"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Double Integrals over Rectangles

Today – Double Integrals!

- The Definite Integral
- The Riemann Integral
- Iterated Integrals
- Fubini's Theorem

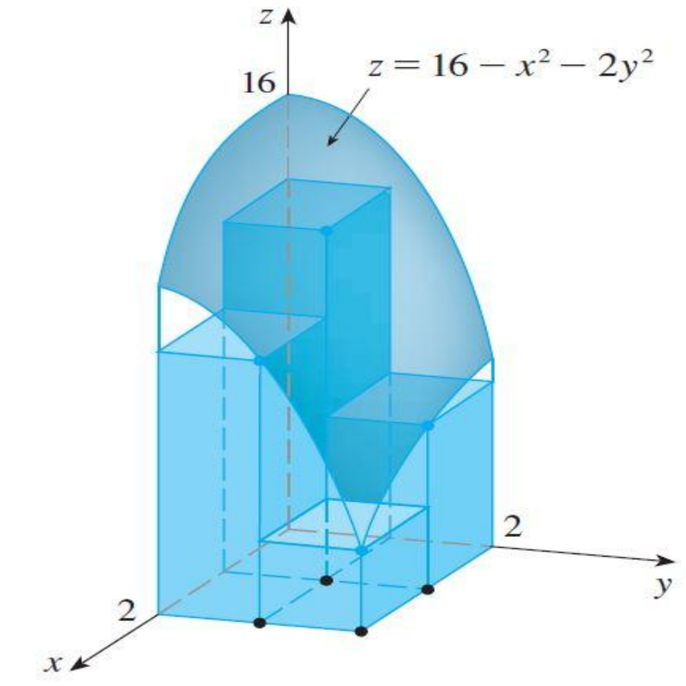
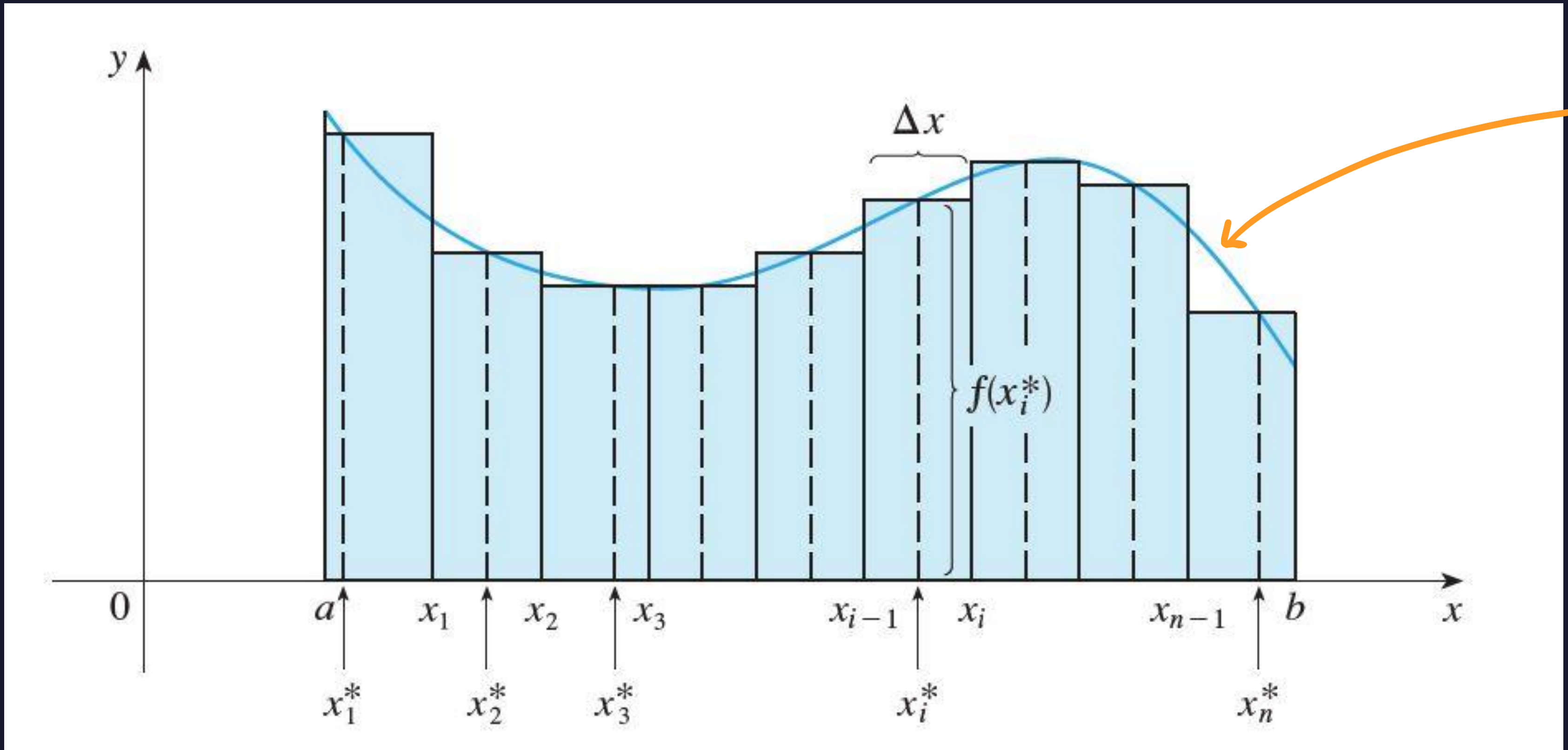


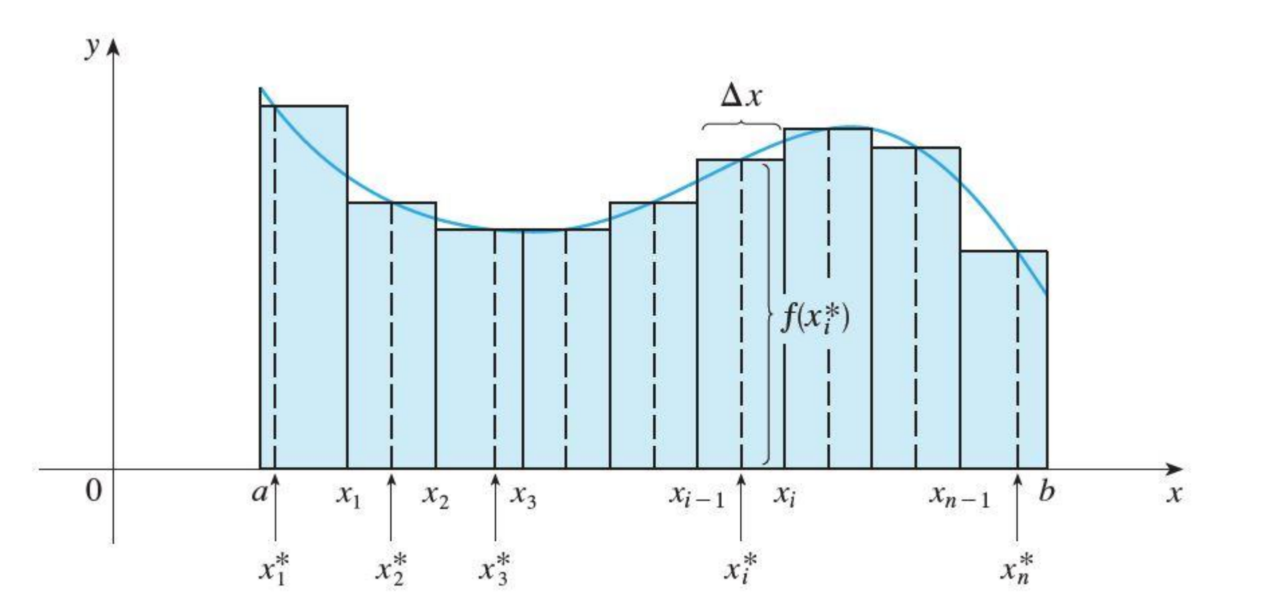
FIGURE 7

The Definite (Riemann) Integral



$y = f(x)$

The Definite (Riemann) Integral



$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x}_{\text{RIEMANN SUM}}$$

The Definite (Riemann) Integral

$z = f(x,y)$

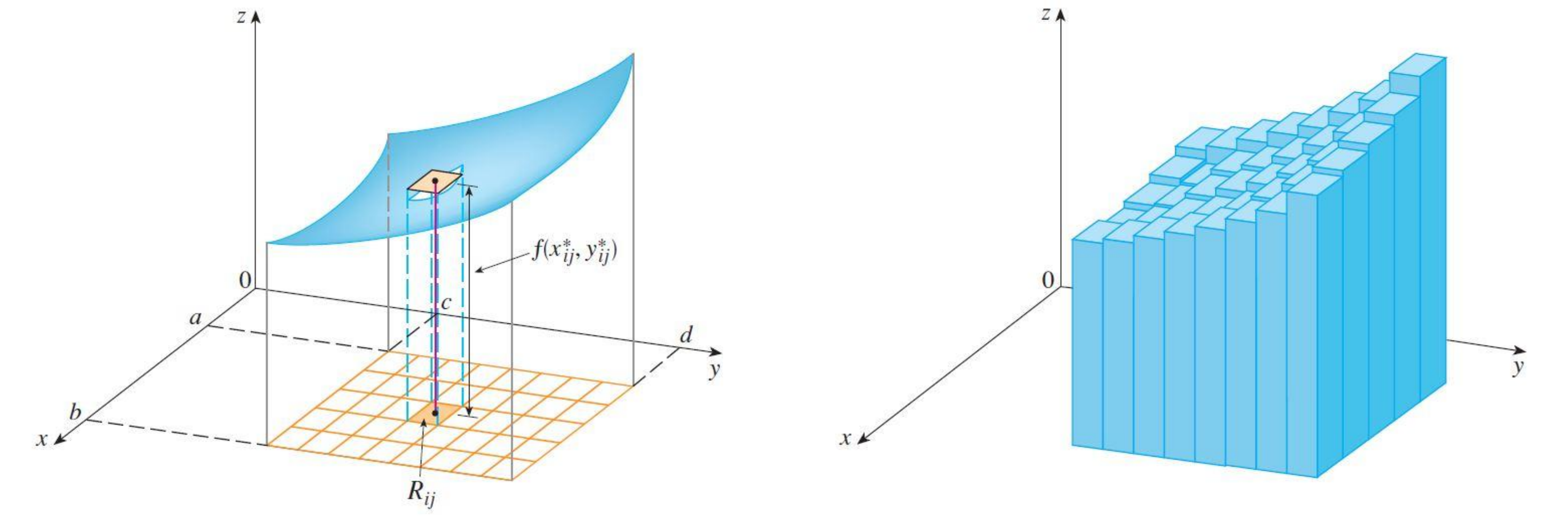


FIGURE 4 FIGURE 5

$R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b \ , \ c \le y \le d\}$

The Definite (Riemann) Integral

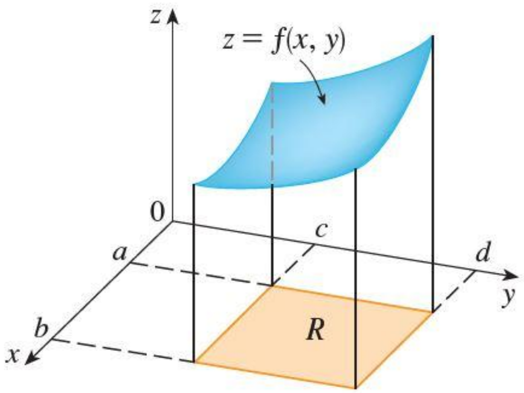


FIGURE 2

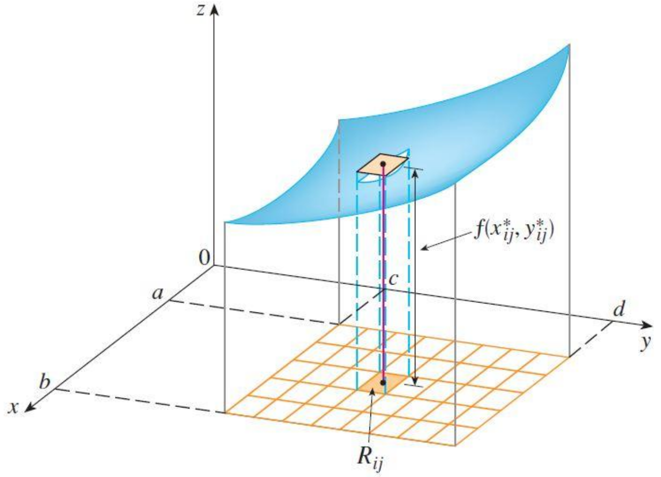


FIGURE 4

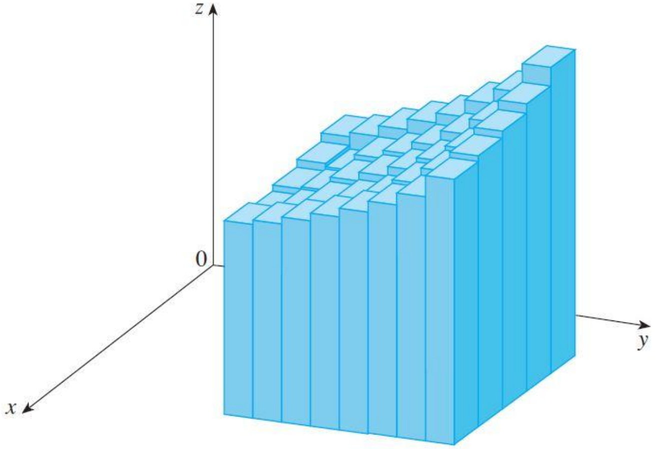


FIGURE 5

$$\iint_R f(x,y)\cdot dA = \lim_{\substack{n\to\infty \\ m\to\infty}} \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_i^*)\overbrace{\Delta x\,\Delta y}^{\Delta A}}_{\text{RIEMANN SUM}}$$

RIEMANN INTEGRAL.

The Definite (Riemann) Integral

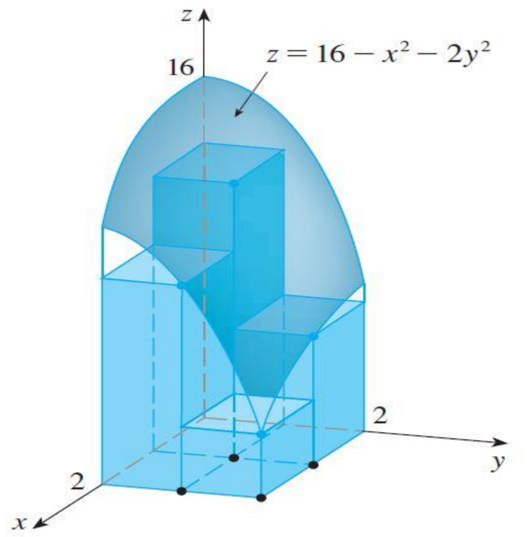
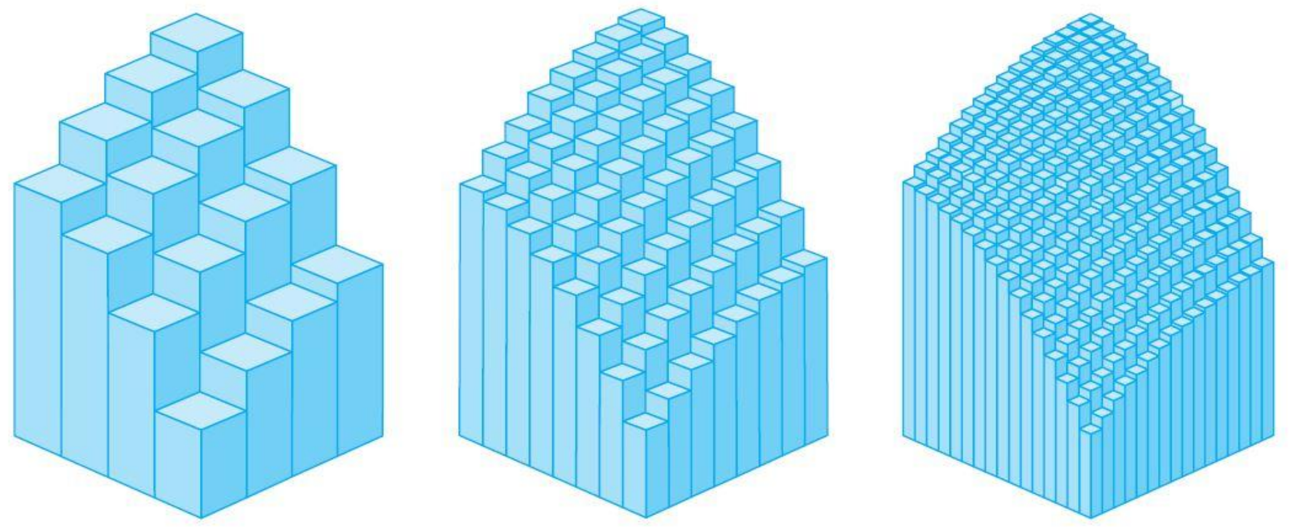


FIGURE 7



(a) $m = n = 4, V \approx 41.5$ (b) $m = n = 8, V \approx 44.875$ (c) $m = n = 16, V \approx 46.46875$

$z = f(x, y) = 16 - x^2 - 2y^2$

$R = [0, 2] \times [0, 2]$

The usual properties of integration still hold for double integrals:

- $\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA.$
- For any constant c,
$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA.$$
- If $f(x,y) \geq g(x,y)$ on the rectangle R, then
$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA.$$

And when letting $m,n \to \infty$, we have $\Delta A \to dA = dx \cdot dy$. Then

$$\iint_R f(x,y)dA = \left(\int_c^d \left(\int_a^b f(x,y)dx\right)dy,\right)$$

this is called an **iterated integral**, and we evaluate its value by computing the innermost integral first and then working the way out. Again, in the case this value represents a volume only if $f(x,y) \geq 0$ on R.

Example: Find the volume under the graph of $f(x,y) = 16 - x^2 - 2y^2$ above the square R = [0,2] x [0,2].

$$\iint_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx \, dy$$

$$\int_0^2 (16 - x^2 - 2y^2)\, dx = 16x - \frac{x^3}{3} - 2y^2 x \Big|_0^2 = 16\cdot 2 - \frac{2^3}{3} - 2y^2\cdot 2 - 16\cdot 0 - \frac{0}{3} - 2y^2\cdot 0$$

$$= 32 - \frac{8}{3} - 4y^2$$

$$= \frac{88}{3} - 4y^2$$

$$\int_0^2 \frac{88}{3} - 4y^2 \, dy = \frac{88}{3} y - 4\frac{y^3}{3} \Big|_0^2 = \frac{88}{3}\cdot 2 - 4\cdot\frac{2^3}{3} - \left(\frac{88}{3}\cdot 0 - 4\frac{0^3}{3}\right) = \boxed{48}$$

Example: Find the volume under the graph of $f(x, y) = 16 - x^2 - 2y^2$ above the square R = [0,2] x [0,2].

Example: Calculate the following iterated integrals

$$\underbrace{\int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx}_{} \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

$$\int_1^2 x^2 y \, dy = x^2 \frac{y^2}{2} \Big|_1^2 = x^2 \frac{4}{2} - x^2 \frac{1^2}{2} = 2x^2 - \frac{x^2}{2} = \frac{3}{2} x^2$$

$$\int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx = \int_0^3 \frac{3}{2} x^2 \, dx = \frac{1}{2} x^3 \Big|_0^3 = \frac{3^3}{2} - \frac{0}{2} = \boxed{\frac{27}{2}}$$

Example: Calculate the following iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \underbrace{\int_1^2 \int_0^3 x^2 y \, dx \, dy}$$

$$\int_0^3 x^2 y \, dx = y \cdot \frac{x^3}{3}\Big|_0^3 = y \cdot \frac{3^3}{3} - y \cdot \frac{0^3}{3} = 9y$$

$$\int_1^2 \left(\int_0^3 x^2 y \, dx\right) dy = \int_1^2 9y \, dy = 9\frac{y^2}{2}\Big|_1^2 = 9 \cdot \frac{2^2}{2} - 9 \cdot \frac{1}{2}$$

$$= 2 \cdot 9 - 9/2 = \frac{3}{2} \cdot 9 = \boxed{\frac{27}{2}}$$

Fubini's Theorem

If f is continuous on the rectangle

$$R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$$

then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Guido Fubini

Born	19 January 1879 Venice
Died	6 June 1943 (aged 64) New York

Example: Evaluate the double integral

$$\iint_R \left(x - 3y^2\right) dA$$

where $R = \{(x, y): 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

$$\int_1^2 \left(\int_0^2 (x - 3y^2)\, dx \right) dy$$

$$\int_0^2 x - 3y^2 \, dx = \frac{x^2}{2} - 3y^2 x \Big|_0^2$$

$$= \frac{4}{2} - 6y^2 = 2 - 6y^2$$

$$\int_1^2 2 - 6y^2 \, dy = 2y - 6\frac{y^3}{3} \Big|_1^2 = 2y - 2y^3 \Big|_1^2 = 4 - 2\cdot 2^3 - (2 - 2) = \boxed{-12}$$

Example: Evaluate the double integral

$$\iint_R (x - 3y^2)\, dA$$

where $R = \{(x, y): 0 \le x \le 2, 1 \le y \le 2\}$.

$$\int_0^2 \int_1^2 x - 3y^2 \, dy \, dx$$

$$\int_1^2 x - 3y^2 \, dy = xy - y^3 \Big|_1^2$$

$$= 2x - 8 - (x - 1) = x - 7$$

$$\shortparallel$$

$$\int_0^2 x - 7 \, dx = \frac{x^2}{2} - 7x \Big|_0^2 = \frac{2^2}{2} - 7 \cdot 2 = 2 - 14 = \boxed{-12}$$

Example: Evaluate the double integral

$$\iint_R y\sin(xy)\,dA$$

where $R = [1,2]\times[0,\pi]$.

$$\cancel{\int_1^2\int_0^\pi y\cdot\sin(xy)\,dy\,dx} \qquad \text{OR} \qquad \overbrace{\int_0^\pi\int_1^2 y\cdot\sin(xy)\,dx\,dy}^{\text{EASIER!}}$$

$$\int_1^2 y\sin(xy)\,dx = y\cdot\left(-\frac{\cos(xy)}{y}\right) = -\cos(xy)\Big|_1^2 = -\cos(2y)+\cos(y)$$

$$\int_0^\pi -\cos(2y)+\cos(y)\,dy = -\frac{\sin(2y)}{2}+\sin(y)\Big|_0^\pi$$

$$= 0+0-(0+0) = \boxed{0}$$

Example: Evaluate the double integral

$$\iint_R y\sin(xy)\,dA$$

where $R = [1,2] \times [0,\pi]$.

When $f(x,y) = g(x) \cdot h(y)$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dxdy = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

- Evaluate the iterated integral

$$\int_1^3 \int_1^5 \frac{\ln(y)}{xy}\,dxdy = \boxed{\ln 5 \cdot \frac{(\ln 3)^2}{2}}$$

$$\iint g(x)\cdot h(y)\,dx\,dy = \int h(y) \cdot \left(\underbrace{\int g(x)\,dx}\right) \cdot dy = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

$$\frac{\ln(y)}{xy} = \frac{1}{x} \cdot \frac{\ln(y)}{y}$$

$$\bullet \int_1^5 \frac{1}{x}\,dx = \ln x \Big|_1^5 = \ln 5 - \ln 1 = \boxed{\ln 5}$$

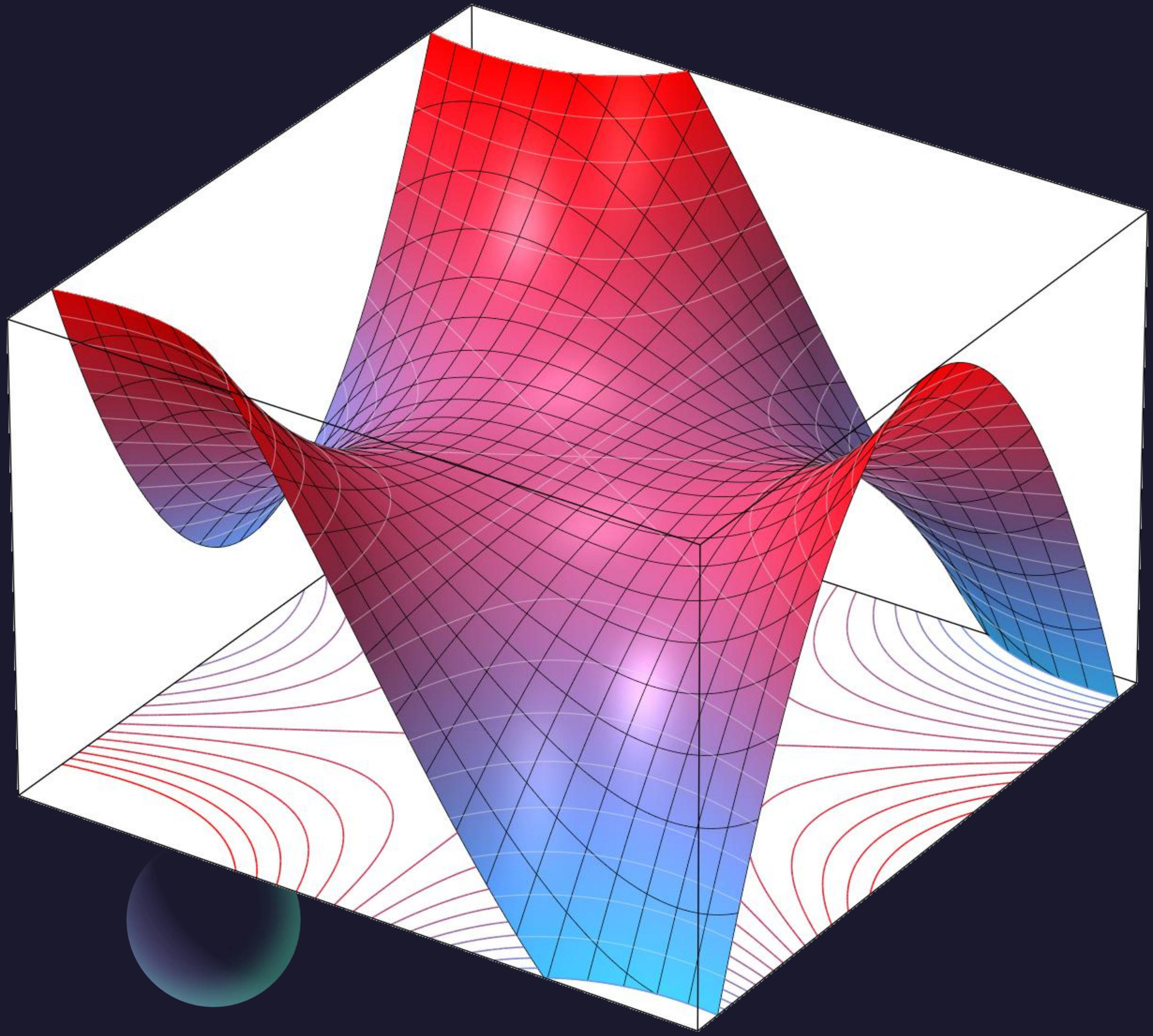
$$\bullet \int_1^3 \frac{\ln(y)}{y}\,dy = \frac{(\ln(y))^2}{2}\Big|_1^3 = \frac{(\ln 3)^2}{2} - \frac{(\overset{0}{\ln 1})^2}{2} = \frac{(\ln 3)^2}{2}$$

$$\left((\ln y)^2\right)' = 2\ln y \cdot \frac{1}{y}$$

Questions?

Thank you

Until next time.



“Calculus 3”

Multi-Variable Calculus

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Double Integrals over Regions

Today – Double Integrals in Regions!

- General Regions
- Regions of Type I and II
- Changing the Order of Integration
- Properties of Double Integrals

Regions of Type I and II

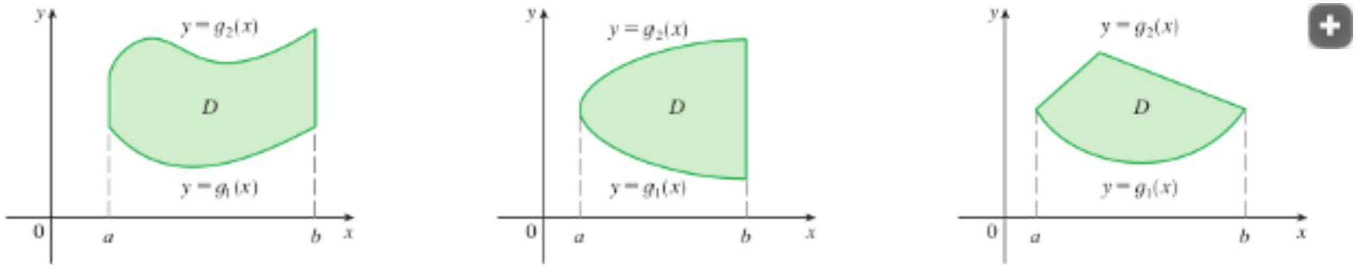
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

Figure 5

Some type I regions



Regions of Type I and II

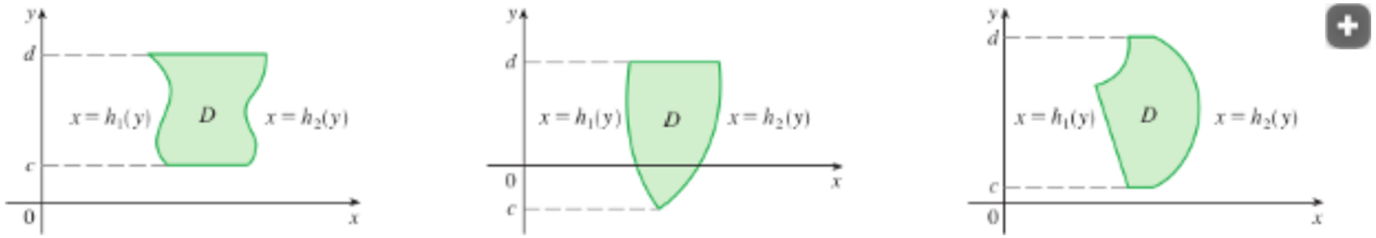
We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

Figure 7

Some type II regions



Integrals over Regions of Type I

$\boxed{3}$ If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Integrals over Regions of Type II

$\boxed{4}$ If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \text{ and } y = 1 + x^2 .$$

[Extra]

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \quad \text{and} \quad y = 1 + x^2 \, .$$

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type I integral.)

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type II integral.)

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Properties of Double Integrals

$$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

$$\iint_D cf(x,y)\, dA = c\iint_D f(x,y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

$\boxed{7}$

$$\iint_D f(x,y)\, dA \geqslant \iint_D g(x,y)\, dA$$

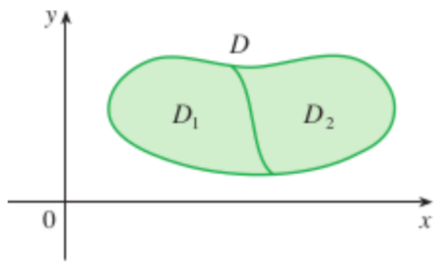
Properties of Double Integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

8

$$\iint\limits_{D} f(x, y)\, dA = \iint\limits_{D_1} f(x, y)\, dA + \iint\limits_{D_2} f(x, y)\, dA$$

Figure 17



Properties of Double Integrals

$$\iint_D 1 \, dA = A(D)$$

10 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y) \, dA \leqslant M \cdot A(D)$$

Example: Estimate the value of the double integral

$$\iint_R e^{-(x^2+y^2)}dA$$

where $R = \{(x, y): x^2 + y^2 \leq 1\}$ is the circle of radius 1.

Questions?

Thank you

Until next time.