

- f. Let $\alpha \neq \lambda$ be given. Show that if $A - \alpha I$ is nonsingular, then $1/(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)^{-1}$ with eigenvector \mathbf{x} .
18. Show that if A is symmetric, then $\|A\|_2 = \rho(A)$.
19. Find matrices A and B for which $\rho(A + B) > \rho(A) + \rho(B)$. (This shows that $\rho(A)$ cannot be a matrix norm.)
20. Show that if $\|\cdot\|$ is any natural norm, then $(\|A^{-1}\|)^{-1} \leq |\lambda| \leq \|A\|$ for any eigenvalue λ of the nonsingular matrix A .

DISCUSSION QUESTIONS

1. Find an application where an eigenvalue of 1 has an important meaning.
2. Discuss the geometrical significance of the spectral radius relative to the eigenvalues of a matrix A .
3. Under what circumstances is the spectral radius of a matrix also an eigenvalue of the matrix?



7.3 The Jacobi and Gauss-Siedel Iterative Techniques

In this section, we describe the Jacobi and the Gauss-Seidel iterative methods, classic methods that date to the late 18th century. Iterative techniques are seldom used for solving linear systems of small dimension since the time required for sufficient accuracy exceeds that required for direct techniques, such as Gaussian elimination. For large systems with a high percentage of 0 entries, however, these techniques are efficient in terms of both computer storage and computation. Systems of this type arise frequently in circuit analysis and in the numerical solution of boundary-value problems and partial-differential equations.

An iterative technique to solve the $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ starts with an initial approximation $\mathbf{x}^{(0)}$ to the solution \mathbf{x} and generates a sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to \mathbf{x} .

Jacobi's Method

The **Jacobi iterative method** is obtained by solving the i th equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^n \left(-\frac{a_{ij}x_j}{a_{ii}} \right) + \frac{b_i}{a_{ii}}, \quad \text{for } i = 1, 2, \dots, n.$$

For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^n (-a_{ij}x_j^{(k-1)}) + b_i \right], \quad \text{for } i = 1, 2, \dots, n. \quad (7.5)$$

Example 1 The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} E_1 : \quad 10x_1 - x_2 + 2x_3 &= 6, \\ E_2 : \quad -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ E_3 : \quad 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ E_4 : \quad 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

Carl Gustav Jacob Jacobi (1804–1851) was initially recognized for his work in the area of number theory and elliptic functions, but his mathematical interests and abilities were very broad. He had a strong personality that was influential in establishing a research-oriented attitude that became the nucleus of a revival of mathematics at German universities in the 19th century.

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to \mathbf{x} starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution We first solve equation E_i for x_i , for each $i = 1, 2, 3, 4$, to obtain

$$\begin{aligned} x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}, \\ x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}, \\ x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}, \\ x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}. \end{aligned}$$

From the initial approximation $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ we have $\mathbf{x}^{(1)}$ given by

$$\begin{aligned} x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} = 0.6000, \\ x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} = 2.2727, \\ x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} = -1.1000, \\ x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} = 1.8750. \end{aligned}$$

Additional iterates, $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 7.1.

Table 7.1

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	-1.1000	-0.8052	-1.0493	-0.9681	-1.0103	-0.9945	-1.0020	-0.9990	-1.0004	-0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

We stopped after 10 iterations because

$$\frac{\|\mathbf{x}^{(10)} - \mathbf{x}^{(9)}\|_\infty}{\|\mathbf{x}^{(10)}\|_\infty} = \frac{8.0 \times 10^{-4}}{1.9998} < 10^{-3}.$$

In fact, $\|\mathbf{x}^{(10)} - \mathbf{x}\|_\infty = 0.0002$. ■

In general, iterative techniques for solving linear systems involve a process that converts the system $A\mathbf{x} = \mathbf{b}$ into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ for some fixed matrix T and vector \mathbf{c} . After the initial vector $\mathbf{x}^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c},$$

for each $k = 1, 2, 3, \dots$. This should be reminiscent of the fixed-point iteration studied in Chapter 2.

The Jacobi method can be written in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ by splitting A into its diagonal and off-diagonal parts. To see this, let D be the diagonal matrix whose diagonal entries are those of A , $-L$ be the strictly lower-triangular part of A , and $-U$ be the strictly upper-triangular part of A . With this notation,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is split into

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ -a_{21} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \\ &= D - L - U. \end{aligned}$$

The equation $A\mathbf{x} = \mathbf{b}$, or $(D - L - U)\mathbf{x} = \mathbf{b}$, is then transformed into

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b},$$

and, if D^{-1} exists, that is, if $a_{ii} \neq 0$ for each i , then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This results in the matrix form of the Jacobi iterative technique:

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, \dots. \quad (7.6)$$

Introducing the notation $T_j = D^{-1}(L + U)$ and $\mathbf{c}_j = D^{-1}\mathbf{b}$ gives the Jacobi technique the form

$$\mathbf{x}^{(k)} = T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j. \quad (7.7)$$

In practice, Eq. (7.5) is used in computation and Eq. (7.7) for theoretical purposes.

Example 2 Express the Jacobi iteration method for the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} E_1 : \quad 10x_1 - x_2 + 2x_3 &= 6, \\ E_2 : \quad -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ E_3 : \quad 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ E_4 : \quad \quad \quad 3x_2 - x_3 + 8x_4 &= 15, \end{aligned}$$

in the form $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Solution We saw in Example 1 that the Jacobi method for this system has the form

$$\begin{aligned} x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5}, \\ x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11}, \\ x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10}, \\ x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8}. \end{aligned}$$

Hence, we have

$$T = \begin{bmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} \frac{3}{5} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{bmatrix}. \quad \blacksquare$$

Algorithm 7.1 implements the Jacobi iterative technique.



Jacobi Iterative Technique

To solve $Ax = b$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n ; the entries a_{ij} , $1 \leq i, j \leq n$ of the matrix A ; the entries b_i , $1 \leq i \leq n$ of \mathbf{b} ; the entries XO_i , $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL ; maximum number of iterations N .

OUTPUT the approximate solution x_1, \dots, x_n or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \leq N$) do Steps 3–6.

Step 3 For $i = 1, \dots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[- \sum_{j=1, j \neq i}^n (a_{ij} XO_j) + b_i \right].$$

Step 4 If $\|\mathbf{x} - \mathbf{XO}\| < TOL$ then OUTPUT (x_1, \dots, x_n);

(The procedure was successful.)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \dots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');

(The procedure was not successful.)

STOP. ■

Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \dots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$. To speed convergence, the equations should be arranged so that a_{ii} is as large as possible. This subject is discussed in more detail later in this chapter.

Another possible stopping criterion in Step 4 is to iterate until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$$

Phillip Ludwig Seidel (1821–1896) worked as an assistant to Jacobi solving problems on systems of linear equations that resulted from Gauss's work on least squares. These equations generally had off-diagonal elements that were much smaller than those on the diagonal, so the iterative methods were particularly effective. The iterative techniques now known as Jacobi and Gauss-Seidel were both known to Gauss before being applied in this situation, but Gauss's results were not often widely communicated.

is smaller than some prescribed tolerance. For this purpose, any convenient norm can be used, the usual being the l_∞ norm.

The Gauss-Seidel Method

A possible improvement in Algorithm 7.1 can be seen by reconsidering Eq. (7.5). The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$. But, for $i > 1$, the components $x_1^{(k)}, \dots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions x_1, \dots, x_{i-1} than are $x_1^{(k-1)}, \dots, x_{i-1}^{(k-1)}$. It seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values. That is, to use

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{j=1}^{i-1} (a_{ij}x_j^{(k)}) - \sum_{j=i+1}^n (a_{ij}x_j^{(k-1)}) + b_i \right], \quad (7.8)$$

for each $i = 1, 2, \dots, n$, instead of Eq. (7.5). This modification is called the **Gauss-Seidel iterative technique** and is illustrated in the following example.

Example 3 Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6, \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25, \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15, \end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Solution The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in Example 1. For the Gauss-Seidel method, we write the system, for each $k = 1, 2, \dots$ as

$$\begin{aligned} x_1^{(k)} &= \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5}, \\ x_2^{(k)} &= \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11}, \\ x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10}, \\ x_4^{(k)} &= -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8}. \end{aligned}$$

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in Table 7.2.

Table 7.2

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	-0.9873	-1.014	-1.0025	-1.0003	-1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4},$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution. Note that Jacobi's method in Example 1 required twice as many iterations for the same accuracy. ■

To write the Gauss-Seidel method in matrix form, multiply both sides of Eq. (7.8) by a_{ii} and collect all k th iterate terms to give

$$a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \cdots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} - \cdots - a_{in}x_n^{(k-1)} + b_i,$$

for each $i = 1, 2, \dots, n$. Writing all n equations gives

$$\begin{aligned} a_{11}x_1^{(k)} &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1, \\ a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2, \\ &\vdots \\ a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} &= b_n; \end{aligned}$$

with the definitions of D , L , and U given previously, we have the Gauss-Seidel method represented by

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

and

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \dots \quad (7.9)$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g. \quad (7.10)$$

For the lower-triangular matrix $D - L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \dots, n$.

Algorithm 7.2 implements the Gauss-Seidel method.



Gauss-Seidel Iterative Method

To solve $Ax = b$ given an initial approximation $x^{(0)}$:

INPUT the number of equations and unknowns n ; the entries a_{ij} , $1 \leq i, j \leq n$ of the matrix A ; the entries b_i , $1 \leq i \leq n$ of b ; the entries XO_i , $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$; tolerance TOL ; maximum number of iterations N .

OUTPUT the approximate solution x_1, \dots, x_n or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \leq N$) do Steps 3–6.

Step 3 For $i = 1, \dots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[- \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^n a_{ij} XO_j + b_i \right].$$

Step 4 If $\|x - \mathbf{XO}\| < TOL$ then **OUTPUT** (x_1, \dots, x_n) ;

(The procedure was successful.)
STOP.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \dots, n$ set $XO_i = x_i$.

Step 7 **OUTPUT** ('Maximum number of iterations exceeded');

(The procedure was not successful.)
STOP.

■

The comments following Algorithm 7.1 regarding reordering and stopping criteria also apply to the Gauss-Seidel Algorithm 7.2.

The results of Examples 1 and 2 appear to imply that the Gauss-Seidel method is superior to the Jacobi method. This is almost always true, but there are linear systems for which the Jacobi method converges and the Gauss-Seidel method does not (See Exercises 9 and 10).

General Iteration Methods

To study the convergence of general iteration techniques, we need to analyze the formula

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k = 1, 2, \dots,$$

where $\mathbf{x}^{(0)}$ is arbitrary. The next lemma and Theorem 7.17 on page 454 provide the key for this study.

Lemma 7.18 If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \dots = \sum_{j=0}^{\infty} T^j.$$

Proof Because $T\mathbf{x} = \lambda\mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of $I - T$. But $|\lambda| \leq \rho(T) < 1$, so $\lambda = 1$ is not an eigenvalue of T , and 0 cannot be an eigenvalue of $I - T$. Hence, $(I - T)^{-1}$ exists.

Let $S_m = I + T + T^2 + \cdots + T^m$. Then

$$(I - T)S_m = (1 + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1},$$

and, since T is convergent, Theorem 7.17 implies that

$$\lim_{m \rightarrow \infty} (I - T)S_m = \lim_{m \rightarrow \infty} (I - T^{m+1}) = I.$$

$$\text{Thus, } (I - T)^{-1} = \lim_{m \rightarrow \infty} S_m = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j. \quad \blacksquare$$

Theorem 7.19 For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k \geq 1, \quad (7.11)$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if $\rho(T) < 1$.

Proof First assume that $\rho(T) < 1$. Then,

$$\begin{aligned} \mathbf{x}^{(k)} &= T\mathbf{x}^{(k-1)} + \mathbf{c} \\ &= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\ &= T^2\mathbf{x}^{(k-2)} + (T + I)\mathbf{c} \\ &\vdots \\ &= T^k\mathbf{x}^{(0)} + (T^{k-1} + \cdots + T + I)\mathbf{c}. \end{aligned}$$

Because $\rho(T) < 1$, Theorem 7.17 implies that T is convergent, and

$$\lim_{k \rightarrow \infty} T^k\mathbf{x}^{(0)} = \mathbf{0}.$$

Lemma 7.18 implies that

$$\lim_{k \rightarrow \infty} \mathbf{x}^{(k)} = \lim_{k \rightarrow \infty} T^k\mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j \right) \mathbf{c} = \mathbf{0} + (I - T)^{-1}\mathbf{c} = (I - T)^{-1}\mathbf{c}.$$

Hence, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the vector $\mathbf{x} \equiv (I - T)^{-1}\mathbf{c}$ and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k \rightarrow \infty} T^k\mathbf{z} = \mathbf{0}$. By Theorem 7.17, this is equivalent to $\rho(T) < 1$.

Let \mathbf{z} be an arbitrary vector and \mathbf{x} be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$. Define $\mathbf{x}^{(0)} = \mathbf{x} - \mathbf{z}$, and, for $k \geq 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$. Then $\{\mathbf{x}^{(k)}\}$ converges to \mathbf{x} . Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{c}) = T(\mathbf{x} - \mathbf{x}^{(k-1)}),$$

so

$$\mathbf{x} - \mathbf{x}^{(k)} = T(\mathbf{x} - \mathbf{x}^{(k-1)}) = T^2(\mathbf{x} - \mathbf{x}^{(k-2)}) = \cdots = T^k(\mathbf{x} - \mathbf{x}^{(0)}) = T^k\mathbf{z}.$$

Hence, $\lim_{k \rightarrow \infty} T^k\mathbf{z} = \lim_{k \rightarrow \infty} T^k(\mathbf{x} - \mathbf{x}^{(0)}) = \lim_{k \rightarrow \infty} (\mathbf{x} - \mathbf{x}^{(k)}) = \mathbf{0}$.

But $\mathbf{z} \in \mathbb{R}^n$ was arbitrary, so by Theorem 7.17, T is convergent and $\rho(T) < 1$. \blacksquare

The proof of the following corollary is similar to the proofs in Corollary 2.5 on page 61. It is considered in Exercise 18.

Corollary 7.20 If $\|T\| < 1$ for any natural matrix norm and \mathbf{c} is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

$$(i) \quad \|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|; \quad (ii) \quad \|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|. \quad \blacksquare$$

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written as

$$\mathbf{x}^{(k)} = T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \quad \mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g$$

using the matrices

$$T_j = D^{-1}(L + U) \quad \text{and} \quad T_g = (D - L)^{-1}U.$$

If $\rho(T_j)$ or $\rho(T_g)$ is less than 1, then the corresponding sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ will converge to the solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$. For example, the Jacobi scheme has

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to \mathbf{x} , then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

This implies that

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b} \quad \text{and} \quad (D - L - U)\mathbf{x} = \mathbf{b}.$$

Since $D - L - U = A$, the solution \mathbf{x} satisfies $A\mathbf{x} = \mathbf{b}$.

We can now give easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods. (To prove convergence for the Jacobi scheme, see Exercise 17, and for the Gauss-Seidel scheme, see [Or2], p. 120.)

Theorem 7.21 If A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$. ■

The relationship of the rapidity of convergence to the spectral radius of the iteration matrix T can be seen from Corollary 7.20. The inequalities hold for any natural matrix norm, so it follows from the statement after Theorem 7.15 on page 452 that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \approx \rho(T)^k \|\mathbf{x}^{(0)} - \mathbf{x}\|. \quad (7.12)$$

Thus, we would like to select the iterative technique with minimal $\rho(T) < 1$ for a particular system $A\mathbf{x} = \mathbf{b}$. No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system. In special cases, however, the answer is known, as is demonstrated in the following theorem. The proof of this result can be found in [Y], pp. 120–127.

Theorem 7.22 (Stein-Rosenberg)

If $a_{ij} \leq 0$, for each $i \neq j$, and $a_{ii} > 0$, for each $i = 1, 2, \dots, n$, then one and only one of the following statements holds:

- (i) $0 \leq \rho(T_g) < \rho(T_j) < 1$; (ii) $1 < \rho(T_j) < \rho(T_g)$;
 (iii) $\rho(T_j) = \rho(T_g) = 0$; (iv) $\rho(T_j) = \rho(T_g) = 1$. ■

For the special case described in Theorem 7.22, we see from part (i) that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method. Part (ii) indicates that when one method diverges, then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $x^{(0)} = \mathbf{0}$:
 - a. $3x_1 - x_2 + x_3 = 1$, $10x_1 - x_2 = 9$,
 - b. $3x_1 + 6x_2 + 2x_3 = 0$, $-x_1 + 10x_2 - 2x_3 = 7$,
 $3x_1 + 3x_2 + 7x_3 = 4$. $-2x_2 + 10x_3 = 6$.
 - c. $10x_1 + 5x_2 = 6$, $4x_1 + x_2 + x_3 + x_5 = 6$,
 - d. $5x_1 + 10x_2 - 4x_3 = 25$, $-x_1 - 3x_2 + x_3 + x_4 = 6$,
 - $-4x_2 + 8x_3 - x_4 = -11$, $2x_1 + x_2 + 5x_3 - x_4 - x_5 = 6$,
 - $-x_3 + 5x_4 = -11$, $-x_1 - x_2 - x_3 + 4x_4 = 6$,
 - $2x_2 - x_3 + x_4 + 4x_5 = 6$.
2. Find the first two iterations of the Jacobi method for the following linear systems, using $x^{(0)} = \mathbf{0}$:
 - a. $4x_1 + x_2 - x_3 = 5$, $-2x_1 + x_2 + \frac{1}{2}x_3 = 4$,
 - b. $-x_1 + 3x_2 + x_3 = -4$, $x_1 - 2x_2 - \frac{1}{2}x_3 = -4$,
 $2x_1 + 2x_2 + 5x_3 = 1$. $x_2 + 2x_3 = 0$.
 - c. $4x_1 + x_2 - x_3 + x_4 = -2$, $4x_1 - x_2 = 0$,
 - d. $x_1 + 4x_2 - x_3 - x_4 = -1$, $-x_1 + 4x_2 - x_3 = 5$,
 - $-x_1 - x_2 + 5x_3 + x_4 = 0$, $-x_2 + 4x_3 = 0$,
 - $x_1 - x_2 + x_3 + 3x_4 = 1$. $+4x_4 - x_5 = 6$,
 - $-x_4 + 4x_5 - x_6 = -2$,
 - $-x_5 + 4x_6 = 6$.
3. Repeat Exercise 1 using the Gauss-Seidel method.
4. Repeat Exercise 2 using the Gauss-Seidel method.
5. Use the Jacobi method to solve the linear systems in Exercise 1, with $TOL = 10^{-3}$ in the l_∞ norm.
6. Use the Jacobi method to solve the linear systems in Exercise 2, with $TOL = 10^{-3}$ in the l_∞ norm.
7. Use the Gauss-Seidel method to solve the linear systems in Exercise 1, with $TOL = 10^{-3}$ in the l_∞ norm.
8. Use the Gauss-Seidel method to solve the linear systems in Exercise 2, with $TOL = 10^{-3}$ in the l_∞ norm.
9. The linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 2x_1 + 2x_2 + 2x_3 &= 4, \\ -x_1 - x_2 + 2x_3 &= -5, \end{aligned}$$

has the solution $(1, 2, -1)^t$.

- a. Show that $\rho(T_j) = \frac{\sqrt{5}}{2} > 1$.
- b. Show that the Jacobi method with $x^{(0)} = \mathbf{0}$ fails to give a good approximation after 25 iterations.

- c. Show that $\rho(T_g) = \frac{1}{2}$.
- d. Use the Gauss-Seidel method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the l_∞ norm.

10. The linear system

$$x_1 + 2x_2 - 2x_3 = 7,$$

$$x_1 + x_2 + x_3 = 2,$$

$$2x_1 + 2x_2 + x_3 = 5$$

has the solution $(1, 2, -1)^t$.

- a. Show that $\rho(T_j) = 0$.
- b. Use the Jacobi method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the l_∞ norm.
- c. Show that $\rho(T_g) = 2$.
- d. Show that the Gauss-Seidel method applied as in part (b) fails to give a good approximation in 25 iterations.

11. The linear system

$$\begin{array}{rcl} x_1 & - & x_3 = 0.2, \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 = -1.425, \\ x_1 - \frac{1}{2}x_2 + x_3 = 2, \end{array}$$

has the solution $(0.9, -0.8, 0.7)^t$.

- a. Is the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{1}{2} & 1 & -\frac{1}{4} \\ 1 & -\frac{1}{2} & 1 \end{bmatrix}$$

strictly diagonally dominant?

- b. Compute the spectral radius of the Gauss-Seidel matrix T_g .
- c. Use the Gauss-Seidel iterative method to approximate the solution to the linear system with a tolerance of 10^{-2} and a maximum of 300 iterations.
- d. What happens in part (c) when the system is changed to the following?

$$\begin{array}{rcl} x_1 & - & 2x_3 = 0.2, \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 = -1.425, \\ x_1 - \frac{1}{2}x_2 + x_3 = 2. \end{array}$$

12. Repeat Exercise 11 using the Jacobi method.

13. Use (a) the Jacobi and (b) the Gauss-Seidel methods to solve the linear system $A\mathbf{x} = \mathbf{b}$ to within 10^{-5} in the l_∞ norm, where the entries of A are

$$a_{i,j} = \begin{cases} 2i, & \text{when } j = i \text{ and } i = 1, 2, \dots, 80, \\ 0.5i, & \text{when } \begin{cases} j = i + 2 \text{ and } i = 1, 2, \dots, 78, \\ j = i - 2 \text{ and } i = 3, 4, \dots, 80, \end{cases} \\ 0.25i, & \text{when } \begin{cases} j = i + 4 \text{ and } i = 1, 2, \dots, 76, \\ j = i - 4 \text{ and } i = 5, 6, \dots, 80, \end{cases} \\ 0, & \text{otherwise,} \end{cases}$$

and those of \mathbf{b} are $b_i = \pi$, for each $i = 1, 2, \dots, 80$.

APPLIED EXERCISES

14. Suppose that an object can be at any one of $n + 1$ equally spaced points x_0, x_1, \dots, x_n . When an object is at location x_i , it is equally likely to move to either x_{i-1} or x_{i+1} and cannot directly move to any other location. Consider the probabilities $\{P_i\}_{i=0}^n$ that an object starting at location x_i will reach the left endpoint x_0 before reaching the right endpoint x_n . Clearly, $P_0 = 1$ and $P_n = 0$. Since the object can move to x_i only from x_{i-1} or x_{i+1} and does so with probability $\frac{1}{2}$ for each of these locations,

$$P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}, \quad \text{for each } i = 1, 2, \dots, n-1.$$

- a. Show that

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ 0 & -\frac{1}{2} & 1 & & & & \\ \vdots & & & \ddots & & & 0 \\ & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- b. Solve this system using $n = 10, 50$, and 100 .
c. Change the probabilities to α and $1 - \alpha$ for movement to the left and right, respectively, and derive the linear system similar to the one in part (a).
d. Repeat part (b) with $\alpha = \frac{1}{3}$.
15. The forces on the bridge truss described in the opening to this chapter satisfy the equations in the following table:

Joint	Horizontal Component	Vertical Component
①	$-F_1 + \frac{\sqrt{2}}{2}f_1 + f_2 = 0$	$\frac{\sqrt{2}}{2}f_1 - F_2 = 0$
②	$-\frac{\sqrt{2}}{2}f_1 + \frac{\sqrt{3}}{2}f_4 = 0$	$-\frac{\sqrt{2}}{2}f_1 - f_3 - \frac{1}{2}f_4 = 0$
③	$-f_2 + f_5 = 0$	$f_3 - 10,000 = 0$
④	$-\frac{\sqrt{3}}{2}f_4 - f_5 = 0$	$\frac{1}{2}f_4 - F_3 = 0$

This linear system can be placed in the matrix form

$$\begin{bmatrix} -1 & 0 & 0 & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10,000 \\ 0 \\ 0 \end{bmatrix}.$$

- a. Explain why the system of equations was reordered.
b. Approximate the solution of the resulting linear system to within 10^{-2} in the l_∞ norm using as initial approximation the vector all of whose entries are 1s with (i) the Jacobi method and (ii) the Gauss-Seidel method.
16. A coaxial cable is made up of a 0.1-inch-square inner conductor and 0.5-inch-square outer conductor. The potential at a point in the cross section of the cable is described by Laplace's equation.

Suppose the inner conductor is kept at 0 volts and the outer conductor is kept at 110 volts. Approximating the potential between the two conductors requires solving the following linear system. (See Exercise 5 of Section 12.1.)

$$\begin{bmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \\ w_{12} \end{bmatrix} = \begin{bmatrix} 220 \\ 110 \\ 110 \\ 220 \\ 110 \\ 110 \\ 110 \\ 110 \\ 220 \\ 110 \\ 110 \\ 220 \end{bmatrix}.$$

- a. Is the matrix strictly diagonally dominant?
- b. Solve the linear system using the Jacobi method with $\mathbf{x}^{(0)} = \mathbf{0}$ and $TOL = 10^{-2}$.
- c. Repeat part (b) using the Gauss-Seidel method.

THEORETICAL EXERCISES

17. Show that if A is strictly diagonally dominant, then $\|T_j\|_\infty < 1$.
18. a. Prove that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\| \quad \text{and} \quad \|\mathbf{x}^{(k)} - \mathbf{x}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|,$$

where T is an $n \times n$ matrix with $\|T\| < 1$ and

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad k = 1, 2, \dots,$$

with $\mathbf{x}^{(0)}$ arbitrary, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

- b. Apply the bounds to Exercise 1, when possible, using the l_∞ norm.
19. Suppose that A is a positive definite.
- a. Show that we can write $A = D - L - L^t$, where D is diagonal with $d_{ii} > 0$ for each $1 \leq i \leq n$ and L is lower triangular. Further, show that $D - L$ is nonsingular.
 - b. Let $T_g = (D - L)^{-1}L^t$ and $P = A - T_g^t AT_g$. Show that P is symmetric.
 - c. Show that T_g can also be written as $T_g = I - (D - L)^{-1}A$.
 - d. Let $Q = (D - L)^{-1}A$. Show that $T_g = I - Q$ and $P = Q^t[AQ^{-1} - A + (Q^t)^{-1}A]Q$.
 - e. Show that $P = Q^t D Q$ and P is positive definite.
 - f. Let λ be an eigenvalue of T_g with eigenvector $\mathbf{x} \neq \mathbf{0}$. Use part (b) to show that $\mathbf{x}^t P \mathbf{x} > 0$ implies that $|\lambda| < 1$.
 - g. Show that T_g is convergent and prove that the Gauss-Seidel method converges.

DISCUSSION QUESTIONS

1. The GMRES method is an iterative method used for solutions of large sparse nonsymmetric linear systems. Compare that method to the iterative methods discussed in this section.
2. Are direct methods such as Gaussian elimination or LU factorization more efficient than indirect methods such as Gauss-Seidel or Jacobi when the size of the system increases significantly?