Exam 1 Practice Problem Solutions - Spring 2026

1. Let $\vec{a} = \langle 1, 1, 4 \rangle$ and $\vec{b} = \langle c, 3, 4 \rangle$, where c is an unknown constant.

   (a) Find the value of c so that $\vec{a}$ and $\vec{b}$ are orthogonal.

> **Solution:** Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if their dot product vanishes, i.e., when $\vec{a} \cdot \vec{b} = 0$. If we compute the dot product of the given vectors we obtain
> $$(1)(c) + (1)(3) + (4)(4) = 0$$
> which implies that $c + 3 + 16 = 0$. Therefore, $c = -19$.

   (b) With the value of c from part (a), find $\vec{a} \times \vec{b}$.

> **Solution:** With $c = -19$, we have $\vec{b} = \langle -19, 3, 4 \rangle$. Now we can compute their cross product:
> $$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 4 \\ -19 & 3 & 4 \end{vmatrix} = \mathbf{i}(4-12) - \mathbf{j}(4-(-76)) + \mathbf{k}(3-(-19)) = \langle -8, -80, 22 \rangle.$$
> So the solution is $\vec{a} \times \vec{b} = \langle -8, -80, 22 \rangle$.

2. Find the equation of a line that passes through $(1, 2, 3)$ and is perpendicular to the plane $x - y + 3z = 5$.

> **Solution:** The direction vector of the line $\vec{v}$ is the normal vector of the plane $\vec{n} = \langle 1, -1, 3 \rangle$, which we obtain from the coefficients in x, y, z of the plane $x - y + 3z = 5$. Using the point $(1, 2, 3)$, the vector equation is:
> $$\vec{r}(t) = \langle 1, 2, 3 \rangle + t\langle 1, -1, 3 \rangle$$
> Alternative forms:
>
> - Parametric: $x = 1 + t, y = 2 - t, z = 3 + 3t$.
> - Symmetric: $x - 1 = 2 - y = \frac{z-3}{3}$.

3. Find the equation of a plane through the origin, $(0, 1, 2)$ and $(3, 0, 1)$.

Solution: Let $P = (0, 0, 0)$ be the origin, and let $Q = (0, 1, 2)$, and $R = (3, 0, 1)$. The vectors from P to Q and from P to R are then given by $\vec{PQ} = \langle 0, 1, 2\rangle$ and $\vec{PR} = \langle 3, 0, 1\rangle$. A normal (i.e., perpendicular) vector to $\vec{PQ}$ and $\vec{PR}$ can be constructed using the cross product $\vec{n} = \vec{PQ} \times \vec{PR}$:

$$\vec{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = \mathbf{i}(1 - 0) - \mathbf{j}(0 - 6) + \mathbf{k}(0 - 3) = \langle 1, 6, -3\rangle$$

Using the origin as the point P and the normal vector $\vec{n}$ we can use the formula for a plane that passes through $P = (0, 0, 0)$ and is perpendicular to $\vec{n} = \langle 1, 6, -3\rangle$ to find the equation of the plane

$$1 \cdot (x - 0) + 6 \cdot (y - 0) - 3 \cdot (z - 0) = 0$$

Thus, the plane we were looking for is $x + 6y - 3z = 0$. (The reader can verify that the points P, Q, and R are in this plane.)

4. Let $f(x, y)$ be a function satisfying $f(4, 3) = 5$ and $\nabla f(4, 3) = \langle 6, 8\rangle$.

(a) Find the equation of the tangent plane to f at $(4, 3)$.

Solution: The formula for a tangent plane at (x_0, y_0, z_0) is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Since $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$, it follows that $f_x(4, 3) = 6$ and $f_y(4, 3) = 8$. Hence, the plane is given by

$$z - 5 = 6(x - 4) + 8(y - 3)$$

which simplifies to the plane $z = 6x + 8y - 43$.

(b) Use the linear approximation of $f(x, y)$ at $(4, 3)$ to approximate $f(5, 2)$.

Solution: The linear approximation of $z = f(x, y)$ at (x_0, y_0) is given by $z = L(x, y)$, the equation of the tangent plane, which we computed to be

$$z = L(x, y) = 5 + 6(x - 4) + 8(y - 3) = 6x + 8y - 43$$

Thus,

$$L(5, 2) = 5 + 6(5 - 4) + 8(2 - 3) = 5 + 6(1) + 8(-1) = 5 + 6 - 8 = 3.$$

So $f(5, 2) \approx 3$.

(c) What is the rate of change of the function at $(4, 3)$ when moving towards the origin?

Solution: The direction vector from $(4, 3)$ to $(0, 0)$ is $\vec{v} = \langle -4, -3 \rangle$. The magnitude is $|\vec{v}| = 5$. Thus, a unit vector in the same direction is given by

$$\vec{u} = \langle -4/5, -3/5 \rangle.$$

Hence the directional derivative in the direction of the unit vector $\vec{u}$ is given by

$$D_{\vec{u}} f(4, 3) = \nabla f \cdot \vec{u} = \langle 6, 8 \rangle \cdot \langle -4/5, -3/5 \rangle = -24/5 - 24/5 = -48/5 = -9.6.$$

(d) Which direction maximizes the rate of change of f at $(4, 3)$?

Solution: The rate of change is maximized in the direction of the gradient: $\nabla f(4, 3) = \langle 6, 8 \rangle$ (or in the direction of the unit vector $\langle 0.6, 0.8 \rangle$).

5. Let $f(x, y) = \sqrt{x^2 + y^2} \cdot \ln(2x)$.

(a) Find the domain of f.

Solution: The square root requires $x^2 + y^2 \geq 0$ (which holds for all real numbers x and y). The natural log requires $2x > 0$ and therefore $x > 0$. Thus, the domain is $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.

(b) Verify by direct computation that $f_{xy} = f_{yx}$.

Solution: Let $f = \sqrt{x^2 + y^2} \ln(2x)$. We compute the first partial derivatives f_x and f_y as follows:

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2 + y^2}} \ln(2x) + \frac{\sqrt{x^2 + y^2}}{x}, \\ f_y &= \frac{y}{\sqrt{x^2 + y^2}} \ln(2x). \end{aligned}$$

Now we compute the mixed second partial derivatives $f_{yx} = (f_y)_x$ and $f_{xy} = (f_x)_y$ as follows

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) = y \ln(2x) \cdot \frac{-x}{(x^2 + y^2)^{3/2}} + \frac{y}{\sqrt{x^2 + y^2}} \cdot \frac{1}{x} = \frac{y}{x\sqrt{x^2 + y^2}} - \frac{xy \ln(2x)}{(x^2 + y^2)^{3/2}} \\ f_{xy} &= \frac{\partial}{\partial y}(f_x) = x \ln(2x) \cdot \frac{-y}{(x^2 + y^2)^{3/2}} + \frac{1}{x} \cdot \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{x\sqrt{x^2 + y^2}} - \frac{xy \ln(2x)}{(x^2 + y^2)^{3/2}}. \end{aligned}$$

Thus, we have verified that $f_{xy} = f_{yx}$.

6. Let $f(x, y) = (x^2 - y^2)e^y$ and let $g(t) = \cos(t)$ and $h(t) = \sin(t)$. Use the chain rule to compute the derivative with respect to t of the function $f(g(t), h(t))$.

Solution: The multivariable one-parameter chain rule says that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

First we compute some partial derivatives:

$$f_x = 2xe^y, f_y = -2ye^y + (x^2 - y^2)e^y = (x^2 - y^2 - 2y)e^y,$$

and one-variable derivatives:

$$\frac{dx}{dt} = -\sin(t), \frac{dy}{dt} = \cos(t).$$

Thus,

$$\frac{df}{dt} = (2\cos te^{\sin t})(-\sin t) + (\cos^2 t - \sin^2 t - 2\sin t)e^{\sin t}(\cos t).$$

7. Let f be a continuous function of two variables which is twice differentiable with the following table of values.

	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{xy}(x,y)$	$f_{yy}(x,y)$
$(-1,2)$	11	0	0	1	5	3
$(1,4)$	-5	1	0	2	0	4
$(-2,-1)$	6	0	0	-3	0	-1
$(-4,-1)$	0	2	2	1	0	1
$(1,-3)$	2	3	0	-2	5	2

(a) Which points are critical points?

Solution: Critical points occur where $f_x = 0$ and $f_y = 0$. Looking at the table, these are A: $(-1, 2)$ and C: $(-2, -1)$.

(b) Classify each critical point.

Solution: Use Second Derivative Test, using the auxiliary value $D = f_{xx}f_{yy} - (f_{xy})^2$.

- For $(-1, 2) : D = (1)(3) - (5)^2 = 3 - 25 = -22$. Since $D = D(-1, 2) < 0$, it is a **Saddle Point**.
- For $(-2, -1) : D(-2, -1) = (-3)(-1) - (0)^2 = 3$. Since $D = D(-2, -1) > 0$ and $f_{xx} = -3 < 0$, it is a **Local Maximum**.

8. Use the method of *Lagrange Multipliers* to find the maximum and the minimum of $f(x,y) = x^2 + y$ over the ellipse $x^2 + 2y^2 = 8$.

Solution: The method of Lagrange Multipliers says that the max and min will happen at those points such that $\nabla f = \lambda \nabla g$. We first compute the gradient vectors:

$$\nabla f = \langle 2x, 1 \rangle, \text{ and } \nabla g = \langle 2x, 4y \rangle.$$

By setting $\nabla f = \lambda \nabla g$ we obtain $\langle 2x, 1\rangle = \lambda\langle 2x, 4y\rangle$. Thus, we obtain two equations:

$$2x = 2x\lambda \text{ and } 1 = 4y\lambda.$$

Since $2x = 2x\lambda$ implies $2x(1-\lambda) = 0$ we have two cases to consider

- Case 1: $x = 0$.
- Case 2: $\lambda = 1$.

If $x = 0$, then $0^2 + 2y^2 = 8$ implies $y^2 = 4$ which in turn implies that $y = \pm 2$. We obtain two points: $(0, 2), (0, -2)$.

If $\lambda = 1$, then $1 = 4y \cdot 1$ implies $y = 1/4$. Moreover the point is on the ellipse so $x^2 + 2(1/4)^2 = 8$ which implies $x^2 + 1/8 = 8$ and so $x^2 = 63/8$. Thus, $x = \pm\sqrt{7.875}$. We obtain another two points $(\pm\sqrt{63/8}, 1/4)$.

Finally we evaluate f at the points we found: $f(0,2) = 2$, $f(0,-2) = -2$, $f(\pm\sqrt{63/8}, 1/4) = 7.875 + 0.25 = 8.125$.

Hence, the **Max value** is 8.125 at $(\pm\sqrt{63/8}, 1/4)$ and the **Min value** is -2 at $(0, -2)$.