Comp 250 Homework3

Question2

for $n>1$.

$$
\begin{aligned}
T(n) &= 2T(n-1)+n \\
&= 2(2T(n-2)+n-1)+n \\
&= 4T(n-2)+2n+n-2 \\
&= 4(2T(n-3)+n-2)+(2+1)n-2 \\
&= 8T(n-3)+(4+2+1)n-(4\times 2+2\times 1) \\
&= 2^k T(n-k)+n\sum_{i=0}^{k-1} 2^{i} - \sum_{i=1}^{k-1} i2^i \\
&= 2^k T(n-k)+n(2^k-1)-2(1+2^{k-1}(k-2))
\end{aligned}
$$

now $k=n-1$

$T(1)=1$

$$
\begin{aligned}
&= 2^{n-1}+n(2^{n-1}-1)-2-2^{n-1}(n-3) \\
&= 2^{n-1}(1+n-n+3)-n-2 \\
&= 4\cdot 2^{n-1}-n-2 \\
&= 2^{n+1}-n-2 .
\end{aligned}
$$

Question3:

Algorithm orderCards(n)

Input: An integer n

Output: Prints the correct card ordering.

//Pseudocode

//Use a double-linked list to implement the method.

```
for i ← n to 1 do{
    //add new node at top
```

```
    Node k ← new Node(i);
    k.next=head;
    head=k;

    //move
    Node k ← tail;
    tail ← k.prev;
    k.next ← head;
    head ← k;
}
Node k ← new Node(1);
k.next=head;
head=k;
```

Question 4

$\log(n!) = \sum_{i=1}^{n} \log i \le n\log n$ for $n \geq 1$.

$\therefore \log(n!) \in O(n\log n)$

"Proof for $\log(n!) \in \Omega(n\log n)$:

~~Induct~~ $\Downarrow$

There are constant $c > 0$ and n_0 such that $\log(n!) \ge cn\log n$ for every $n \ge n_0$.

Induction hypothesis: $\log((n-1)!) \ge c(n-1)\log(n-1)$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

for any $x > 0$, $e^x > 1 + x$.

$e^{\frac{1}{n-1}} > 1 + \frac{1}{n-1} = \frac{n}{n-1}$

$\frac{1}{n-1}\log e > \log\left(\frac{n}{n-1}\right)$

$= \log n - \log(n-1)$.

$\log(n-1) > \log n - \log e/(n-1)$

$\log(n!) = \log(n(n-1)!)$

$= \log n + \log((n-1)!)$

$\ge \log n + c(n-1)\log(n-1)$

$> \log n + c(n-1)\left(\log n - \frac{\log e}{n-1}\right)$

$= \log n + c(n-1)\log n - c\log e$

$= (1-c)\log n + cn\log n - c\log e$.

for $c = \frac{1}{2}$, $n \ge 3$. $\log n > \log e$.

$(1-c)\log n - c\log e > 0$.

$\therefore \log(n!) \ge cn\log n$.

Base case: $n = 2$.

$\log(2!) = 1 \ge \frac{1}{2\cdot 2}\log 2$

$\therefore$ for every $n \ge n_0 = 2$, $\log(n!) \in \Omega(n\log n)$.

$\because \log(n!) \in O(n\log n)$ & $\log(n!) \in \Omega(n\log n)$.

$\therefore \log(n!) \in \Theta(n\log n)$.