“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 6

Any Reminders? Any Questions?

- Class ends at 3:15.
- Slides are being posted on GitHub!
  https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- Request videos!!

Questions?

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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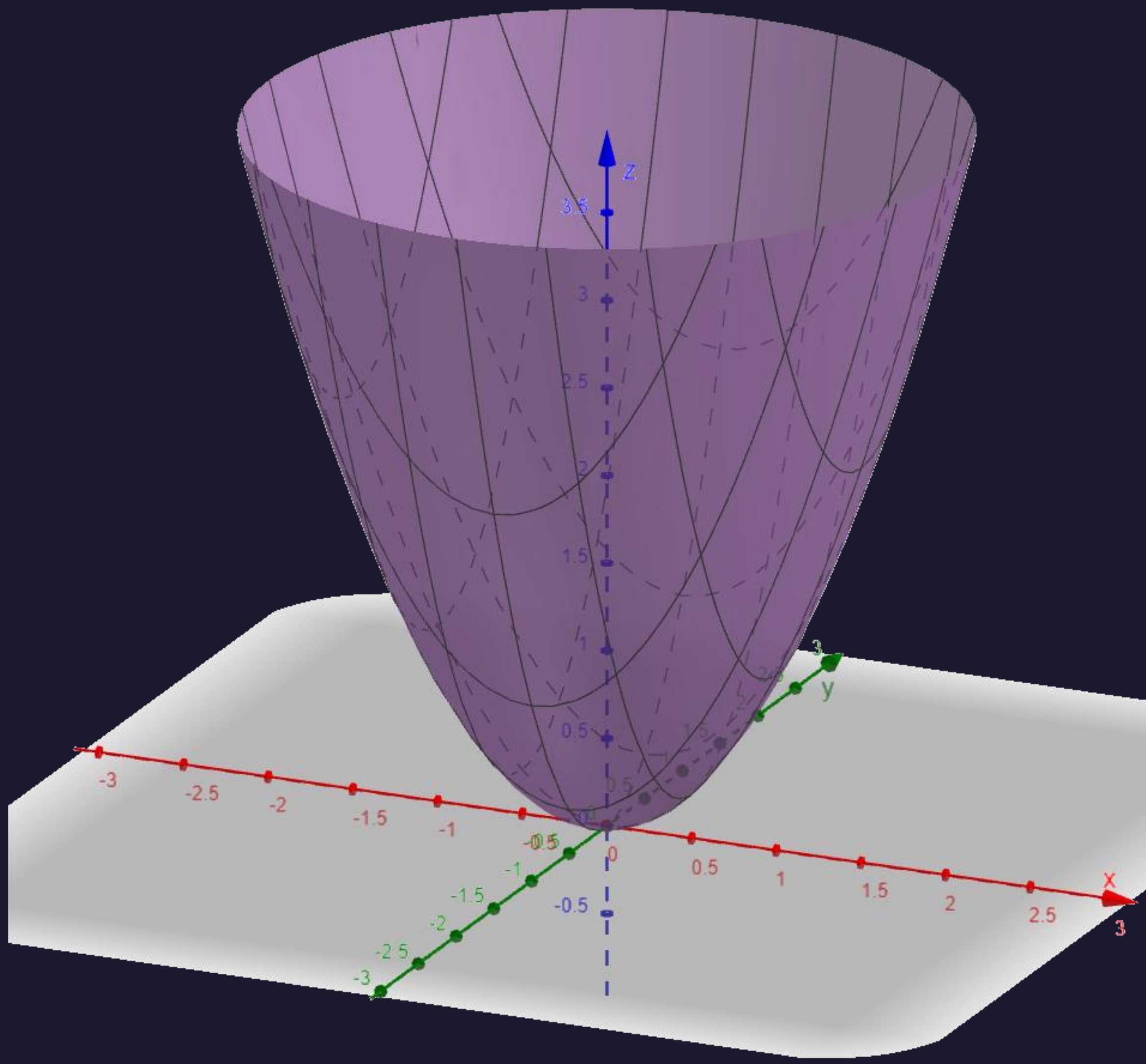
Partial Derivatives - Examples

Today – Derivatives!

- Partial Derivatives
- Interpretation
- Higher Derivatives
- PDEs



Partial Derivatives – The Limit Definition

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Partial Derivatives – Notation

$\frac{df}{dx}$

If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Example: Find the partial derivatives of $f(x,y) = 4 - x^2 - y^2$ at (1,1) and interpret those as slopes.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(4 - x^2 - y^2\right) = 0 - 2x + 0 = -2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(4 - x^2 - y^2\right) = 0 + 0 - 2y$$

$$\frac{\partial f}{\partial x}(1,1) = \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = (-2x)\Big|_{(1,1)} = -2 \quad , \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = -2$$

Example: Find the partial derivatives of $f(x,y) = 4 - x^2 - y^2$ at (1,1) and interpret those as slopes. [Extra space]

$$z = 4 - x^2 - y^2$$

$$= 4 - (x^2 + y^2)$$

$$z = 4 - 1^2 - 1^2 = 2$$

$$z = x^2 + y^2$$

$(1,1,2)$

$$-2 = \left.\frac{\partial f}{\partial y}\right|_{(1,1)}$$

$$\leftarrow \text{slope} = -2 = \left.\frac{\partial f}{\partial x}\right|_{(1,1)}$$

Example: Find the partial derivatives of $f(x,y) = x \cdot \ln(y^2 - x)$ at (3,2).

PROD RULE

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x \cdot \ln(y^2 - x)\right) \overset{\downarrow}{=} 1 \cdot \ln(y^2 - x) + x \cdot \frac{-1}{y^2 - x}$$

$$\boxed{(\ln x)' = \frac{1}{x}}$$

$$= \ln(y^2 - x) - \frac{x}{y^2 - x}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(3,2)} = 0 - \frac{3}{1} = -3$$

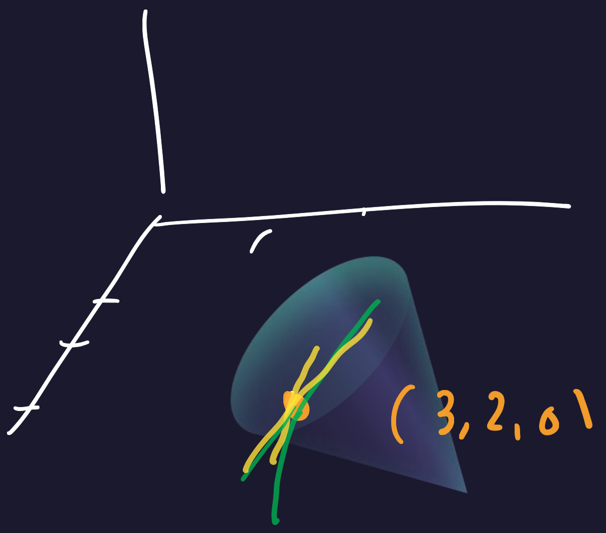
$(2^2 - 3 = 1)$

$\ln 1 = 0$

Example: Find the partial derivatives of $f(x,y) = x \cdot \ln(y^2 - x)$ at (3,2). [Extra space]

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x \cdot \ln(y^2 - x)\right) = x \cdot \frac{2y}{y^2 - x} = \frac{2xy}{y^2 - x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(3,2)} = \frac{2 \cdot 3 \cdot 2}{1} = 12$$



Example: Find the partial derivatives of the Body-Mass-Index formula

$$B(m,h) = m/\mathrm{h}^2 \quad \text{at} \quad \mathrm{m} = 64\mathrm{kg} \quad \text{and} \quad \mathrm{h} = 1.68\mathrm{m}.$$

$$\frac{\partial B}{\partial m} = \frac{1}{h^2} \quad \text{eval} \Rightarrow \left.\frac{\partial B}{\partial m}\right|_{(\ ,\ )} > 0$$

$$\frac{d}{dh}\left(\frac{1}{h^2}\right) = \frac{-2}{h^3}$$

$$\frac{\partial B}{\partial h} = \frac{-2m}{h^3} \quad \text{eval} \Rightarrow \left.\frac{\partial B}{\partial h}\right|_{(\ ,\ )} < 0$$

$$\frac{d}{dh}\left(h^n\right) = n \cdot h^{n-1}$$

$$\frac{d}{dh}\left(h^{-2}\right) = -2 \cdot h^{-3}$$

Example: Find the partial derivatives of the Body-Mass-Index formula

$B(m, h) = m/\mathrm{h}^2$ at m = 64kg and h = 1.68m. [Extra]

Example: Find the partial derivatives of $f(x, y, z) = \sin(x^2 + y^2 + z^2)$ at (1,2,3).

$$\frac{\partial f}{\partial x} = \cos(x^2+y^2+z^2) \cdot 2x$$

$$\frac{\partial f}{\partial y} = \cos(x^2+y^2+z^2) \cdot 2y$$

$$\frac{\partial f}{\partial z} = \cos(x^2+y^2+z^2) \cdot 2z$$

Example: Find the partial derivatives of $f(x, y, z) = \sin(x^2 + y^2 + z^2)$ at $(1,2,3)$.

[Extra space]

Higher Partial Derivatives

$$z = f(x, y)$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}(x, y)\right) = \frac{\partial^2 f}{\partial y\, \partial x} = f_{yx}$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial^2 f}{\partial y\, \partial x} = f_{yx},\ f_{xy},\ f_{yy}$$

Example: Find the second partial derivatives of

$$f(x, y) = 4x^2y - x^3 - y^2$$

$$\frac{\partial f}{\partial x} = 8xy - 3x^2 \rightarrow f_{xx} = 8y - 6x$$

$$\rightarrow f_{yx} = 8x$$

$$\frac{\partial f}{\partial y} = 4x^2 - 2y \rightarrow f_{yy} = -2$$

$$\rightarrow f_{xy} = 8x$$

$$f$$

$$f_{xx} = (f_x)_x \qquad f_{xy} = (f_x)_y$$

Example: Find the second partial derivatives of

$$f(x, y) = 4x^2y - x^3 - y^2$$

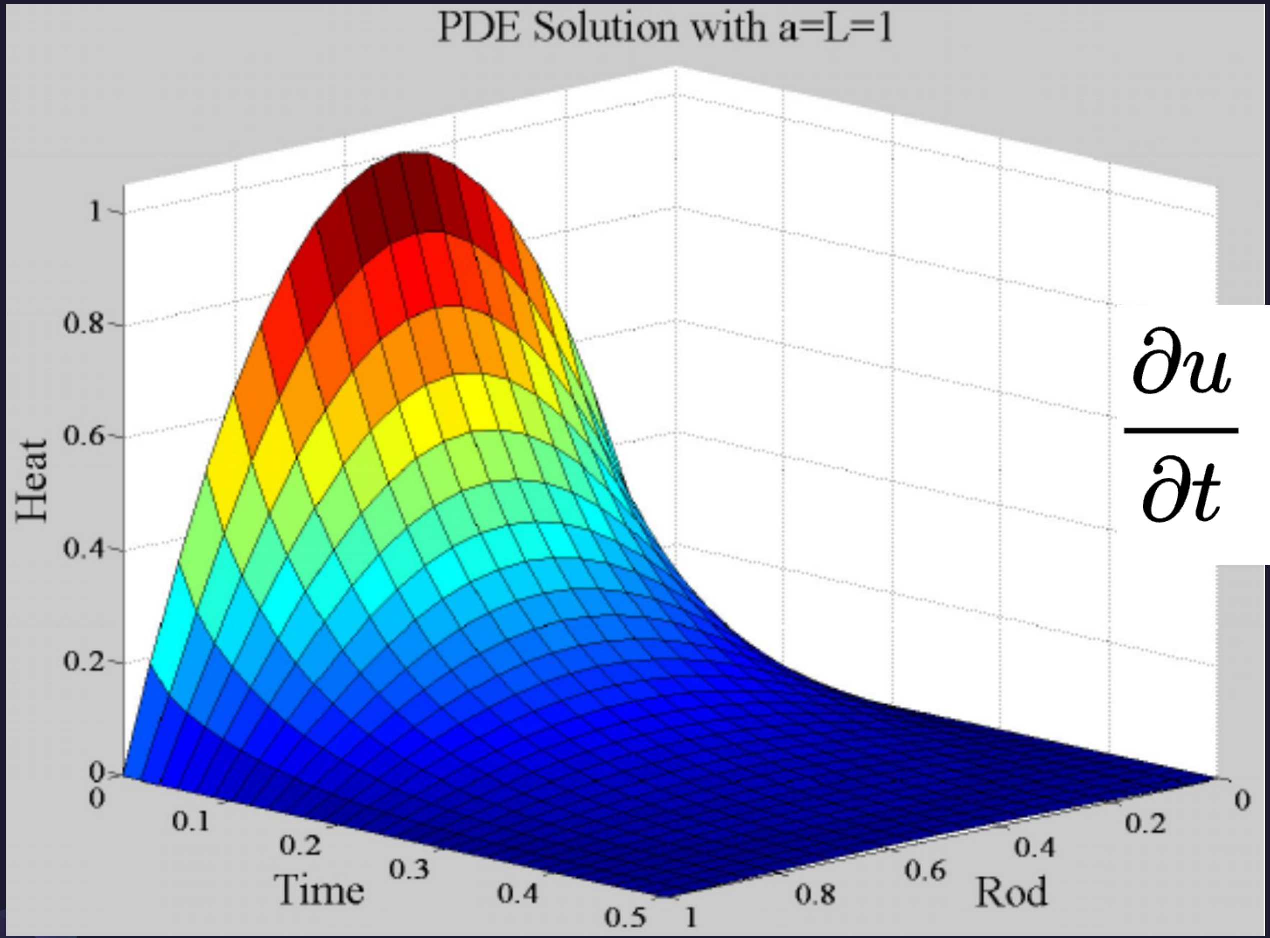
[Extra space]

$\underline{\text{FIX!}}$

$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = f_{xy} = (f_x)_y$$

Partial Differential Equations



Example: The Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Example: Show that the function $w(x,t) = \sin(x - a \cdot t)$ satisfies the wave equation:

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$$

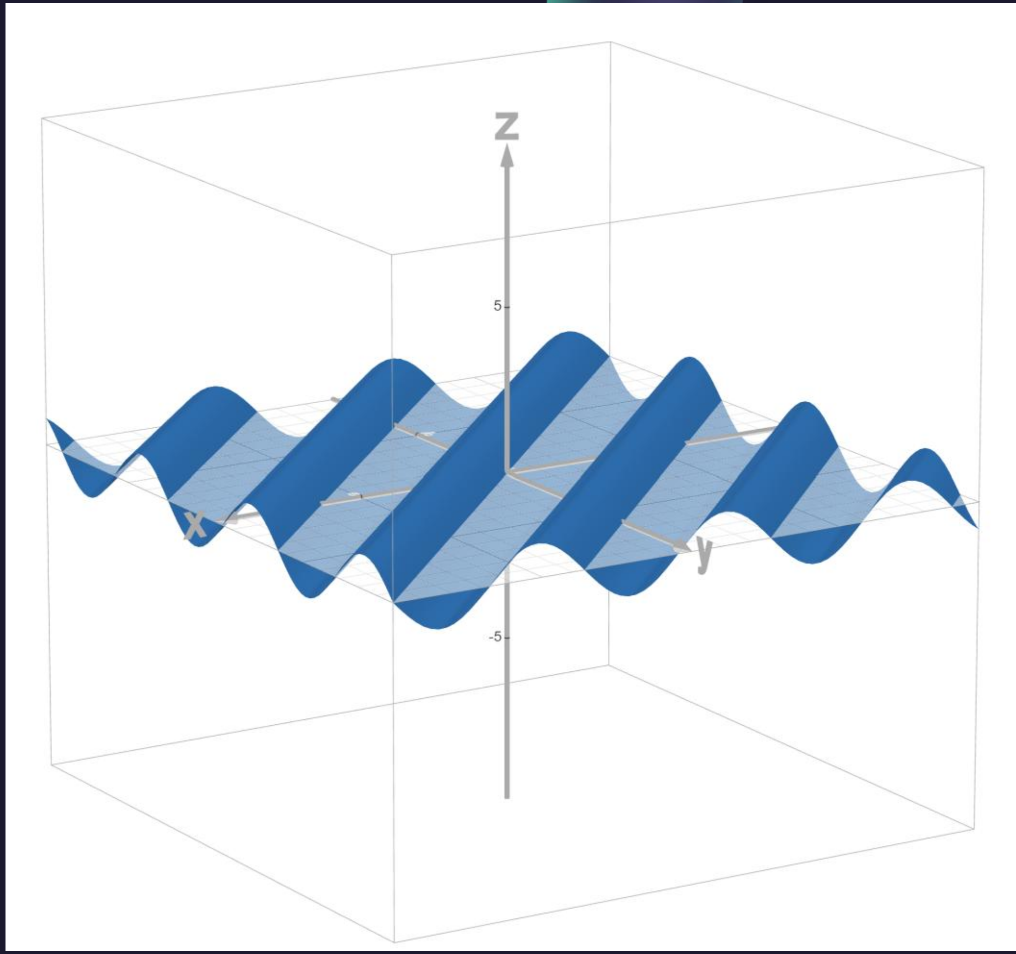
$$\frac{\partial w}{\partial x} = \cos(x - a \cdot t) \Rightarrow \frac{\partial^2 w}{\partial x^2} = -\sin(x - at) \quad ✓$$

$$\frac{\partial w}{\partial t} = -a \cdot \cos(x - at) \Rightarrow \frac{\partial^2 w}{\partial t^2} = -a \cdot (-\sin(x - at)) \cdot (-a)$$

$$= -a^2 \cdot \sin(x - at) \quad ✓$$

Example: Show that the function w(x,t) = sin(x – a·t) satisfies the wave equation:

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$$



Questions?

ALVARO: Start the recording!

“Calculus 3”

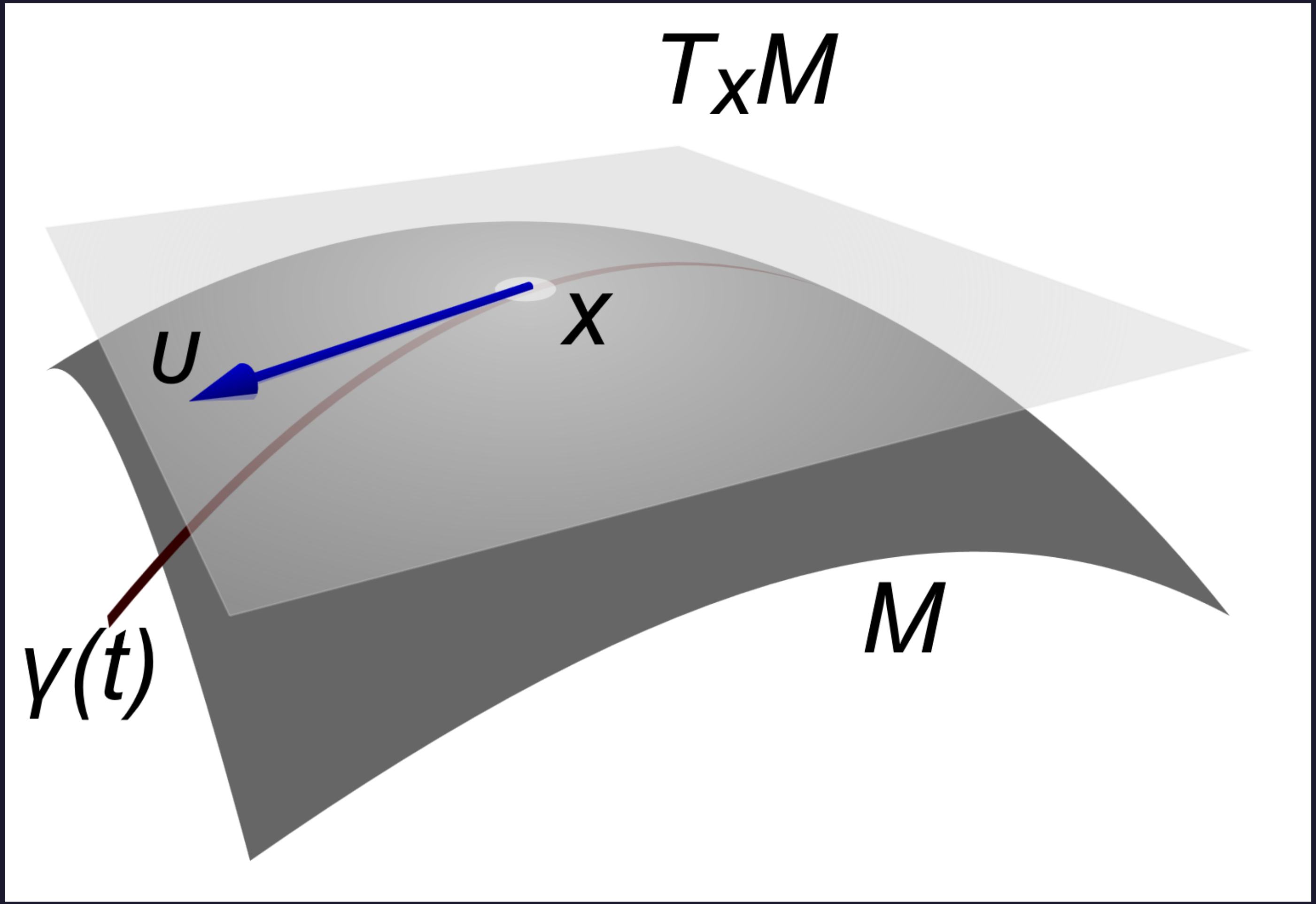
Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Tangent Planes

Today – Tangent Planes!

- Equation
- Linear Approximations
- Differentiability
- Differentials



Equation of a Tangent Plane

$z = f(x,y)$ at (x_0, y_0, z_0)

$z_0 = f(x_0, y_0)$ at $(x_0, y_0, f(x_0, y_0))$

$$z - z_0 = \overset{?}{m_1}(x - x_0) + \overset{?}{m_2}(y - y_0)$$

Fix $y = y_0 \Rightarrow z - z_0 = m_1 \cdot (x - x_0) \Rightarrow m_1 = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$

Fix $x = x_0 \Rightarrow z - z_0 = m_2 (y - y_0) \Rightarrow m_2 = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$

Equation of a Tangent Plane

2 Equation of a Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note the similarity between the equation of a tangent plane and the equation of a tangent line:

$$y - y_0 = f'(x_0)(x - x_0)$$

Example: Find the tangent plane at (3,2,5) to the hyperbolic paraboloid

$$z = x^2 - y^2 \qquad z_0 = 3^2 - 2^2 = 5$$

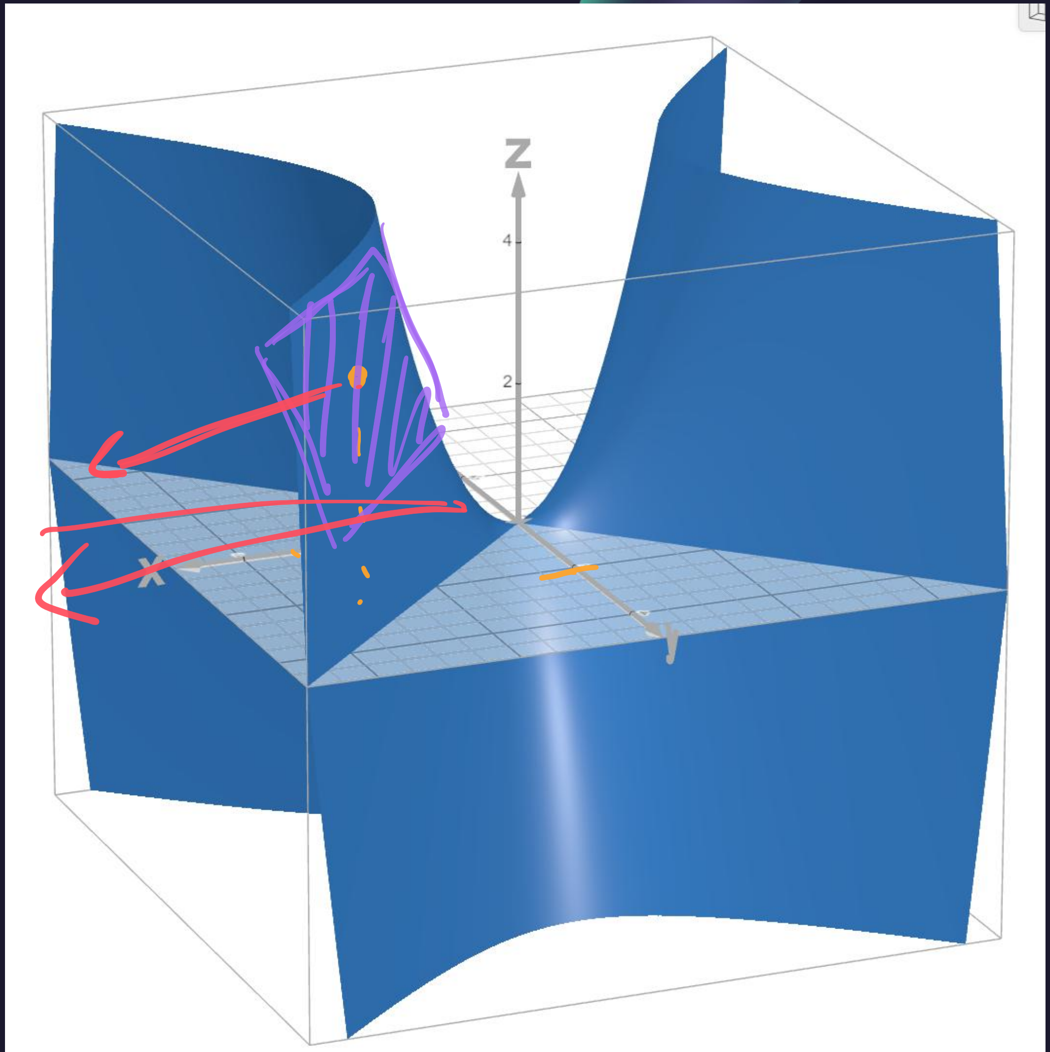
$$z - z_0 = \left.\frac{\partial f}{\partial x}\right|_{(3,2)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(3,2)} (y - y_0)$$

$$\frac{\partial f}{\partial x} = 2x \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(3,2)} = 6$$

$$\frac{\partial f}{\partial y} = -2y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(3,2)} = -4$$

$$\boxed{z - 5 = 6\cdot(x-3) - 4\cdot(y-2)}$$

$$n = (6, -4, -1)$$



Example: Find the tangent plane at (3,2,5) to the hyperbolic paraboloid

$$z = x^2 - y^2$$

[Extra space]

$$z - 5 = 6(x-3) - 4(y-2)$$

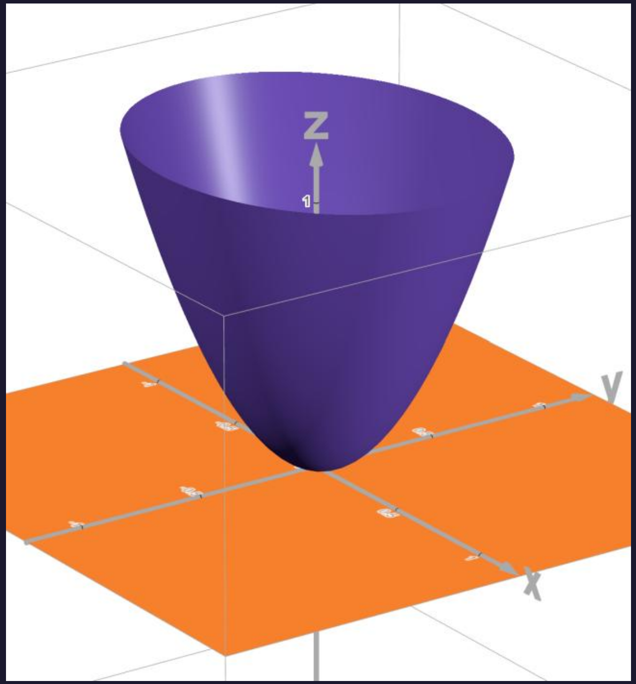
$$6(x-3) - 4(y-2) - z + 5 = 0$$

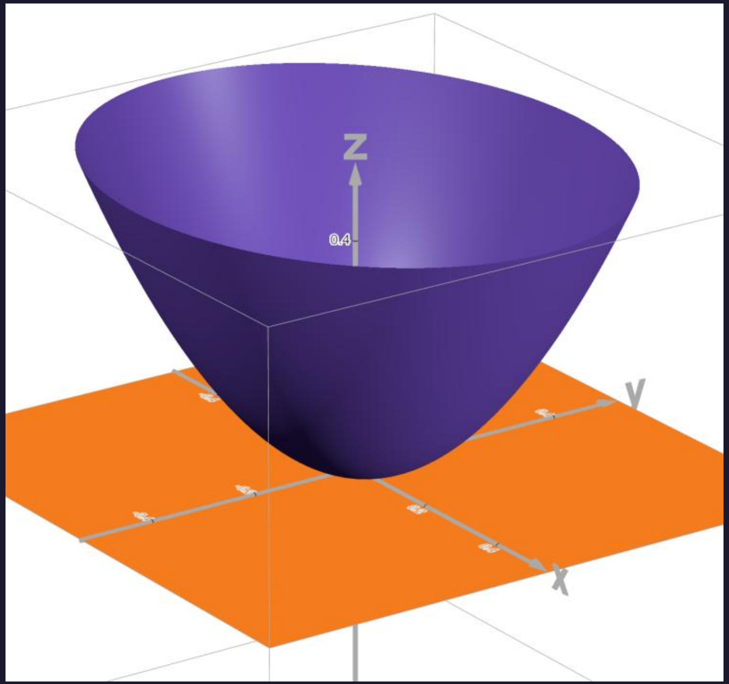
$$6x - 4y - z - 18 + 8 + 5$$

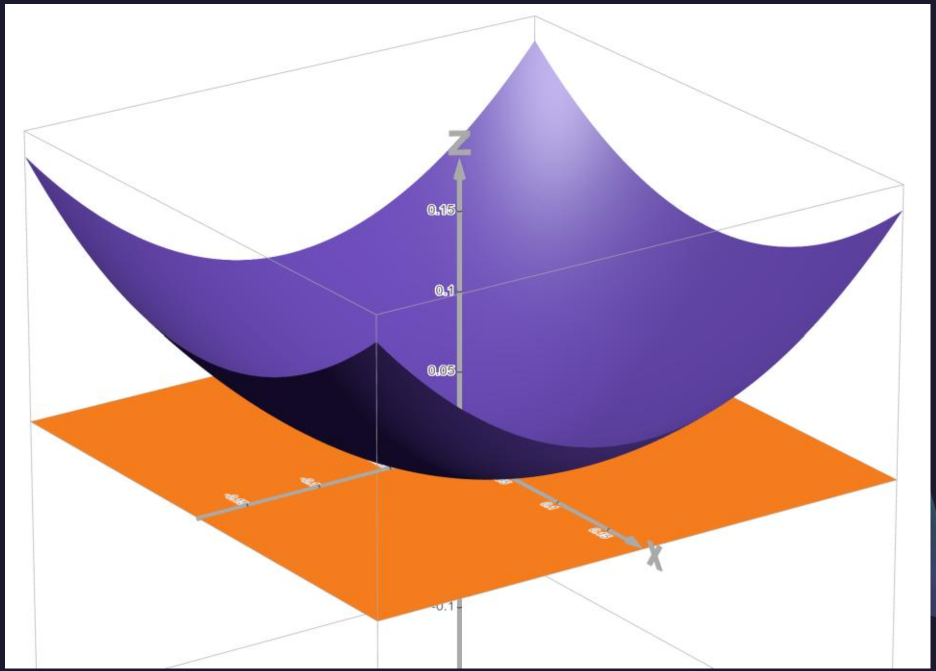
$$\boxed{6x - 4y - z - 5 = 0}$$

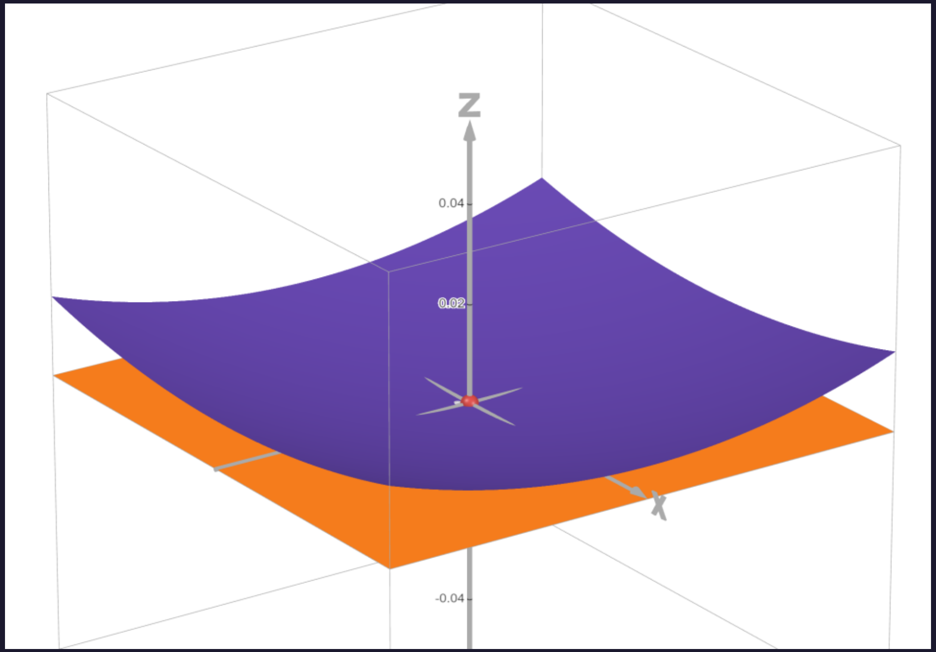
$(6,-4,-1)$

Linear Approximations

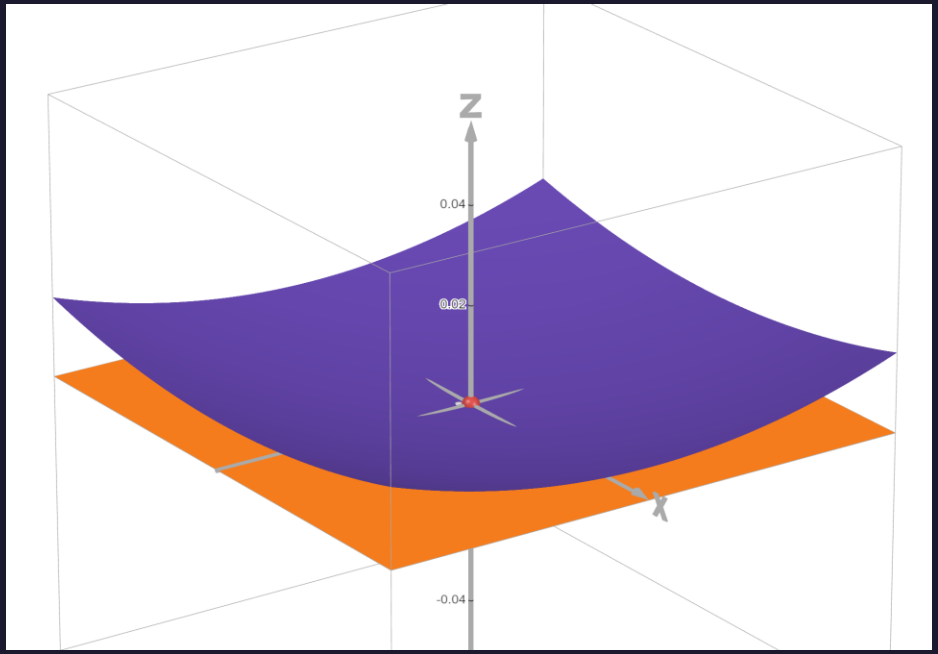








Linear Approximations



$$z = f(x,y)$$

$$z_0 = f(x_0,y_0)$$

$$z - z_0 = f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$$

$$z = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)}$$

$$f(x,y) \approx L(x,y)$$

Example: Find the linear approximation at (x,y)=(3,-1) of

$$f(x,y) = 2x^2 - xy + 3y^2$$

$\frac{\partial f}{\partial x} = 4x - y \xrightarrow{(3,-1)} 13$

$\frac{\partial f}{\partial y} = -x + 6y \rightarrow -9$

$z_0 = f(3,-1) = 2\cdot 3^2 - 3\cdot(-1) + 3\cdot(-1)^2 = 18 + 3 + 3 = 24$

$z - 24 = 13\cdot(x-3) - 9\cdot(y+1)$

$L(x,y) = 24 + 13\cdot(x-3) - 9\cdot(y+1)$

$f(3.1, -0.9) \approx L(3.1, -0.9)$

Example: Find the linear approximation at (x,y)=(3,-1) of

$$f(x,y) = 2x^2 - xy + 3y^2$$

[Extra space]

Example: Find the linear approximation at (x,y)=(2,3) and use it to approximate $\sqrt{21}$, where

$$f(x,y) = \sqrt{x^2+4y}$$

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+4y}} = \frac{x}{\sqrt{x^2+4y}} \qquad \left.\frac{\partial f}{\partial x}\right|_{(2,3)} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{4}{2\sqrt{x^2+4y}} = \frac{2}{\sqrt{x^2+4y}} \qquad \left.\frac{\partial f}{\partial y}\right|_{(2,3)} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$z_0 = f(2,3) = 4$$

Example: Find the linear approximation at (x,y)=(2,3) and use it to approximate $\sqrt{21}$, where

$$f(x,y) = \sqrt{x^2+4y}$$

[Extra space]

$$z - 4 = \frac{1}{2}\cdot(x-2) + \frac{1}{2}(y-3)$$

$$f(x,y) \approx 4 + \frac{1}{2}(x-2) + \frac{1}{2}(y-3)$$

$$f(3,3) = \sqrt{21} \approx L(3,3) = 4 + \frac{1}{2}(3-2) + \frac{1}{2}\underbrace{(3-3)}_{}^{0}$$

$$= 4 + \frac{1}{2} = \boxed{4.5}$$

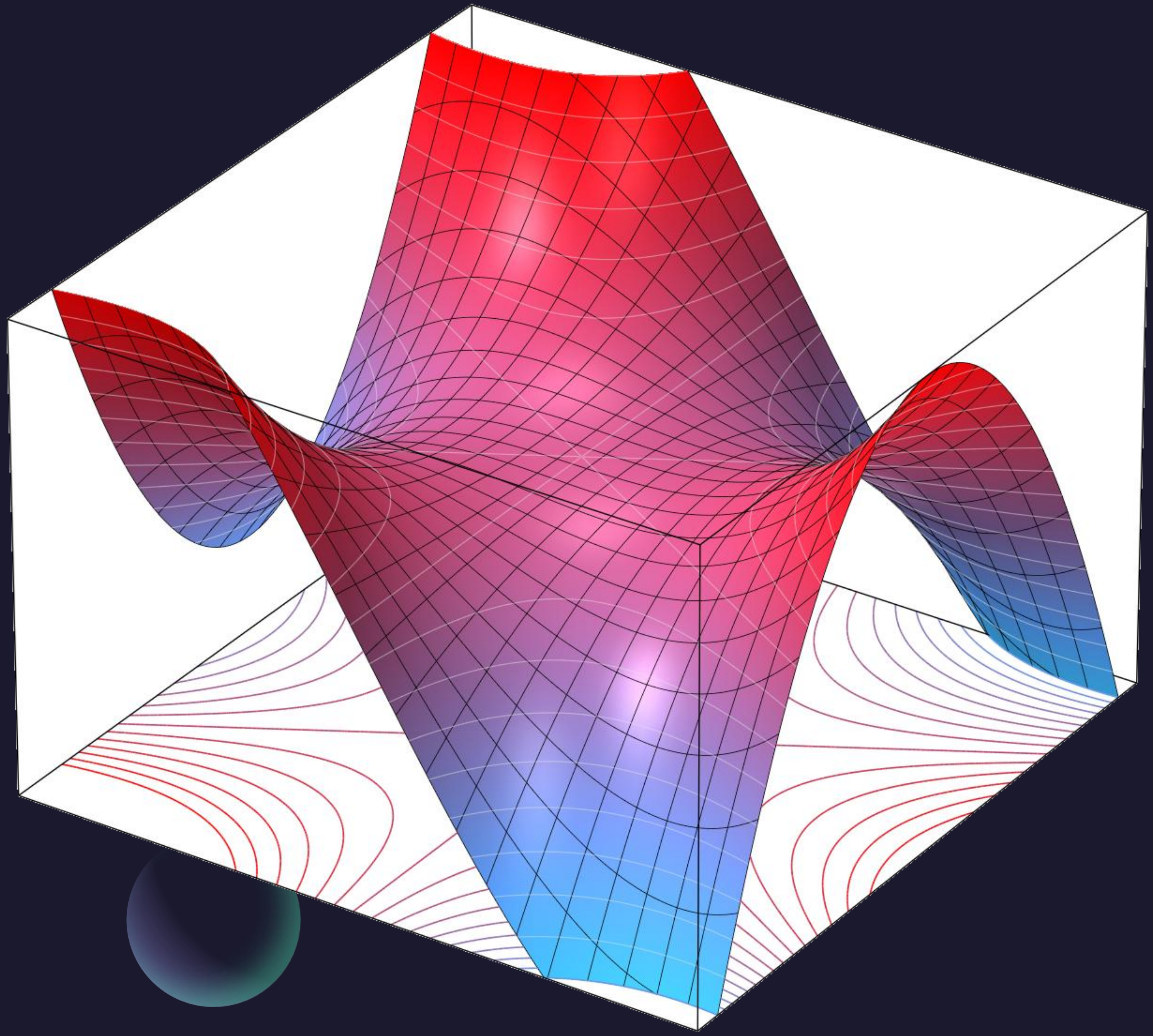
$$z = \frac{3}{2} + \frac{x}{2} + \frac{y}{2}$$

$\sqrt{21}$ = 4.58257569496…

Questions?

Thank you

Until next time.



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“Calculus 3”

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The Chain Rule

Today – The Chain Rule!

- The Single Variable Case $f(g(t))$
- Chain Rule with One Parameter $f(g(t), h(t))$
- Chain Rule with Two Parameters

$$f(g(s,t), h(s,t))$$

The Good Ol' Chain Rule

$$f(x) \quad , x = g(t)$$

$$\frac{d}{dt}\left(f(g(t))\right) = f'(g(t)) \cdot g'(t)$$

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

Example: Find the derivative of f(g(t)) with respect to t where

$$f(x) = \sin(x) \qquad \text{and} \qquad g(t) = t^2 + 1$$

Example: Find the derivative of f(g(t)) with respect to t where

$$f(x) = \sin(x) \qquad \text{and} \qquad g(t) = t^2 + 1$$

[Extra space]

The New Chain Rule – Case 1

1 The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example: Find the derivative of f(g(t),h(t)) with respect to t where

$$f(x,y) = x^2 + y \quad \text{and} \qquad g(t) = 3t^4 + 1, \quad h(t) = 3t$$

Example: Find the derivative of f(g(t),h(t)) with respect to t where

$f(x,y) = x^2 + y$ and $g(t) = 3t^4 + 1, \quad h(t) = 3t$ [Extra]

The New Chain Rule – Case 2

2 The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$$f(x,y) = x^2y^3 \quad \text{and} \quad g(s,t) = s \cdot \cos(t)\ ,\ \ h(s,t) = s \cdot e^{2t}$$

Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$$f(x,y) = x^2y^3 \quad \text{and} \quad g(s,t) = s \cdot \cos(t)\ ,\ \ h(s,t) = s \cdot e^{2t}$$

[Extra]

Questions?

Thank you

Until next time.