of 2. □

Here, we use the assumed properties of the interval graphs to improve on the upper bound on the number of colors needed to color a suiteable interval graph G. The upper bound we start with is given in [3], which is $2t(\omega - 1)$.

Lemma 4. *For a t-interval graph with clique number ω, $\chi(G) \leq t\omega$.*

Proof. Restrict the interval representation $\mathcal{I}$ of G to only the classes with one meeting, and call it $\mathcal{I}_1$ with clique number ω_1. As $\mathcal{I}_1$ is a 1-interval graph, the result from [4] guarantees that there is a valid ω_1 coloring.

Then, move on to the classes with two meetings. The induced subgraph is G_2, the interval representation is $\mathcal{I}_2$, and the clique size is ω_2. We will first show that there is a 1-interval representation $\mathcal{I}_2'$ that produces the same graph as $\mathcal{I}_2$. This interval representation is exactly the one created by limiting $\mathcal{I}_2$ to the first meetings of each class for $\mathcal{I}_2'$. G_2 has an edge between classes c_i and c_j if and only if, by the first property, there is some k with classes $c_{i,k}$ and $c_{j,k}$ overlap. By the second property, these two classes overlap if and only if all meetings of these classes overlap, as they both meet twice. Therefore, this is equivalent to only classes $c_{i,1}$ and $c_{j,1}$ overlap. Therefore, we can represent G_2 with the 1-interval graph $\mathcal{I}_2'$, and we can color it with ω_2 new colors.

We can continue this with the classes of up to t meetings, meaning that we use at most $\omega_1 + \omega_2 + \ldots + \omega_t$ colors with this procedure. Since each $\mathcal{I}_i$ is part of $\mathcal{I}$, each of these ω_i are bounded above by ω. This means that $\chi(G) \leq t\omega$. □

3.1 Bounding the Independent Neighbors

A key property of standard 1-interval graphs shown in [2] is that every interval graph has a simplicial vertex, meaning that the neighbors of this vertex form a clique. This implies that every 1-interval graph has a vertex with zero independent neighbors. With a greedy coloring scheme, we can pick out this simplicial vertex v, color it, and then move on to the next simplicial vertex in $G \setminus \{v\}$.

In contrast,

4 Algorithms

This section provides a summary of the currently implemented approximation algorithms as well as some ideas for future algorithms. Consider graphs that have the two properties defined in Section 1. We define a procedure for removing intervals correlating to independent from the interval representation in order to decrease the depth by one. Define $c_{i,j}$ to be meeting j of class c_i.

4.1 Depth Reduction

```
reduceDepth(I, k):
  set deleted = false, valid = true for all intervals in I
  order interval cliques left to right
  if size of I_i,j == k:
    if c_i,j has valid == true
      set c_i,j and any other c_i meeting as deleted = true
      mark any intersecting interval with any c_i meeting as valid = false
  else if there is c_x,y with valid = true in I_i,j:
    pick valid c_x,y
    set c_x,y and any other c_x meeting as deleted = true
    mark any intersecting interval with any c_x meeting as valid = false
  else:
    skip I_i,j
```

We can perform this procedure ω times to achieve a d-coloring of the graph. By Theorem 3, this algorithm cannot perform optimally in general.

4.2 Left-to-right Coloring

This procedure gives a valid ω coloring of the interval graph. Valid colors are $1, \dots, \omega$.

```
lrColoring():
  scan interval representation left to right
  if class c_x,y does not have a color and a color is available:
    pick the lowest available color
    color all other meetings of c_x the same
  else if no color is available:
    give all meetings of c_x color 0
```

4.3 Ratio Coloring

This is a possible (not yet implemented) approach to graph coloring. Rank the classes based on the ratio of the number of size d interval cliques to the number of class meetings. This will result in an algorithm that favors infrequent classes that cause many conflicts.

4.4 Independent Set

We implemented the algorithm from [1] to test the performance of the linear program on these types of interval graphs.

References

[1] R. Bar-Yehuda et al. "Scheduling Split Intervals". In: *SIAM Journal on Computing* 36.1 (2006), pp. 1–15. DOI: 10.1137/S0097539703437843. eprint: https://doi.org/10.1137/S0097539703437843. URL: https://doi.org/10.1137/S0097539703437843.

[2] Derek Corneil, Stephan Olariu, and Lorna Stewart. "The LBFS Structure and Recognition of Interval Graphs". In: *SIAM J. Discrete Math.* 23 (Jan. 2009), pp. 1905–1953. DOI: 10.1137/S0895480100373455.

[3] A. Gyárfás. "On the chromatic number of multiple interval graphs and overlap graphs". In: *Discrete Mathematics* 55.2 (1985), pp. 161–166. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(85)90044-5. URL: https://www.sciencedirect.com/science/article/pii/0012365X85900445.

[4] Stephan Olariu. "An optimal greedy heuristic to color interval graphs". In: *Information Processing Letters* 37.1 (1991), pp. 21–25. ISSN: 0020-0190. DOI: https://doi.org/10.1016/0020-0190(91)90245-D. URL: https://www.sciencedirect.com/science/article/pii/002001909190245D.