

“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 8

1

Any Reminders? Any Questions?

- I will be away on
 - Monday 2/16 --- no office hours that day
 - Tuesday 2/17 --- I will send videos to watch instead of class
- I will be back teaching in-person on Thursday 2/19
- I will do some review for the midterm during Thursday's class
- I will have regular office hours 2/19 – 3:30-4:30
- I will have additional office hours 2/19 – 4:30-5:30
- Calc 3 Calc Night: MONT 104 at 6:30-8:30pm on Thursdays!
- Exam I is on Friday, Feb 20th

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EXAM 1 -- Friday, February 20th

Exam Covers:

- **Chapter 12**
 - Sections 12.1 – 12.6
- **Chapter 14**
 - Sections 14.1, 14.3 – 14.8



(NEW) Exam Study Guide and Practice Problems in HuskyCT

3



ALVARO: Start the recording!



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“Calculus 3”

Multi-Variable Calculus

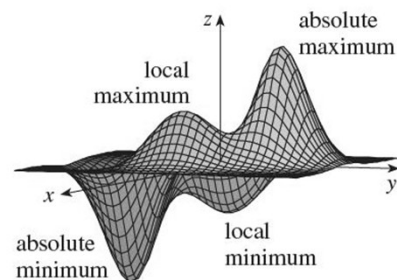
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More on Maximum and Minimum Values

5

Today – Maximum and Minimum Values!

- Local Max and Min Values
- Second Derivative Test
- Absolute Max and Min Values



6

Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

A **critical point** for a function $f(x, y)$ is a point (a, b) where

$$\nabla f(a, b) = \vec{0},$$

that is $f_x(a, b) = 0, f_y(a, b) = 0$.

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Example: Find all the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

8

Local Max and Min Values: Second Derivative Test

3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is a saddle point of f .

WARNING! IF $D = 0$, THE TEST IS INCONCLUSIVE.

9

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find and classify the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

10

[Extra]

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find and classify the critical points for the function

$$f(x, y) = x^2 + y^2 - 2x - 6y + 14$$

11

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find and classify the critical points for the function

$$f(x, y) = x^2 + 4xy + y^2$$

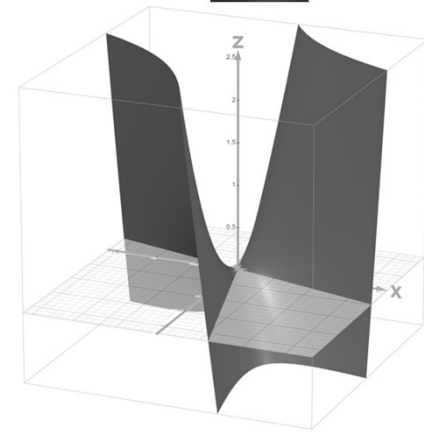
12

[Extra]

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find and classify the critical points for the function

$$f(x, y) = x^2 + 4xy + y^2$$



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Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D .
- **absolute minimum** value of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D .

8 Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in \mathbb{R}^2 , then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

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Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the



- **absolute maximum** value of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D .
- **absolute minimum** value of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D .

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

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$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find the absolute maximum and minimum values of

$$f(x, y) = xy^2$$

in the region $D = \{(x, y): x^2 + y^2 \leq 3\}$.

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[Extra]

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Example: Find the absolute maximum and minimum values of

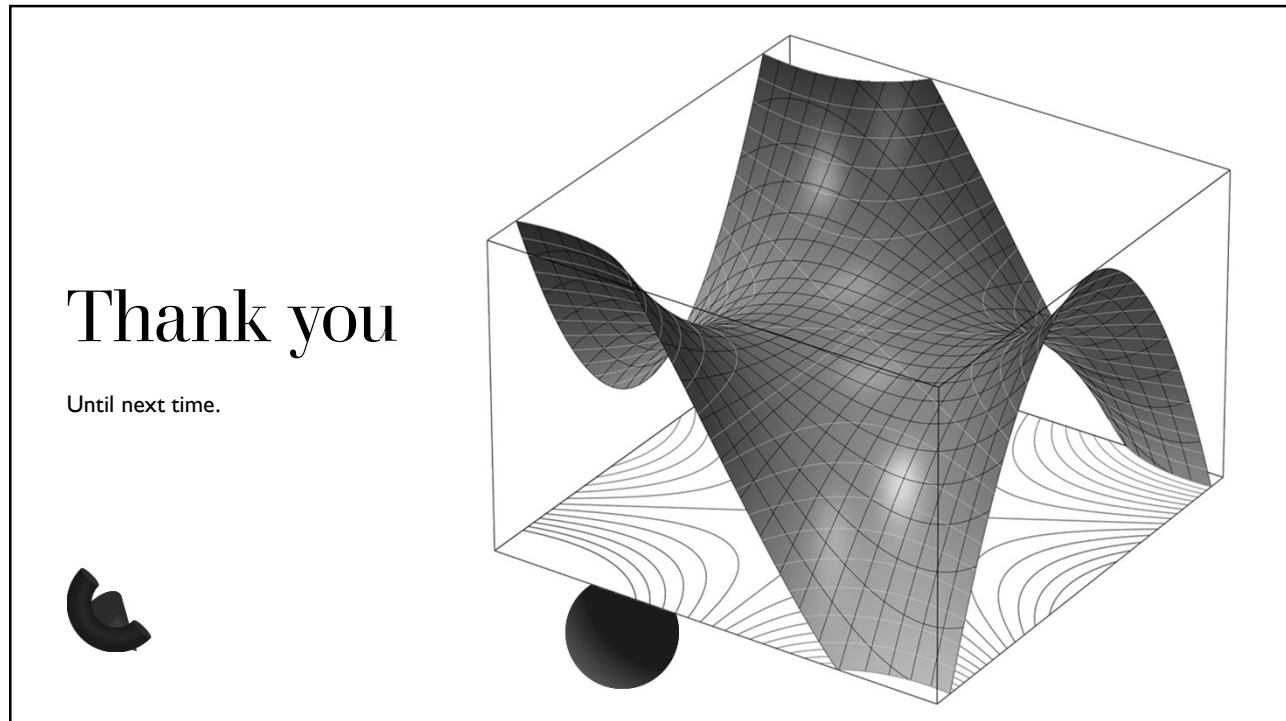
$$f(x, y) = xy^2$$

in the region $D = \{(x, y): x^2 + y^2 \leq 3\}$.

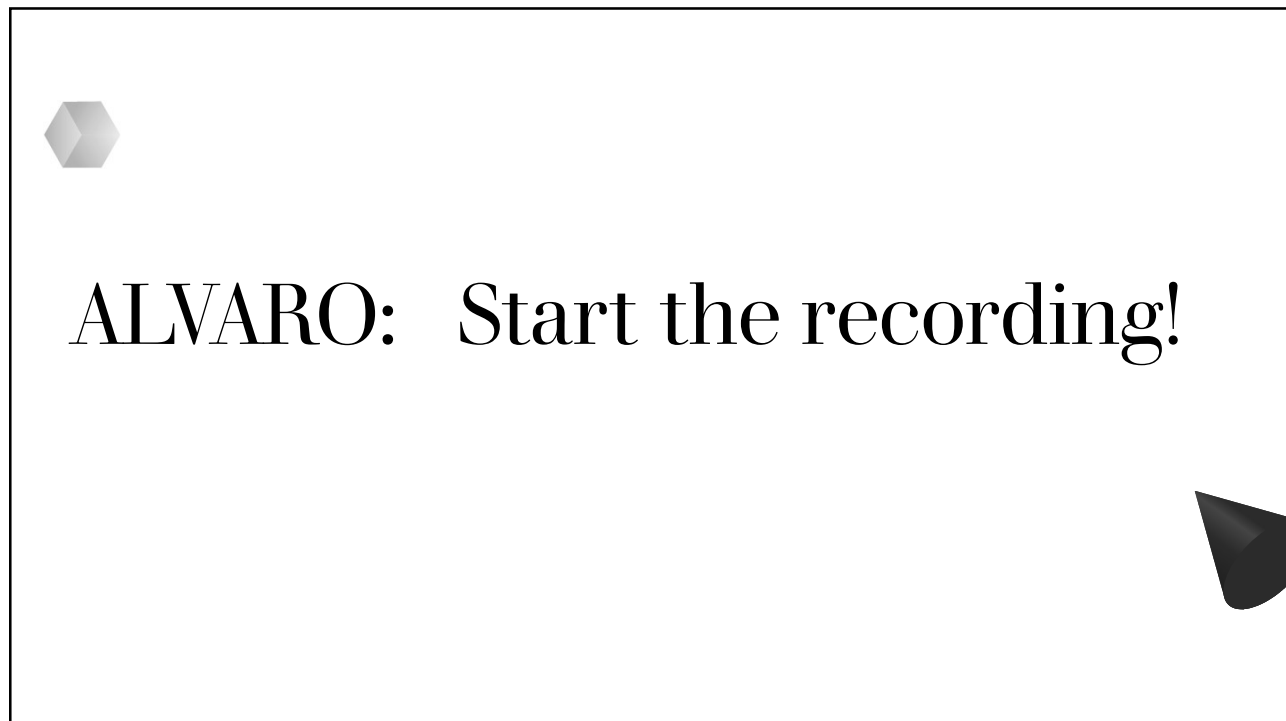
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Questions?

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19



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Multi-Variable Calculus

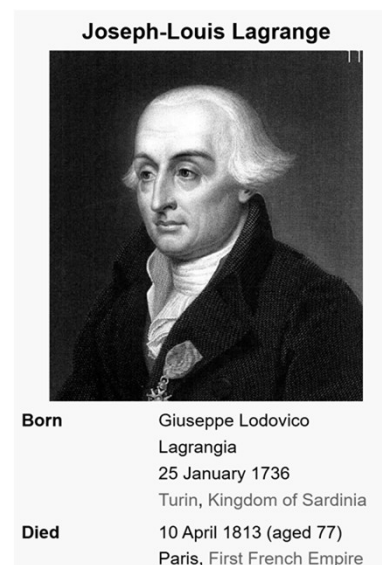
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Lagrange Multipliers

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Today – “Lagrange Multipliers!”

- The Method
- One Constraint
- Examples



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Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$
on the circle $x^2 + y^2 = 1$.

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Recall: Properties of the Gradient Vector

Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

The gradient vector in two dimensions, $\nabla F = (f_x, f_y)$ is normal to any level curves of $f(x, y)$ at any point, indicating the maximum rate of change.

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Example: Let $f(x, y) = 4 - x^2 - y^2$

- (a) Find the normal vector to the graph of $f(x, y)$ at $(1, 1, 2)$
- (b) Find the tangent plane to the graph of $f(x, y)$ at $(1, 1, 2)$
- (c) Find the normal vector to the cross section $z = 0$ at $(1, 1)$

25

[Extra]

Example: Let $f(x, y) = 4 - x^2 - y^2$

- (a) Find the normal vector to the graph of $f(x, y)$ at $(1, 1, 2)$
- (b) Find the tangent plane to the graph of $f(x, y)$ at $(1, 1, 2)$
- (c) Find the normal vector to the cross section $z = 0$ at $(1, 1)$

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[Extra]

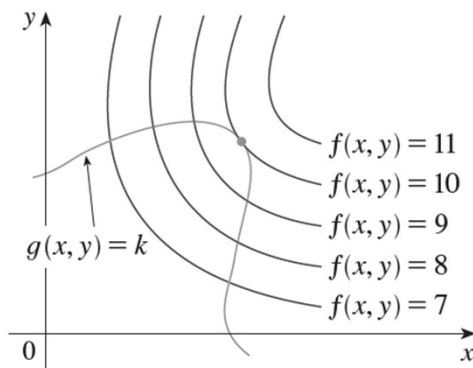
Example: Let $f(x, y) = 4 - x^2 - y^2$

- (a) Find the normal vector to the graph of $f(x, y)$ at $(1, 1, 2)$
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- (c) Find the normal vector to the cross section $z = 0$ at $(1, 1)$

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The “Lagrange Multipliers” Method

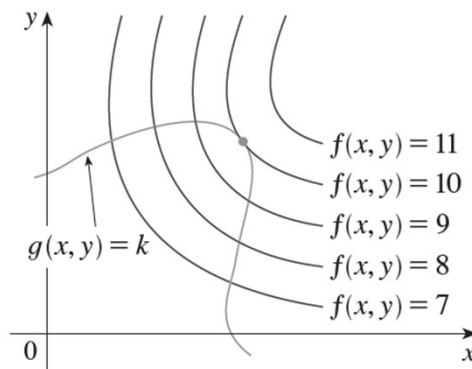
GOAL : Maximize $z = f(x, y)$ on the curve $g(x, y) = k$.



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The “Lagrange Multipliers” Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.



The value of $f(x,y)$ on the curve $g(x,y)=k$ will be maximized at some point (x_0, y_0) such that

$\nabla f(x_0, y_0)$ is parallel to $\nabla g(x_0, y_0)$

or equivalently a point (x_0, y_0) such that there is a constant λ with

$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0)$

and $g(x_0, y_0) = k$.

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The “Lagrange Multipliers” Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.

SOLVE:

$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0)$$

$$g(x_0, y_0) = k.$$

30

$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0) \quad \text{and} \quad g(x_0, y_0) = k.$$

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$
on the circle $x^2 + y^2 = 1$.

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$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0) \quad \text{and} \quad g(x_0, y_0) = k.$$

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$
on the circle $x^2 + y^2 = 1$.

[Extra]

32

$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0) \quad \text{and} \quad g(x_0, y_0) = k.$$

Example: Find the extreme values of $f(x, y) = x^2 + y^2$
on the curve $xy = 1$.



33

$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0) \quad \text{and} \quad g(x_0, y_0) = k.$$

[Extra]

Example: Find the extreme values of $f(x, y) = x^2 + y^2$
on the curve $xy = 1$.



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$$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0) \text{ and } g(x_0, y_0) = k.$$

Example: Find the largest area of a rectangle with fixed perimeter equal to p .

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The “Lagrange Multipliers” Method

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z , and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f ; the smallest is the minimum value of f .

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$$\nabla f(x_0, y_0, z_0) = \lambda \cdot \nabla g(x_0, y_0, z_0) \text{ and } g(x_0, y_0, z_0) = k.$$

Example: Find the dimensions of the closed box with the largest volume and fixed surface area S .

37

$$\nabla f(x_0, y_0, z_0) = \lambda \cdot \nabla g(x_0, y_0, z_0) \text{ and } g(x_0, y_0, z_0) = k.$$

Example: Find the dimensions of the closed box with the largest volume and fixed surface area S .

[Extra]

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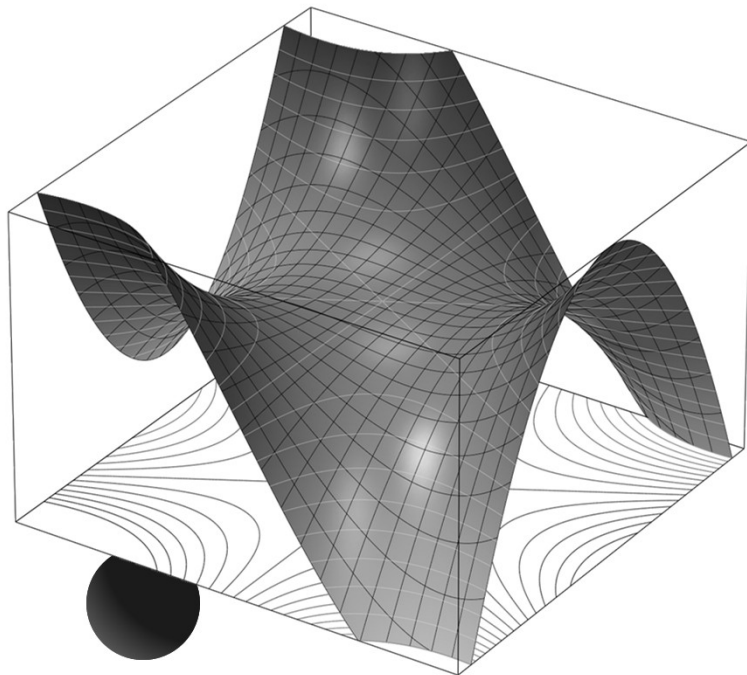
Questions?



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Thank you

Until next time.



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ALVARO: Start the recording!



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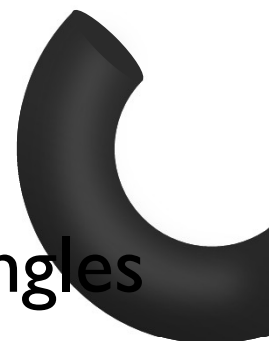
“Calculus 3”



Multi-Variable Calculus

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Double Integrals over Rectangles



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Today – Double Integrals!

- The Definite Integral
- The Riemann Integral
- Iterated Integrals
- Fubini's Theorem

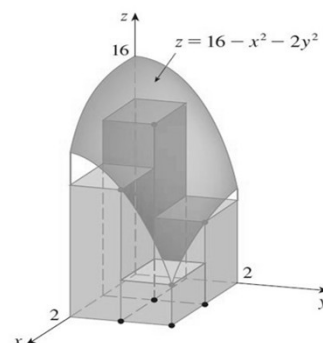
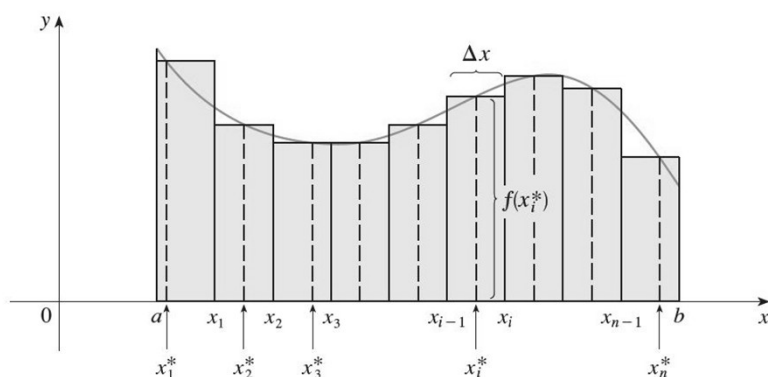


FIGURE 7

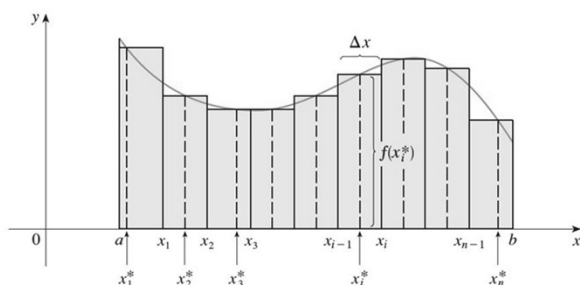
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The Definite (Riemann) Integral



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The Definite (Riemann) Integral



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The Definite (Riemann) Integral

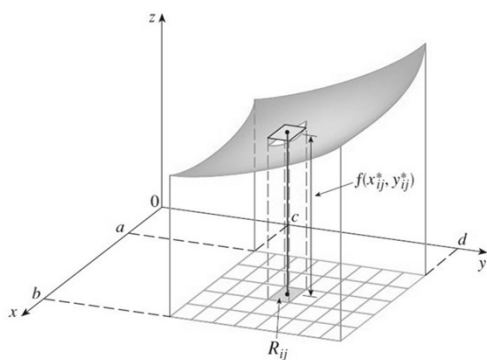


FIGURE 4

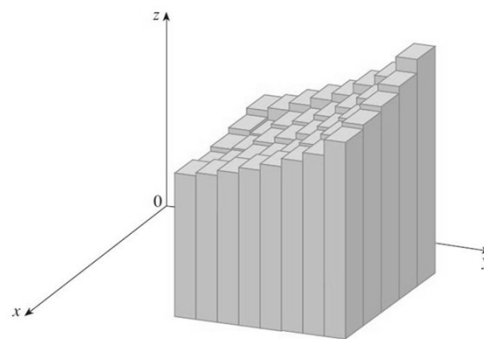


FIGURE 5

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The Definite (Riemann) Integral

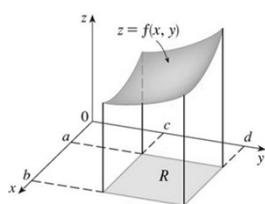


FIGURE 2

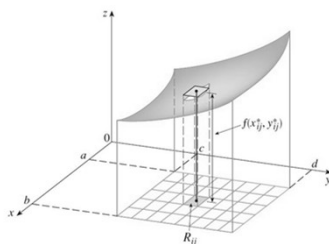


FIGURE 4

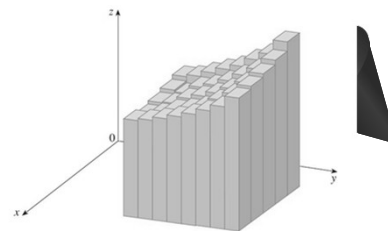


FIGURE 5

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The Definite (Riemann) Integral

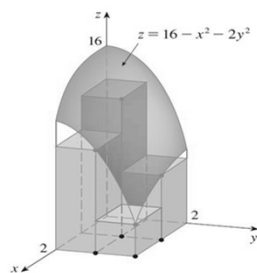
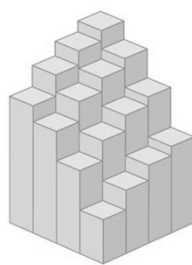
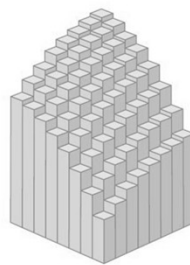
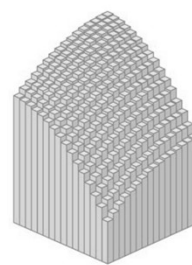


FIGURE 7

(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

48

The usual properties of integration still hold for double integrals:

► $\iint_R [f(x, y) \pm g(x, y)] dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA.$

► For any constant c ,

$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA.$$

► If $f(x, y) \geq g(x, y)$ on the rectangle R , then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA.$$

And when letting $m, n \rightarrow \infty$, we have $\Delta A \rightarrow dA = dx \cdot dy$. Then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy,$$

this is called an **iterated integral**, and we evaluate its value by computing the innermost integral first and then working the way out. Again, in the case this value represents a volume only if $f(x, y) \geq 0$ on R .

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Example: Find the volume under the graph of $f(x, y) = 16 - x^2 - 2y^2$ above the square $R = [0, 2] \times [0, 2]$.

50

[Extra]

Example: Find the volume under the graph of $f(x, y) = 16 - x^2 - 2y^2$ above the square $R = [0, 2] \times [0, 2]$.

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Example: Calculate the following iterated integrals
 $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ and $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

52

[Extra]

Example: Calculate the following iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

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Fubini's Theorem

If f is continuous on the rectangle

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Guido Fubini



Born	19 January 1879 Venice
Died	6 June 1943 (aged 64) New York

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Example: Evaluate the double integral

$$\iint_R (x - 3y^2) dA$$

where $R = \{(x, y): 0 \leq x \leq 2, 1 \leq y \leq 2\}$.



55

[Extra]

Example: Evaluate the double integral

$$\iint_R (x - 3y^2) dA$$

where $R = \{(x, y): 0 \leq x \leq 2, 1 \leq y \leq 2\}$.



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Example: Evaluate the double integral

$$\iint_R y \sin(xy) \, dA$$

where $R = [1, 2] \times [0, \pi]$.



57

Example: Evaluate the double integral

$$\iint_R y \sin(xy) \, dA$$

where $R = [1, 2] \times [0, \pi]$.



[Extra]

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When $f(x, y) = g(x) \cdot h(y)$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b g(x)h(y) dx dy = \int_a^b g(x) dx \cdot \int_c^d h(y) dy$$

► Evaluate the iterated integral

$$\int_1^3 \int_1^5 \frac{\ln(y)}{xy} dx dy$$

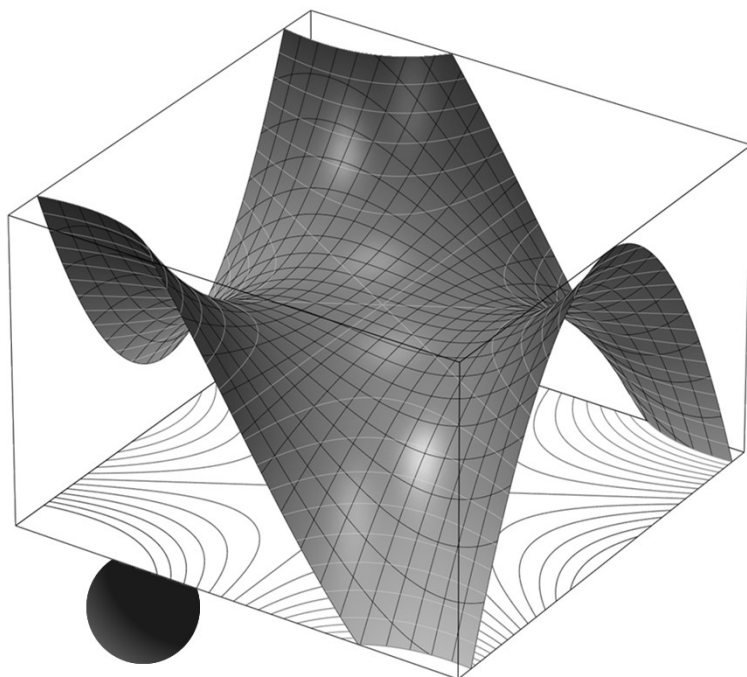
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Questions?

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Thank you

Until next time.



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“Calculus 3”

Multi-Variable Calculus

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Double Integrals over Regions



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Today – Double Integrals in Regions!

- General Regions
- Regions of Type I and II
- Changing the Order of Integration
- Properties of Double Integrals

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Regions of Type I and II

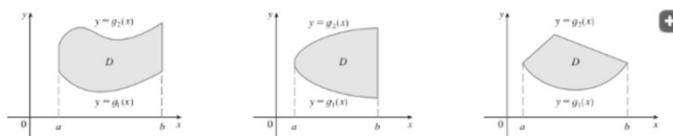
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x , that is,

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in [Figure 5](#).

Figure 5

Some type I regions



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Regions of Type I and II

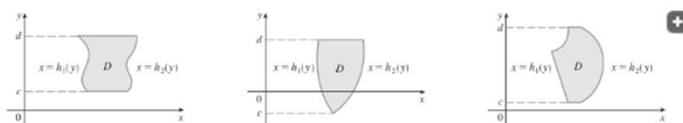
We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in [Figure 7](#).

Figure 7

Some type II regions



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Integrals over Regions of Type I

3 If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$$

66

Integrals over Regions of Type II

4 If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy$$

67

Example: Evaluate the double integral

$$\iint_R (x + 2y) \, dA$$

where R is the region bounded by the parabolas
 $y = 2x^2$ and $y = 1 + x^2$.

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[Extra]

Example: Evaluate the double integral

$$\iint_R (x + 2y) \, dA$$

where R is the region bounded by the parabolas
 $y = 2x^2$ and $y = 1 + x^2$.

69

Example: Find the volume of the solid that lies under the paraboloid
 $z = x^2 + y^2$ and above the region in the xy -plane bounded by
the line $y = 2x$ and the parabola $y = x^2$. (As a Type I integral.)

70

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

71

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type II integral.)

72

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy -plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

73

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2) dy dx$$

74

[Extra]

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2) dy dx$$

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Properties of Double Integrals

$$\iint_D [f(x, y) + g(x, y)] dA = \iint_D f(x, y) dA + \iint_D g(x, y) dA$$

$$\iint_D c f(x, y) dA = c \iint_D f(x, y) dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D , then

7

$$\iint_D f(x, y) dA \geq \iint_D g(x, y) dA$$

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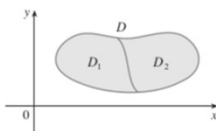
Properties of Double Integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see [Figure 17](#)), then

8

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA$$

Figure 17



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Properties of Double Integrals

$$\iint_D 1 \, dA = A(D)$$

10 If $m \leq f(x, y) \leq M$ for all (x, y) in D , then

$$m \cdot A(D) \leq \iint_D f(x, y) \, dA \leq M \cdot A(D)$$

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[Extra]

Example: Estimate the value of the double integral

$$\iint_R e^{-(x^2+y^2)} dA$$

where $R = \{(x, y): x^2 + y^2 \leq 1\}$ is the circle of radius 1.

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Questions?

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