

## 2.2 Fixed-Point Iteration

A *fixed point* for a function is a number at which the value of the function does not change when the function is applied.

**Definition 2.2** The number  $p$  is a **fixed point** for a given function  $g$  if  $g(p) = p$ . ■

Fixed-point results occur in many areas of mathematics and are a major tool of economists for proving results concerning equilibria. Although the idea behind the technique is old, the terminology was first used by the Dutch mathematician L. E. J. Brouwer (1882–1962) in the early 1900s.

In this section, we consider the problem of finding solutions to fixed-point problems and the connection between the fixed-point problems and the root-finding problems we wish to solve. Root-finding problems and fixed-point problems are equivalent classes in the following sense:

- Given a root-finding problem  $f(p) = 0$ , we can define functions  $g$  with a fixed point at  $p$  in a number of ways, for example, as

$$g(x) = x - f(x) \quad \text{or as} \quad g(x) = x + 3f(x).$$

- Conversely, if the function  $g$  has a fixed point at  $p$ , then the function defined by

$$f(x) = x - g(x)$$

has a zero at  $p$ .

Although the problems we wish to solve are in the root-finding form, the fixed-point form is easier to analyze, and certain fixed-point choices lead to very powerful root-finding techniques.

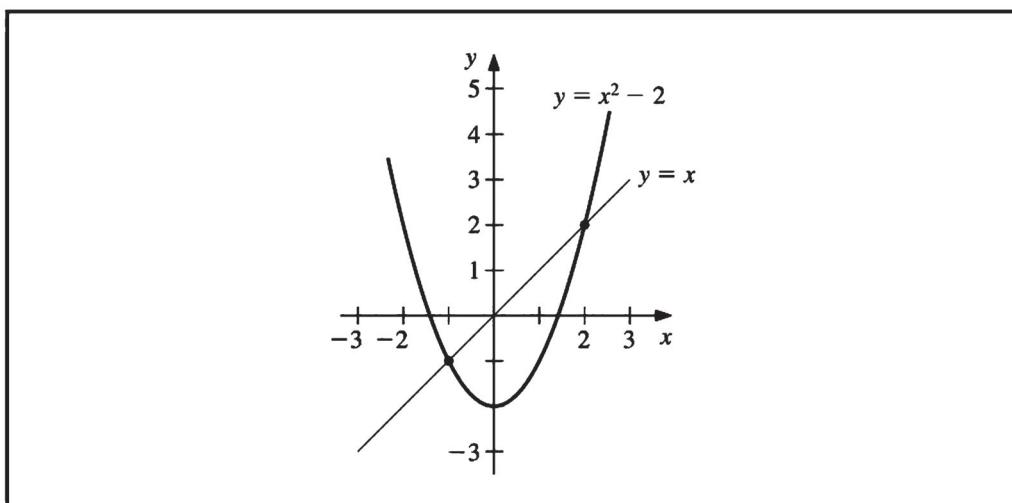
We first need to become comfortable with this new type of problem and to decide when a function has a fixed point and how the fixed points can be approximated to within a specified accuracy.

**Example 1** Determine any fixed points of the function  $g(x) = x^2 - 2$ .

**Solution** A fixed point  $p$  for  $g$  has the property that

$$p = g(p) = p^2 - 2, \quad \text{which implies that } 0 = p^2 - p - 2 = (p + 1)(p - 2).$$

A fixed point for  $g$  occurs precisely when the graph of  $y = g(x)$  intersects the graph of  $y = x$ , so  $g$  has two fixed points, one at  $p = -1$  and the other at  $p = 2$ . These are shown in Figure 2.2. ■

**Figure 2.2**

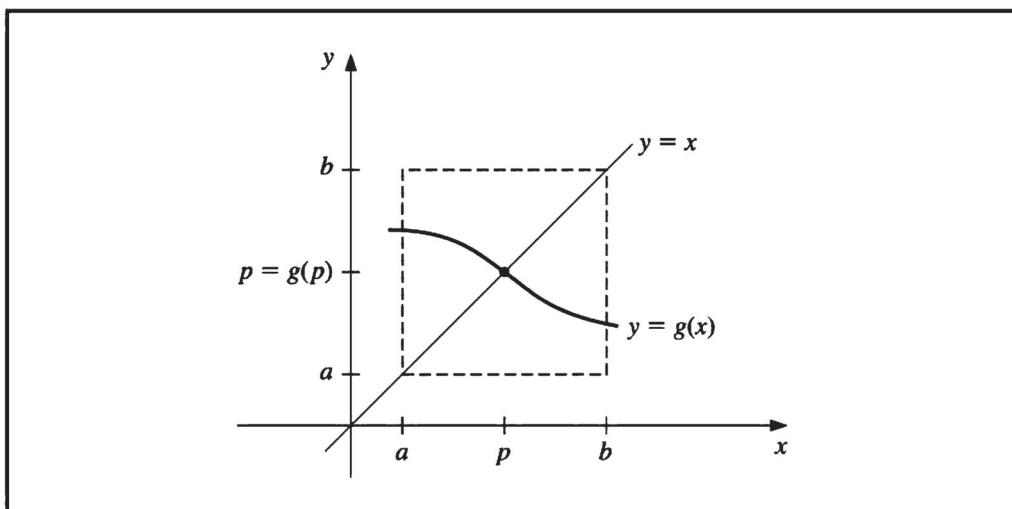
The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

**Theorem 2.3**

- (i) If  $g \in C[a, b]$  and  $g(x) \in [a, b]$  for all  $x \in [a, b]$ , then  $g$  has at least one fixed point in  $[a, b]$ .
- (ii) If, in addition,  $g'(x)$  exists on  $(a, b)$  and a positive constant  $k < 1$  exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b),$$

then there is exactly one fixed point in  $[a, b]$ . (See Figure 2.3.)

**Figure 2.3**

**Proof**

- (i) If  $g(a) = a$  or  $g(b) = b$ , then  $g$  has a fixed point at an endpoint. If not, then  $g(a) > a$  and  $g(b) < b$ . The function  $h(x) = g(x) - x$  is continuous on  $[a, b]$ , with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

The Intermediate Value Theorem implies that there exists  $p \in (a, b)$  for which  $h(p) = 0$ . This number  $p$  is a fixed point for  $g$  because

$$0 = h(p) = g(p) - p \quad \text{implies that} \quad g(p) = p.$$

- (ii) Suppose, in addition, that  $|g'(x)| \leq k < 1$  and that  $p$  and  $q$  are both fixed points in  $[a, b]$ . If  $p \neq q$ , then the Mean Value Theorem implies that a number  $\xi$  exists between  $p$  and  $q$  and hence in  $[a, b]$  with

$$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

Thus,

$$|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \leq k|p - q| < |p - q|,$$

which is a contradiction. This contradiction must come from the only supposition,  $p \neq q$ . Hence,  $p = q$ , and the fixed point in  $[a, b]$  is unique. ■

**Example 2** Show that  $g(x) = (x^2 - 1)/3$  has a unique fixed point on the interval  $[-1, 1]$ .

**Solution** The maximum and minimum values of  $g(x)$  for  $x$  in  $[-1, 1]$  must occur either when  $x$  is an endpoint of the interval or when the derivative is 0. Since  $g'(x) = 2x/3$ , the function  $g$  is continuous, and  $g'(x)$  exists on  $[-1, 1]$ . The maximum and minimum values of  $g(x)$  occur at  $x = -1$ ,  $x = 0$ , or  $x = 1$ . But  $g(-1) = 0$ ,  $g(1) = 0$ , and  $g(0) = -1/3$ , so an absolute maximum for  $g(x)$  on  $[-1, 1]$  occurs at  $x = -1$  and  $x = 1$  and an absolute minimum at  $x = 0$ .

Moreover,

$$|g'(x)| = \left| \frac{2x}{3} \right| \leq \frac{2}{3}, \quad \text{for all } x \in (-1, 1).$$

So  $g$  satisfies all the hypotheses of Theorem 2.3 and has a unique fixed point in  $[-1, 1]$ . ■

For the function in Example 2, the unique fixed point  $p$  in the interval  $[-1, 1]$  can be determined algebraically. If

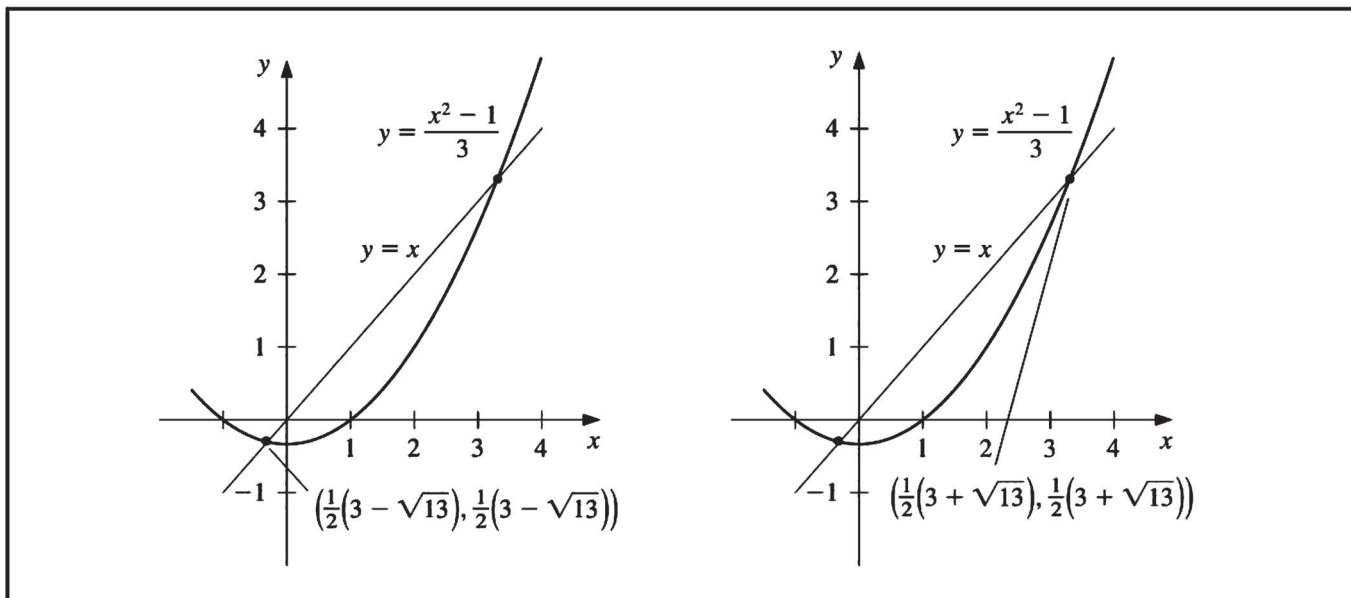
$$p = g(p) = \frac{p^2 - 1}{3}, \quad \text{then} \quad p^2 - 3p - 1 = 0,$$

which, by the quadratic formula, implies, as shown on the left graph in Figure 2.4, that

$$p = \frac{1}{2}(3 - \sqrt{13}).$$

Note that  $g$  also has a unique fixed point  $p = \frac{1}{2}(3 + \sqrt{13})$  for the interval  $[3, 4]$ . However,  $g(4) = 5$  and  $g'(4) = \frac{8}{3} > 1$ , so  $g$  does not satisfy the hypotheses of Theorem 2.3 on  $[3, 4]$ . This demonstrates that the hypotheses of Theorem 2.3 are sufficient to guarantee a unique fixed point but are not necessary. (See the graph on the right in Figure 2.4.)

Figure 2.4



**Example 3** Show that Theorem 2.3 does not ensure a unique fixed point of  $g(x) = 3^{-x}$  on the interval  $[0, 1]$ , even though a unique fixed point on this interval does exist.

**Solution**  $g'(x) = -3^{-x} \ln 3 < 0$  on  $[0, 1]$ , the function  $g$  is strictly decreasing on  $[0, 1]$ . So

$$g(1) = \frac{1}{3} \leq g(x) \leq 1 = g(0), \quad \text{for } 0 \leq x \leq 1.$$

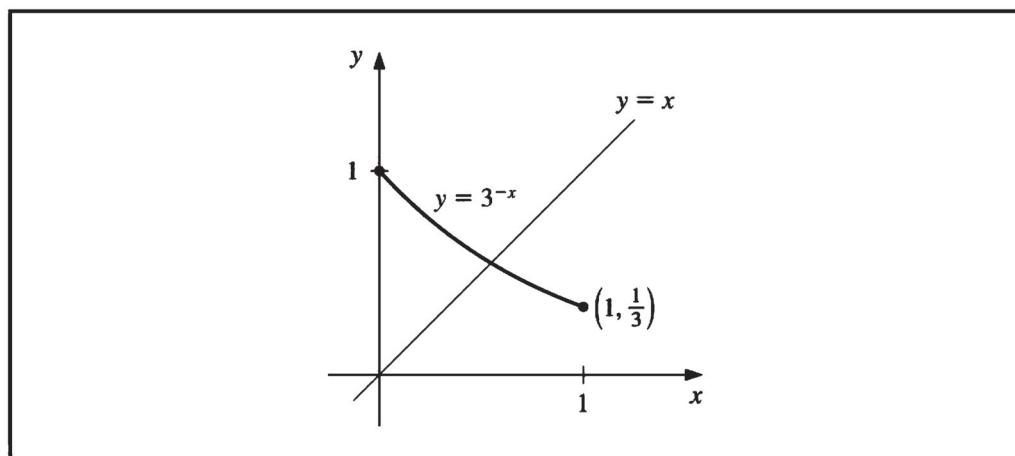
Thus, for  $x \in [0, 1]$ , we have  $g(x) \in [0, 1]$ . The first part of Theorem 2.3 ensures that there is at least one fixed point in  $[0, 1]$ .

However,

$$g'(0) = -\ln 3 = -1.098612289,$$

so  $|g'(x)| \not\leq 1$  on  $(0, 1)$ , and Theorem 2.3 cannot be used to determine uniqueness. But  $g$  is always decreasing, and it is clear from Figure 2.5 that the fixed point must be unique. ■

Figure 2.5



## Fixed-Point Iteration

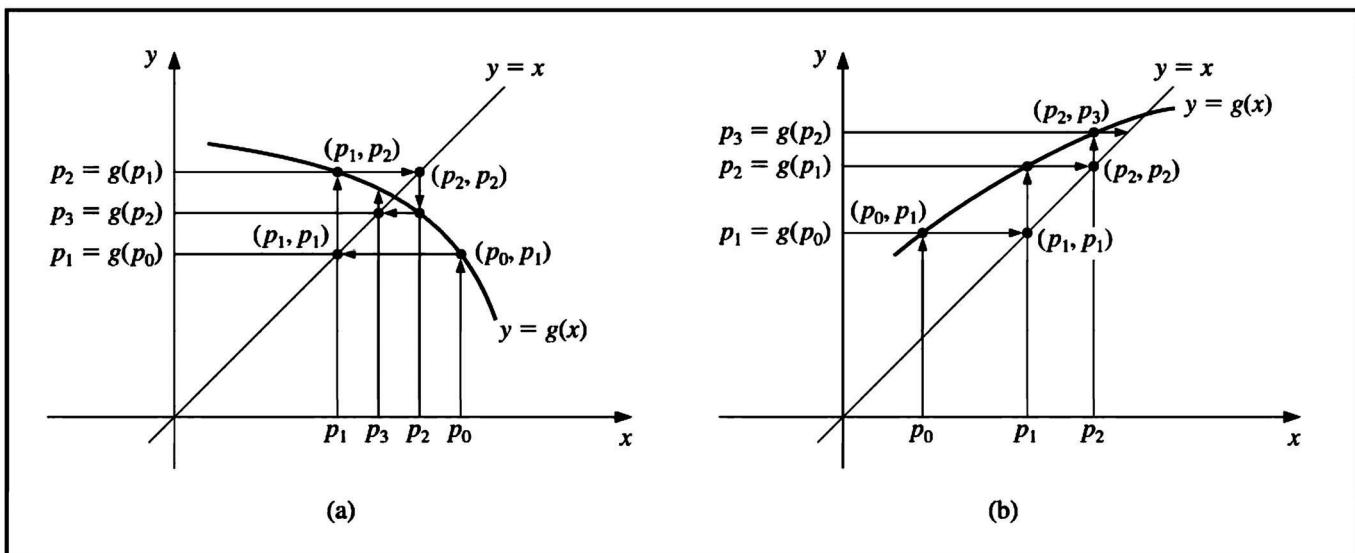
We cannot explicitly determine the fixed point in Example 3 because we have no way to solve for  $p$  in the equation  $p = g(p) = 3^{-p}$ . We can, however, determine approximations to this fixed point to any specified degree of accuracy. We will now consider how this can be done.

To approximate the fixed point of a function  $g$ , we choose an initial approximation  $p_0$  and generate the sequence  $\{p_n\}_{n=0}^{\infty}$  by letting  $p_n = g(p_{n-1})$ , for each  $n \geq 1$ . If the sequence converges to  $p$  and  $g$  is continuous, then

$$p = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} g(p_{n-1}) = g\left(\lim_{n \rightarrow \infty} p_{n-1}\right) = g(p),$$

and a solution to  $x = g(x)$  is obtained. This technique is called **fixed-point**, or **functional iteration**. The procedure is illustrated in Figure 2.6 and detailed in Algorithm 2.2.

**Figure 2.6**



## Fixed-Point Iteration

To find a solution to  $p = g(p)$  given an initial approximation  $p_0$ :

**INPUT** initial approximation  $p_0$ ; tolerance  $TOL$ ; maximum number of iterations  $N_0$ .

**OUTPUT** approximate solution  $p$  or message of failure.

**Step 1** Set  $i = 1$ .

**Step 2** While  $i \leq N_0$  do Steps 3–6.

**Step 3** Set  $p = g(p_0)$ . (Compute  $p_i$ .)

**Step 4** If  $|p - p_0| < TOL$  then

OUTPUT ( $p$ ); (The procedure was successful.)

STOP.

**Step 5** Set  $i = i + 1$ .

**Step 6** Set  $p_0 = p$ . (Update  $p_0$ .)

**Step 7** OUTPUT ('The method failed after  $N_0$  iterations,  $N_0 =$ ',  $N_0$ );  
*(The procedure was unsuccessful.)*  
**STOP.**

The following illustrates some features of functional iteration.

**Illustration** The equation  $x^3 + 4x^2 - 10 = 0$  has a unique root in  $[1, 2]$ . There are many ways to change the equation to the fixed-point form  $x = g(x)$  using simple algebraic manipulation. For example, to obtain the function  $g$  described in part (c), we can manipulate the equation  $x^3 + 4x^2 - 10 = 0$  as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm \frac{1}{2}(10 - x^3)^{1/2}.$$

To obtain a positive solution,  $g_3(x)$  is chosen. It is not important for you to derive the functions shown here, but you should verify that the fixed point of each is actually a solution to the original equation,  $x^3 + 4x^2 - 10 = 0$ .

$$\begin{array}{ll} \text{(a)} & x = g_1(x) = x - x^3 - 4x^2 + 10 \\ & \text{(b)} \quad x = g_2(x) = \left( \frac{10}{x} - 4x \right)^{1/2} \\ \text{(c)} & x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2} \\ & \text{(d)} \quad x = g_4(x) = \left( \frac{10}{4+x} \right)^{1/2} \\ \text{(e)} & x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \end{array}$$

With  $p_0 = 1.5$ , Table 2.2 lists the results of the fixed-point iteration for all five choices of  $g$ .

**Table 2.2**

<i>n</i>	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	$1.03 \times 10^8$		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

The actual root is 1.365230013, as was noted in Example 1 of Section 2.1. Comparing the results to the Bisection Algorithm given in that example, it can be seen that excellent results have been obtained for choices (d) and (e) (the Bisection method requires 27 iterations for this accuracy). It is interesting to note that choice (a) was divergent and that choice (b) became undefined because it involved the square root of a negative number. ■

Although the various functions we have given are fixed-point problems for the same root-finding problem, they differ vastly as techniques for approximating the solution to the root-finding problem. Their purpose is to illustrate what needs to be answered:

- Question: How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

The following theorem and its corollary give us some clues concerning the paths we should pursue and, perhaps more important, some we should reject.

**Theorem 2.4 (Fixed-Point Theorem)**

Let  $g \in C[a, b]$  be such that  $g(x) \in [a, b]$ , for all  $x$  in  $[a, b]$ . Suppose, in addition, that  $g'$  exists on  $(a, b)$  and that a constant  $0 < k < 1$  exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then, for any number  $p_0$  in  $[a, b]$ , the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point  $p$  in  $[a, b]$ .

**Proof** Theorem 2.3 implies that a unique point  $p$  exists in  $[a, b]$  with  $g(p) = p$ . Since  $g$  maps  $[a, b]$  into itself, the sequence  $\{p_n\}_{n=0}^{\infty}$  is defined for all  $n \geq 0$ , and  $p_n \in [a, b]$  for all  $n$ . Using the fact that  $|g'(x)| \leq k$  and the Mean Value Theorem 1.8, we have, for each  $n$ ,

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)||p_{n-1} - p| \leq k|p_{n-1} - p|,$$

where  $\xi_n \in (a, b)$ . Applying this inequality inductively gives

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots \leq k^n|p_0 - p|. \quad (2.4)$$

Since  $0 < k < 1$ , we have  $\lim_{n \rightarrow \infty} k^n = 0$  and

$$\lim_{n \rightarrow \infty} |p_n - p| \leq \lim_{n \rightarrow \infty} k^n|p_0 - p| = 0.$$

Hence,  $\{p_n\}_{n=0}^{\infty}$  converges to  $p$ . ■

**Corollary 2.5** If  $g$  satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using  $p_n$  to approximate  $p$  are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} \quad (2.5)$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \quad \text{for all } n \geq 1. \quad (2.6)$$

**Proof** Because  $p \in [a, b]$ , the first bound follows from Inequality (2.4):

$$|p_n - p| \leq k^n|p_0 - p| \leq k^n \max\{p_0 - a, b - p_0\}.$$

For  $n \geq 1$ , the procedure used in the proof of Theorem 2.4 implies that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq k|p_n - p_{n-1}| \leq \cdots \leq k^n|p_1 - p_0|.$$

Thus, for  $m > n \geq 1$ ,

$$\begin{aligned}|p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \cdots + p_{n+1} - p_n| \\&\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n| \\&\leq k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \cdots + k^n|p_1 - p_0| \\&= k^n|p_1 - p_0|(1 + k + k^2 + \cdots + k^{m-n-1}).\end{aligned}$$

By Theorem 2.3,  $\lim_{m \rightarrow \infty} p_m = p$ , so

$$|p - p_n| = \lim_{m \rightarrow \infty} |p_m - p_n| \leq \lim_{m \rightarrow \infty} k^n|p_1 - p_0| \sum_{i=0}^{m-n-1} k^i \leq k^n|p_1 - p_0| \sum_{i=0}^{\infty} k^i.$$

But  $\sum_{i=0}^{\infty} k^i$  is a geometric series with ratio  $k$  and  $0 < k < 1$ . This sequence converges to  $1/(1-k)$ , which gives the second bound:

$$|p - p_n| \leq \frac{k^n}{1-k}|p_1 - p_0|. \quad \blacksquare$$

Both inequalities in the corollary relate the rate at which  $\{p_n\}_{n=0}^{\infty}$  converges to the bound  $k$  on the first derivative. The rate of convergence depends on the factor  $k^n$ . The smaller the value of  $k$ , the faster the convergence. However, the convergence may be very slow if  $k$  is close to 1.

**Illustration** Let us reconsider the various fixed-point schemes described in the preceding illustration in light of the Fixed-Point Theorem 2.4 and its Corollary 2.5.

- (a) For  $g_1(x) = x - x^3 - 4x^2 + 10$ , we have  $g_1(1) = 6$  and  $g_1(2) = -12$ , so  $g_1$  does not map  $[1, 2]$  into itself. Moreover,  $g'_1(x) = 1 - 3x^2 - 8x$ , so  $|g'_1(x)| > 1$  for all  $x$  in  $[1, 2]$ . Although Theorem 2.4 does not guarantee that the method must fail for this choice of  $g$ , there is no reason to expect convergence.
- (b) With  $g_2(x) = [(10/x) - 4x]^{1/2}$ , we can see that  $g_2$  does not map  $[1, 2]$  into  $[1, 2]$ , and the sequence  $\{p_n\}_{n=0}^{\infty}$  is not defined when  $p_0 = 1.5$ . Moreover, there is no interval containing  $p \approx 1.365$  such that  $|g'_2(x)| < 1$  because  $|g'_2(p)| \approx 3.4$ . There is no reason to expect that this method will converge.
- (c) For the function  $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$ , we have

$$g'_3(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0 \quad \text{on } [1, 2],$$

so  $g_3$  is strictly decreasing on  $[1, 2]$ . However,  $|g'_3(2)| \approx 2.12$ , so the condition  $|g'_3(x)| \leq k < 1$  fails on  $[1, 2]$ . A closer examination of the sequence  $\{p_n\}_{n=0}^{\infty}$  starting with  $p_0 = 1.5$  shows that it suffices to consider the interval  $[1, 1.5]$  instead of  $[1, 2]$ . On this interval, it is still true that  $g'_3(x) < 0$  and  $g_3$  is strictly decreasing, but, additionally,

$$1 < 1.28 \approx g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5,$$

for all  $x \in [1, 1.5]$ . This shows that  $g_3$  maps the interval  $[1, 1.5]$  into itself. It is also true that  $|g'_3(x)| \leq |g'_3(1.5)| \approx 0.66$  on this interval, so Theorem 2.4 confirms the convergence of which we were already aware.

(d) For  $g_4(x) = (10/(4+x))^{1/2}$ , we have

$$|g'_4(x)| = \left| \frac{-5}{\sqrt{10(4+x)^{3/2}}} \right| \leq \frac{5}{\sqrt{10(5)^{3/2}}} < 0.15, \quad \text{for all } x \in [1, 2].$$

The bound on the magnitude of  $g'_4(x)$  is much smaller than the bound (found in (c)) on the magnitude of  $g'_3(x)$ , which explains the more rapid convergence using  $g_4$ .

(e) The sequence defined by

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

converges much more rapidly than our other choices. In the next sections, we will see where this choice came from and why it is so effective. ■

From what we have seen, the

- Question: How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

might have the

- Answer: Manipulate the root-finding problem into a fixed point problem that satisfies the conditions of Fixed-Point Theorem 2.4 and has a derivative that is as small as possible near the fixed point.

In the next sections, we will examine this in more detail.

## EXERCISE SET 2.2

1. Use algebraic manipulation to show that each of the following functions has a fixed point at  $p$  precisely when  $f(p) = 0$ , where  $f(x) = x^4 + 2x^2 - x - 3$ .
  - $g_1(x) = (3 + x - 2x^2)^{1/4}$
  - $g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{1/2}$
  - $g_3(x) = \left(\frac{x + 3}{x^2 + 2}\right)^{1/2}$
  - $g_4(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$
2. a. Perform four iterations, if possible, on each of the functions  $g$  defined in Exercise 1. Let  $p_0 = 1$  and  $p_{n+1} = g(p_n)$ , for  $n = 0, 1, 2, 3$ .  
b. Which function do you think gives the best approximation to the solution?
3. Let  $f(x) = x^3 - 2x + 1$ . To solve  $f(x) = 0$ , the following four fixed-point problems are proposed. Derive each fixed point method and compute  $p_1, p_2, p_3, p_4$ . Which methods seem to be appropriate?
  - $x = \frac{1}{2}(x^3 + 1), \quad p_0 = \frac{1}{2}$
  - $x = \frac{2}{x} - \frac{1}{x^2}, \quad p_0 = \frac{1}{2}$
  - $x = \sqrt[3]{2 - \frac{1}{x}}, \quad p_0 = \frac{1}{2}$
  - $x = -\sqrt[3]{1 - 2x}, \quad p_0 = \frac{1}{2}$
4. Let  $f(x) = x^4 + 3x^2 - 2$ . To solve  $f(x) = 0$ , the following four fixed-point problems are proposed. Derive each fixed point method and compute  $p_1, p_2, p_3$ , and  $p_4$ . Which methods seem to be appropriate?

a.  $x = \sqrt{\frac{2-x^4}{3}}, p_0 = 1$

c.  $x = \frac{2-x^4}{3x}, p_0 = 1$

b.  $x = \sqrt[4]{2-3x^2}, p_0 = 1$

d.  $x = \sqrt[3]{\frac{2-3x^2}{x}}, p_0 = 1$

5. The following four methods are proposed to compute  $21^{1/3}$ . Rank them in order, based on their apparent speed of convergence, assuming  $p_0 = 1$ .

a.  $p_n = \frac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b.  $p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c.  $p_n = p_{n-1} - \frac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d.  $p_n = \left( \frac{21}{p_{n-1}} \right)^{1/2}$

6. The following four methods are proposed to compute  $7^{1/5}$ . Rank them in order, based on their apparent speed of convergence, assuming  $p_0 = 1$ .

a.  $p_n = p_{n-1} \left( 1 + \frac{7-p_{n-1}^5}{p_{n-1}^2} \right)^3$

b.  $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{p_{n-1}^2}$

c.  $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

d.  $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$

7. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-2}$  for  $x^4 - 3x^2 - 3 = 0$  on  $[1, 2]$ . Use  $p_0 = 1$ .

8. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-2}$  for  $x^3 - x - 1 = 0$  on  $[1, 2]$ . Use  $p_0 = 1$ .

9. Use Theorem 2.3 to show that  $g(x) = \pi + 0.5 \sin(x/2)$  has a unique fixed point on  $[0, 2\pi]$ . Use fixed-point iteration to find an approximation to the fixed point that is accurate to within  $10^{-2}$ . Use Corollary 2.5 to estimate the number of iterations required to achieve  $10^{-2}$  accuracy and compare this theoretical estimate to the number actually needed.

10. Use Theorem 2.3 to show that  $g(x) = 2^{-x}$  has a unique fixed point on  $[\frac{1}{3}, 1]$ . Use fixed-point iteration to find an approximation to the fixed point accurate to within  $10^{-4}$ . Use Corollary 2.5 to estimate the number of iterations required to achieve  $10^{-4}$  accuracy and compare this theoretical estimate to the number actually needed.

11. Use a fixed-point iteration method to find an approximation to  $\sqrt{3}$  that is accurate to within  $10^{-4}$ . Compare your result and the number of iterations required with the answer obtained in Exercise 14 of Section 2.1.

12. Use a fixed-point iteration method to find an approximation to  $\sqrt[3]{25}$  that is accurate to within  $10^{-4}$ . Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

13. For each of the following equations, determine an interval  $[a, b]$  on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within  $10^{-5}$  and perform the calculations.

a.  $x = \frac{2-e^x+x^2}{3}$

b.  $x = \frac{5}{x^2} + 2$

c.  $x = (e^x/3)^{1/2}$

d.  $x = 5^{-x}$

e.  $x = 6^{-x}$

f.  $x = 0.5(\sin x + \cos x)$

14. For each of the following equations, use the given interval or determine an interval  $[a, b]$  on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within  $10^{-5}$  and perform the calculations.

a.  $2 + \sin x - x = 0$  use  $[2, 3]$

b.  $x^3 - 2x - 5 = 0$  use  $[2, 3]$

c.  $3x^2 - e^x = 0$

d.  $x - \cos x = 0$

15. Find all the zeros of  $f(x) = x^2 + 10 \cos x$  by using the fixed-point iteration method for an appropriate iteration function  $g$ . Find the zeros accurate to within  $10^{-4}$ .

16. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-4}$  for  $x = \tan x$ , for  $x$  in  $[4, 5]$ .

17. Use a fixed-point iteration method to determine a solution accurate to within  $10^{-2}$  for  $x = 2 \sin(\pi x) + x = 0$ , for  $x$  on  $[1, 2]$ . Use  $p_0 = 1$ .