"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 6

1

Any Reminders? Any Questions?

- Class ends at 3:15.
- Slides are being posted on GitHub! https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- Request videos!!

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Questions?

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ALVARO: Start the recording!

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Partial Derivatives - Examples

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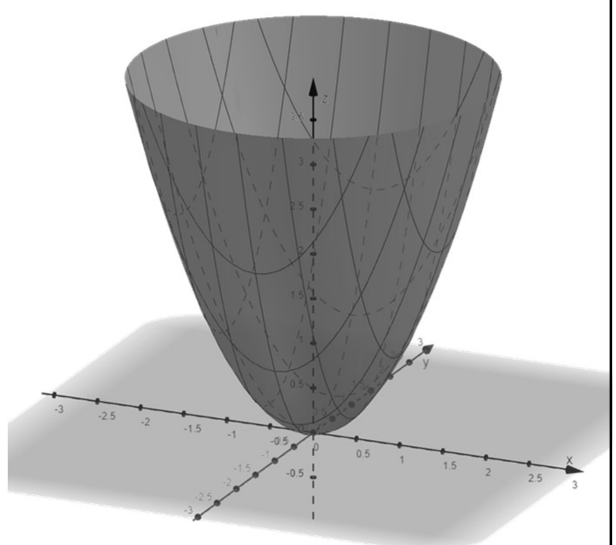
Today – Derivatives!

- Partial Derivatives
- Interpretation
- Higher Derivatives
- PDEs



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Partial Derivatives – The Limit Definition

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

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Partial Derivatives – Notation

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

8

Example: Find the partial derivatives of $f(x, y) = 4 - x^2 - y^2$ at (1,1) and interpret those as slopes.

9

Example: Find the partial derivatives of $f(x, y) = 4 - x^2 - y^2$ at (1,1) and interpret those as slopes. [Extra space]

10

Example: Find the partial derivatives of $f(x,y) = x \cdot \ln(y^2 - x)$ at (3,2).

11

Example: Find the partial derivatives of $f(x,y) = x \cdot \ln(y^2 - x)$ at (3,2).

[Extra space]

12

Example: Find the partial derivatives of the Body-Mass-Index formula
$B(m,h) = m/h^2$ at $m = 64$kg and $h = 1.68$m.

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Example: Find the partial derivatives of the Body-Mass-Index formula
$B(m,h) = m/h^2$ at $m = 64$kg and $h = 1.68$m. [Extra]

14

Example: Find the partial derivatives of $f(x, y, z) = \sin(x^2 + y^2 + z^2)$ at (1,2,3).

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Example: Find the partial derivatives of $f(x, y, z) = \sin(x^2 + y^2 + z^2)$ at (1,2,3). [Extra space]

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Higher Partial Derivatives

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Example: Find the second partial derivatives of

$$f(x,y) = 4x^2y - x^3 - y^2$$

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Example: Find the second partial derivatives of [Extra space]

$$f(x,y) = 4x^2y - x^3 - y^2$$

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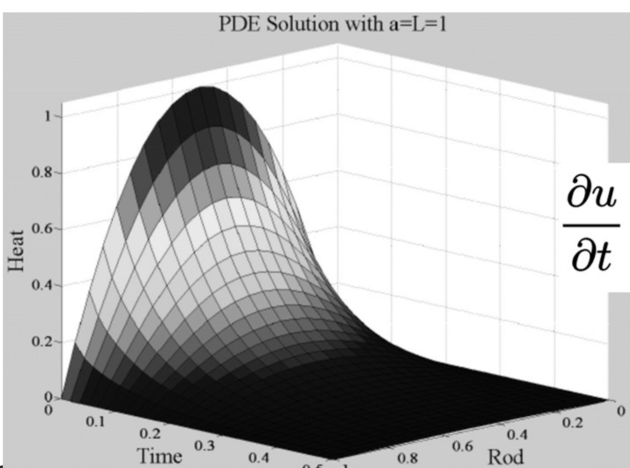
Partial Differential Equations



Example: The Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

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Example: Show that the function w(x,t) = sin(x – a·t) satisfies the wave equation:

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$$

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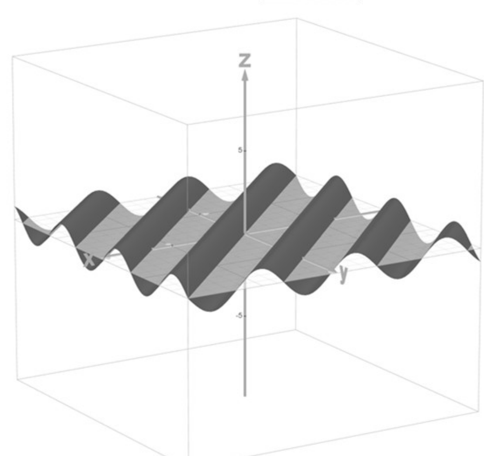
$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$$



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Questions?

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Tangent Planes

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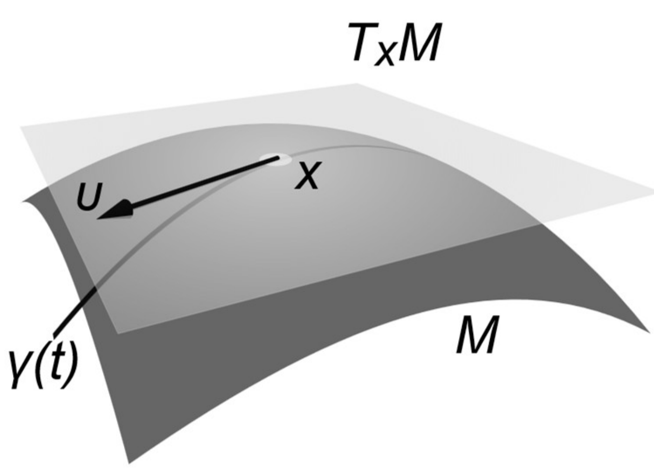
Today – Tangent Planes!

- Equation
- Linear Approximations
- Differentiability
- Differentials



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Equation of a Tangent Plane

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Equation of a Tangent Plane

2 Equation of a Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

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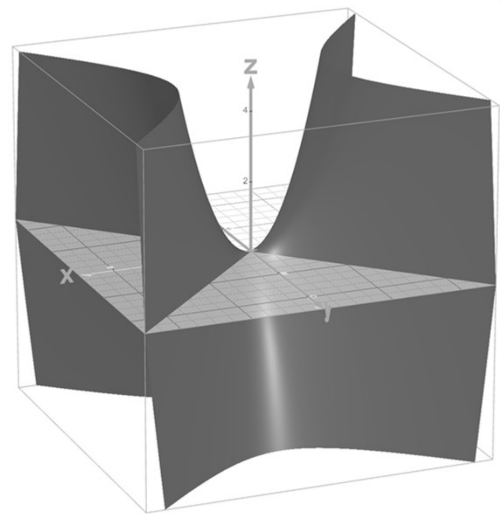
Example: Find the tangent plane at (3,2,5) to the hyperbolic paraboloid

$$z = x^2 - y^2$$



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Example: Find the tangent plane at (3,2,5) to the hyperbolic paraboloid

$$z = x^2 - y^2$$

[Extra space]

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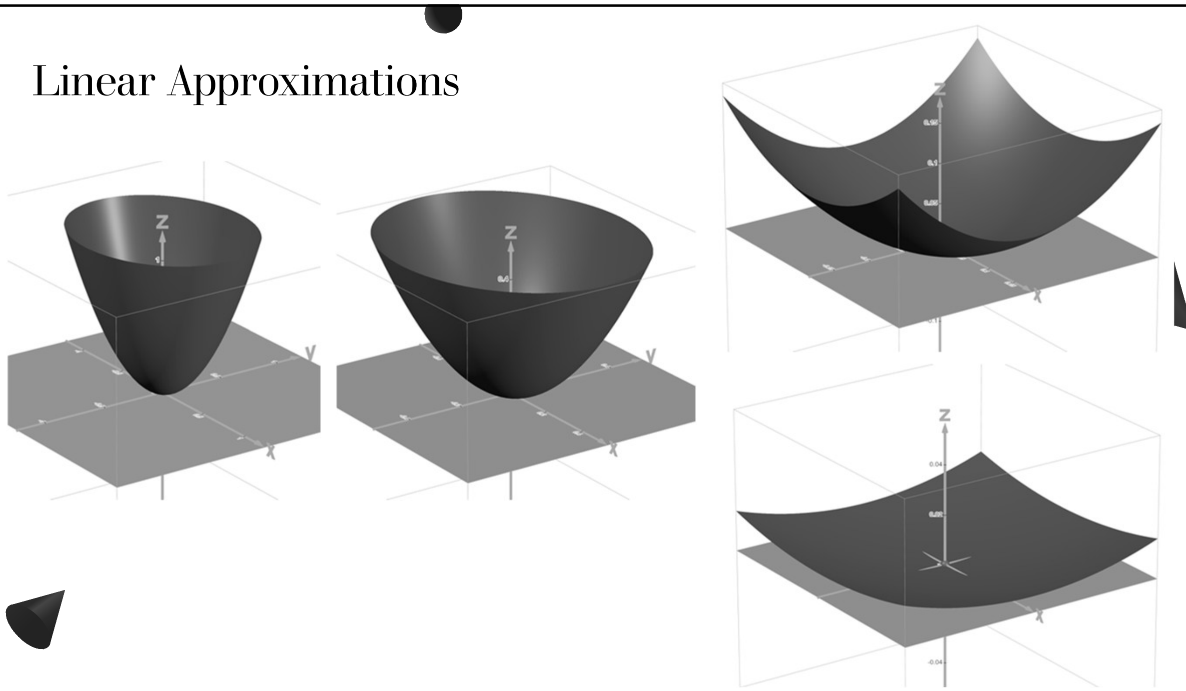
Linear Approximations



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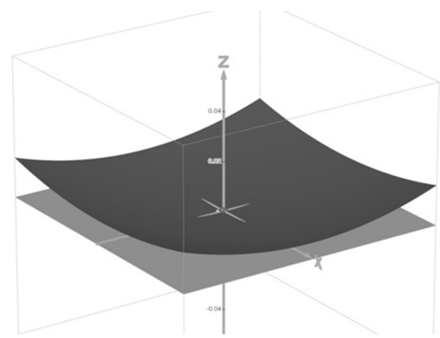
Linear Approximations



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Example: Find the linear approximation at (x,y)=(3,-1) of

$$f(x, y) = 2x^2 - xy + 3y^2$$

33

Example: Find the linear approximation at (x,y)=(3,-1) of

$$f(x, y) = 2x^2 - xy + 3y^2$$

[Extra space]

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Example: Find the linear approximation at (x,y)=(2,3) and use it to approximate $\sqrt{21}$, where

$$f(x, y) = \sqrt{x^2 + 4y}$$

35

Example: Find the linear approximation at (x,y)=(2,3) and use it to approximate $\sqrt{21}$, where

$$f(x, y) = \sqrt{x^2 + 4y}$$

[Extra space]

$z = \frac{3}{2} + \frac{x}{2} + \frac{y}{2}$

$\sqrt{21} = 4.58257569496\ldots$

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Questions?

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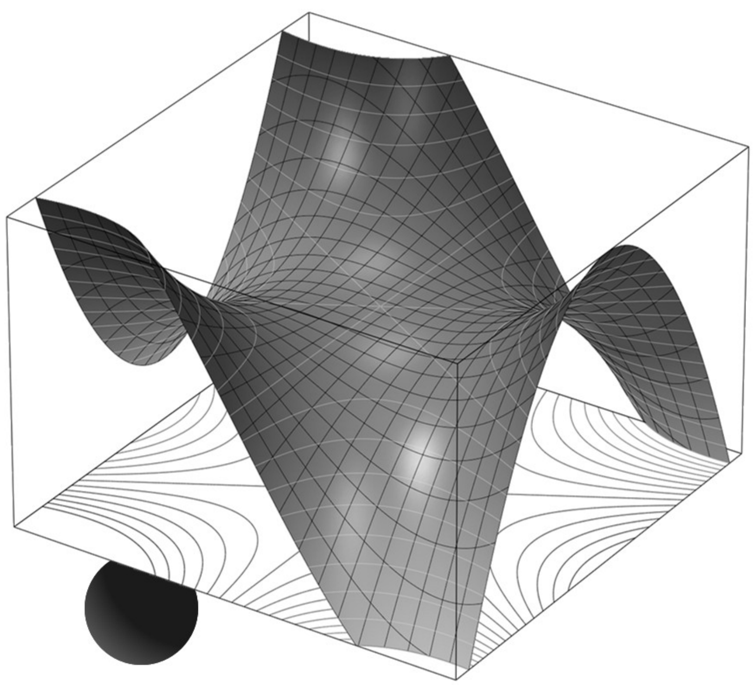
Thank you

Until next time.



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The Chain Rule

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Today – The Chain Rule!

- The Single Variable Case
- Chain Rule with One Parameter
- Chain Rule with Two Parameters

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The Good Ol' Chain Rule

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Example: Find the derivative of f(g(t)) with respect to t where

$f(x) = \sin(x)$ and $g(t) = t^2 + 1$

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Example: Find the derivative of f(g(t)) with respect to t where

$f(x) = \sin(x)$ and $g(t) = t^2 + 1$ [Extra space]

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The New Chain Rule – Case 1

1 The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

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Example: Find the derivative of f(g(t),h(t)) with respect to t where

$f(x, y) = x^2 + y$ and $g(t) = 3t^4 + 1$, $h(t) = 3t$

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Example: Find the derivative of f(g(t),h(t)) with respect to t where

$f(x, y) = x^2 + y$ and $g(t) = 3t^4 + 1$, $h(t) = 3t$ [Extra]

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The New Chain Rule – Case 2

2 The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

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Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$f(x,y) = x^2y^3$ and $g(s,t) = s \cdot \cos(t)$, $h(s,t) = s \cdot e^{2t}$

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Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$f(x,y) = x^2y^3$ and $g(s,t) = s \cdot \cos(t)$, $h(s,t) = s \cdot e^{2t}$

[Extra]

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Questions?

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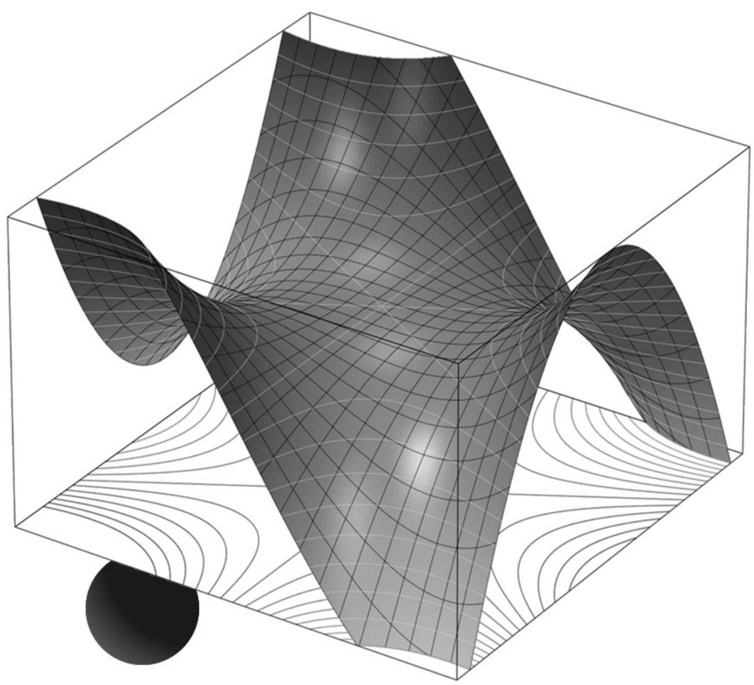
Thank you

Until next time.



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