"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 7

Any Reminders? Any Questions?

- Class ends at 3:15.
- Slides are being posted on GitHub!
  https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- Request videos!!

EXAM 1 -- Friday, February 20th

Exam Covers:

- **Chapter 12**
  - Sections 12.1 – 12.6
- **Chapter 14**
  - Sections 14.1, 14.3 – 14.8

Questions?

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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The Chain Rule

Today – The Chain Rule!

- The Single Variable Case
- Chain Rule with One Parameter
- Chain Rule with Two Parameters

The Good Ol' Chain Rule

$$y = f(x) \qquad x = g(t)$$

$$\frac{d}{dt}\Big(f(g(t))\Big) = \frac{df}{dx}(g(t)) \cdot \frac{dg}{dt}$$

$$\big(f(g(t))\big)' = f'(g(t)) \cdot g'(t)$$

Example: Find the derivative of f(g(t)) with respect to t where

$$f(x) = \sin(x) \quad \text{and} \quad g(t) = t^2 + 1$$

$$\left(f(g(t))\right)' = f'(g(t)) \cdot g'(t)$$

$$\left(f(g(t))\right)' = \cos(t^2+1) \cdot (2t)$$

$$= \boxed{2t \cdot \cos(t^2+1)}$$

Example: Find the derivative of f(g(t)) with respect to t where

$$f(x) = \sin(x) \qquad \text{and} \qquad g(t) = t^2 + 1$$

[Extra space]

The New Chain Rule – Case 1

1 The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{d}{dt}\left(f(g(t), h(t))\right)$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial f}{\partial x}(g,h) \cdot \frac{dg}{dt} + \frac{\partial f}{\partial y}(g,h) \cdot \frac{dh}{dt}$$

Example: Find the derivative of f(g(t),h(t)) with respect to t where

$$f(x,y) = x^2 + y \quad \text{and} \quad x = g(t) = 3t^4 + 1, \quad y = h(t) = 3t$$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 1 \qquad \frac{d}{dt} g = 12 \cdot t^3 \qquad \frac{d}{dt} h = 3$$

$$\frac{d}{dt}\left(f(g,h)\right) = 2 \cdot (3t^4+1) \cdot 12 \cdot t^3 + 1 \cdot 3$$

$$= \boxed{24t^3 \cdot (3t^4+1) + 3}$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example: Find the derivative of f(g(t),h(t)) with respect to t where

$$f(x,y) = x^2 + y \quad \text{and} \qquad g(t) = 3t^4 + 1, \quad h(t) = 3t$$

[Extra]

$$f(g,h) = (3t^4+1)^2 + 3t$$

$$\frac{d}{dt}(f(g,h)) = 2\cdot(3t^4+1)\cdot 12\cdot t^3 + 3$$

$$= \boxed{24t^3(3t^4+1) + 3}$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The New Chain Rule – Case 2

2 The Chain Rule (Case 2)

$f(g(s,t), h(s,t))$

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

z

Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$$f(x,y) = x^2y^3 \quad \text{and} \quad x = g(s,t) = s\cdot\cos(t)\ ,\ y = h(s,t) = s\cdot e^{2t}$$

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

$$\begin{cases} \dfrac{\partial g}{\partial s} = \cos(t) \\[2ex] \dfrac{\partial g}{\partial t} = -s\cdot\sin t \end{cases} \qquad \begin{cases} \dfrac{\partial h}{\partial s} = e^{2t} \\[2ex] \dfrac{\partial h}{\partial t} = 2s\cdot e^{2t} \end{cases}$$

$$\frac{\partial z}{\partial s} = \underbrace{2\cdot(s\cdot\cos(t))\cdot(s\cdot e^{2t})^3\cdot\cos(t)}$$

$$\underbrace{+3\cdot(s\cos t)^2\cdot(s\cdot e^{2t})^2\cdot e^{2t}}$$

$$\frac{\partial z}{\partial t} = \underbrace{\dots}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Find the derivatives of f(g(s,t),h(s,t)) with respect to s and t where

$f(x,y) = x^2y^3$ and $g(s,t) = s \cdot \cos(t)$, $h(s,t) = s \cdot e^{2t}$

[Extra]

Questions?

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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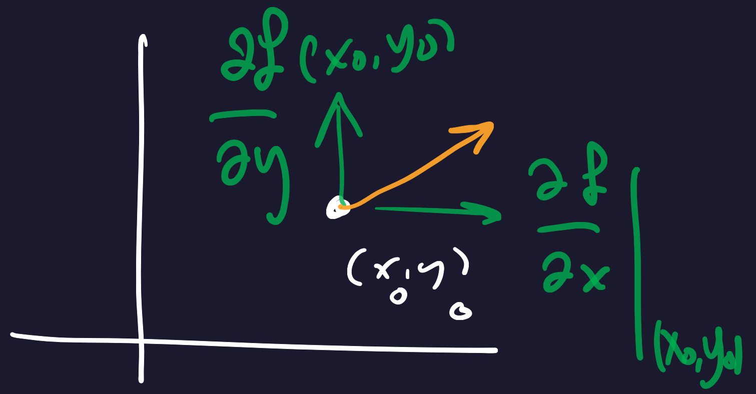
Directional Derivatives

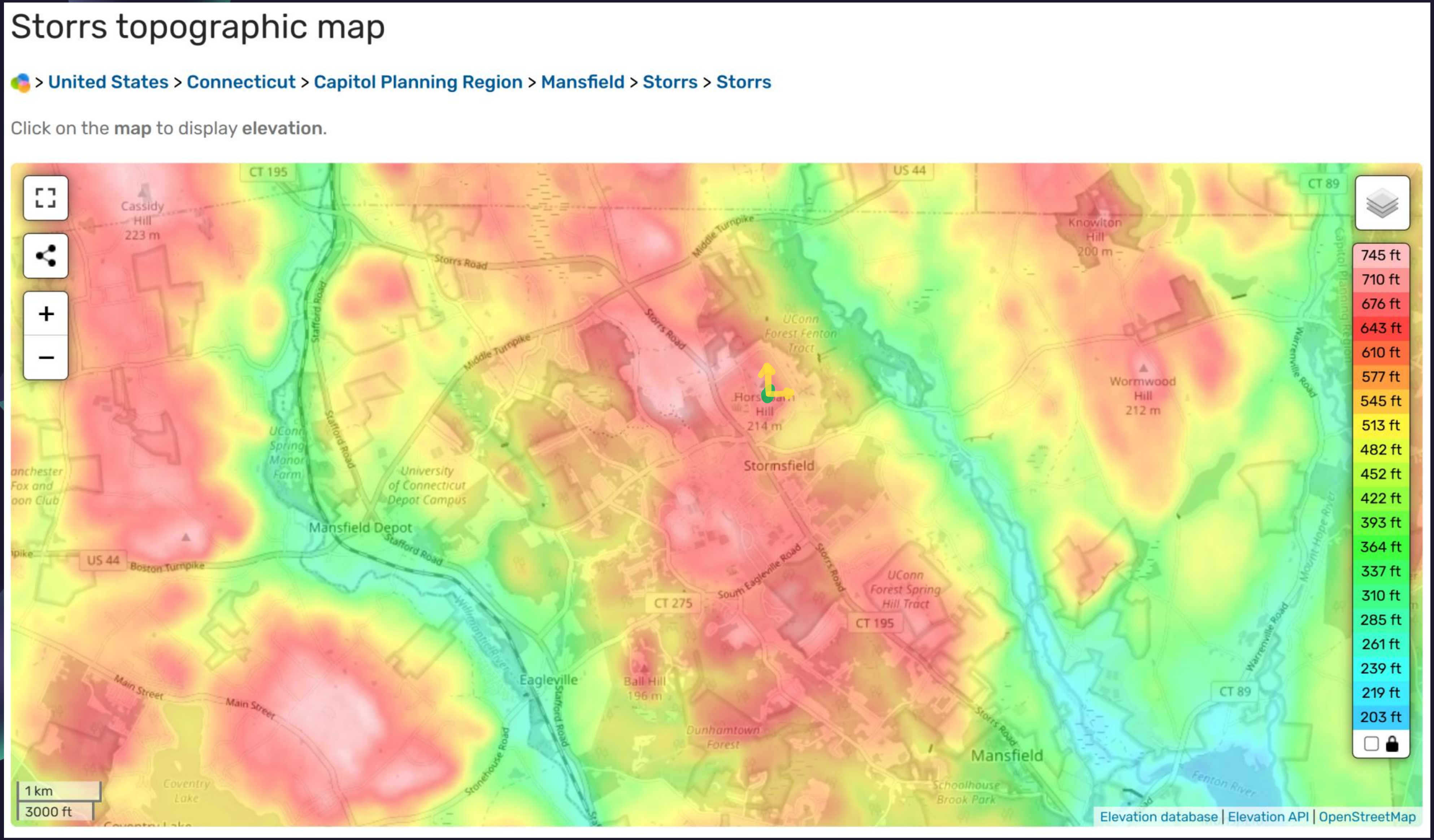
Today – Directional Derivatives!



- Directional Derivatives
- Gradient Vector
- Maximizing the Directional Derivative

$z = f(x, y)$

What is the steepest path down from Horsebarn Hill?



Definition of Directional Derivative

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

How to Compute the Directional Derivative

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathrm{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

3 Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$\left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}, \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \right) \cdot (a, b)$$

$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b \quad /\!/$$

$$D_{\mathrm{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

Example: Find the directional derivative in the direction of u = (1,-1) for

$$f(x,y) = x^3 - 3xy + 4y^2 \qquad \text{at } (x_0, y_0) = (1,2).$$

$u = (1,-1) \quad |u| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad u_1 = \frac{1}{\sqrt{2}} u = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$\frac{\partial f}{\partial x} = 3x^2 - 3y \quad \left.\frac{\partial f}{\partial x}\right|_{(1,2)} = -3$

$\frac{\partial f}{\partial y} = -3x + 8y \quad \left.\frac{\partial f}{\partial y}\right|_{(1,2)} = 13$

A unit vector in the direction of u is given

$u_1 = \frac{u}{|u|}$

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

Example: Find the directional derivative in the direction of u = (1,-1) for

$$f(x,y) = x^3 - 3xy + 4y^2 \qquad \text{at } (x_0, y_0) = (1,2).$$

$$u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad \left.\frac{\partial f}{\partial x}\right|_{(1,2)} = -3, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,2)} = 13$$

$$D_u f = -3 \cdot \frac{1}{\sqrt{2}} + 13 \cdot \left(-\frac{1}{\sqrt{2}}\right) = \boxed{-\frac{16}{\sqrt{2}}}$$

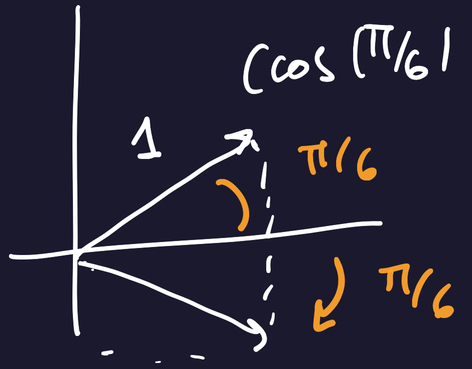
[Extra]

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

Example: Find the directional derivative in the direction given by the angle pi/6 measured clockwise from the x-axis, of the function:

$$f(x,y) = \ln(x^2+y^2) \quad \text{at} \quad (x_0, y_0) = (1,0).$$

$(\cos(\pi/6), \sin(\pi/6))$



$$u_1 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \qquad \left.\frac{\partial f}{\partial x}\right|_{(1,0)} = \frac{2}{1} = 2 \qquad \frac{\partial f}{\partial y} = \left.\frac{2y}{x^2+y^2}\right|_{(1,0)} = 0$$

$$D_u f = 2 \cdot \frac{\sqrt{3}}{2} + 0 \cdot \left(-\frac{1}{2}\right) = \boxed{\sqrt{3}}$$

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

Example: Find the directional derivative in the direction given by the angle π/6 measured clockwise from the x-axis, of the function:

$$f(x,y) = \ln(x^2 + y^2) \quad \text{at} \quad (x_0, y_0) = (1,0).$$

[Extra]

The Gradient Vector

3 Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathrm{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = 3i + 4j of the function:

$$f(x,y) = x^2y^3 - x \cdot \cos(y) \quad \text{at} \quad (x_0, y_0) = (1,0).$$

$|u| = \sqrt{3^2 + 4^2} = 5 \qquad u_1 = \frac{3}{5} i + \frac{4}{5} j$

$\frac{\partial f}{\partial x} = 2xy^3 - \cos(y) \Big|_{(1,0)} = -1$

$\frac{\partial f}{\partial y} = 3x^2y^2 + x\sin(y) \Big|_{(1,0)} = 0$

$D_u f = \nabla f \Big|_{(1,0)} \cdot \left(\frac{3}{5}, \frac{4}{5} \right)$

$= (-1, 0) \cdot \left(\frac{3}{5}, \frac{4}{5} \right)$

$= \boxed{-3/5}$

[Extra]

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = 3i + 4j of the function:

$$f(x,y) = x^2y^3 - x \cdot \cos(y) \quad \text{at} \quad (x_0, y_0) = (1,0).$$

Maximizing the Directional Derivative

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Fixed (x_0, y_0)

$$|D_u f| = |\nabla f|_{(x_0,y_0)} \cdot u|$$

$$= |\nabla f| \cdot \underbrace{|u|}_{=1} \cdot \cos\theta$$

$$= |\nabla f| \cdot \cos\theta \qquad -1 \le \underbrace{\cos\theta \le 1}$$

$$\theta = 0$$

MAX $\Rightarrow \nabla f \parallel u$!!

Maximizing the Directional Derivative

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

15 Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathrm{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Example: Find the directional derivative in the direction of u = (1,1) and also the maximal directional derivative of the function:

$$f(x, y) = x^2 \ln(y) \qquad \text{at } (x_0, y_0) = (3, 1).$$

$u = (1,1)$, $u_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\frac{\partial f}{\partial x} = 2x \ln y \Big|_{(3,1)} = 0$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} \Big|_{(3,1)} = 9$$

$$D_{u_1} f = (0, 9) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \underbrace{9/\sqrt{2}}$$

$D_u f$ in the direction of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

[Extra]

Example: Find the directional derivative in the direction of u = (1,1) and also the maximal directional derivative of the function:

$$f(x,y) = x^2 \ln(y) \qquad \text{at } (x_0, y_0) = (3,1).$$

Max'l dir. Derivative will be in the direction of $\nabla f\big|_{(3,1)} = (0,9) \rightsquigarrow u = (0,1)$

$|(0,9)| = 9$

$\left(\frac{0}{9}, \frac{9}{9}\right) = (0,1)$

$D_u = (0,9)\cdot(0,1) = \boxed{9}$ MAX

Properties of the Gradient Vector

$0 = f(x, y) - z$

Suppose a surface is defined by $z = f(x, y)$. We can view it as level surface $t = 0$ of a higher-dimensional function $t = F(x, y, z)$ defined by

$$F(x, y, z) = f(x, y) - z$$

Then $\nabla F = (f_x, f_y, -1)$ is the normal vector to the tangent plane of $f(x, y) - z = 0$.

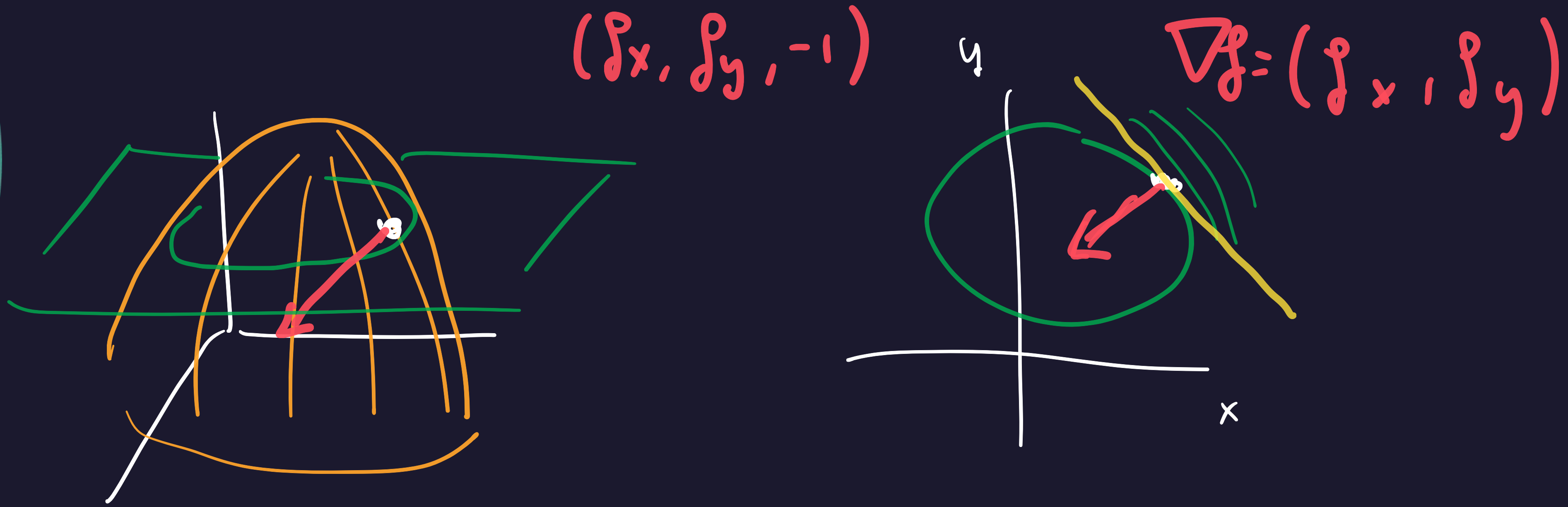
Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

The gradient vector in two dimensions, $\nabla F = (f_x, f_y)$ is normal to any level curves of $f(x,y)$ at any point, indicating the maximum rate of change.

Properties of the Gradient Vector

Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

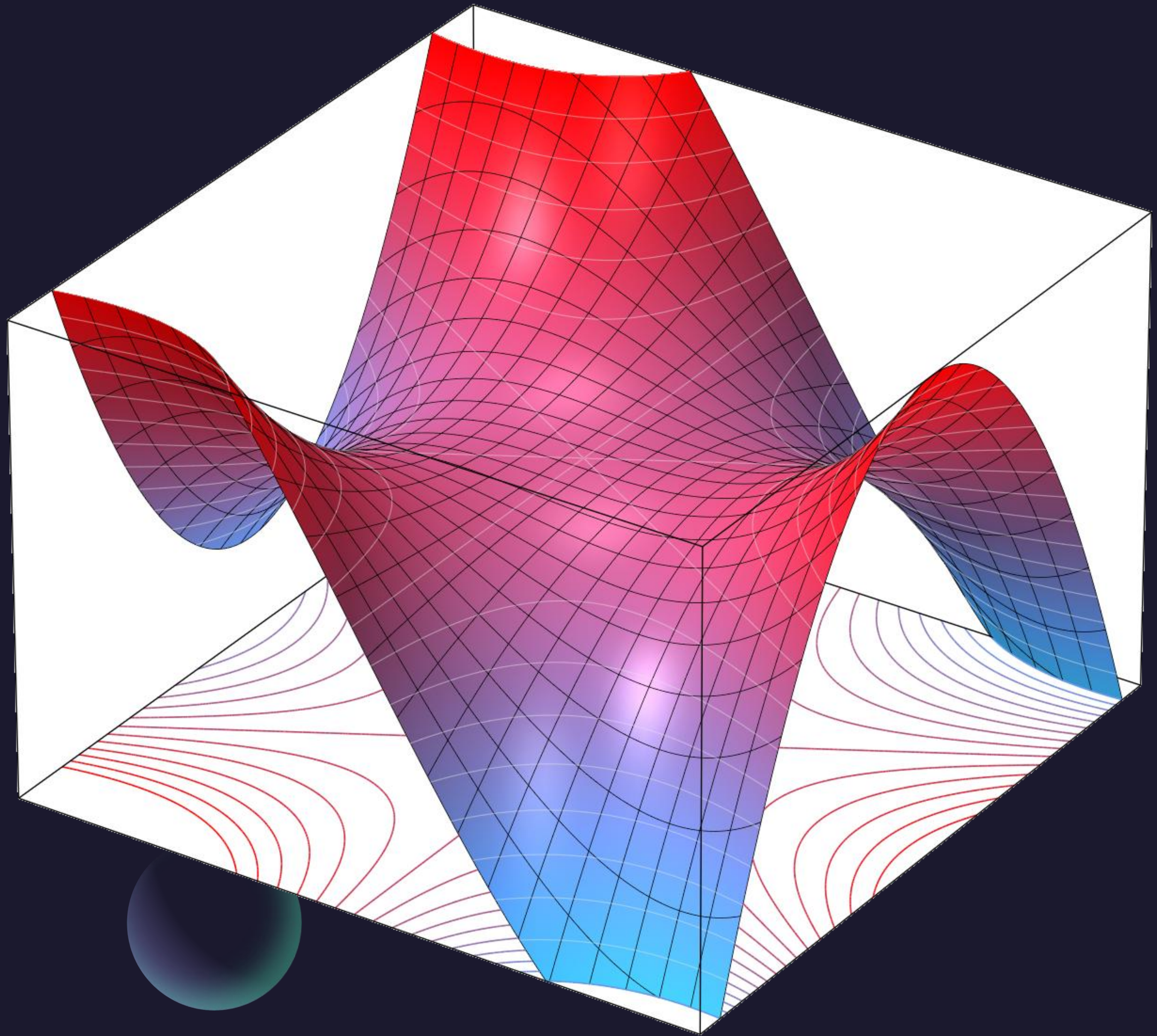
The gradient vector in two dimensions, $\nabla F = (f_x, f_y)$ is normal to any level curves of $f(x,y)$ at any point, indicating the maximum rate of change.



Questions?

Thank you

Until next time.



ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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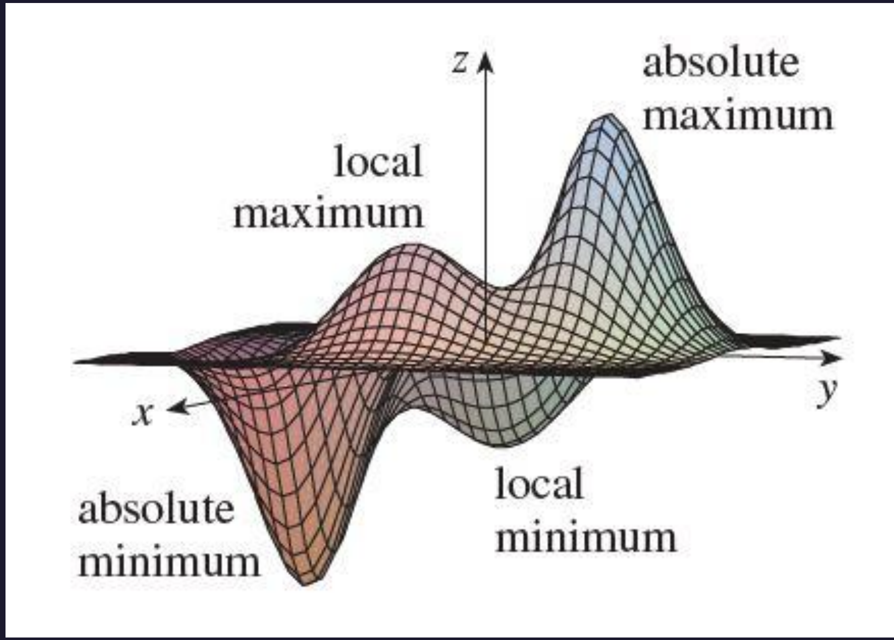
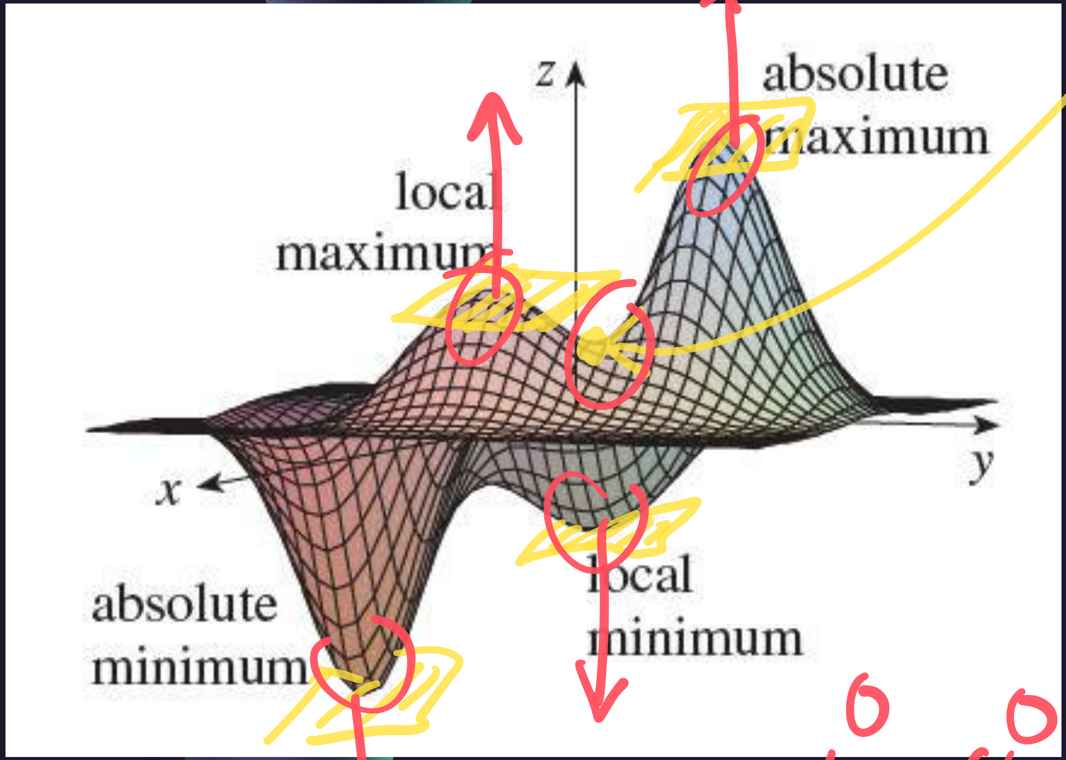
Maximum and Minimum Values

Today – Maximum and Minimum Values!

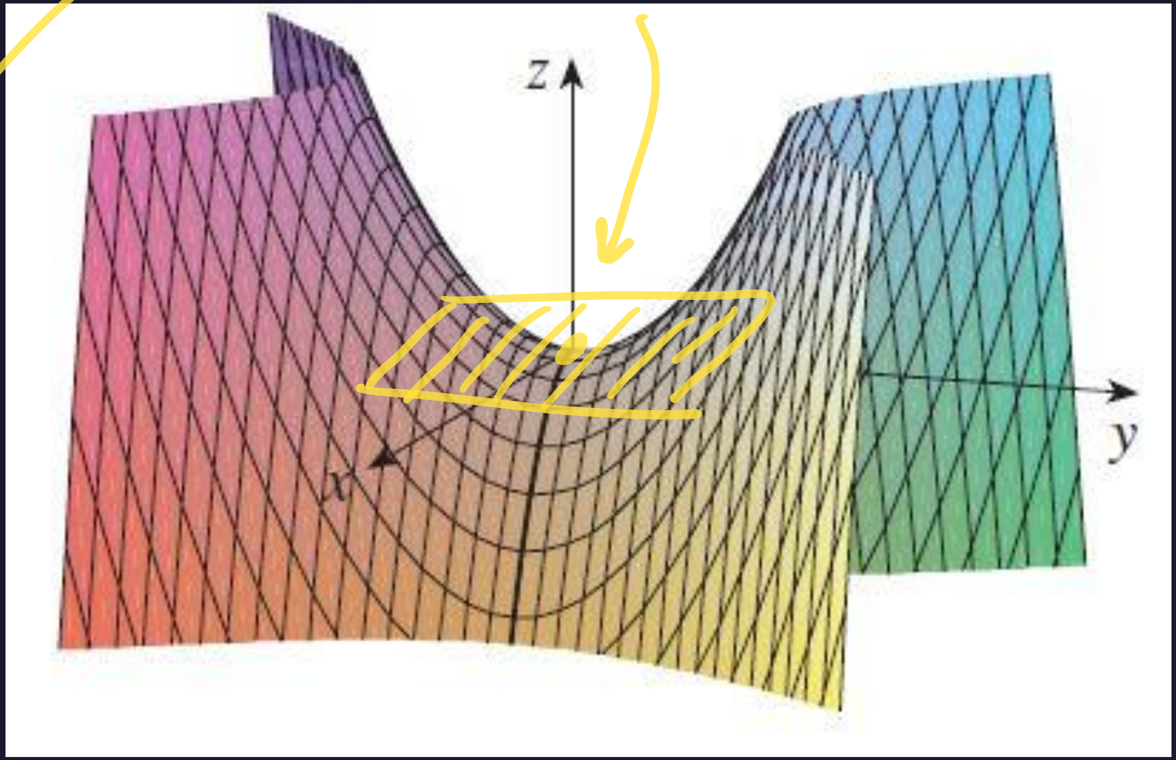
- Local Max and Min Values
- Second Derivative Test
- Absolute Max and Min Values



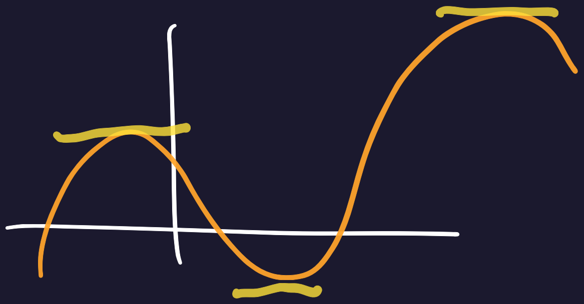
Local Max and Min Values



saddle

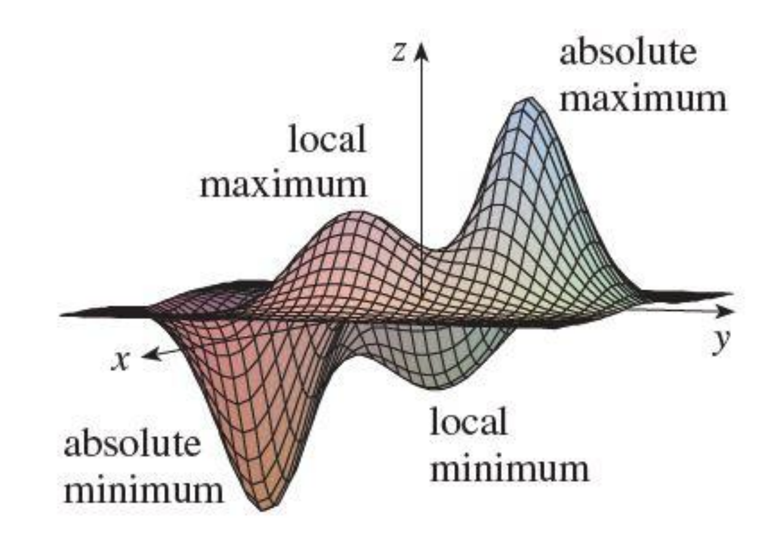


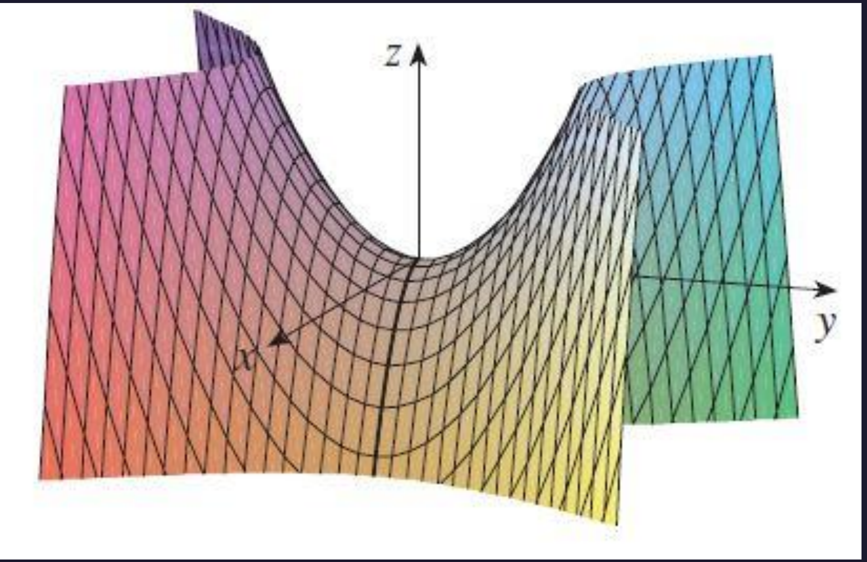
$$\nabla F = (\overset{0}{f_x}, \overset{0}{f_y}, -1) \parallel (0,0,-1)$$



Local Max and Min Values

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

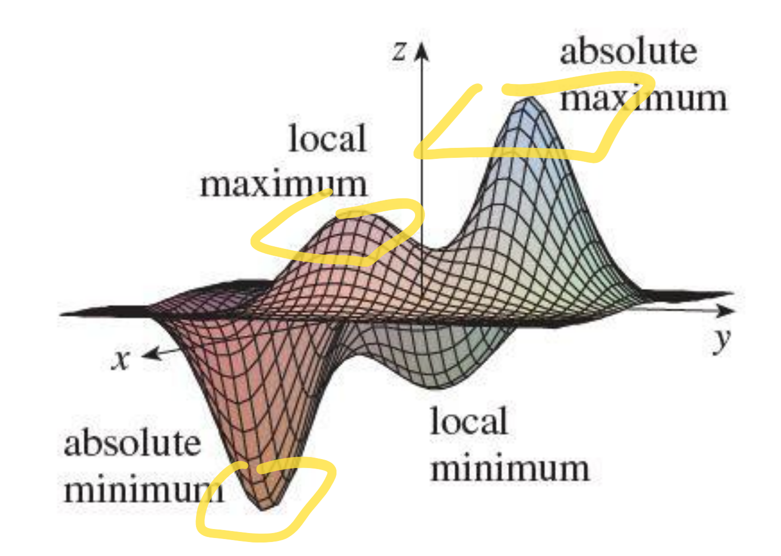


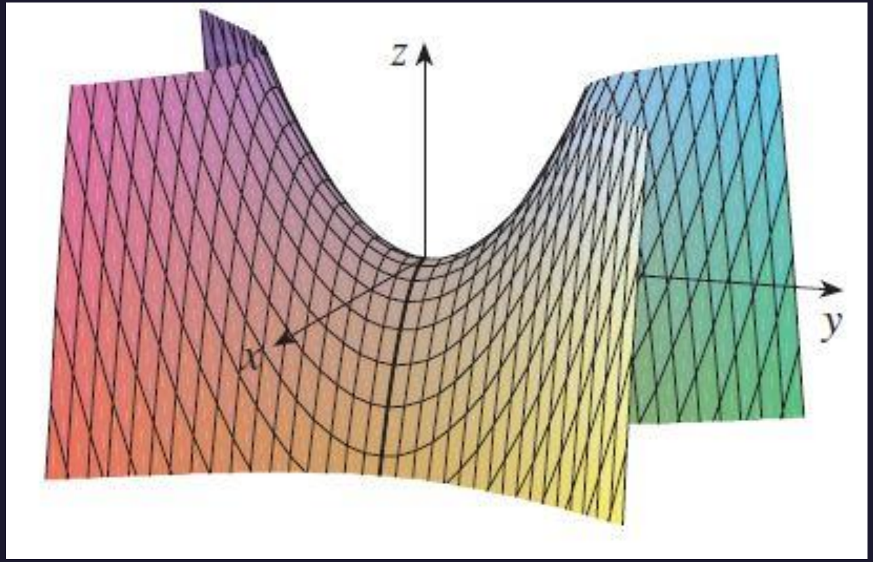


Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.





Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x\,(a, b) = 0$ and $f_y\,(a, b) = 0$.

A **critical point** for a function $f(x, y)$ is a point (a, b) where

$$\nabla f(a, b) = \vec{0},$$

that is $f_x(a, b) = 0$, $f_y(a, b) = 0$.

Example: Find all the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

$\nabla f = \vec{0}$? $\qquad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x} = 2x - 2 = 0 \Rightarrow x = 1 \\ \frac{\partial f}{\partial y} = 2y - 6 = 0 \Rightarrow y = 3 \end{array} \right\}$$

one critical pt

$$\boxed{P = (1, 3)}$$

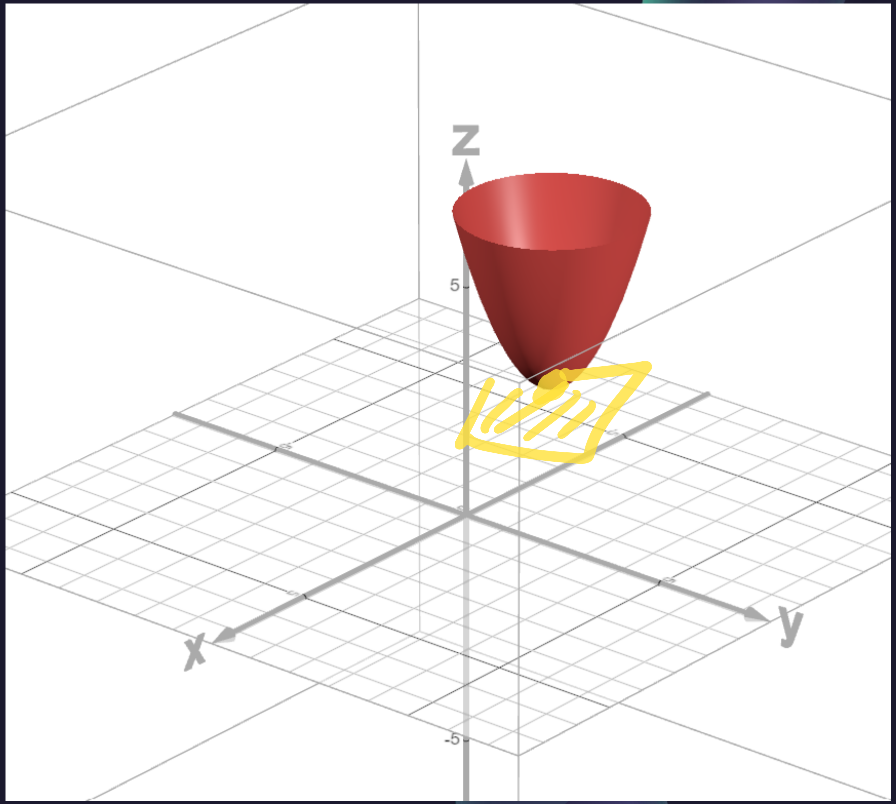
Example: Find all the critical points for the function

$$z = f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

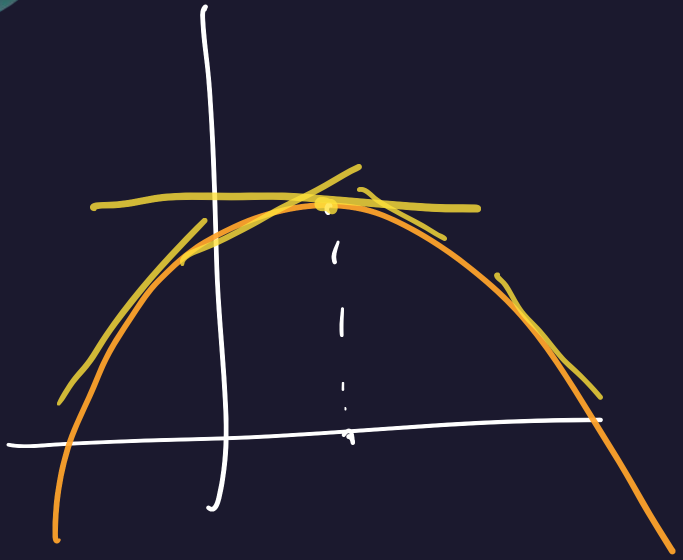
$$z = (x-1)^2 + (y-3)^2 + 4$$

$$f(1,3) = 4$$

GLOBAL/LOCAL MIN

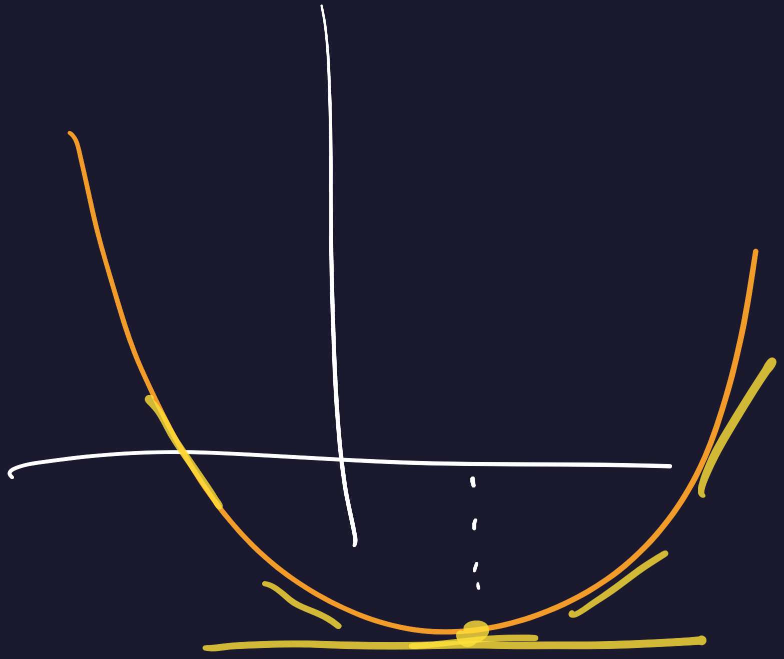


Local Max and Min Values: Second Derivative Test



$f'' < 0$

$f' = 0$



$f'' > 0$

$f' = 0$

Local Max and Min Values: Second Derivative Test

3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$\boxed{D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$\left(ac - \frac{b^2}{4}\right)$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is a saddle point of f.

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

[Extra]

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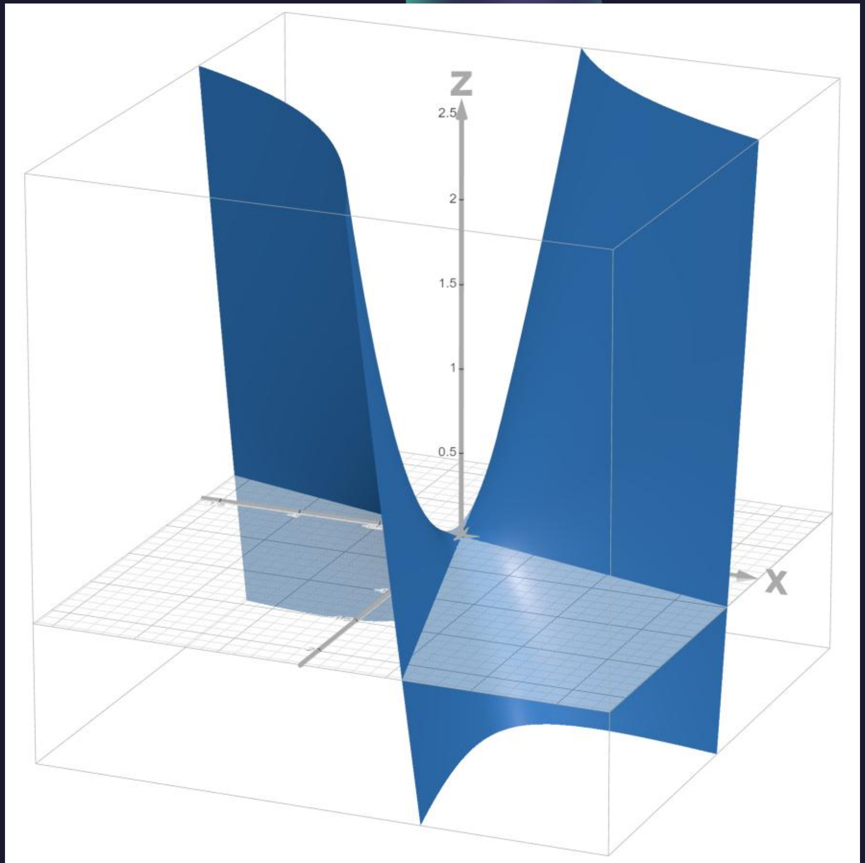

Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

8 Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Absolute Max and Min Values

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$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find the absolute maximum and minimum of

$$f(x,y) = xy^2$$

in the region $D = \{(x,y) \colon x^2 + y^2 \leq 3\}$.

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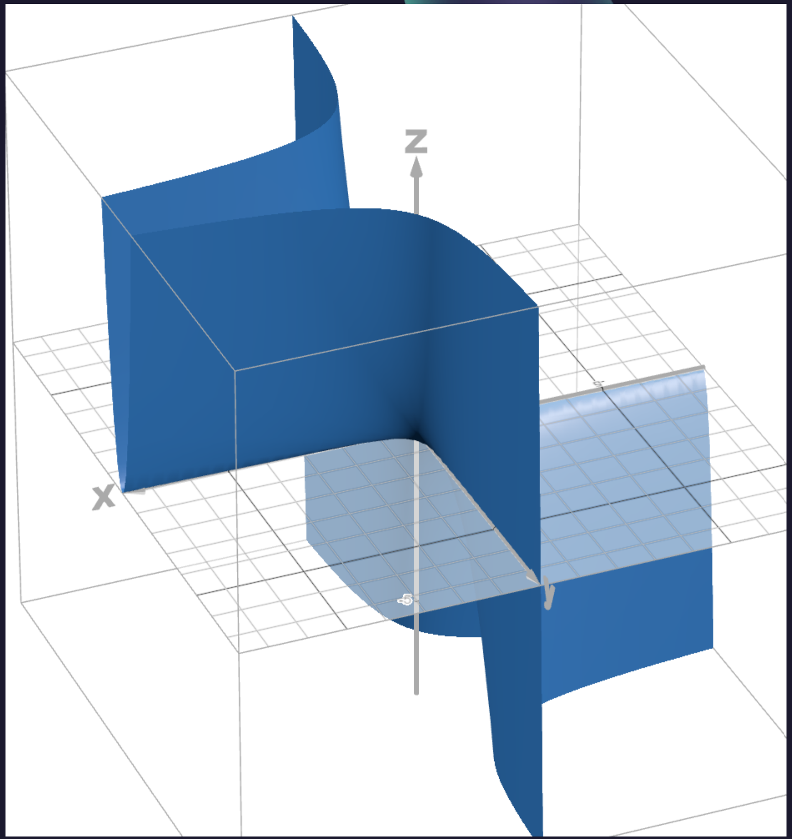
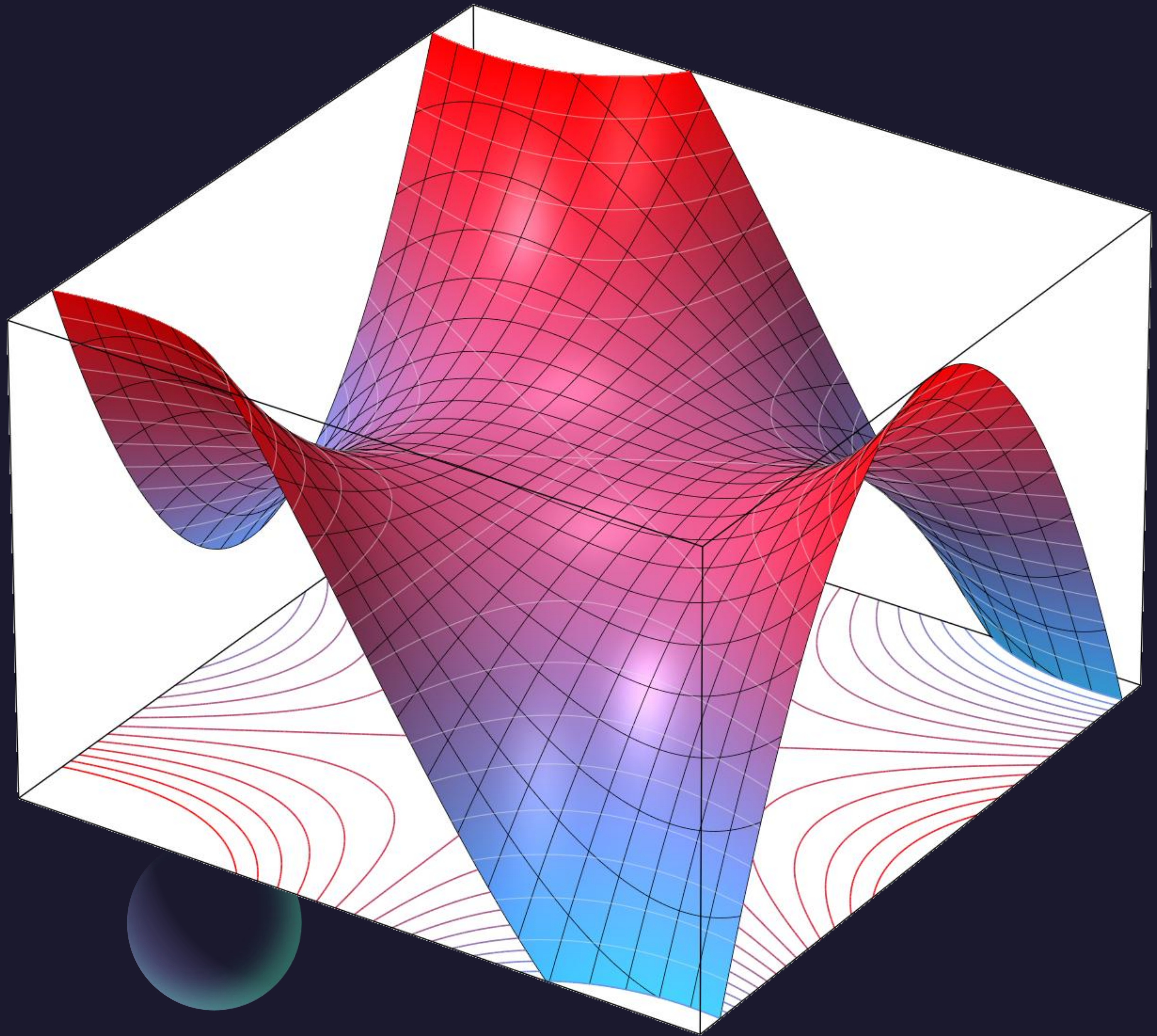


Questions?

Thank you

Until next time.



ALVARO: Start the recording!

"Calculus 3"

Multi-Variable Calculus

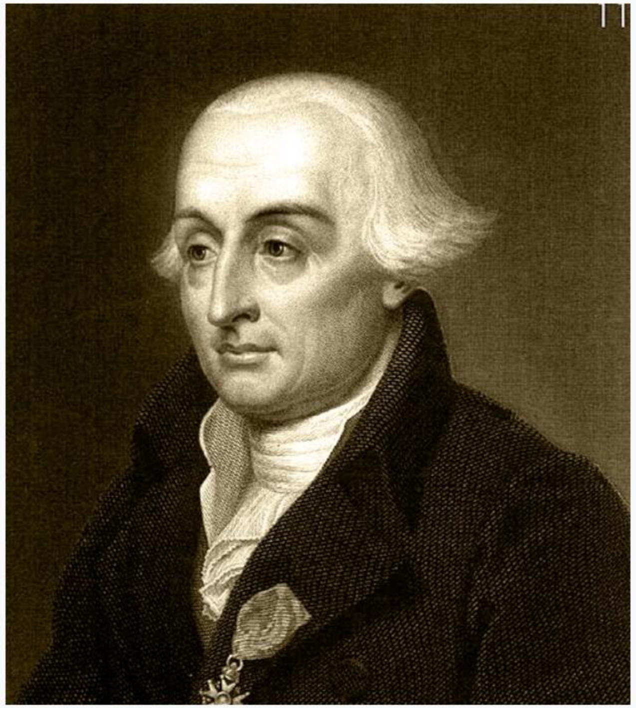
Instructor: Álvaro Lozano-Robledo

Lagrange Multipliers

Today – "Lagrange Multipliers!"

- The Method
- One Constraint
- Examples

Joseph-Louis Lagrange

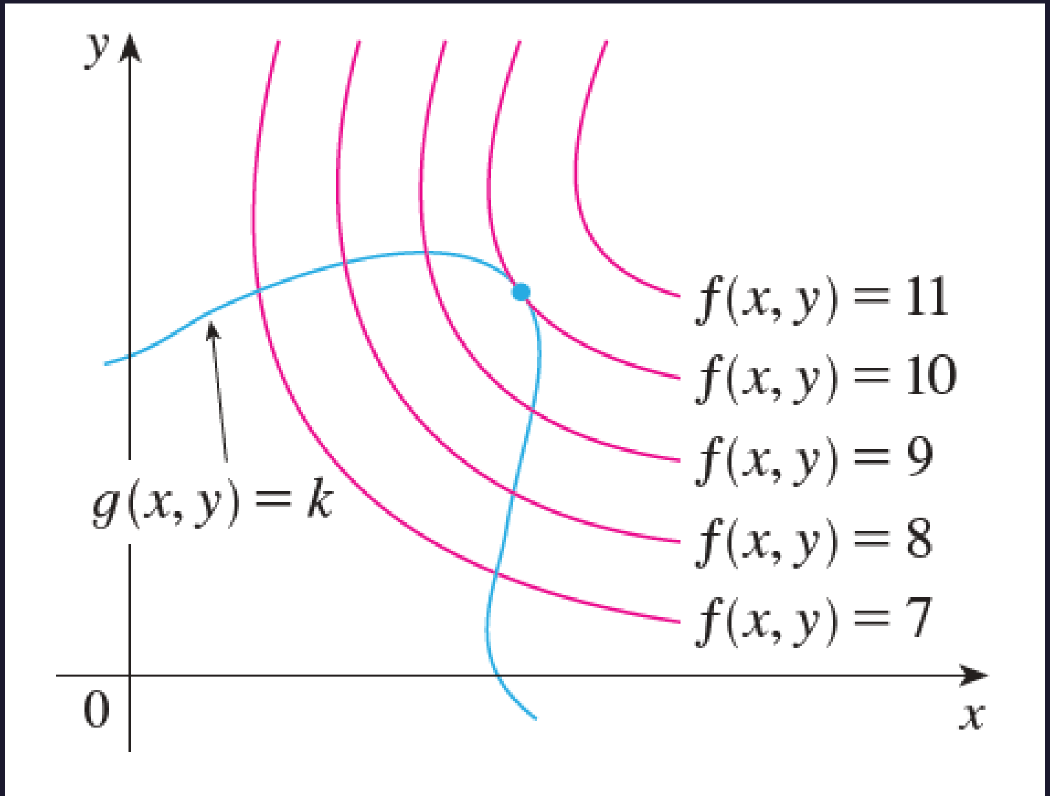


Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin, Kingdom of Sardinia
Died	10 April 1813 (aged 77) Paris, First French Empire

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

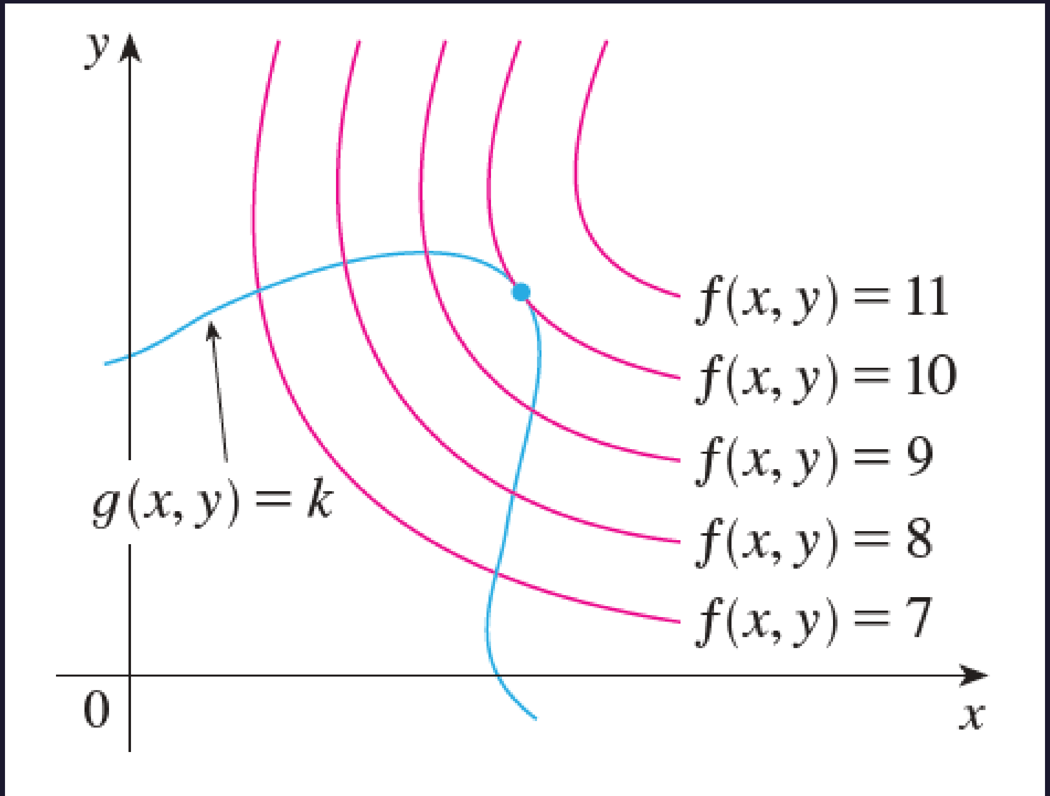
The "Lagrange Multipliers" Method

GOAL : Maximize z = f(x,y)   on the curve   g(x,y) = k .



The "Lagrange Multipliers" Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.



The value of $f(x,y)$ on the curve $g(x,y)=k$ will be maximized at some point (x_0,y_0) such that

$\nabla f\,(x_0,y_0)$ is parallel to $\nabla g\,(x_0,y_0)$

or equivalently a point (x_0,y_0) such that there is a constant λ with

$\nabla f\,(x_0,y_0) = \lambda \cdot \nabla g\,(x_0,y_0)$

and $g(x_0,y_0) = k$.

$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0)$ and $g(x_0, y_0) = k$.

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0) \quad \text{and} \quad g(x_0, y_0) = k\ .$$

[Extra]

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0) \quad \text{and} \quad g(x_0,y_0) = k\,.$$

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

The “Lagrange Multipliers” Method

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Questions?

Thank you

Until next time.