“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 8

Any Reminders? Any Questions?

- I will be away on
  - Monday 2/16 --- no office hours that day
  - Tuesday 2/17 --- I will send videos to watch instead of class

- I will be back teaching in-person on Thursday 2/19
- I will do some review for the midterm during Thursday's class
- I will have regular office hours 2/19 – 3:30-4:30
- I will have additional office hours 2/19 – 4:30-5:30
- Calc 3 Calc Night: MONT 104 at 6:30-8:30pm on Thursdays!

- Exam 1 is on Friday, Feb 20th

EXAM 1 -- Friday, February 20th

Exam Covers:

- **Chapter 12**
  - Sections 12.1 – 12.6
- **Chapter 14**
  - Sections 14.1, 14.3 – 14.8

(NEW) Exam Study Guide and Practice Problems in HuskyCT

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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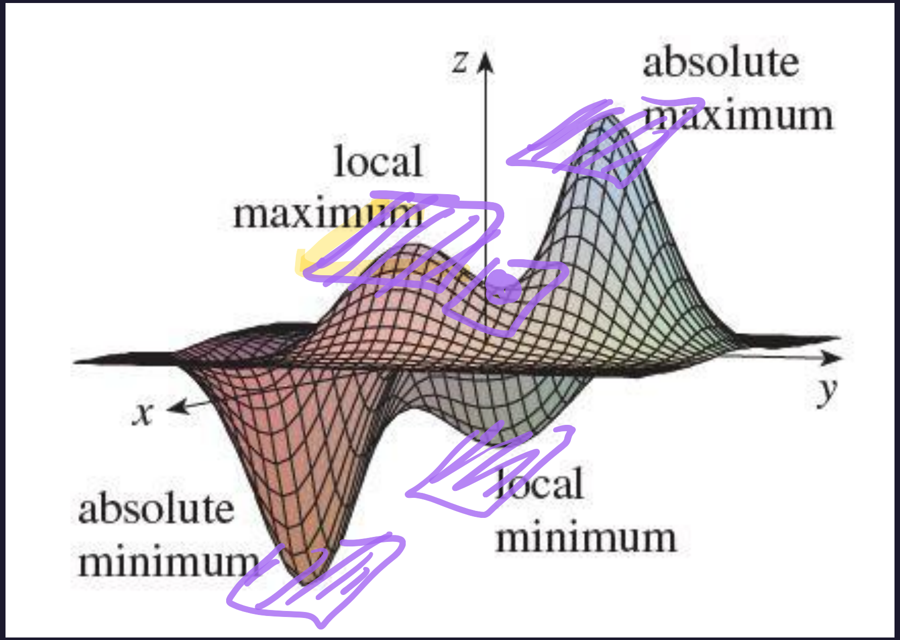
More on Maximum and Minimum Values

Today – Maximum and Minimum Values!

- Local Max and Min Values
- Second Derivative Test
- Absolute Max and Min Values



Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x\,(a, b) = 0$ and $f_y\,(a, b) = 0$.

A **critical point** for a function $f(x, y)$ is a point (a, b) where

$$\nabla f(a, b) = \vec{0},$$

that is $f_x(a, b) = 0, f_y(a, b) = 0$.

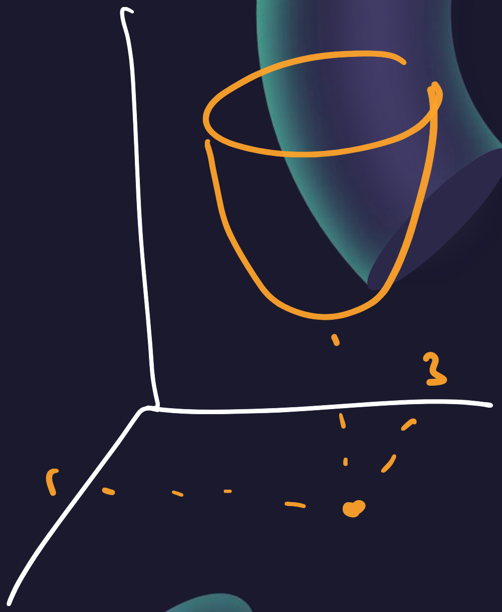
Example: Find all the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \Rightarrow x = 1$$

$$\frac{\partial f}{\partial y} = 2y - 6 = 0 \Rightarrow y = 3$$

$$P = (1, 3)$$



Local Max and Min Values: Second Derivative Test

3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$(f_{xy} = f_{yx})$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f.

WARNING! IF D = 0 , THE TEST IS INCONCLUSIVE.

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

$P = (1,3)$ was critical.

$f_x = 2x - 2$ $\to$ $f_{xx} = 2$, $f_{xy} = 0$

$f_y = 2y - 6$ $\to$ $f_{yy} = 2$, $f_{yx} = 0$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$
$$= 2 \cdot 2 - 0^2$$
$$= 4 > 0$$
$$f_{xx} = 2 > 0$$

$\left.\begin{array}{l} D > 0 \\ f_{xx} > 0 \end{array}\right\}$ LOCAL MIN AT (1,3).

[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + 4xy + y^2$$

$$\nabla f = 0 \begin{cases} f_x = 2x + 4y = 0 \\ f_y = 4x + 2y = 0 \end{cases}$$

$f_x = 0 \rightarrow y = -\frac{x}{2}$, $f_y = 0 \rightarrow y = -\frac{2x}{4}$

$\rightarrow 4x + 2\left(-\frac{x}{2}\right)$

$= 4x - x = 3x = 0$

$\Rightarrow x = 0$

$y = 0$

ONE CRITICAL PT.

$$\boxed{(0,0)}$$

[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + 4xy + y^2$$

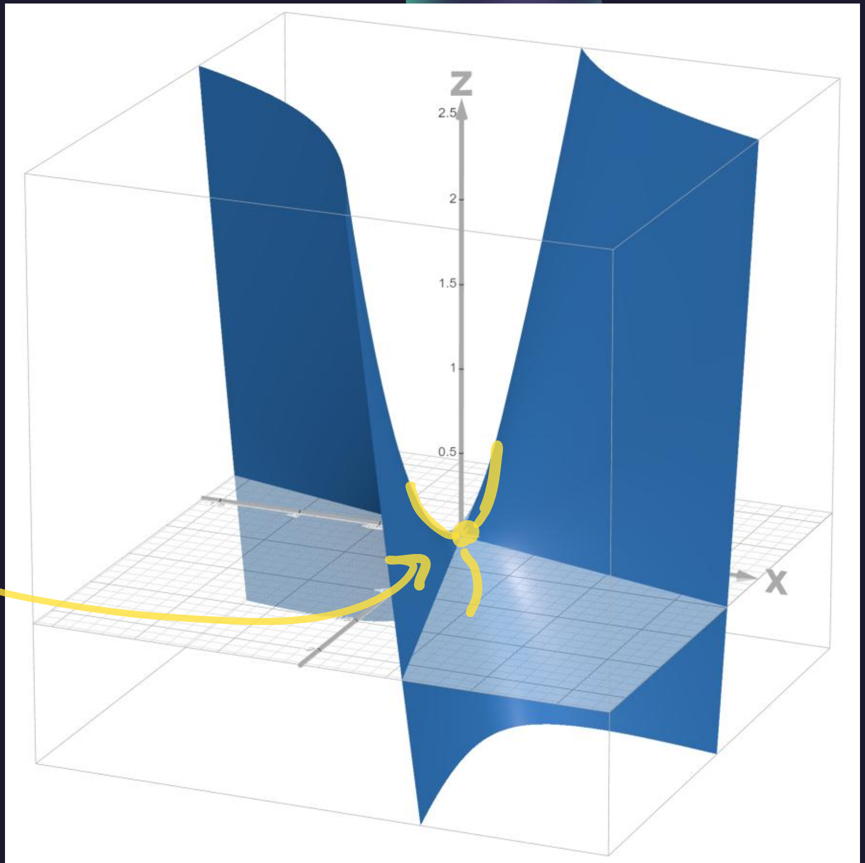
$f_x = 2x + 4y \rightarrow f_{xx} = 2$, $\quad f_{xy} = 4$

$f_y = 4x + 2y \rightarrow f_{yy} = 2$, $\quad f_{yx} = 4$

$(0,0)$

SADDLE POINT!

$D = 2 \cdot 2 - 4^2 = 4 - 16 = -12 < 0$



Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

8 Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

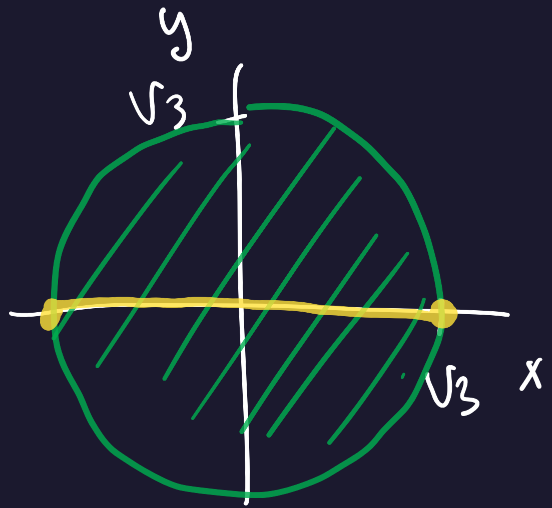
$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find the absolute maximum and minimum values of

$$f(x,y) = xy^2$$

$R^2 \Rightarrow R = \sqrt{3}$

in the region $D = \{(x,y): x^2 + y^2 \leq 3\}$.



(1) CRIT. PTS.

$$\begin{cases} f_x = y^2 = 0 \\ f_y = 2xy = 0 \end{cases}$$

$y = 0$

$\Rightarrow (x, 0)$ CRITICAL! FOR ANY x!

$$f(x,0) = x \cdot 0^2 = \boxed{0}$$

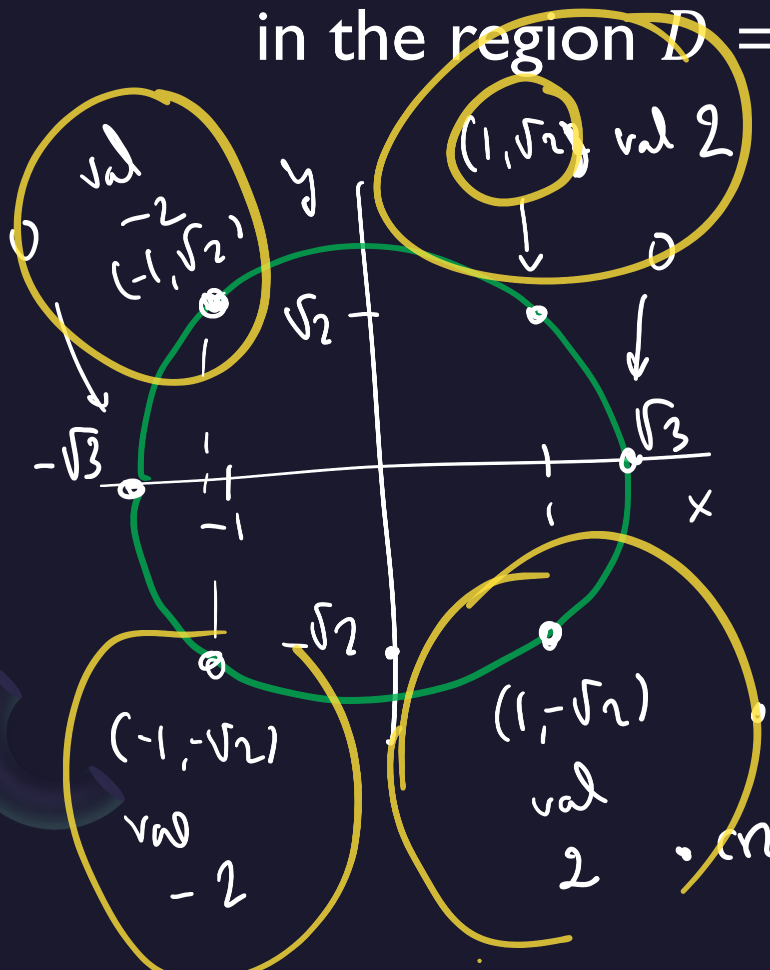
[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find the absolute maximum and minimum values of

$$f(x,y) = xy^2$$

in the region $D = \{(x,y): x^2 + y^2 \le 3\}$.



$3x - x^3$

$\shortparallel$

$f(x) = x(3 - x^2)$

$-\sqrt{3} \le x \le \sqrt{3}$

$f(x,y) = xy^2 = x(3 - x^2)$

on $x^2 + y^2 = 3 \Rightarrow y^2 = 3 - x^2$

$f'(x) = 3 - 3x^2 = 0 \Rightarrow x = \pm 1 \Rightarrow y = \pm\sqrt{2}$

- BOUNDARY : $x = \pm\sqrt{3}$, $f(\pm\sqrt{3}) = 0$
- CRIT PTS : $x = \pm 1$, $y = \pm\sqrt{2}$, $\boxed{f(x,y) = \pm 2}$

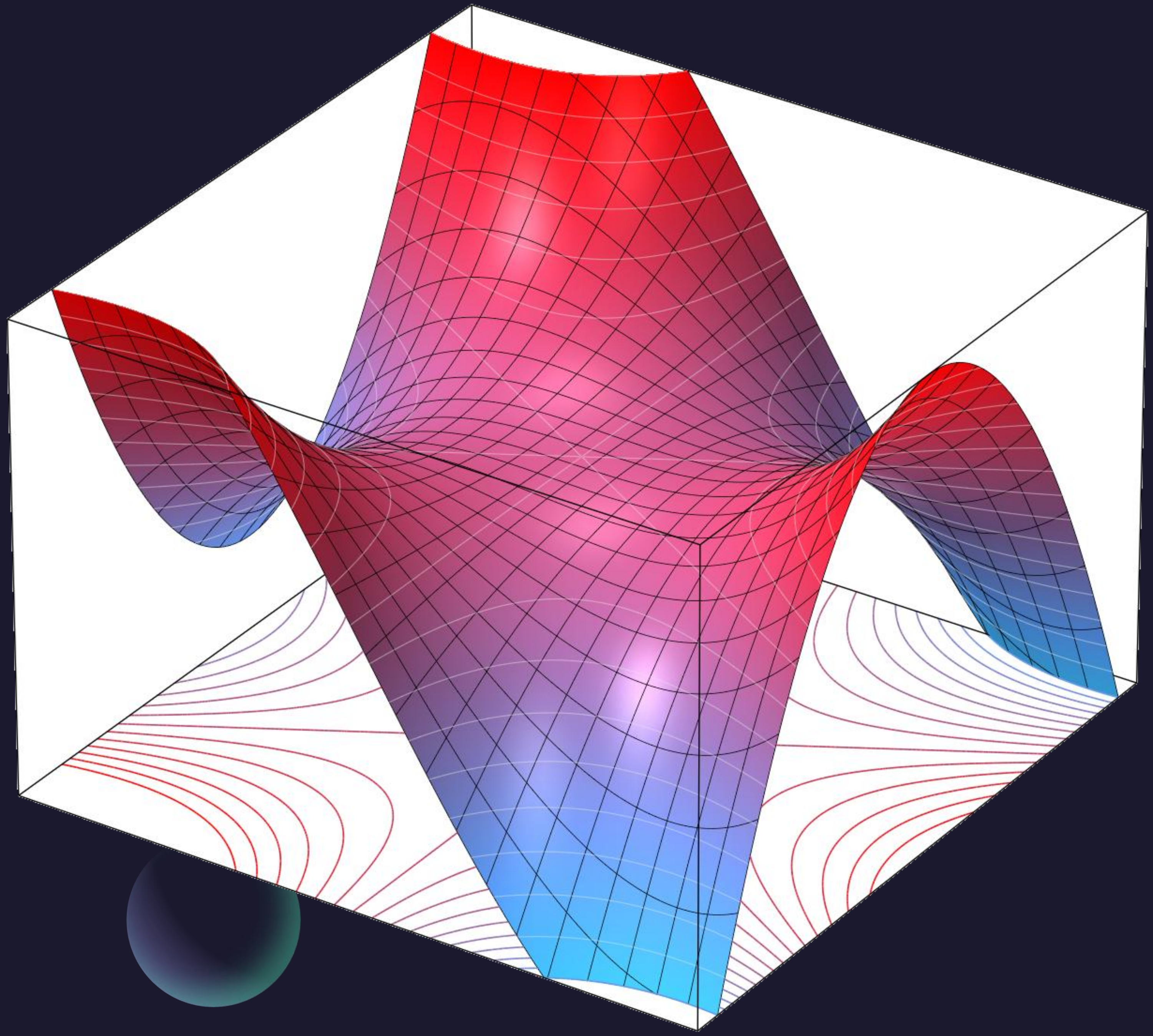
MAX = 2

MIN = −2

Questions?

Thank you

Until next time.



ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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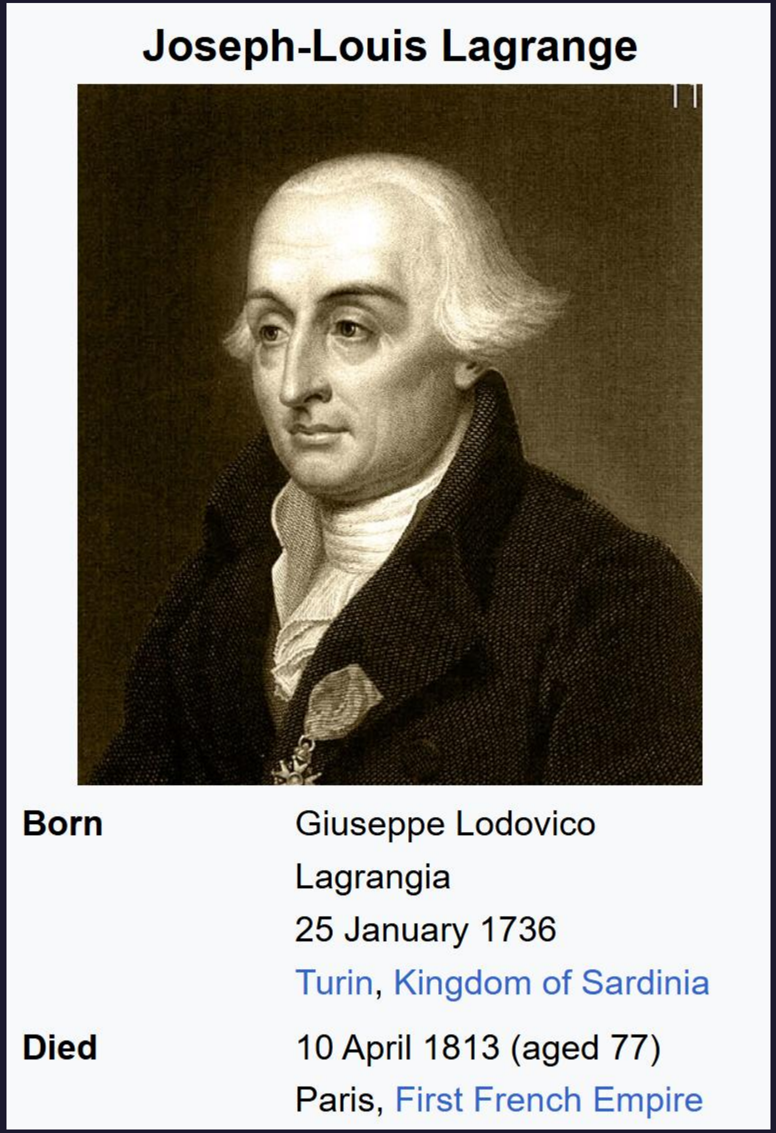
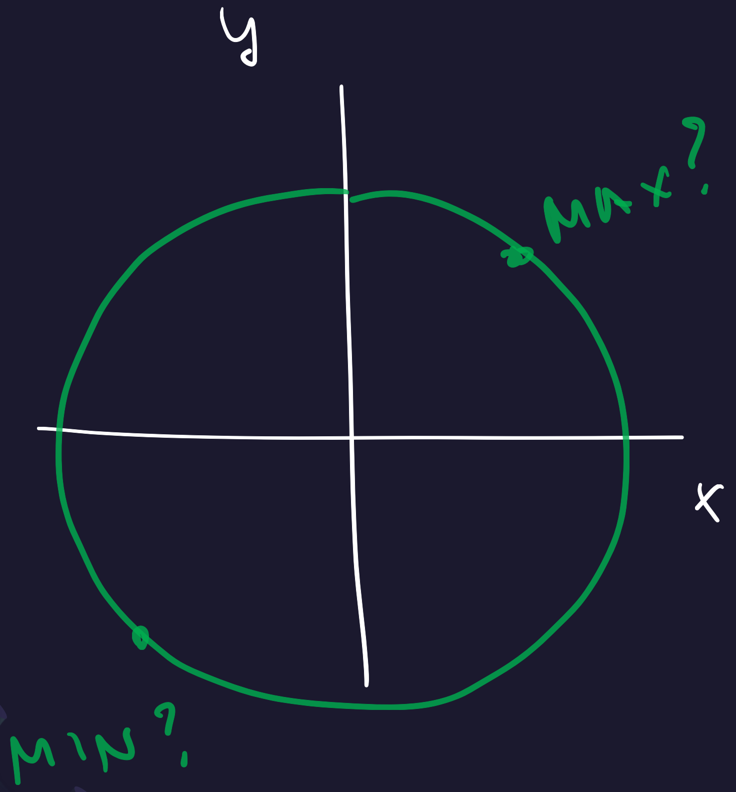
Lagrange Multipliers

Today – "Lagrange Multipliers!"

- The Method
- One Constraint
- Examples



Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

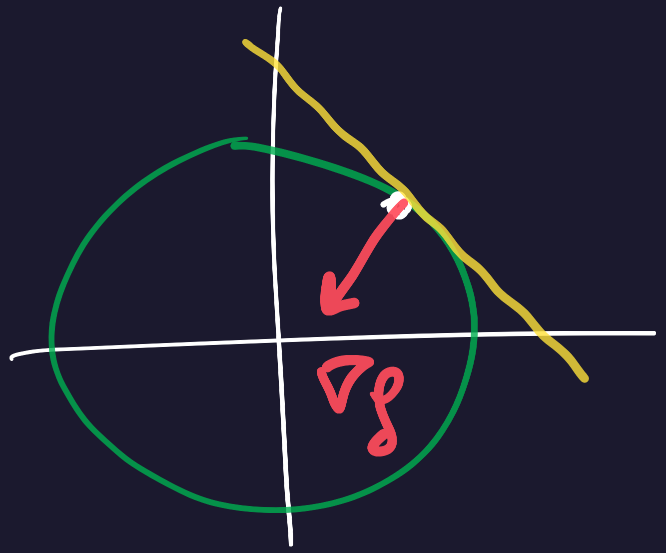


Recall: Properties of the Gradient Vector

$f(x,y) - z = 0$

Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

The gradient vector in two dimensions, $\overset{\nabla f}{\cancel{\nabla F}} = (f_x, f_y)$ is normal to any level curves of $f(x,y)$ at any point, indicating the maximum rate of change.



Example: Let $f(x,y) = 4 - x^2 - y^2$

(a) Find the normal vector to the graph of $f(x,y)$ at $(1,1,2)$

(b) Find the tangent plane to the graph of $f(x,y)$ at $(1,1,2)$

(c) Find the normal vector to the cross section $z = 0$ at $\cancel{(1,1)}$ $(\sqrt{2},\sqrt{2})$

$$f_x = -2x$$

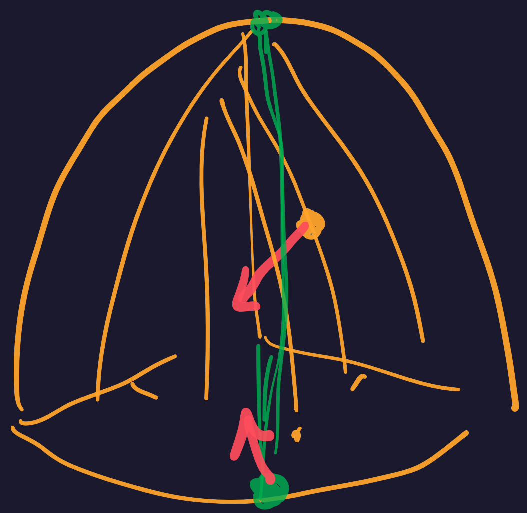
$$f_y = -2y$$

(a) $$\nabla F = (f_x, f_y, -1)$$

$$= (-2x, -2y, -1)\Big|_{(1,1,2)} = (-2,-2,-1)$$

(b) $$f_x\big|_p \cdot (x - x_0) + f_y\big|_p \cdot (y - y_0) - (z - z_0) = 0$$

$$-2(x-1) + (-2)\cdot(y-1) - (z-2) = 0$$



Example: Let $f(x,y) = 4 - x^2 - y^2$

(a) Find the normal vector to the graph of $f(x,y)$ at $(1,1,2)$

(b) Find the tangent plane to the graph of $f(x,y)$ at $(1,1,2)$

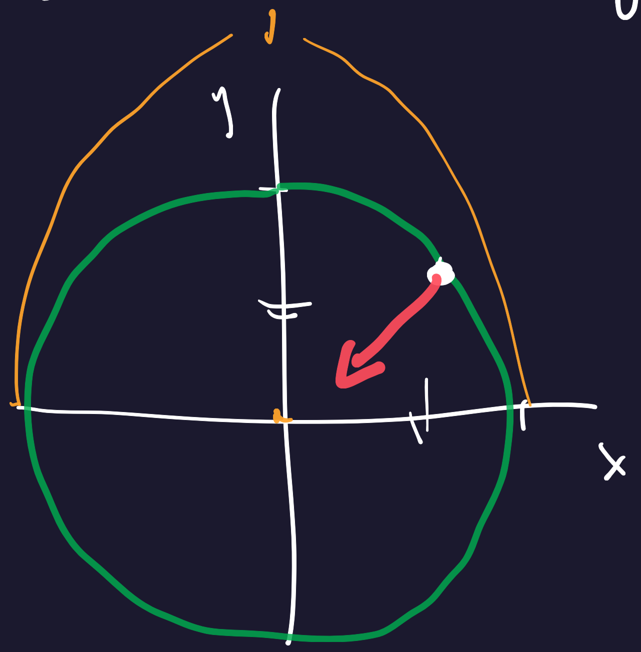
(c) Find the normal vector to the cross section $z = 0$ at $\cancel{(1,1)}$ $(\sqrt{2}, \sqrt{2})$

$z = 0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$

$x^2 + y^2 = x^2 + x^2 = 4$

$= 2x^2 = 4$

$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$



$\nabla f = (f_x, f_y)\Big|_{(\sqrt{2},\sqrt{2})}$

$= (-2x, -2y)\Big|_{(\sqrt{2},\sqrt{2})} = (-2\sqrt{2}, -2\sqrt{2})$

[Extra]

Example: Let $f(x,y) = 4 - x^2 - y^2$

(a) Find the normal vector to the graph of $f(x,y)$ at $(1,1,2)$

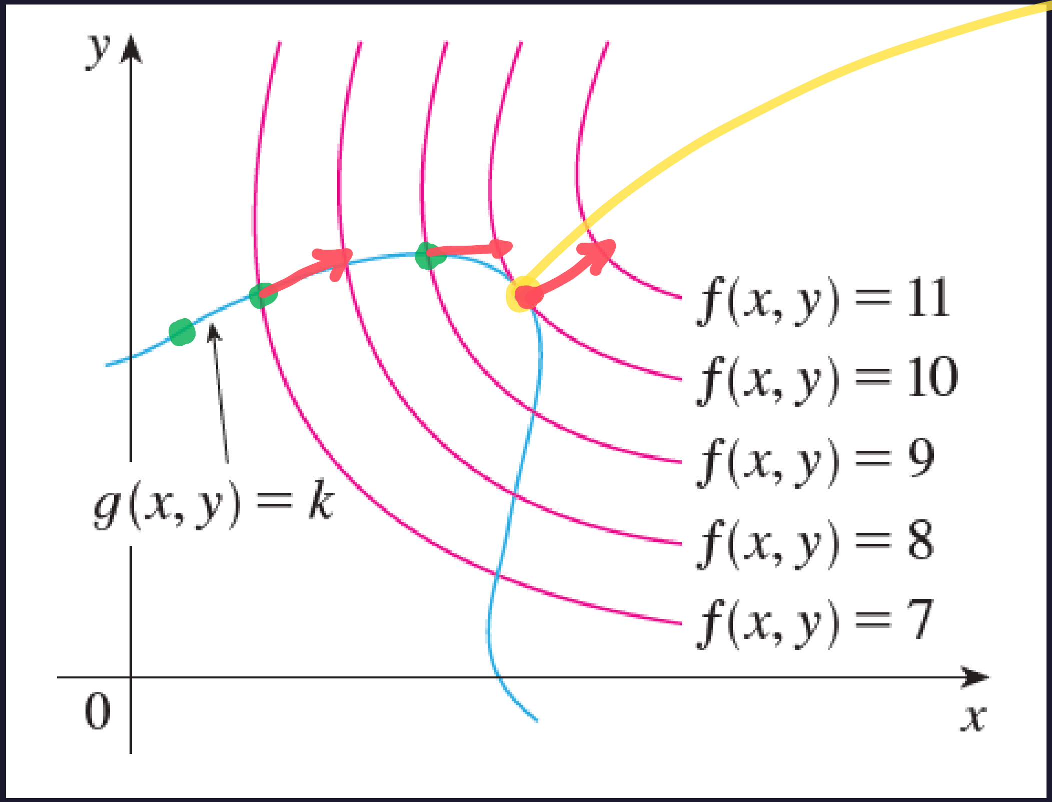
(b) Find the tangent plane to the graph of $f(x,y)$ at $(1,1,2)$

(c) Find the normal vector to the cross section $z = 0$ at $(1,1)$

The "Lagrange Multipliers" Method

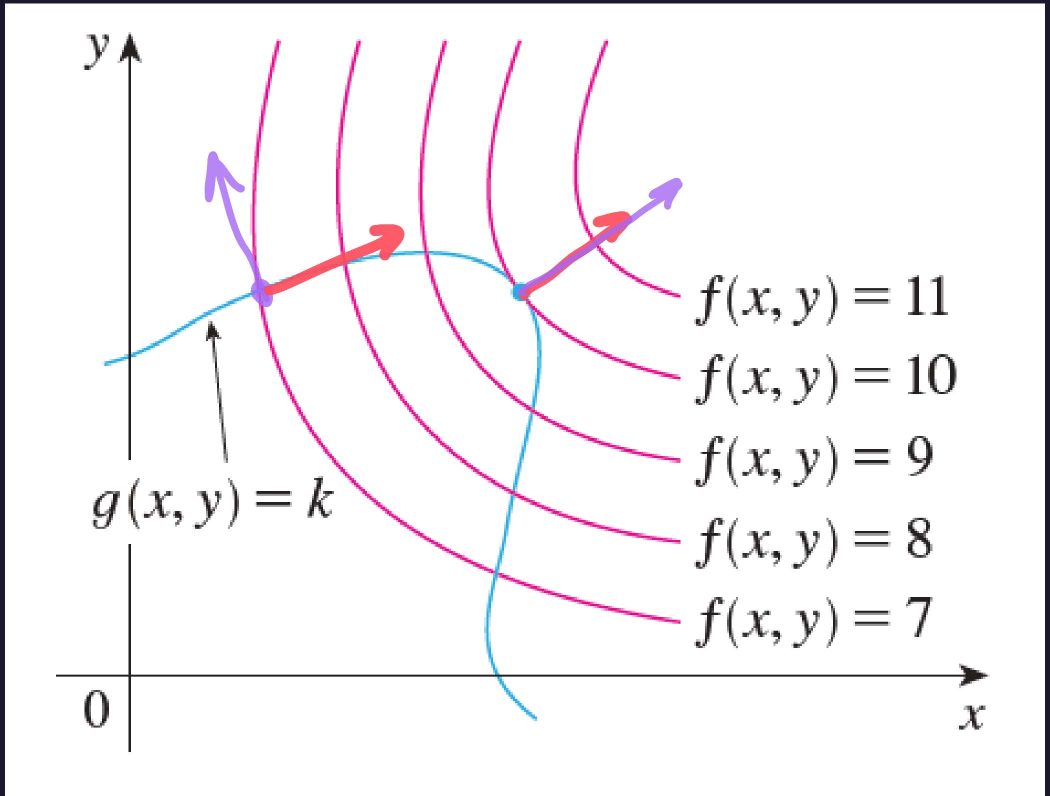
GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.

MAX



The "Lagrange Multipliers" Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.



The value of $f(x,y)$ on the curve $g(x,y)=k$ will be maximized at some point (x_0,y_0) such that

$\nabla f\ (x_0,y_0)$ is parallel to $\nabla g\ (x_0,y_0)$

or equivalently a point (x_0,y_0) such that there is a constant λ with

$$\begin{cases} \nabla f\ (x_0,y_0) = \lambda \cdot \nabla g\ (x_0,y_0) \\ \text{and } g(x_0,y_0) = k\ . \end{cases}$$

The "Lagrange Multipliers" Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.

SOLVE:

$$\begin{cases} \nabla f\,(x_0,y_0) = \lambda \cdot \nabla g\,(x_0,y_0) \\ g(x_0,y_0) = k\,. \end{cases}$$

$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$. $\leftarrow$ $g(x,y) = 1$, $g(x,y) = x^2 + y^2$

$$\nabla f = (2x, 4y)$$

$$\nabla g = (2x, 2y)$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} (2x, 4y) = \lambda (2x, 2y) \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} 2x = 2x\lambda \\ 4y = 2y\lambda \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} 2x(1-\lambda) = 0 \\ 2y(2-\lambda) = 0 \\ x^2 + y^2 = 1 \end{cases}$$

CASES

- $x = 0$, $y = \pm 1$, $\lambda = 2$
- $y = 0$, $x = \pm 1$, $\lambda = 1$
- $\lambda = 1$, $2y = 0 \Rightarrow y = 0$, $x = \pm 1$
- $\lambda = 2$, $2x = 0 \Rightarrow x = 0$, $y = \pm 1$

$(0, \pm 1)$ or $(\pm 1, 0)$

$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

CANDIDATES: $(0,\pm 1)$, $(\pm 1, 0)$

$\underbrace{f(0,\pm 1) = 2}_{\text{MAX}}$, $\underbrace{f(\pm 1, 0) = 1}_{\text{MIN}}$

MAX $(0,\pm 1)$

MIN $(\pm 1, 0)$

$$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0) \quad \text{and} \quad g(x_0,y_0) = k\,.$$

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

$$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0) \quad \text{and} \quad g(x_0,y_0) = k.$$

Example: Find the largest area of a rectangle with fixed perimeter equal to p.

The "Lagrange Multipliers" Method

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

$$\nabla f(x_0,y_0,z_0) = \lambda \cdot \nabla g(x_0,y_0,z_0) \quad \text{and} \quad g(x_0,y_0,z_0) = k\ .$$

Example: Find the dimensions of the closed box with the largest volume and fixed surface area S.

$\nabla f(x_0,y_0,z_0) = \lambda \cdot \nabla g(x_0,y_0,z_0)$ and $g(x_0,y_0,z_0) = k$.

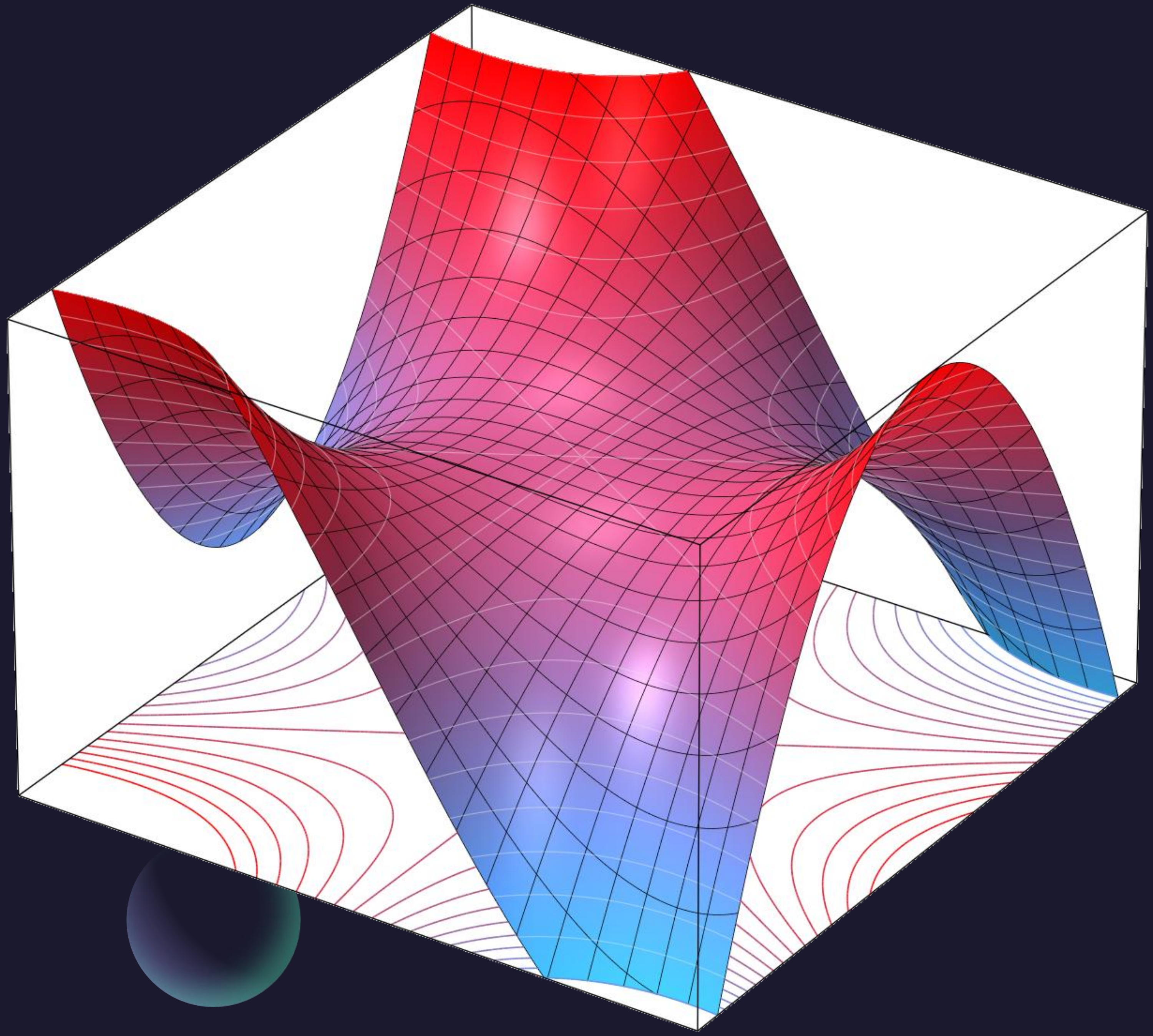
[Extra]

Example: Find the dimensions of the closed box with the largest volume and fixed surface area S.

Questions?

Thank you

Until next time.



ALVARO: Start the recording!

"Calculus 3"

Multi-Variable Calculus

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Double Integrals over Rectangles

Today – Double Integrals!

- The Definite Integral
- The Riemann Integral
- Iterated Integrals
- Fubini's Theorem

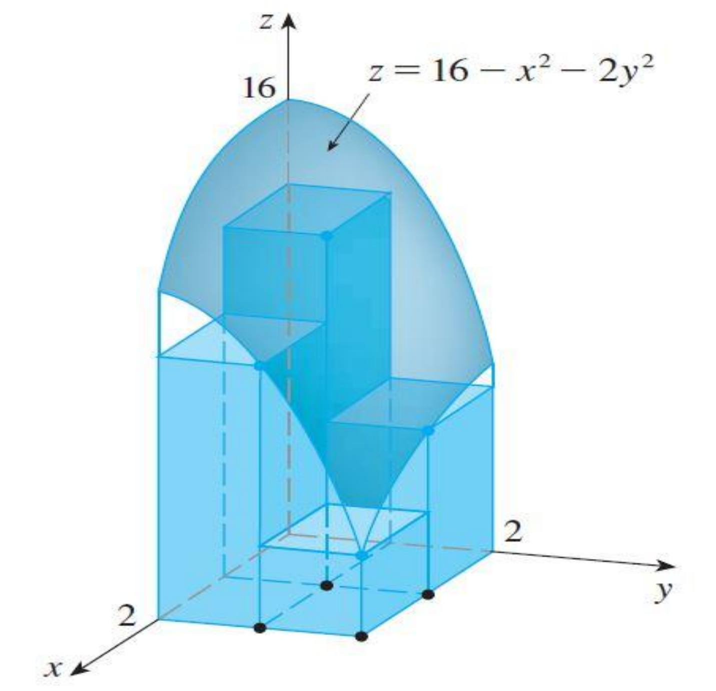
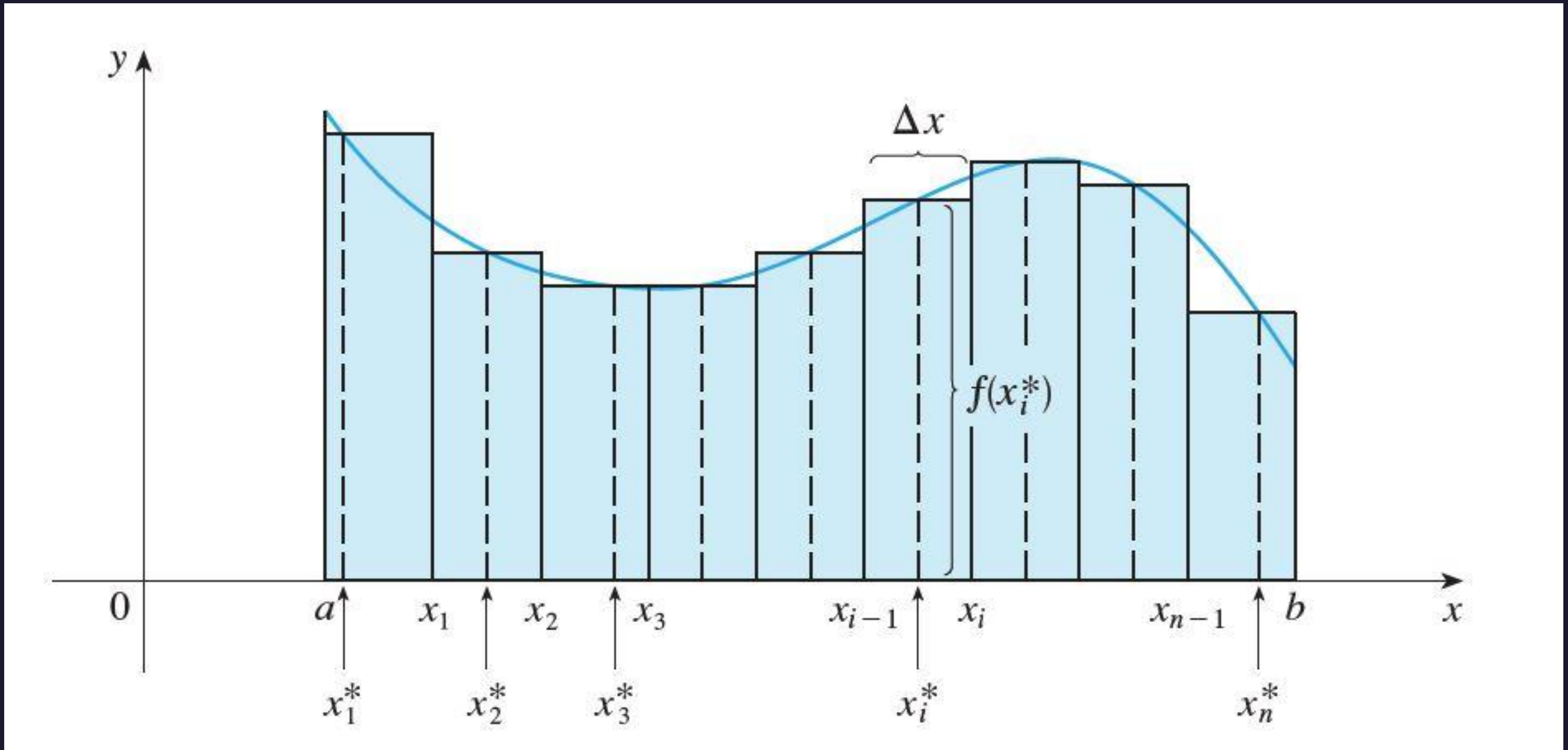
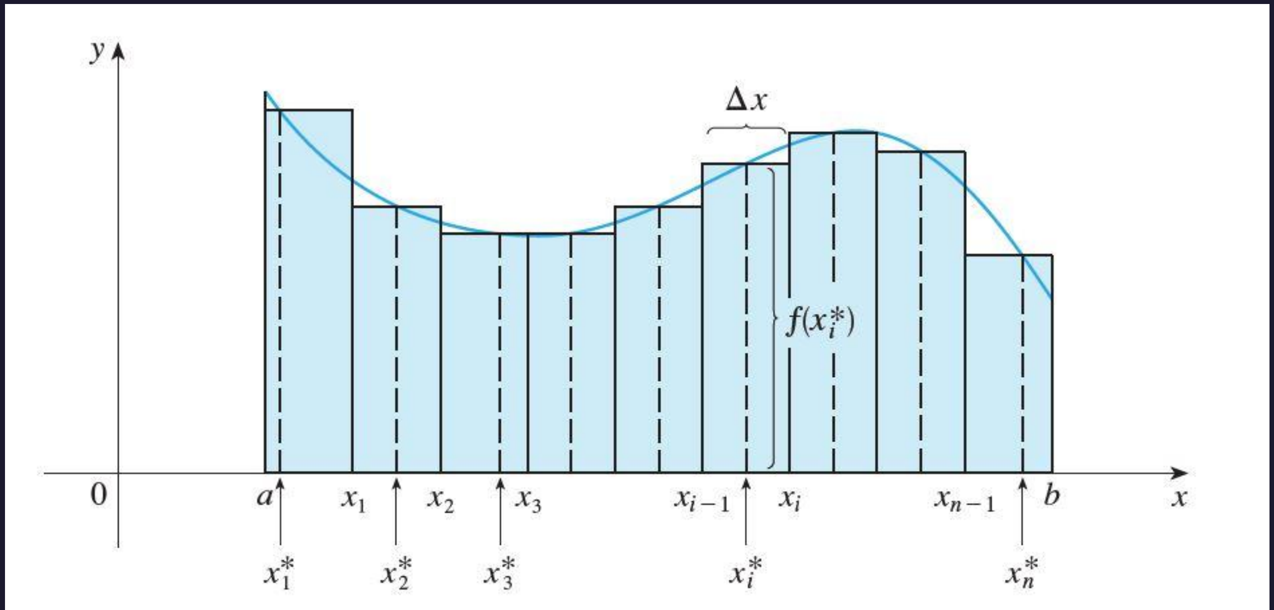


FIGURE 7

The Definite (Riemann) Integral



The Definite (Riemann) Integral



The Definite (Riemann) Integral

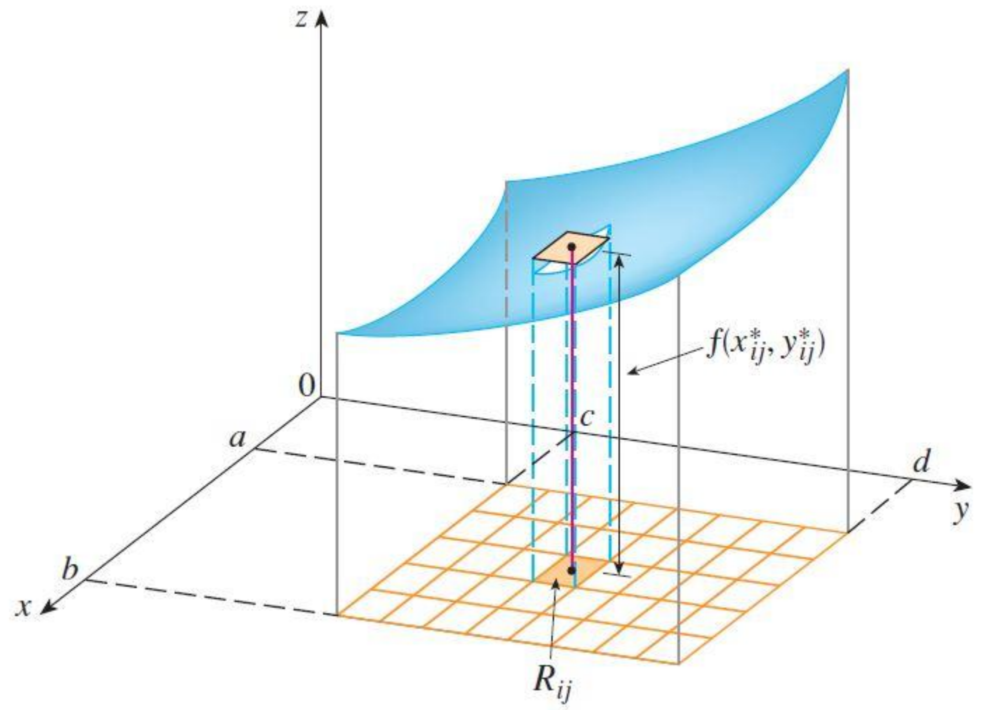


FIGURE 4

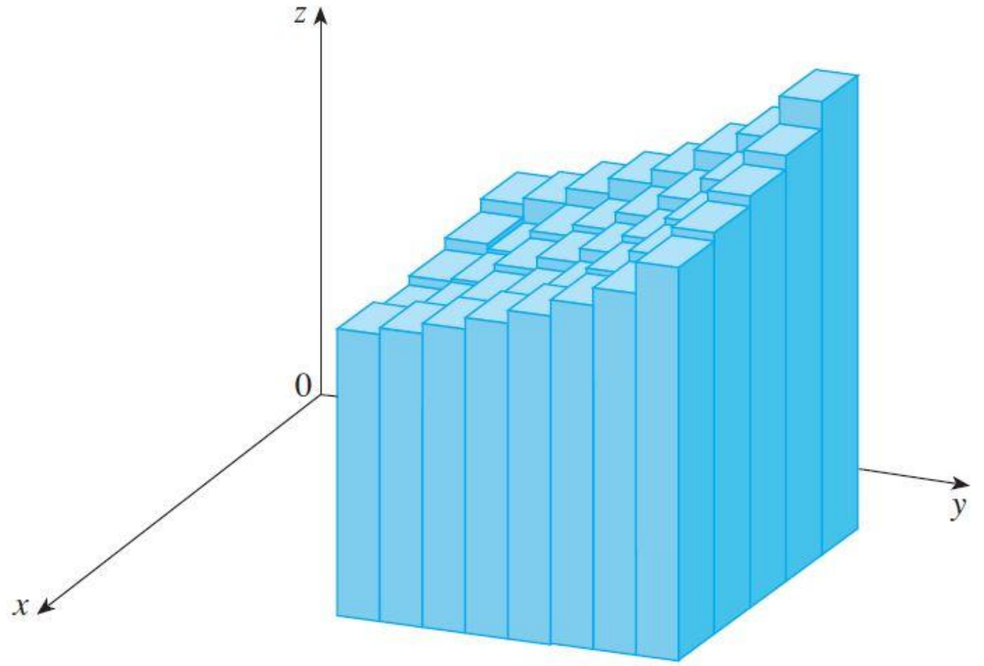


FIGURE 5

The Definite (Riemann) Integral

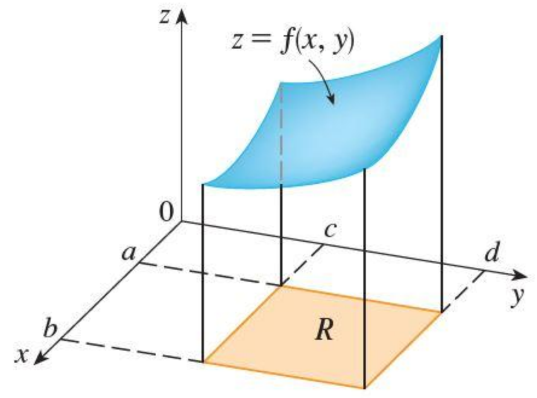


FIGURE 2

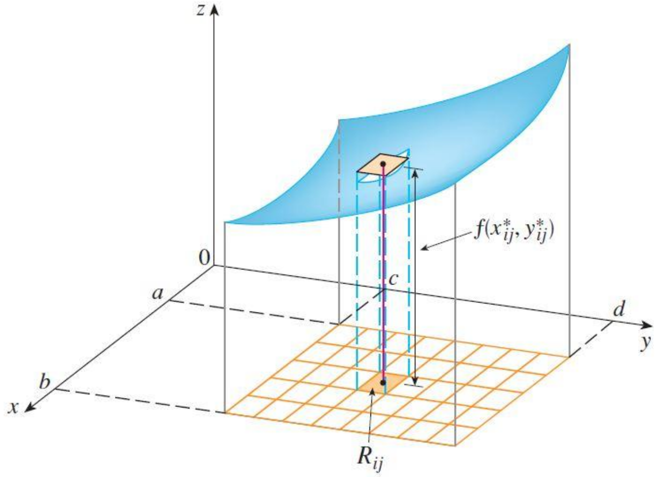


FIGURE 4

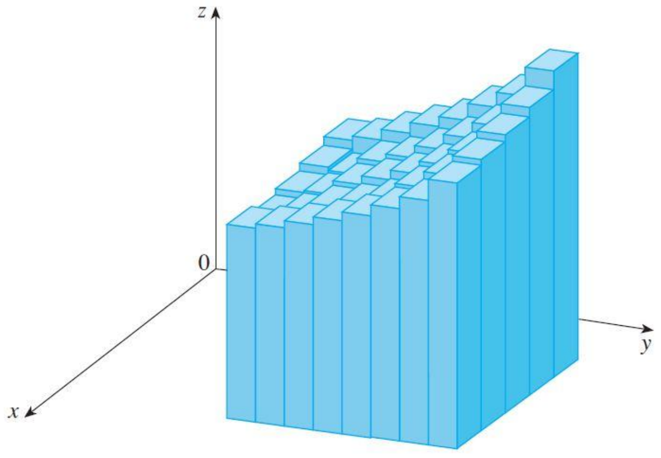


FIGURE 5

The Definite (Riemann) Integral

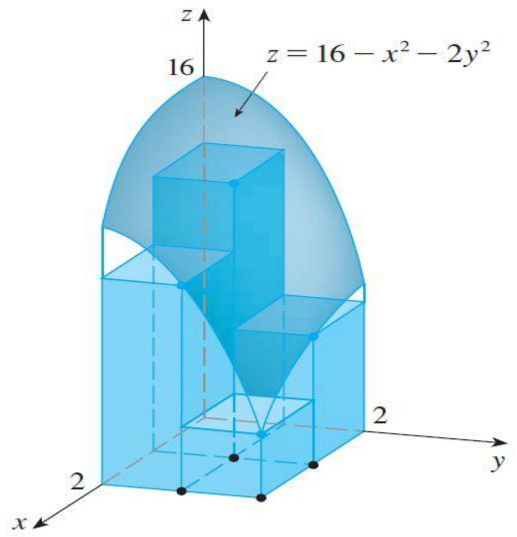
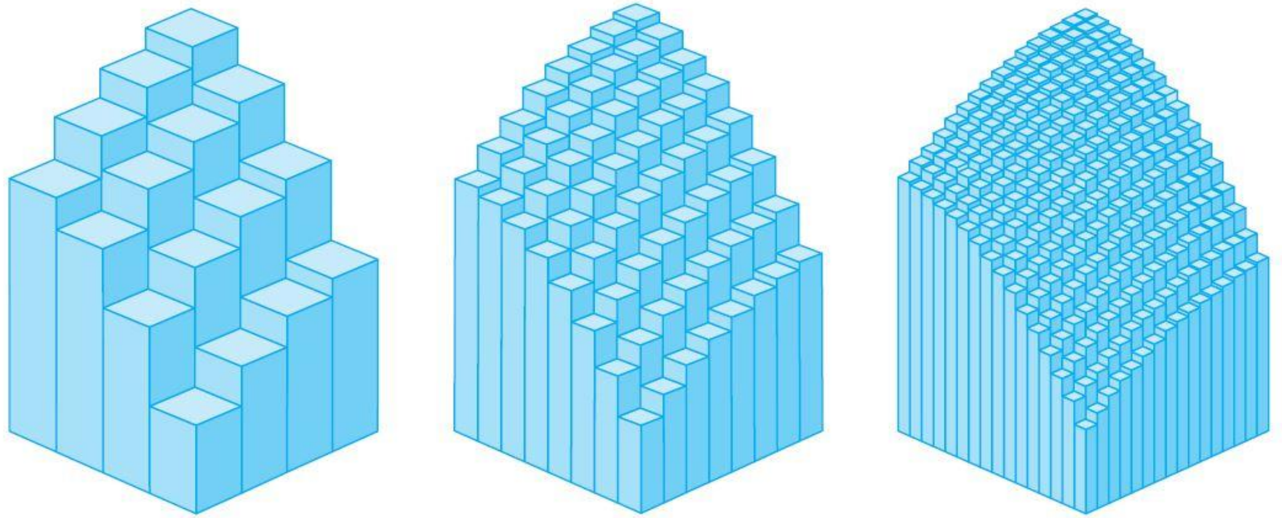


FIGURE 7



(a) $m = n = 4, V \approx 41.5$ (b) $m = n = 8, V \approx 44.875$ (c) $m = n = 16, V \approx 46.46875$

The usual properties of integration still hold for double integrals:

- $\displaystyle\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA.$
- For any constant c,

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA.$$

- If $f(x,y) \geq g(x,y)$ on the rectangle R, then

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA.$$

And when letting $m, n \to \infty$, we have $\Delta A \to dA = dx \cdot dy$. Then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy,$$

this is called an **iterated integral**, and we evaluate its value by computing the innermost integral first and then working the way out. Again, in the case this value represents a volume only if $f(x,y) \geq 0$ on R.

Example: Find the volume under the graph of $f(x, y) = 16 - x^2 - 2y^2$ above the square R = [0,2] x [0,2].

[Extra]

Example: Find the volume under the graph of $f(x, y) = 16 - x^2 - 2y^2$ above the square R = [0,2] x [0,2].

Example: Calculate the following iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

Example: Calculate the following iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

Fubini's Theorem

If f is continuous on the rectangle

$$R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$$

then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Guido Fubini

Born	19 January 1879 Venice
Died	6 June 1943 (aged 64) New York

Example: Evaluate the double integral

$$\iint_R \left(x - 3y^2\right) dA$$

where $R = \{(x, y): 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

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Example: Evaluate the double integral

$$\iint_R y \sin(xy) \, dA$$

where $R = [1,2] \times [0, \pi]$.

Example: Evaluate the double integral

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where $R = [1,2] \times [0, \pi]$.

When $f(x, y) = g(x) \cdot h(y)$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b g(x)h(y) dxdy = \int_a^b g(x) dx \cdot \int_c^d h(y) dy$$

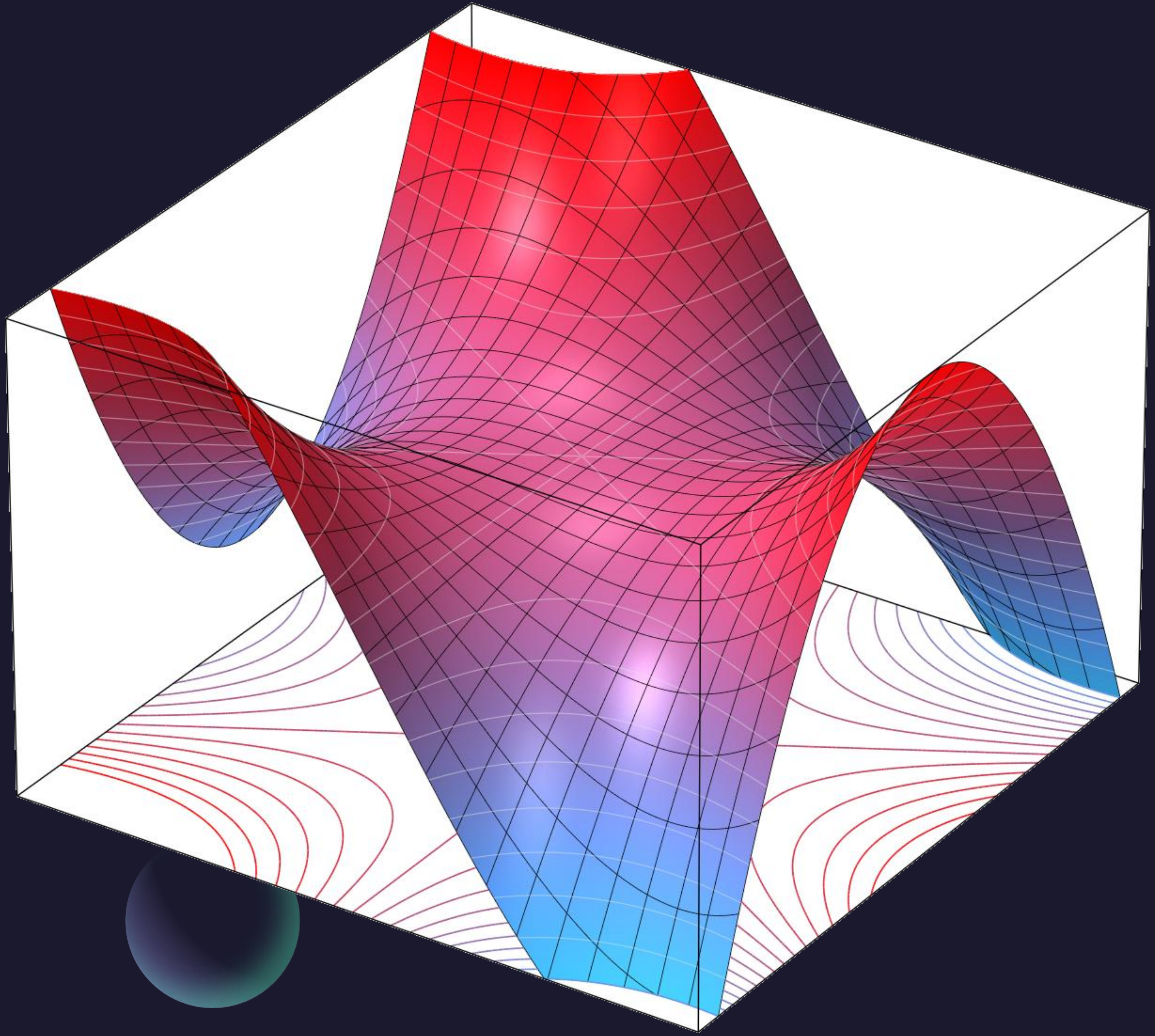
- Evaluate the iterated integral

$$\int_1^3 \int_1^5 \frac{\ln(y)}{xy} dxdy$$

Questions?

Thank you

Until next time.



“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Double Integrals over Regions

Today – Double Integrals in Regions!

- General Regions
- Regions of Type I and II
- Changing the Order of Integration
- Properties of Double Integrals

Regions of Type I and II

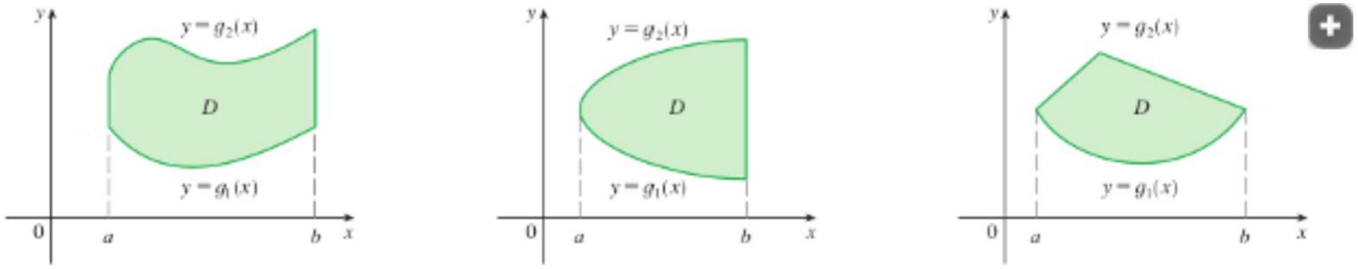
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

Figure 5

Some type I regions



Regions of Type I and II

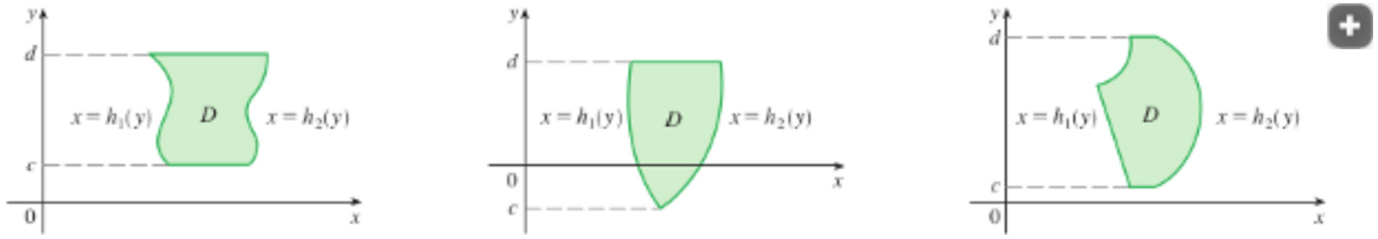
We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

Figure 7

Some type II regions



Integrals over Regions of Type I

$\boxed{3}$ If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ \ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Integrals over Regions of Type II

$\boxed{4}$ If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \ \text{ and } \ y = 1 + x^2 .$$

Example: Evaluate the double integral

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Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type I integral.)

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type II integral.)

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Properties of Double Integrals

$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA \qquad \text{where } c \text{ is a constant}$$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

$\boxed{7}$

$$\iint_D f(x,y)\,dA \geqslant \iint_D g(x,y)\,dA$$

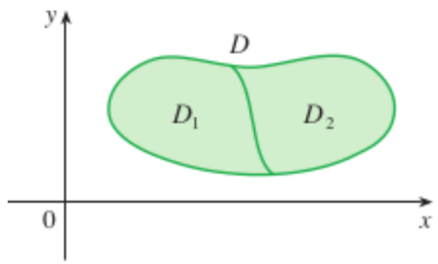
Properties of Double Integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

$\boxed{8}$

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

Figure 17



Properties of Double Integrals

$$\iint_D 1\, dA = A(D)$$

10 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y)\, dA \leqslant M \cdot A(D)$$

[Extra]

Example: Estimate the value of the double integral

$$\iint_R e^{-(x^2+y^2)}\,dA$$

where $R = \{(x, y): x^2 + y^2 \leq 1\}$ is the circle of radius 1.

Questions?

Thank you

Until next time.