“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 10

Any Reminders? Any Questions?

- I will have regular office hours 2/19 – 3:30-4:30
- I will have additional office hours 2/19 – 4:30-5:30
- Calc 3 Calc Night: MONT 104 at 6:30-8:30pm on Thursdays!

- Exam 1 is on Friday, Feb 20th
- Exam practice questions/exam and solutions on HuskyCT

EXAM 1 -- Friday, February 20th

Exam Covers:

- **Chapter 12**
  - Sections 12.1 – 12.6
- **Chapter 14**
  - Sections 14.1, 14.3 – 14.8

ALVARO: Start the recording!

"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Double Integrals over Regions

Today – Double Integrals in Regions!

- General Regions
- Regions of Type I and II
- Changing the Order of Integration
- Properties of Double Integrals

Example: (Warm up) Calculate the following iterated integral

$$\int_0^1 \int_0^2 (2xy + 2y + 1)\, dy\, dx$$

Regions of Type I and II

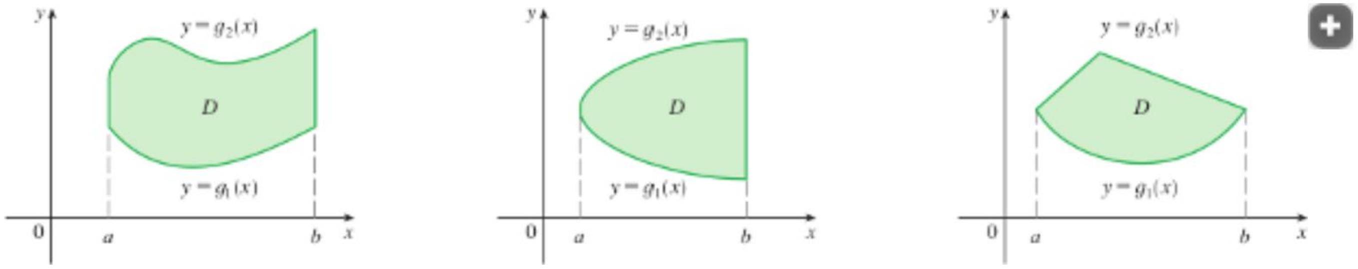
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

Figure 5

Some type I regions



Regions of Type I and II

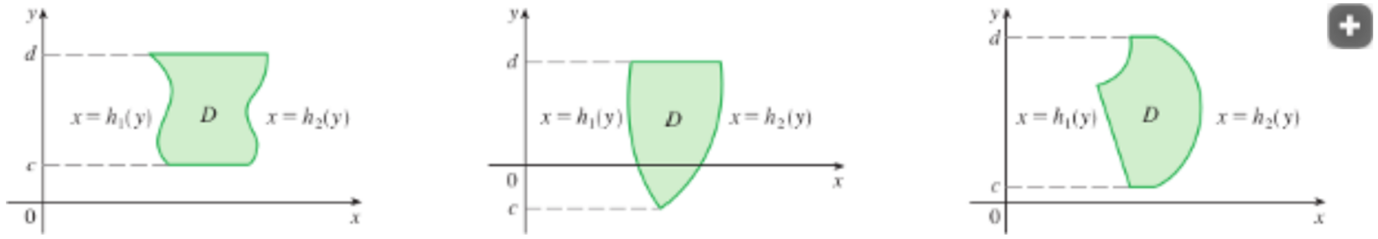
We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

Figure 7

Some type II regions



Integrals over Regions of Type I

$\boxed{3}$ If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Integrals over Regions of Type II

4 If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ \ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \text{ and } y = 1 + x^2 .$$

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \text{ and } y = 1 + x^2 .$$

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type I integral.)

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type II integral.)

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

[Extra]

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Properties of Double Integrals

$$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

$$\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

7

$$\iint_D f(x,y)\, dA \geqslant \iint_D g(x,y)\, dA$$

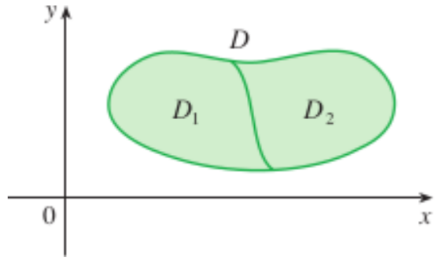
Properties of Double Integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

8

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

Figure 17



Properties of Double Integrals

$$\iint_D 1 \, dA = A(D)$$

$\boxed{10}$ If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y) \, dA \leqslant M \cdot A(D)$$

[Extra]

Example: Estimate the value of the double integral

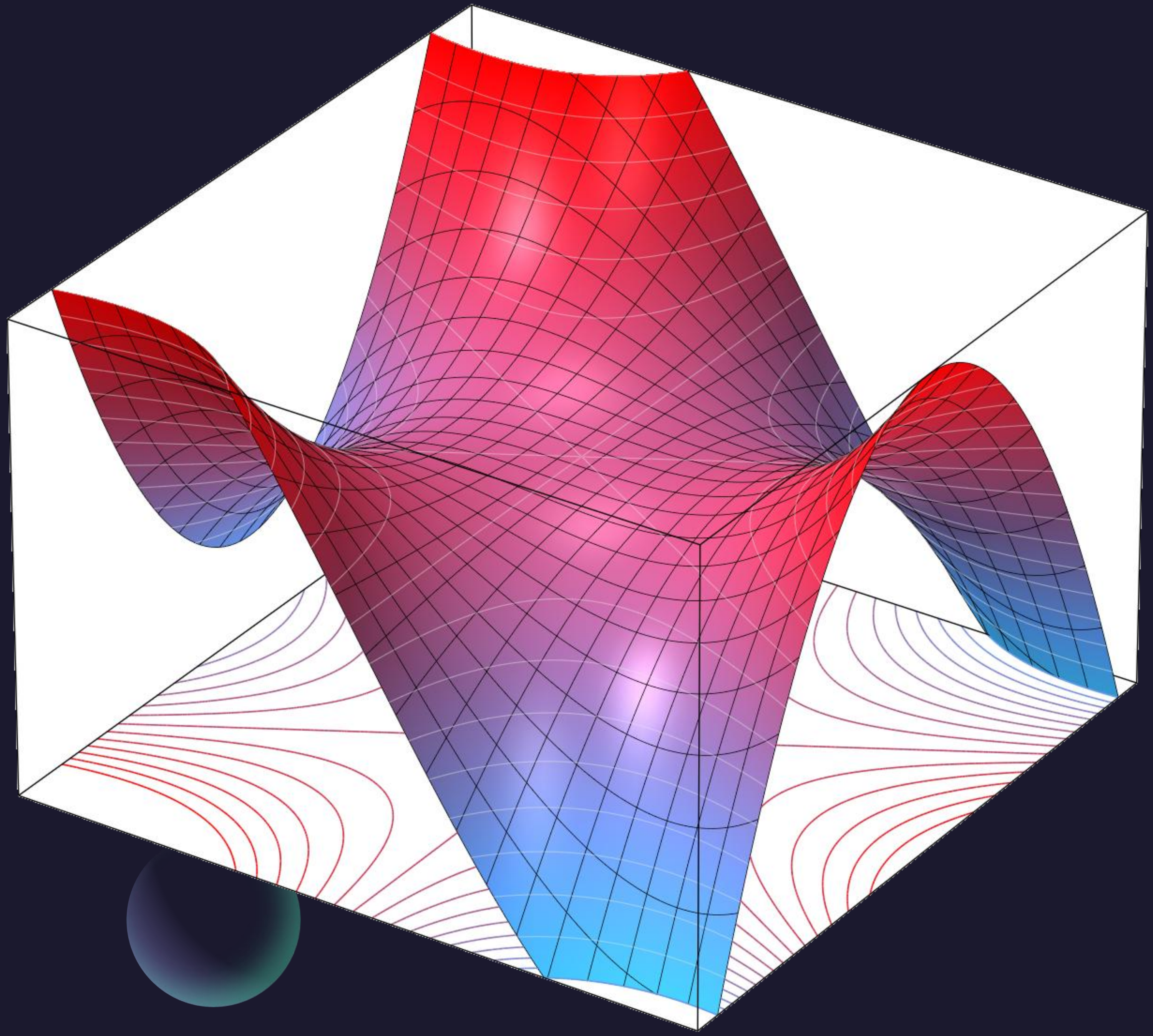
$$\iint_R e^{-(x^2+y^2)}dA$$

where $R = \{(x, y): x^2 + y^2 \leq 1\}$ is the circle of radius 1.

Questions?

Thank you

Until next time.



ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Exam 1 : Review

1. Let $\vec{a} = \langle 1, 1, 4 \rangle$ and $\vec{b} = \langle c, 3, 4 \rangle$, where c is an unknown constant.

   (a) Find the value of c so that $\vec{a}$ and $\vec{b}$ are orthogonal.

   (b) With the value of c from part (a), find $\vec{a} \times \vec{b}$.

[Extra]

1. Let $\vec{a} = \langle 1, 1, 4 \rangle$ and $\vec{b} = \langle c, 3, 4 \rangle$, where c is an unknown constant.

   (a) Find the value of c so that $\vec{a}$ and $\vec{b}$ are orthogonal.

   (b) With the value of c from part (a), find $\vec{a} \times \vec{b}$.

2. Find the equation of a line that passes through $(1, 2, 3)$ and is perpendicular to the plane $x - y + 3z = 5$.

[Extra]

2. Find the equation of a line that passes through $(1, 2, 3)$ and is perpendicular to the plane $x - y + 3z = 5$.

3. Find the equation of a plane through the origin, $(0, 1, 2)$ and $(3, 0, 1)$.

[Extra]

3. Find the equation of a plane through the origin, $(0,1,2)$ and $(3,0,1)$.

4. Let $f(x, y)$ be a function satisfying $f(4, 3) = 5$ and $\nabla f(4, 3) = \langle 6, 8 \rangle$.

(a) Find the equation of the tangent plane to f at $(4, 3)$.

(b) Use the linear approximation of $f(x, y)$ at $(4, 3)$ to approximate $f(5, 2)$.

(c) What is the rate of change of the function at $(4, 3)$ when moving towards the origin?

(d) Which direction maximizes the rate of change of f at $(4, 3)$?

[Extra]

4. Let $f(x,y)$ be a function satisfying $f(4,3) = 5$ and $\nabla f(4,3) = \langle 6, 8 \rangle$.

   (a) Find the equation of the tangent plane to f at $(4,3)$.

   (b) Use the linear approximation of $f(x,y)$ at $(4,3)$ to approximate $f(5,2)$.

   (c) What is the rate of change of the function at $(4,3)$ when moving towards the origin?

   (d) Which direction maximizes the rate of change of f at $(4,3)$?

5. Let $f(x, y) = \sqrt{x^2 + y^2} \cdot \ln(2x)$.

(a) Find the domain of f.

(b) Verify by direct computation that $f_{xy} = f_{yx}$ (also known as *Clairaut's Theorem*).

[Extra]

5. Let $f(x, y) = \sqrt{x^2 + y^2} \cdot \ln(2x)$.

   (a) Find the domain of f.

   (b) Verify by direct computation that $f_{xy} = f_{yx}$ (also known as *Clairaut's Theorem*).

6. Let $f(x, y) = (x^2 - y^2)e^y$ and let $g(t) = \cos(t)$ and $h(t) = \sin(t)$. Use the chain rule to compute the derivative with respect to t of the function $f(g(t), h(t))$.

[Extra]

6. Let $f(x, y) = (x^2 - y^2)e^y$ and let $g(t) = \cos(t)$ and $h(t) = \sin(t)$. Use the chain rule to compute the derivative with respect to t of the function $f(g(t), h(t))$.

7. Let f be a continuous function of two variables which is twice differentiable with the following table of values.

	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{xy}(x,y)$	$f_{yy}(x,y)$
$(-1,2)$	11	0	0	1	5	3
$(1,4)$	-5	1	0	2	0	4
$(-2,-1)$	6	0	0	-3	0	-1
$(-4,-1)$	0	2	2	1	0	1
$(1,-3)$	2	3	0	-2	5	2

(a) Which points are critical points? **Select ALL that apply.**

[Extra]

(b) Classify each critical point as a local maximum, local minimum or saddle point or explain why there is not enough information to tell.

8. Use the method of *Lagrange Multipliers* to find the maximum and the minimum of $f(x, y) = x^2 + y$ over the ellipse $x^2 + 2y^2 = 8$.

[Extra]

8. Use the method of *Lagrange Multipliers* to find the maximum and the minimum of $f(x, y) = x^2 + y$ over the ellipse $x^2 + 2y^2 = 8$.

Questions?

Thank you

Until next time.