Do not edit
How to change the design

Who will win the Superbowl LX?

The Slido app must be installed on every computer you're presenting from

slido

"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 4

Any Reminders? Any Questions?

- Class ends at 3:15.
- Slides are being posted on GitHub!
  https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- All requests for make-up quizzes need to go to your TA
- Second quiz (Friday) will be on previous week's material

Today – Lines and Planes!

- Lines
  - Parametric equations of a line
  - Symmetric equation
  - Line segments
- Planes
  - Vector equation
  - Scalar equation
  - Distance to a plane

Questions?

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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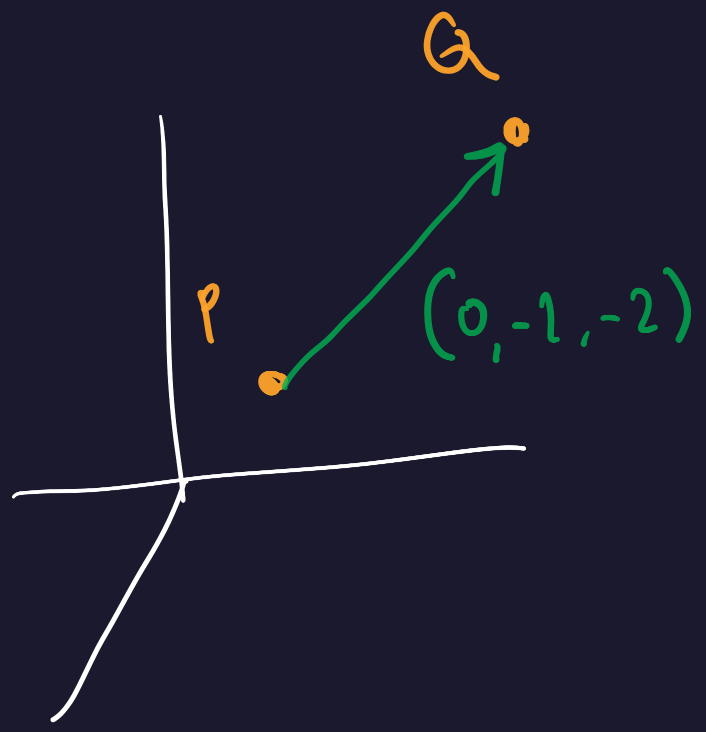
Review of Lines and Planes

Example: Find the parametric equation of a line that passes through $P_0 = (1, 0, 2)$ in the direction of $v = (-1, 1, 1)$.

$$\vec{P} = \vec{P_0} + t \cdot v$$

$$= \boxed{\begin{cases} x = 1 + t \cdot (-1) & = 1 - t \\ y = 0 + t \cdot 1 & = t \\ z = 2 + t \cdot 1 & = 2 + t \end{cases}}$$

Example: Find the parametric equation for the segment that goes from $P = (1, 2, 3)$ to $Q = (1, 0, 1)$.



$$S = (1-t)\cdot\vec{P} + t\cdot\vec{Q}$$

for $0 \leq t \leq 1$

$$= (1-t)\cdot(1,2,3) + t\cdot(1,0,1)$$

$$= \begin{cases} x = (1-t)\cdot 1 + t = 1 \\ y = (1-t)\cdot 2 + 0\cdot t = 2-2t \\ z = (1-t)\cdot 3 + 1\cdot t = 3-2t \end{cases}$$

$$0 \leq t \leq 1$$

Example: Show that the lines L_1 and L_2 intersect, and find the point of intersection.

$$L_1 : x = -2+t, y = 2-2t, z = -1+3t \quad \text{and} \quad L_2 : x = -2-s, y = 1+s, z = -2s.$$

$$\begin{cases} x = -2+t = -2-s \\ y = 2-2t = 1+s \\ z = -1+3t = -2s \end{cases} \rightsquigarrow \ldots$$

Example (LET'S FIX IT!): Investigate the relative position of the lines

L_1 : $(1+t, -2t, -1+3t)$ and L_2 : $(2+t, -2-2t, 2+3t)$.

$v_1 = (1, -2, 3)$ $\qquad$ $v_2 = (1, -2, 3)$

• A pt on L_1 at $t=0$

$P = (1, 0, -1)$

• P on L_2?

$x = 2 + t = 1$ $\quad$ $t = -1$

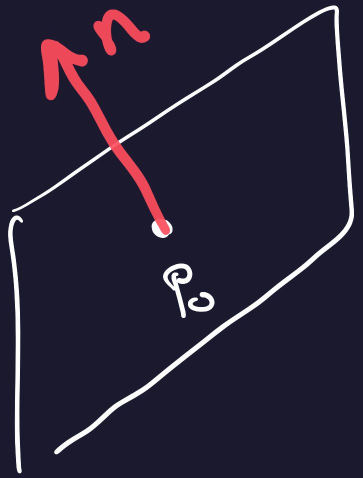
$y = -2 - 2t = 0$ $\quad$ $-2 - 2\cdot(-1) = 0$ ✓

$z = 2 + 3t = -1$ $\quad$ $2 + 3(-1) = -1$ ✓

Yes!

$\Rightarrow$ IDENTICAL!

Example: Find the equation of a plane that goes through $P_0 = (1, 2, 3)$ and it is perpendicular to $n = (1, -1, 2)$.



$$(r - r_0) \cdot n = 0$$

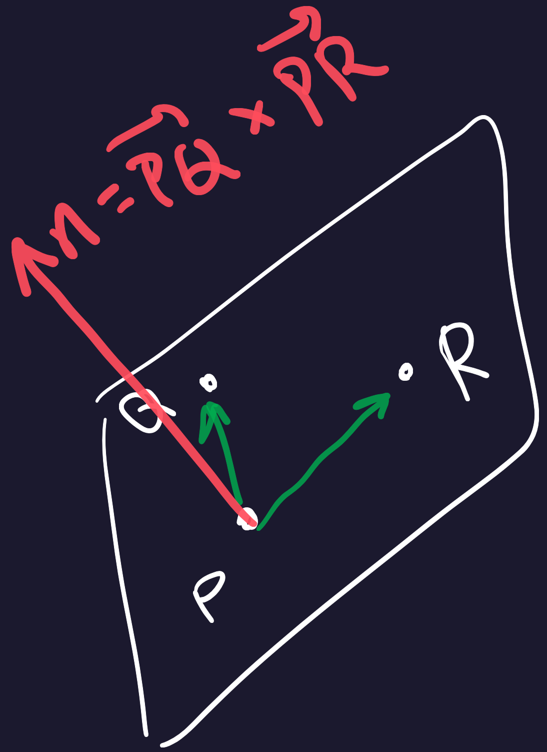
$$(x-1,\ y-2,\ z-3) \cdot (1, -1, 2) = 0$$

$$\boxed{1 \cdot (x-1) - (y-2) + 2 \cdot (z-3) = 0}$$

$$x - 1 - y + 2 + 2z - 6 = 0$$

$$\boxed{x - y + 2z - 5 = 0}$$

Example: Find the equation of a plane that goes through
$P = (1, 2, 3)$, $Q = (1, 0, 1)$, and $R = (0, -1, 1)$.



$$\vec{PQ} = \vec{Q} - \vec{P} = (0, -2, -2)$$
$$\vec{PR} = \vec{R} - \vec{P} = (-1, -3, -2)$$

$$n = \begin{vmatrix} i & j & k \\ 0 & -2 & -2 \\ -1 & -3 & -2 \end{vmatrix} = \dots = (a, b, c)$$

$$a \cdot (x-1) + b \cdot (y-2) + c(z-3) = 0$$

The planes $2(x - 2) - 4(y + 1) - 2(z - 1) = 0$ and $x - 2y - z = 3$ are...

Example: The planes

$2(x-2)-4(y+1)-2(z-1)=0$ and $x-2y-z=3$

are…

$n_1=(2,-4,-2)$ $\qquad$ $n_2=(1,-2,-1)$

$n_1 = 2\cdot n_2 \Rightarrow$ parallel or identical.

$2x-4-4y-4-2z+2=0$

$2x-4y-2z=6$

$\boxed{x-2y-z=3}$

Same!

Identical!

Questions?

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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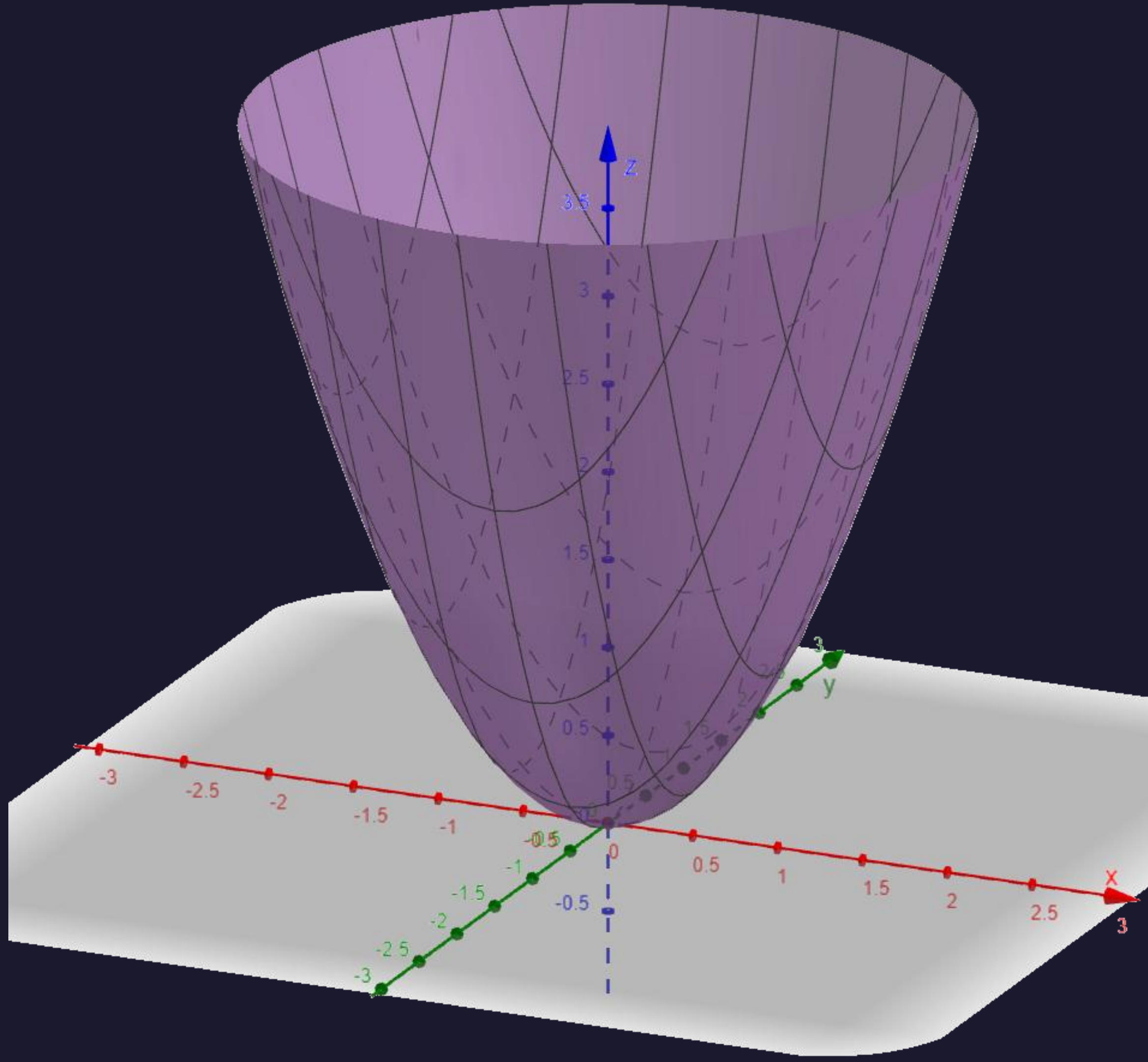
“Cylinders” and Quadrics

Today – Quadrics!

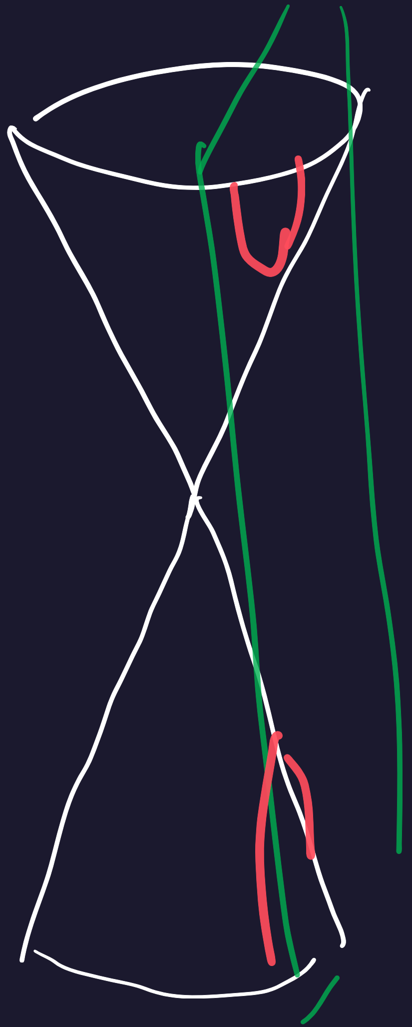
- “Cylinders”
- Quadric Surfaces
- Ellipsoids, Paraboloids, Hyperboloids.
- Sketching a Quadric Surface



“Cylinders”

A “cylinder” is a surface that consists of all lines (called “rulings”) that are parallel to a given line and pass through a given plane curve.

Recall: Circles, Ellipses, Parabolas and Hyperbolas (Conic Sections)



$$x^2 + y^2 = R^2$$

CIRCLES

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = R^2$$

ELLIPSES

$$y = ax^2 + bx + c$$

PARABOLAS

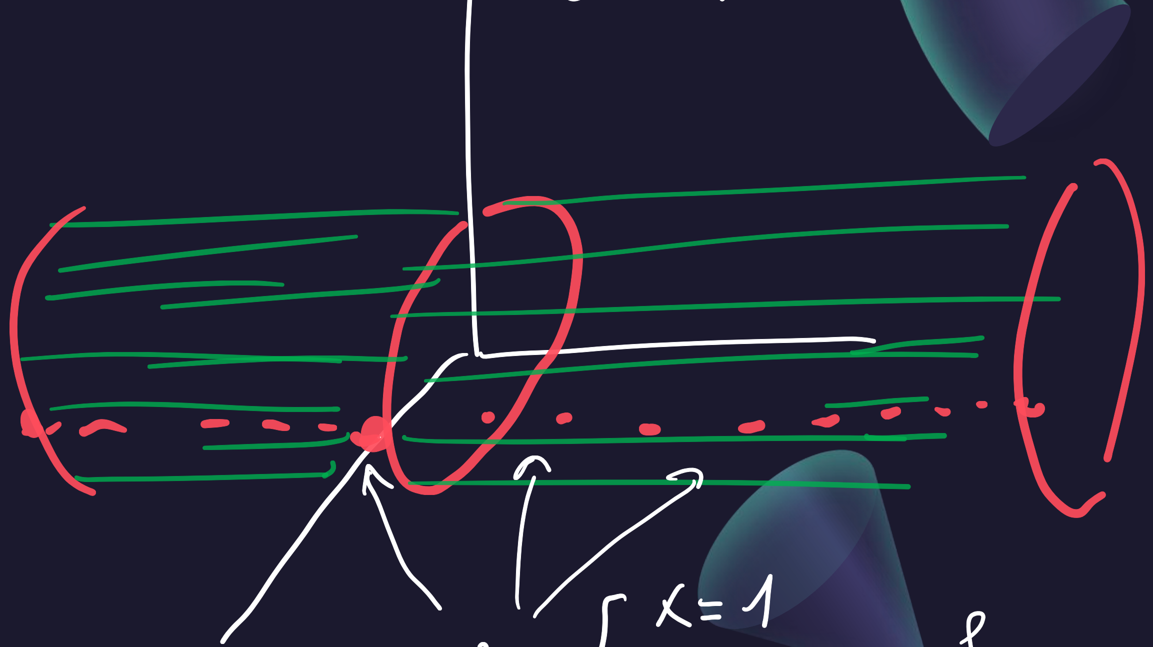
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = R$$

HYPERB.

Example: Sketch the surfaces $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

$$S = \{(x, y, z) : x^2 + y^2 = 1\}$$

$$S = \{(x, y, z) : x^2 + z^2 = 1\}$$

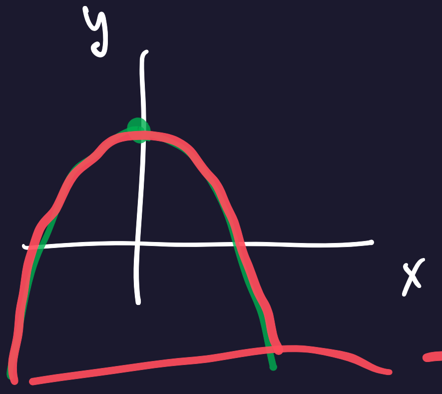


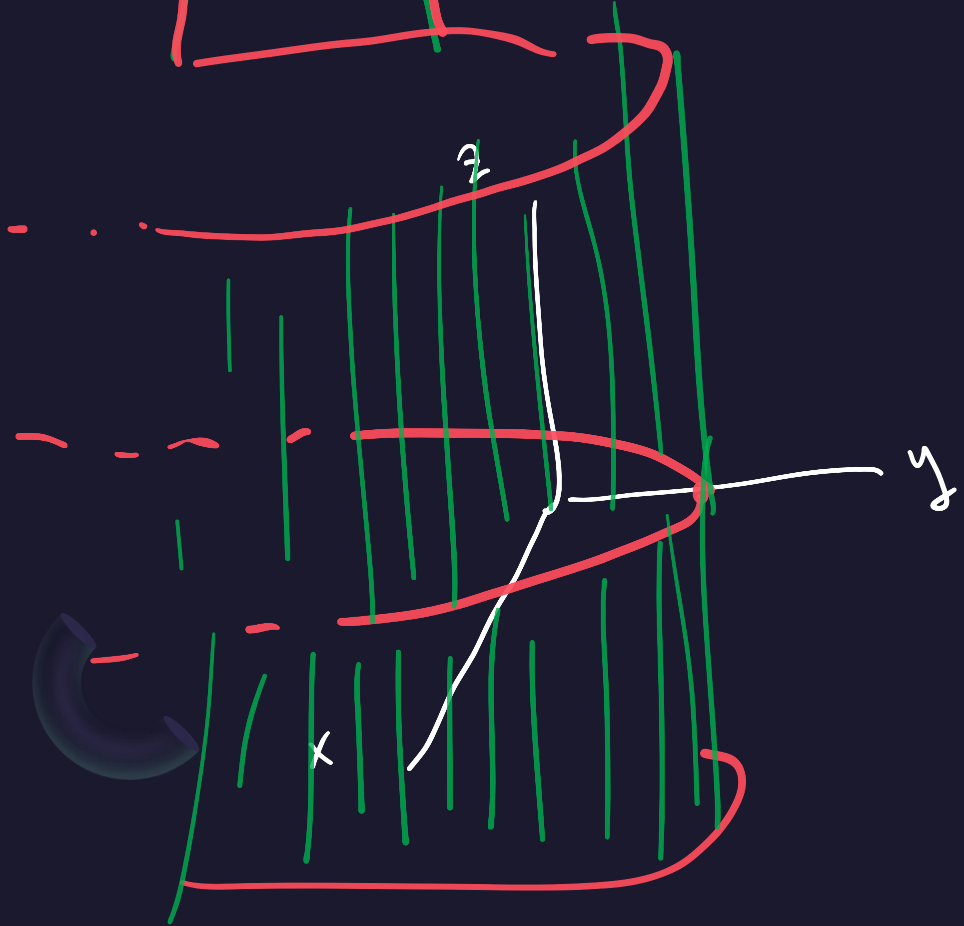
$$\{(1, y, 0) : y \in \mathbb{R}\} = \begin{cases} x = 1 \\ y = t \\ z = 0 \end{cases} \quad \text{for any } t \in \mathbb{R}$$

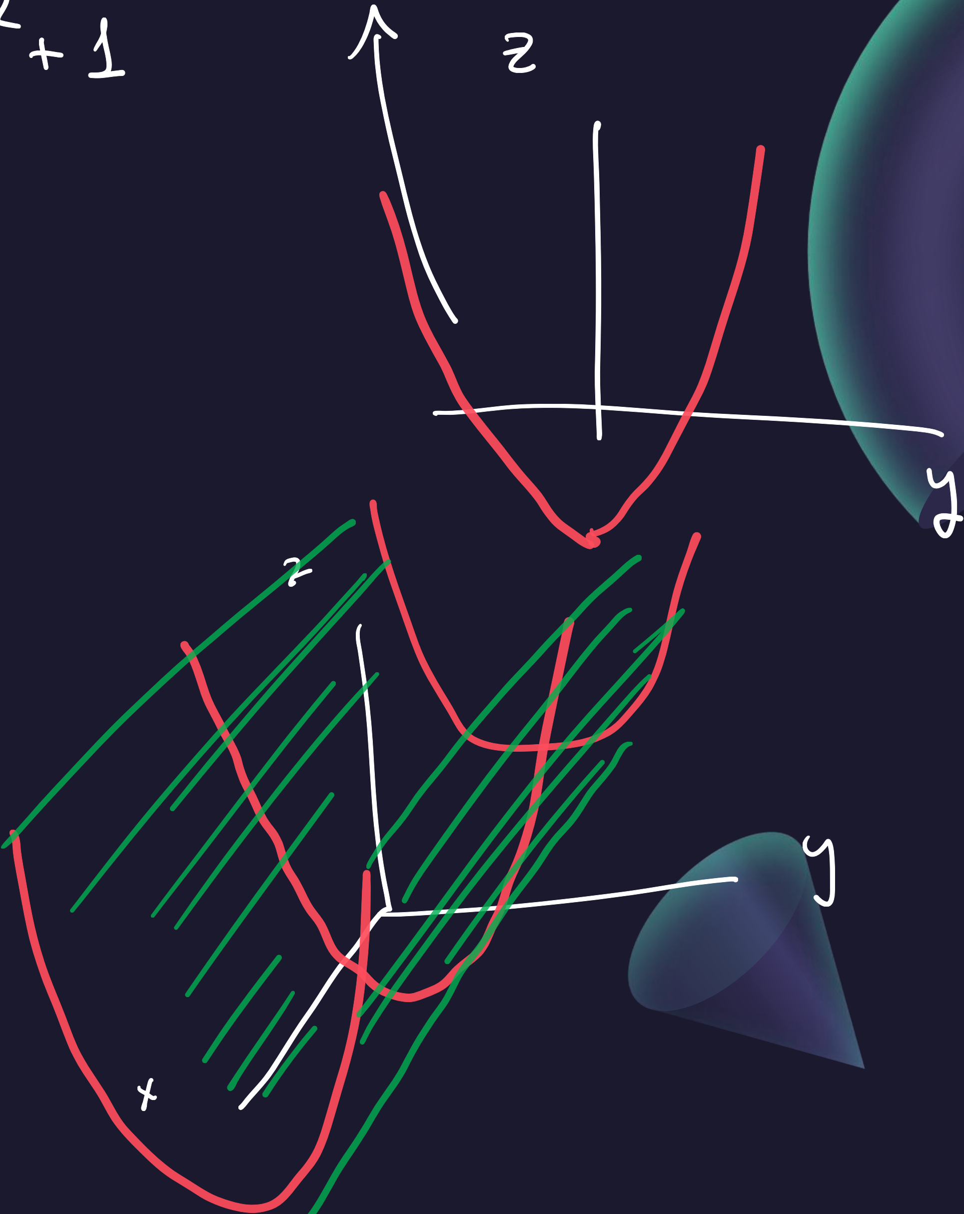
Example: Sketch the surfaces $y + x^2 = 1$ and $z - y^2 = -1.$

$z = y^2 - 1$

$y = -x^2 + 1$





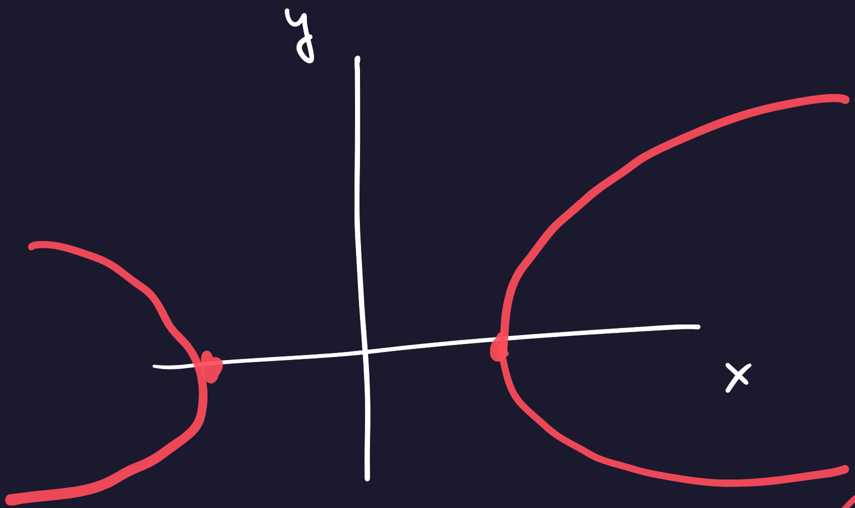


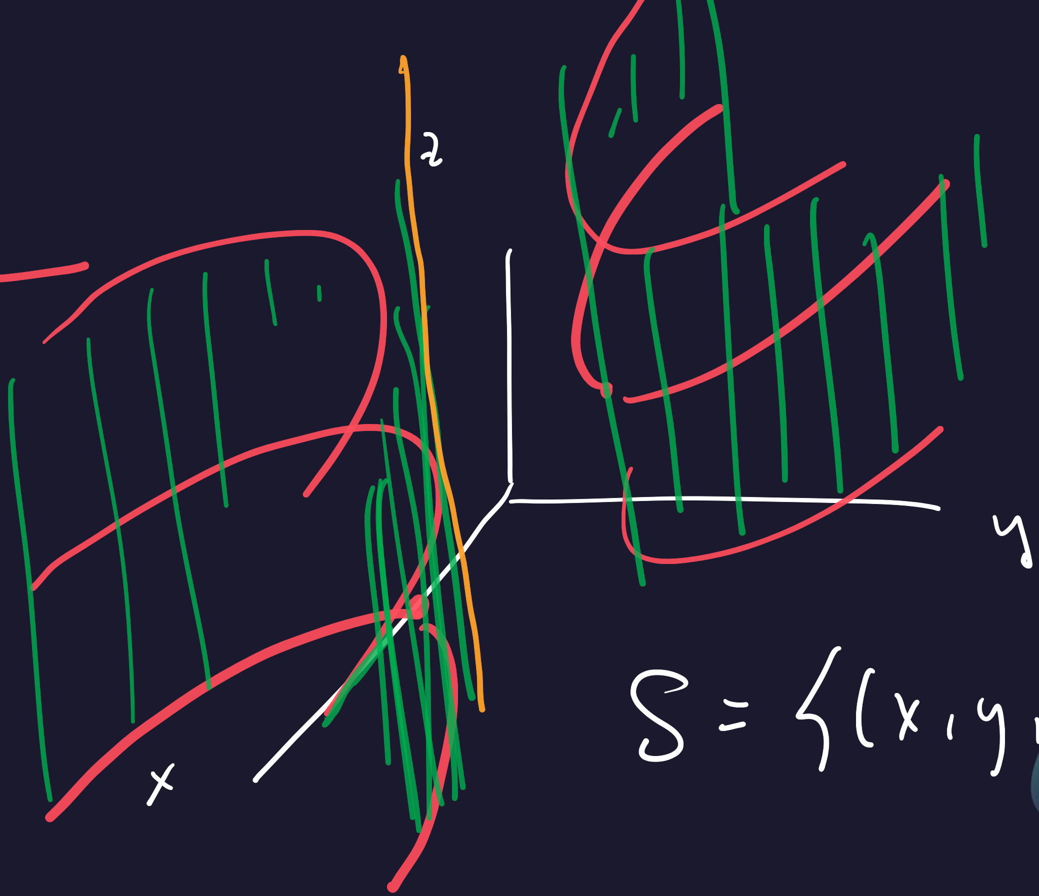
The equation x^2 = 1 + y^2 describes...

Example: Sketch the surface $x^2 - y^2 = 1$.

$$y^2 = x^2 - 1$$

$$y = \pm\sqrt{x^2 - 1}$$





$$S = \{(x, y, z) : x^2 - y^2 = 1\}$$

Example: Sketch the surface

$$x^2 + (y - z)^2 = 1 \quad \text{, i.e.,} \quad x^2 + y^2 - 2yz + z^2 = 1.$$

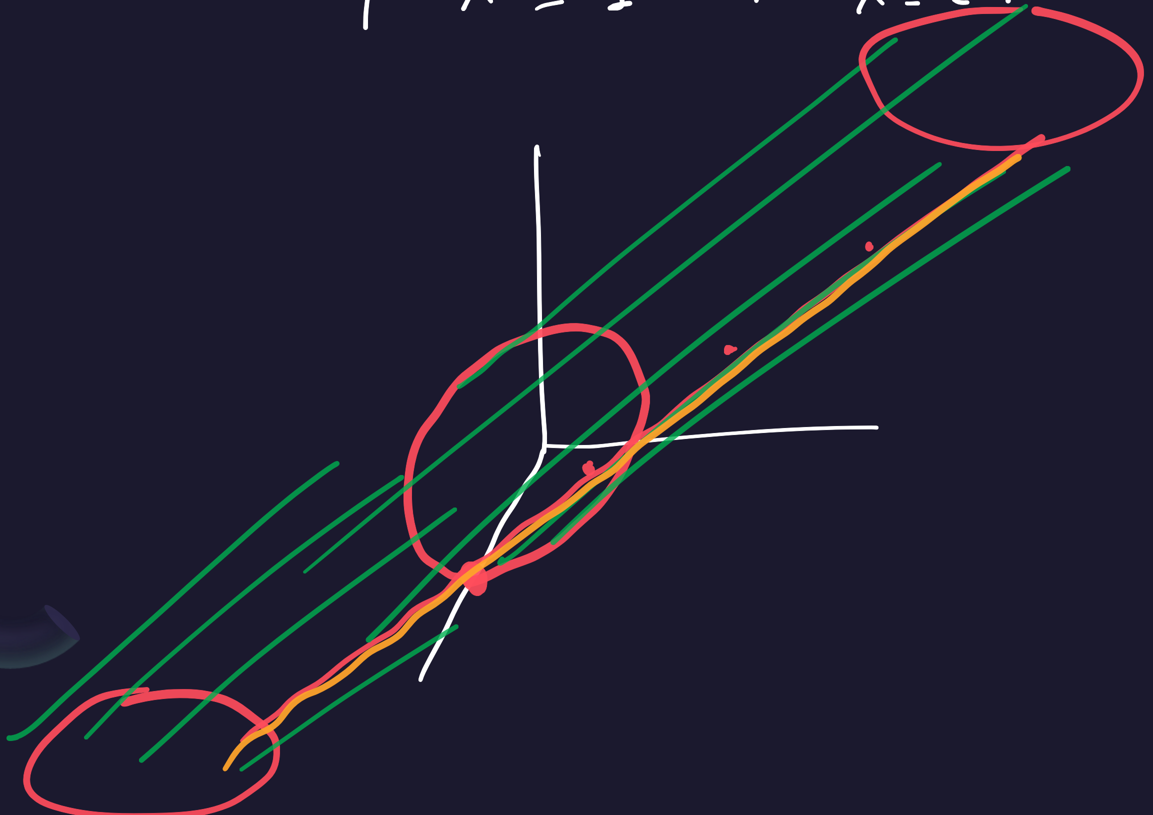
$$\underbrace{(y-z)}_{=0}$$

$$\begin{cases} y = z \\ x^2 = 1 \Rightarrow x = \pm 1 \end{cases}$$

$$(\pm 1, y, y)$$

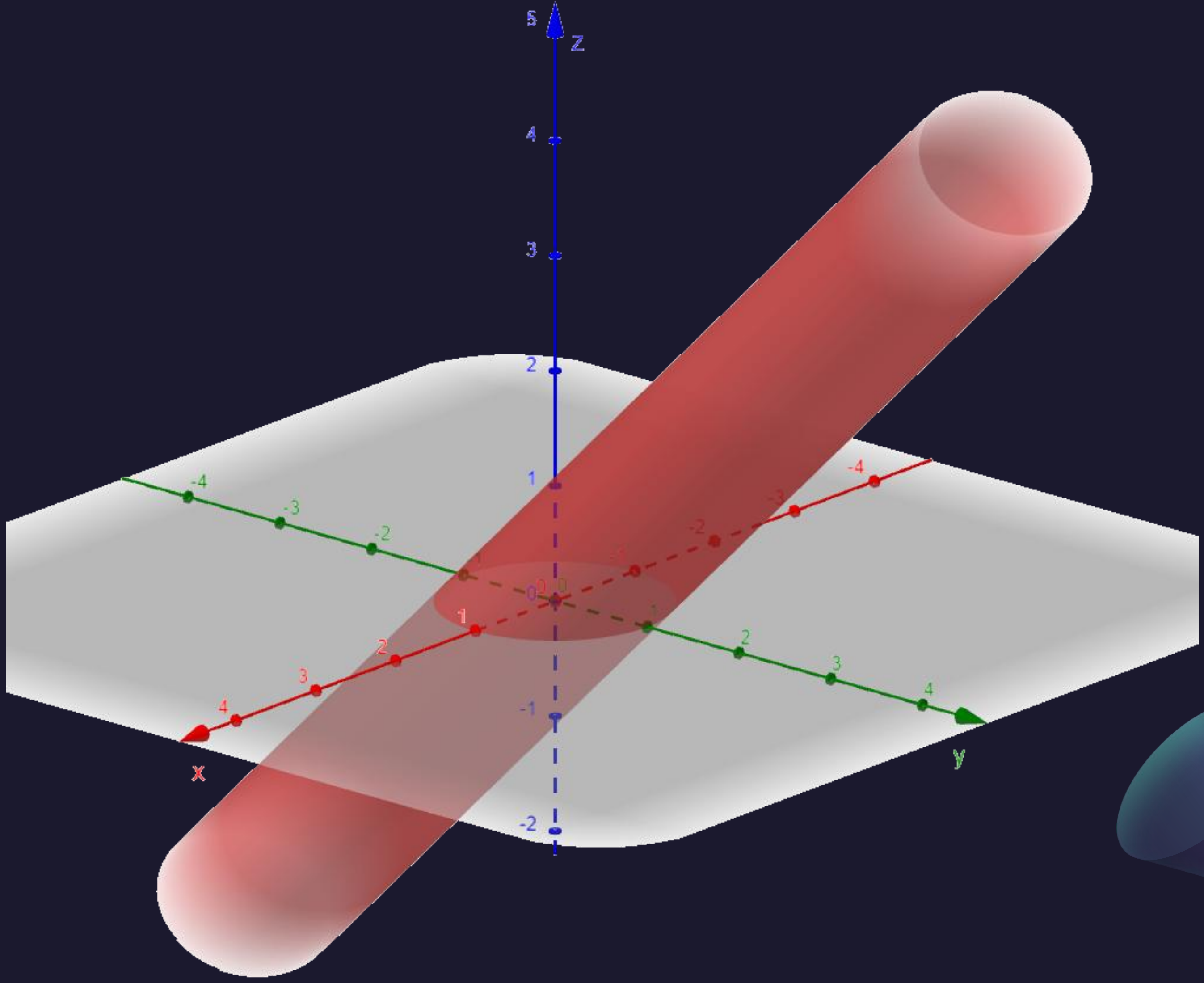
$$\begin{cases} z = 0 \\ x^2 + (y - z)^2 = 1 \end{cases}$$

$$\hookrightarrow \quad x^2 + y^2 = 1$$



Example: Sketch the surface

$$x^2 + (y - z)^2 = 1 \quad , \text{ i.e.,} \quad x^2 + y^2 - 2yz + z^2 = 1.$$



Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two *standard forms*

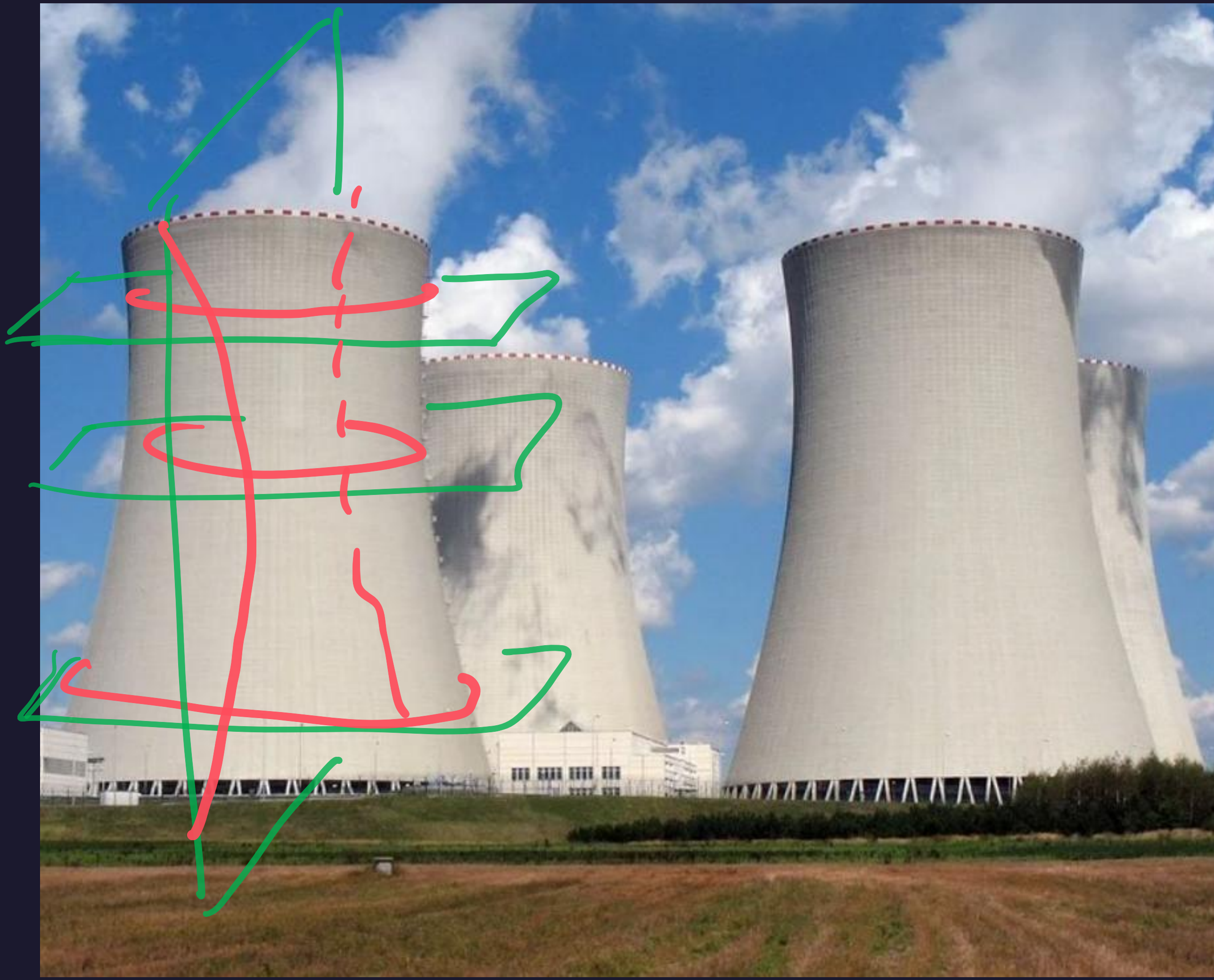
$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

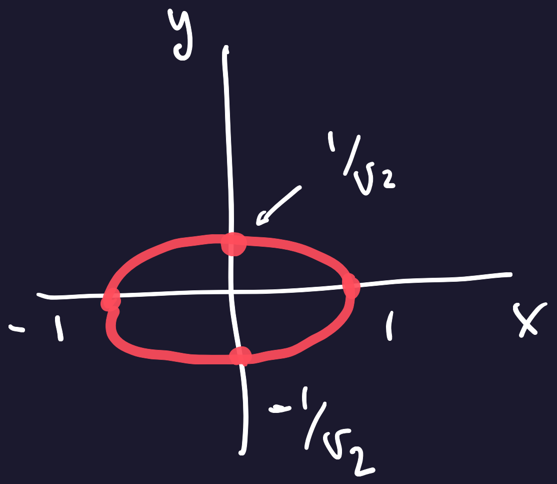
How to sketch a quadric surface?

Traces or Cross Sections of a Surface

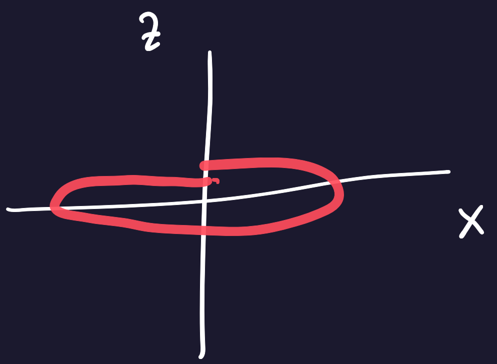


Example: Sketch the surface $x^2 + 2y^2 + 3z^2 = 1$.

$z = 0 \longrightarrow x^2 + 2y^2 = 1$

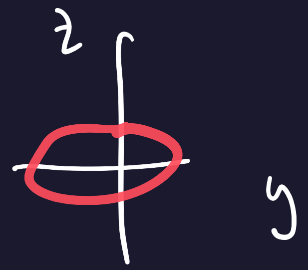


$y = 0 \qquad x^2 + 3z^2 = 1$



$x = 0$

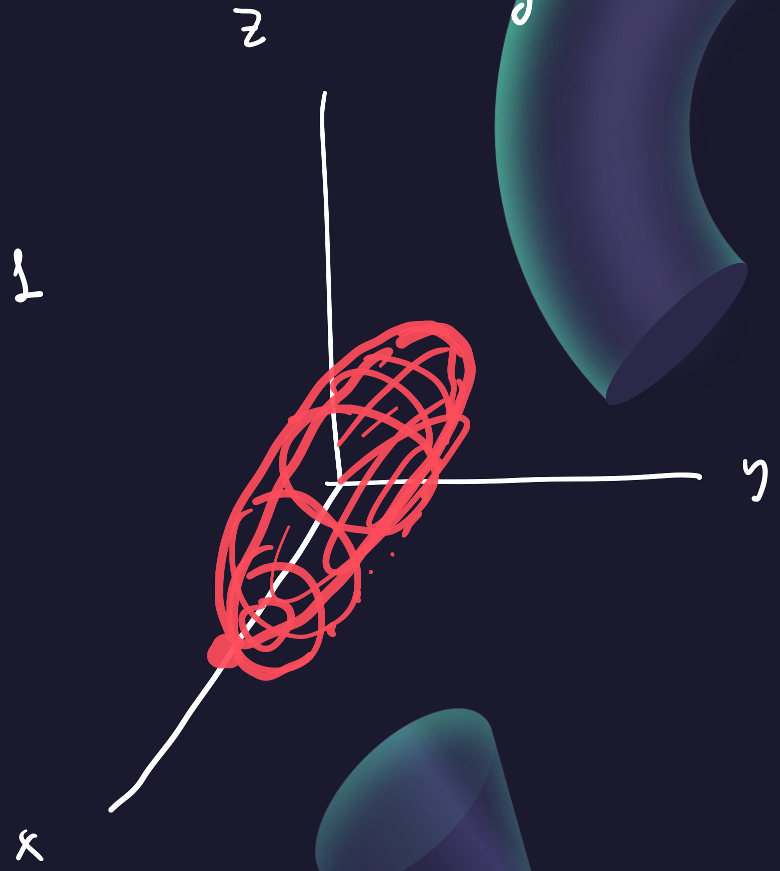
$2y^2 + 3z^2 = 1$



$x = 1 \qquad 2y^2 + 3z^2 = 0$

$\Rightarrow y = 0 \quad z = 0$

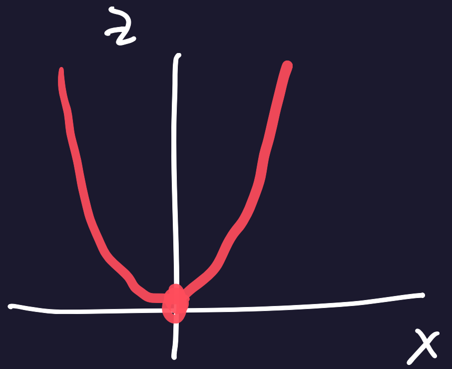
$x = \frac{1}{2} \qquad 2y^2 + 3z^2 = \frac{3}{4}$



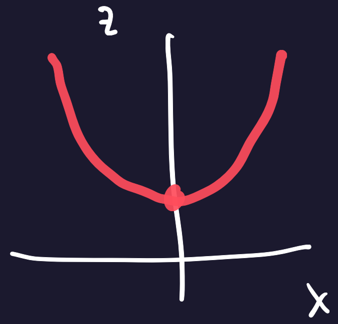
Example: Sketch the surface $z = 4x^2 + y^2$.

$z = 1 \quad 4x^2 + y^2 = 1$

$y = 0 \quad z = 4x^2$

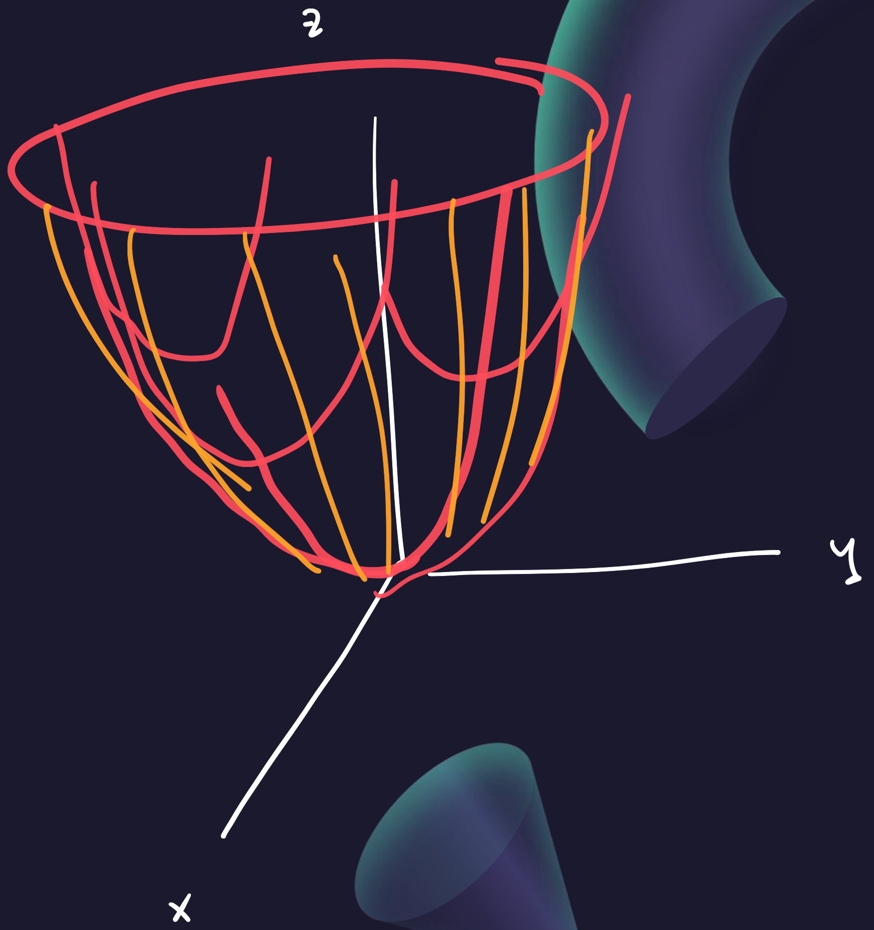


$y = \pm 1 \quad z = 4x^2 + 1$



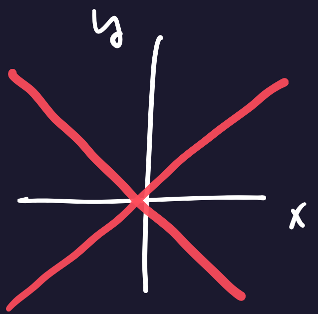
$x = 0 \quad z = y^2$

$x = 1 \quad z = y^2 + 4$



Example: Sketch the surface $z = x^2 - y^2$.

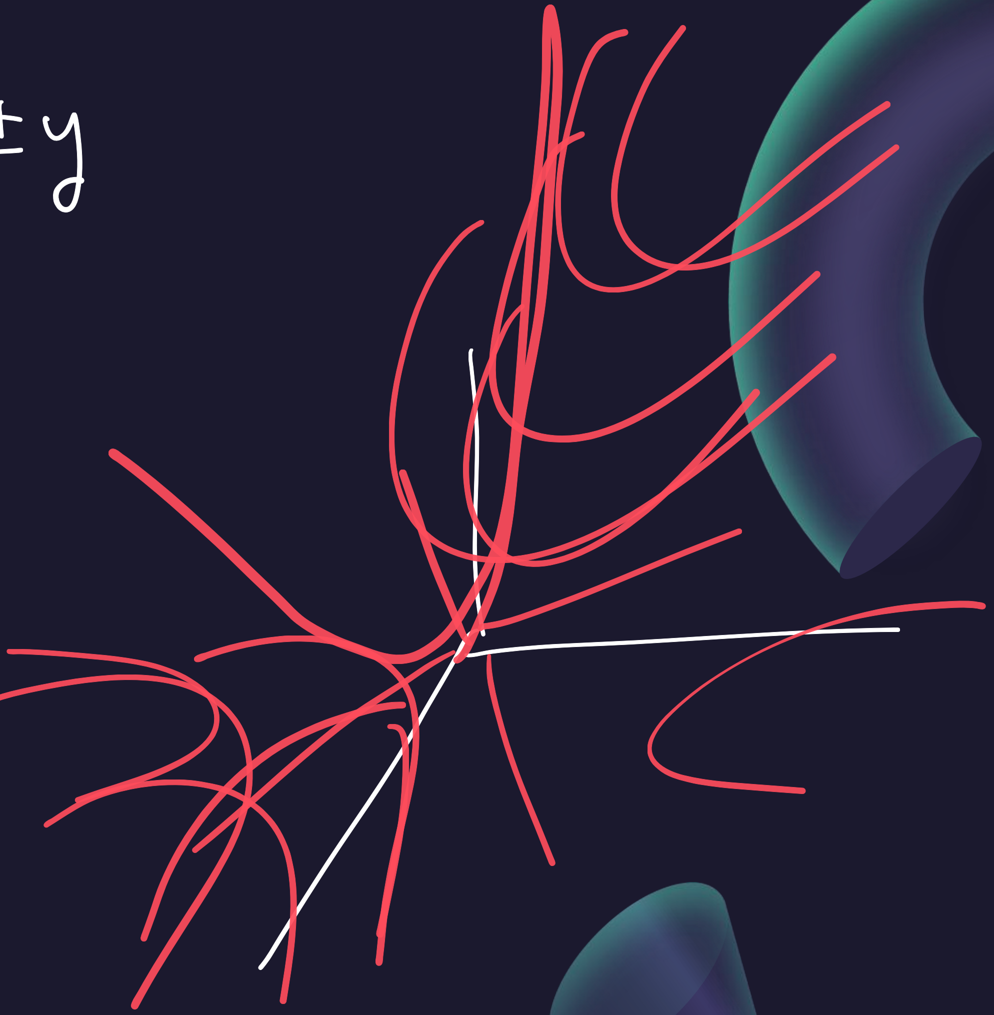
$z=0 \qquad 0 = x^2 - y^2 \Rightarrow x = \pm y$



$z=1 \qquad 1 = x^2 - y^2$

$z=2 \qquad 2 = x^2 - y^2$

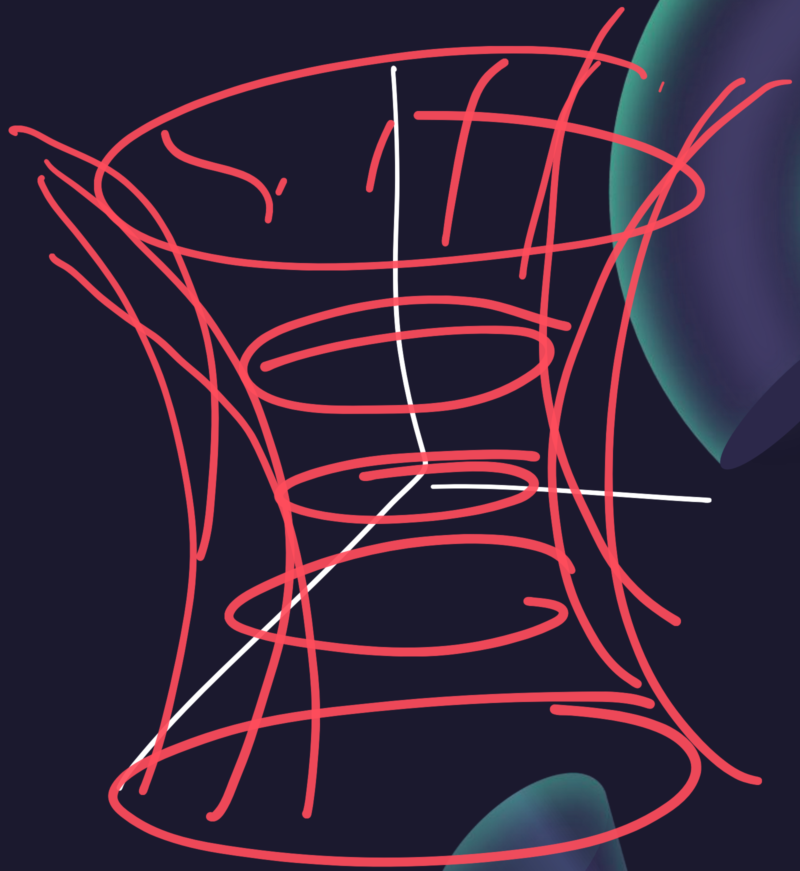
$y=0 \qquad z = x^2$



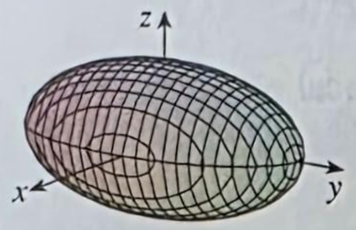
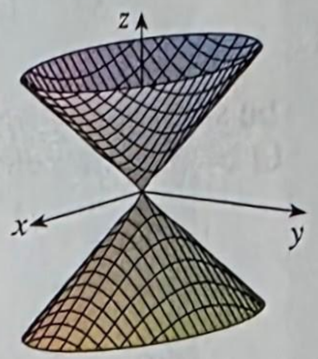
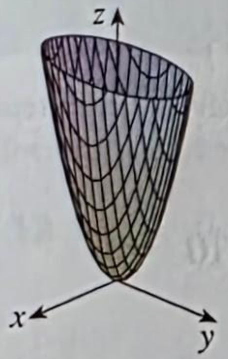
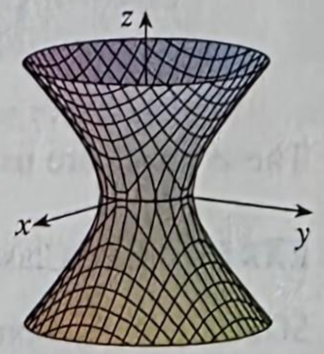
Example: Sketch the surface $x^2 + y^2 - z^2 = 1.$

$y = 0 \qquad x^2 - z^2 = 1$

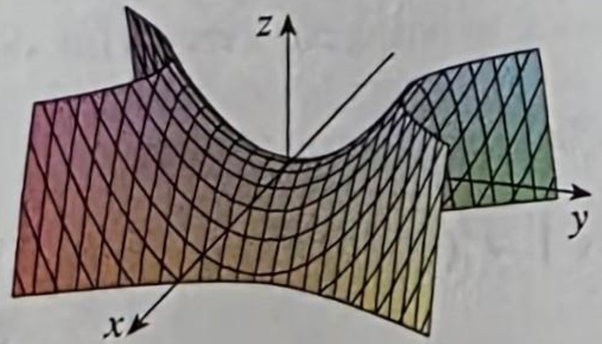
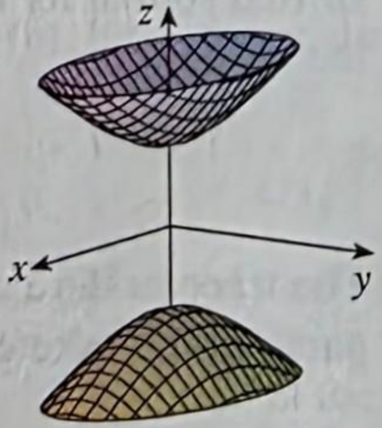
$x = 0 \qquad y^2 - z^2 = 1$



Types of Quadrics (1)

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.

Types of Quadrics (2)

Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

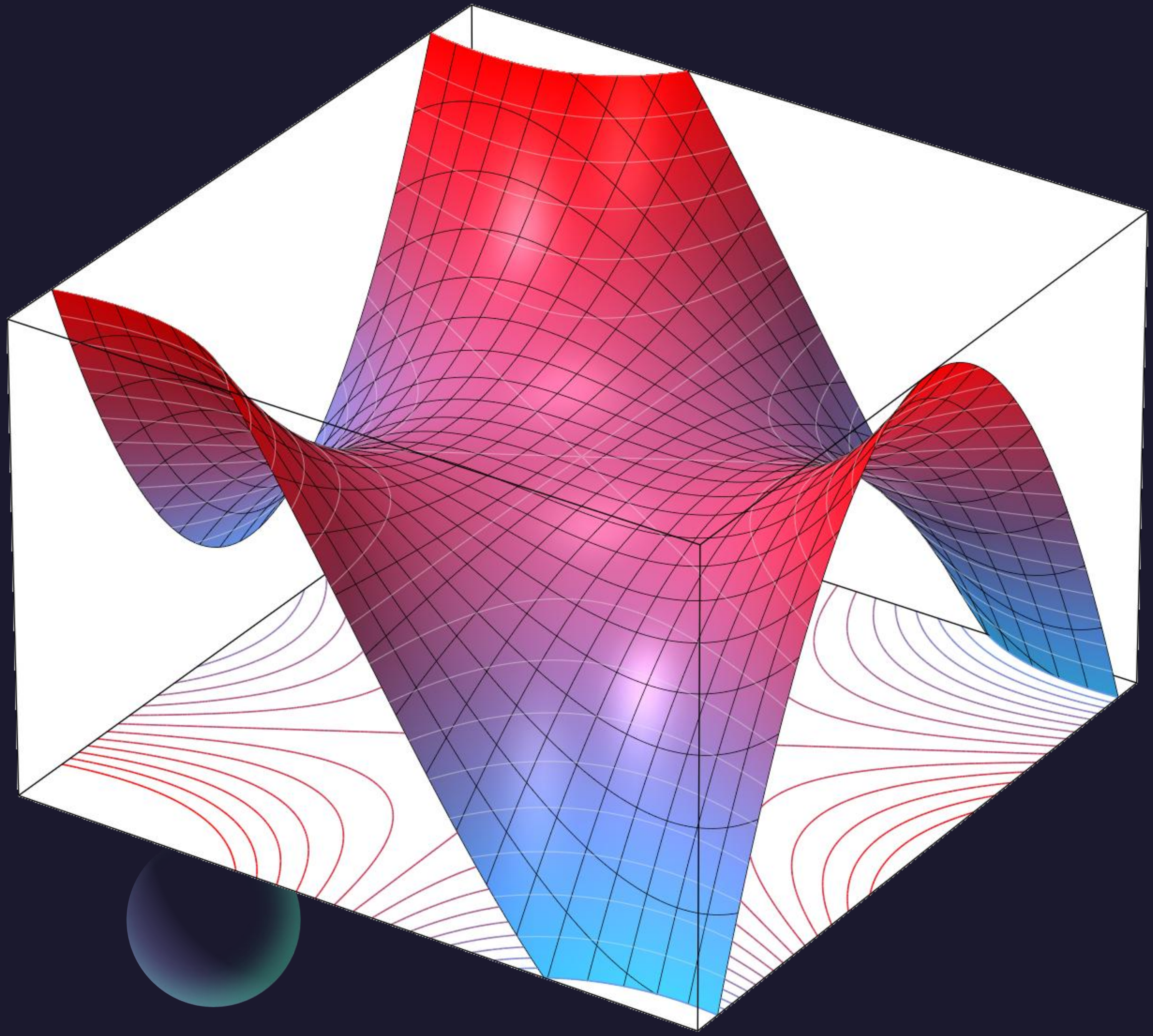
Example: Sketch the surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

Example: Sketch the surface $x^2 + 2z^2 - 6x - y + 10 = 0$. (Extra space)

Questions?

Thank you

Until next time.



ALVARO: Start the recording!

“Calculus 3”

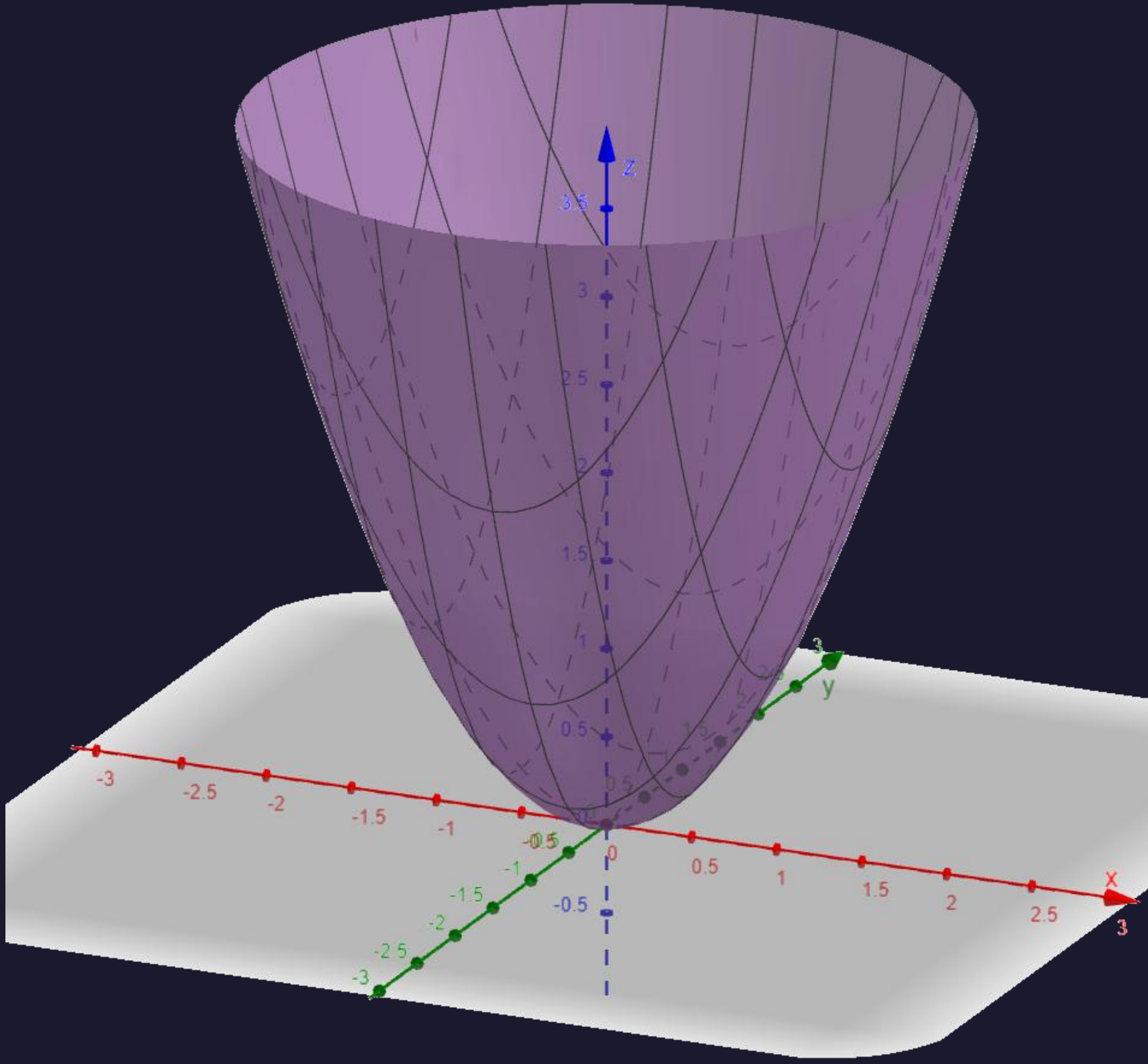
Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

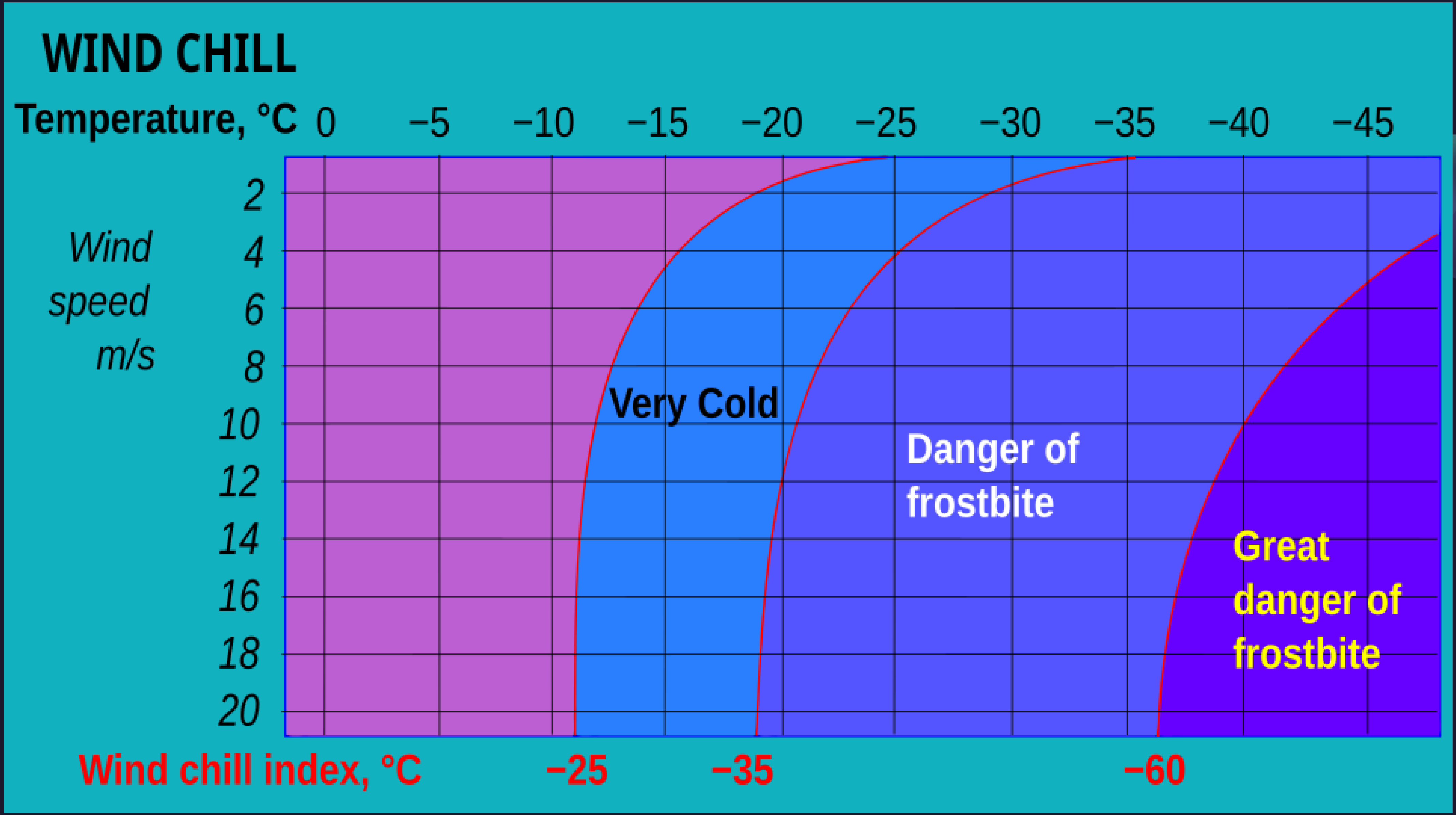
Functions of Several Variables

Today – Functions!

- Functions of Two Variables
- Domain and Range
- Graphs
- Level Curves
- Functions of More Than Two Variables



Functions of Two Variables



Functions of Two Variables

The standard wind chill formula for Environment Canada is:[3]

$$T_{\mathrm{wc}} = 13.12 + 0.6215 T_{\mathrm{a}} - 11.37 v^{+0.16} + 0.3965 T_{\mathrm{a}} v^{+0.16},$$

where T_{wc} is the wind chill index, based on the Celsius temperature scale; T_{a} is the air temperature in degrees Celsius; and v is the wind speed at 10 m (33 ft) standard anemometer height, in kilometres per hour.[11]

When the temperature is −20 °C (−4 °F) and the wind speed is 5 km/h (3 mph), the wind chill index is −24. If the temperature remains at −20 °C and the wind speed increases to 30 km/h (19 mph), the wind chill index falls to −33.

The equivalent formula in US customary units is:[12][3]

$$T_{\mathrm{wc}} = 35.74 + 0.6215 T_{\mathrm{a}} - 35.75 v^{+0.16} + 0.4275 T_{\mathrm{a}} v^{+0.16},$$

where T_{wc} is the wind chill index, based on the Fahrenheit scale; T_{a} is the air temperature in degrees Fahrenheit; and v is the wind speed in miles per hour.[13]

Functions of Two Variables, Domain, and Range

Definition

A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is,

$$\{f(x, y) \mid (x, y) \in D\}.$$

Functions of Two Variables, Domain, and Range

Example: Sketch the domain of the function, and evaluate at (3,2)

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

Example: Sketch the domain of the function, and evaluate at (3,2)

$$f(x, y) = x \cdot \ln(y^2 - x)$$

Example: Find the domain and range of the function

$$f(x, y) = \sqrt{9 - x^2 - y^2}$$

Questions?

Thank you

Until next time.