“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 7

1

Any Reminders? Any Questions?

- Class ends at 3:15.
- Slides are being posted on GitHub! https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- Request videos!!

2

EXAM 1 -- Friday, February 20th

Exam Covers:

- **Chapter 12**
  - Sections 12.1 – 12.6
- **Chapter 14**
  - Sections 14.1, 14.3 – 14.8

3

Questions?

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

4

ALVARO: Start the recording!

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"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

The Chain Rule

6

Today – The Chain Rule!

- The Single Variable Case
- Chain Rule with One Parameter
- Chain Rule with Two Parameters

7

The Good Ol' Chain Rule

8

Example: Find the derivative of f(g(t)) with respect to t where

$f(x) = \sin(x)$ and $g(t) = t^2 + 1$

9

Example: Find the derivative of f(g(t)) with respect to t where

$f(x) = \sin(x)$ and $g(t) = t^2 + 1$ [Extra space]

10

The New Chain Rule – Case 1

1 The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

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Example: Find the derivative of f(g(t),h(t)) with respect to t where

$f(x,y) = x^2 + y$ and $g(t) = 3t^4 + 1,\quad h(t) = 3t$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

12

Example: Find the derivative of f(g(t),h(t)) with respect to t where

$f(x,y) = x^2 + y$ and $g(t) = 3t^4 + 1$, $h(t) = 3t$ [Extra]

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

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The New Chain Rule – Case 2

2 The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

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$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$f(x, y) = x^2y^3$ and $g(s, t) = s \cdot \cos(t)$, $h(s, t) = s \cdot e^{2t}$

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$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$f(x, y) = x^2y^3$ and $g(s, t) = s \cdot \cos(t)$, $h(s, t) = s \cdot e^{2t}$

[Extra]

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Questions?

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ALVARO: Start the recording!

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"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Directional Derivatives

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Today – Directional Derivatives!

- Directional Derivatives
- Gradient Vector
- Maximizing the Directional Derivative

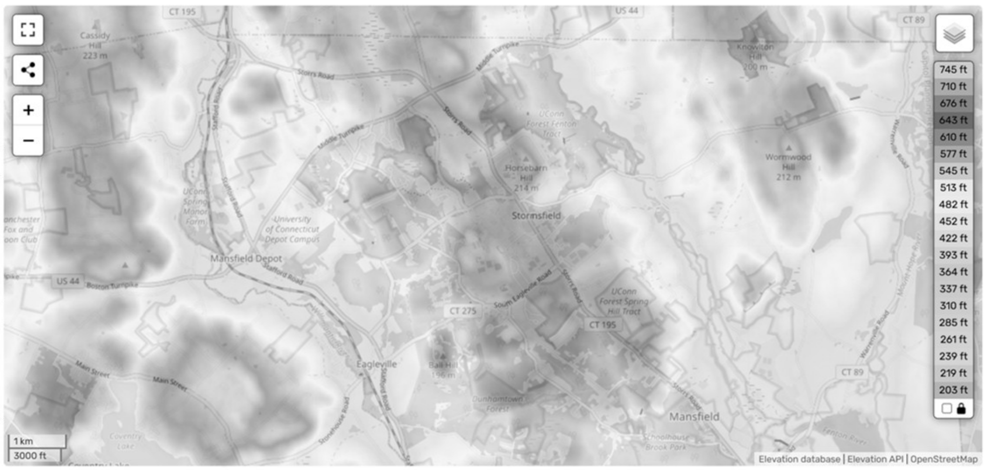
20

What is the steepest path down from Horsebarn Hill?

Storrs topographic map

> United States > Connecticut > Capitol Planning Region > Mansfield > Storrs > Storrs

Click on the **map** to display **elevation**.



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Definition of Directional Derivative

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathrm{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

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How to Compute the Directional Derivative

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathrm{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

3 Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

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$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Example: Find the directional derivative in the direction of u = (1,-1) for

$f(x, y) = x^3 - 3xy + 4y^2$ at $(x_0, y_0) = (1,2)$.

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$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Example: Find the directional derivative in the direction of u = (1,-1) for

$f(x,y) = x^3 - 3xy + 4y^2$ at (x_0, y_0) = (1,2).

[Extra]

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$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Example: Find the directional derivative in the direction given by the angle pi/6 measured clockwise from the x-axis, of the function:

$f(x,y) = \ln(x^2 + y^2)$ at (x_0, y_0) = (1,0).

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$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Example: Find the directional derivative in the direction given by the angle π/6 measured clockwise from the x-axis, of the function:

$f(x, y) = \ln(x^2 + y^2)$ at $(x_0, y_0) = (1,0)$.

[Extra]

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The Gradient Vector

3 Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\, \mathbf{i} + \frac{\partial f}{\partial y}\, \mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

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$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = 3i + 4j of the function:

$$f(x, y) = x^2 y^3 - x \cdot \cos(y) \quad \text{at } (x_0, y_0) = (1,0).$$

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[Extra]

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = 3i + 4j of the function:

$$f(x, y) = x^2 y^3 - x \cdot \cos(y) \quad \text{at } (x_0, y_0) = (1,0).$$

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Maximizing the Directional Derivative

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

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Maximizing the Directional Derivative

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

15 Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

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$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = (1,1) and also the maximal directional derivative of the function:

$f(x, y) = x^2 \ln(y)$ at $(x_0, y_0) = (3,1)$.

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[Extra]

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = (1,1) and also the maximal directional derivative of the function:

$f(x, y) = x^2 \ln(y)$ at $(x_0, y_0) = (3,1)$.

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Properties of the Gradient Vector

Suppose a surface is defined by $z = f(x, y)$. We can view it as level surface $t = 0$ of a higher-dimensional function $t = F(x, y, z)$ defined by

$$F(x, y, z) = f(x, y) - z$$

Then $\nabla F = (f_x, f_y, -1)$ is the normal vector to the tangent plane of $f(x, y) - z = 0$.

Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

The gradient vector in two dimensions, $\nabla F = (f_x, f_y)$ is normal to any level curves of $f(x,y)$ at any point, indicating the maximum rate of change.

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Properties of the Gradient Vector

Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

The gradient vector in two dimensions, $\nabla F = (f_x, f_y)$ is normal to any level curves of $f(x,y)$ at any point, indicating the maximum rate of change.

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Questions?

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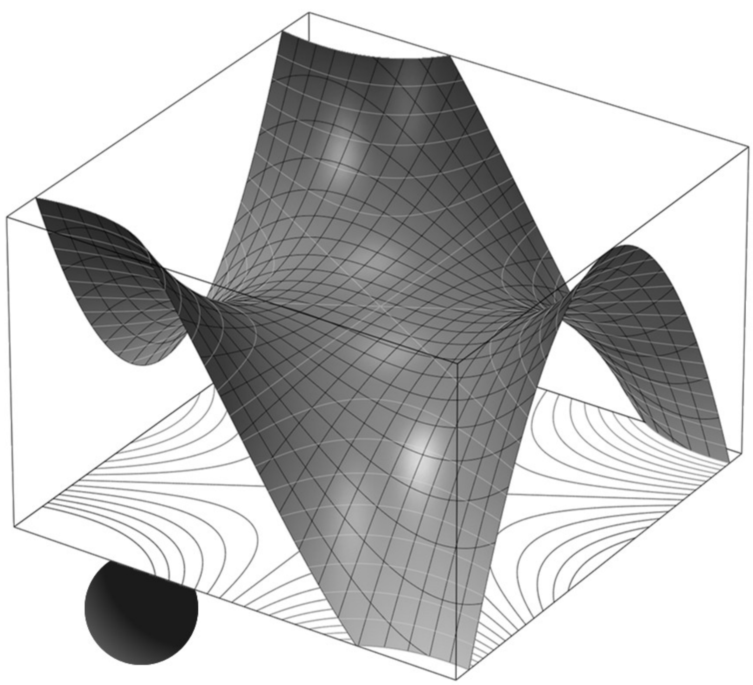
Thank you

Until next time.



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"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Maximum and Minimum Values

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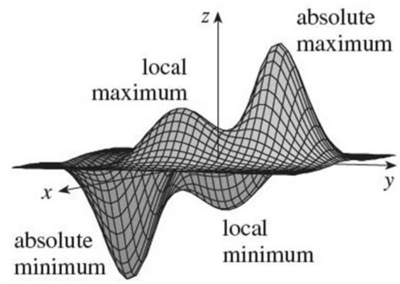
Today – Maximum and Minimum Values!

- Local Max and Min Values
- Second Derivative Test
- Absolute Max and Min Values



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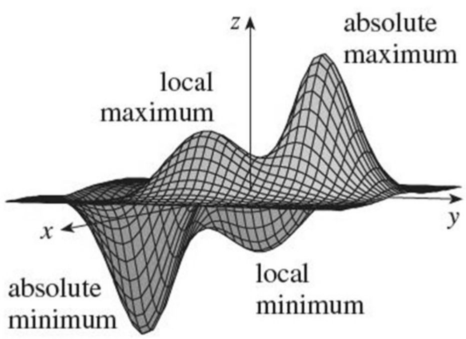
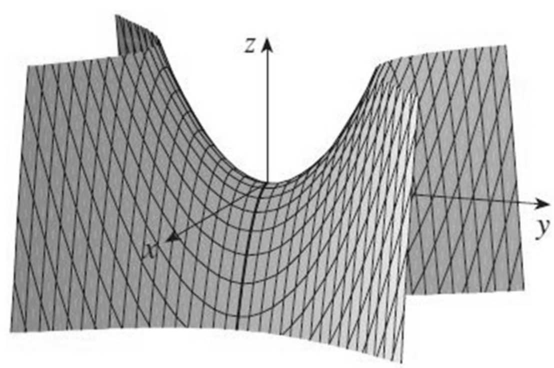
Local Max and Min Values





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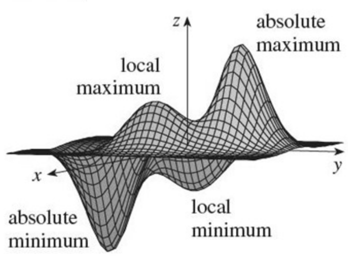
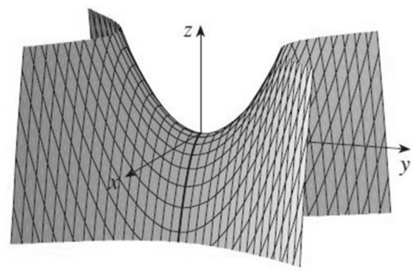
Local Max and Min Values

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.





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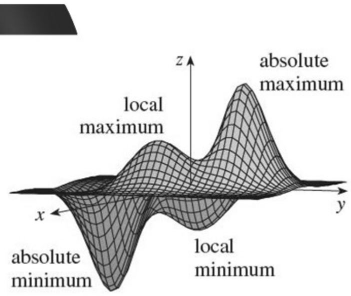
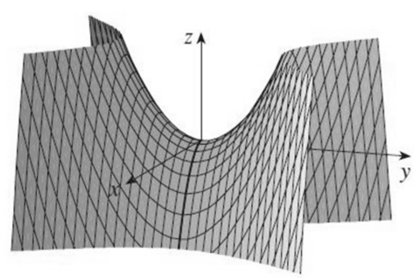
Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.





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Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x\,(a, b) = 0$ and $f_y\,(a, b) = 0$.

A **critical point** for a function $f(x, y)$ is a point (a, b) where

$$\nabla f(a, b) = \vec{0},$$

that is $f_x(a, b) = 0, f_y(a, b) = 0$.

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Example: Find all the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

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[Extra]

Example: Find all the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

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Local Max and Min Values: Second Derivative Test

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Local Max and Min Values: Second Derivative Test

3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f.

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$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find and classify the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

50

[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

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$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + 4xy + y^2$$

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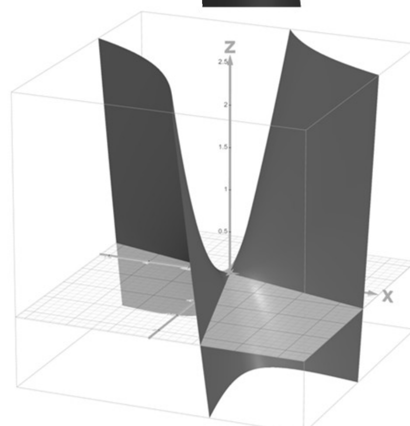
[Extra]

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find and classify the critical points for the function

$$f(x, y) = x^2 + 4xy + y^2$$



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Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

8 Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

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Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

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$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find the absolute maximum and minimum of

$$f(x, y) = xy^2$$

in the region $D = \{(x, y)\colon x^2 + y^2 \leq 3\}$.

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[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find the absolute maximum and minimum of

$$f(x,y) = xy^2$$

in the region $D = \{(x,y):\ x^2 + y^2 \leq 3\}$.

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Questions?

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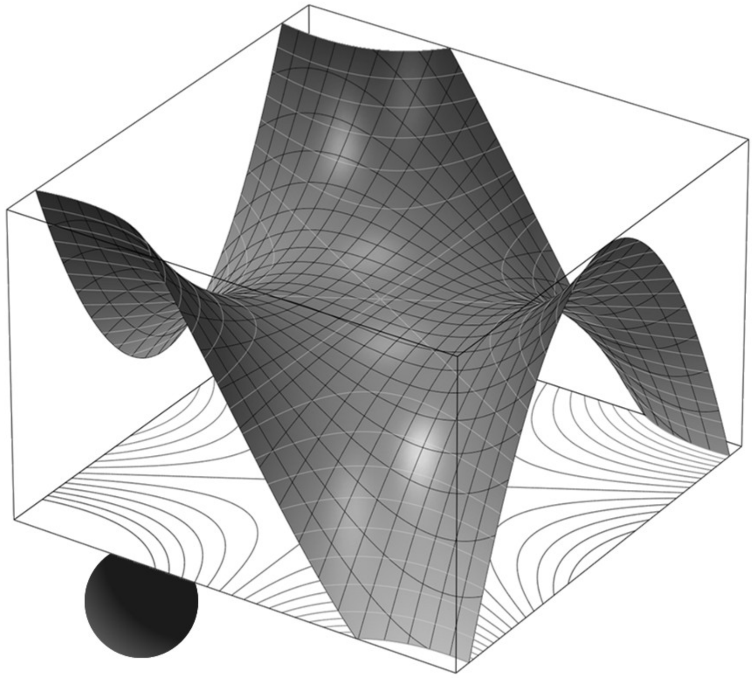
Thank you

Until next time.



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ALVARO: Start the recording!

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“Calculus 3”

Multi-Variable Calculus

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Lagrange Multipliers

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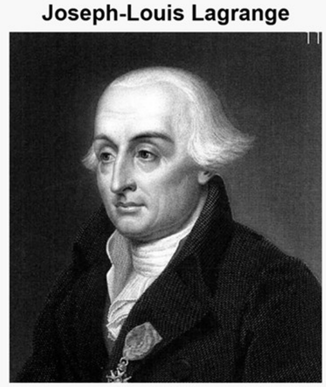
Today – “Lagrange Multipliers!”

- The Method
- One Constraint
- Examples

Joseph-Louis Lagrange



Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin, Kingdom of Sardinia
Died	10 April 1813 (aged 77) Paris, First French Empire

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Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

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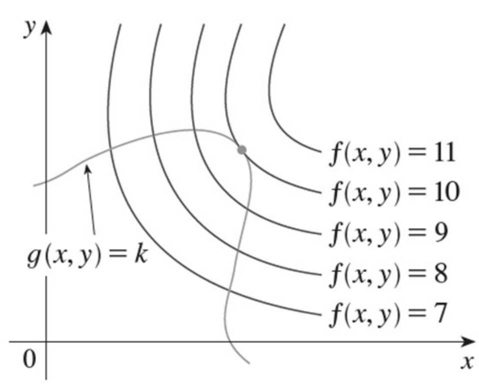
The "Lagrange Multipliers" Method

GOAL : Maximize z = f(x,y) on the curve g(x,y) = k .



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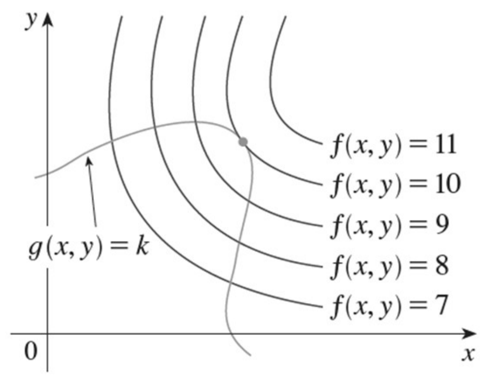
The “Lagrange Multipliers” Method

GOAL : Maximize z = f(x,y) on the curve g(x,y) = k .



The value of f(x,y) on the curve g(x,y)=k will be maximized at some point (x_0,y_0) such that

$\nabla f\ (x_0,y_0)$ is parallel to $\nabla g\ (x_0,y_0)$

or equivalently a point (x_0,y_0) such that there is a constant λ with

$\nabla f\ (x_0,y_0) = \lambda \cdot \nabla g\ (x_0,y_0)$

and $g(x_0,y_0) = k$.

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$\nabla f\ (x_0,y_0) = \lambda \cdot \nabla g\ (x_0,y_0)$ and $g(x_0,y_0) = k$.

Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

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$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

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$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

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$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

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The "Lagrange Multipliers" Method

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

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Questions?

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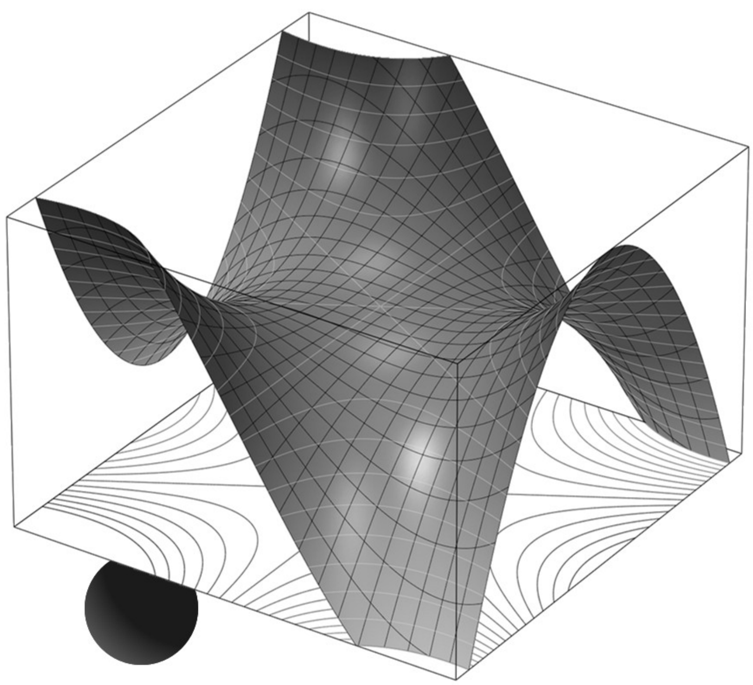
Thank you

Until next time.



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