Do you use "AI" (ChatGPT, Gemini,...) for your math coursework?

“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 2

Any Reminders? Any Questions?

- Slides will be posted on GitHub! https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- First two worksheets are available in HuskyCT (extra credit)
- First quiz (Friday) will be on derivatives and integrals

Questions?

ALVARO: Start the recording!

"Calculus 3"

Multi-Variable Calculus

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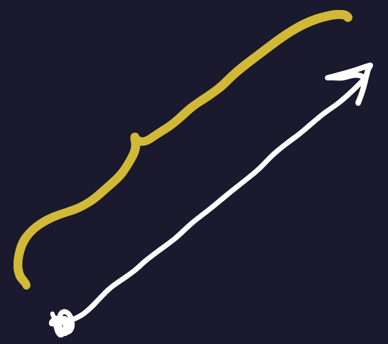
More on Vectors

Today – Finish Vectors!

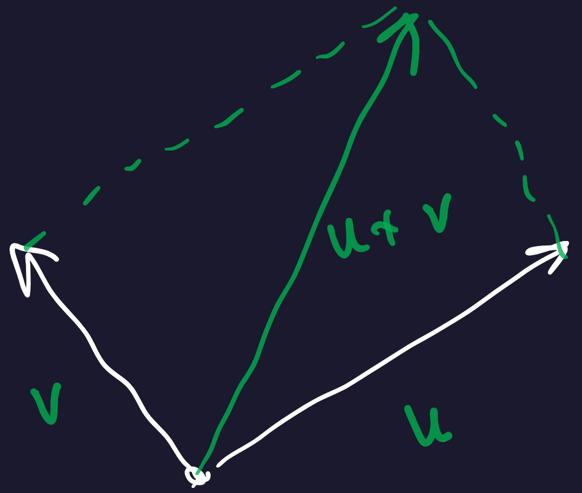
- Vectors
  - Vector addition and scalar multiplication
  - Components and length
  - Properties
  - Applications

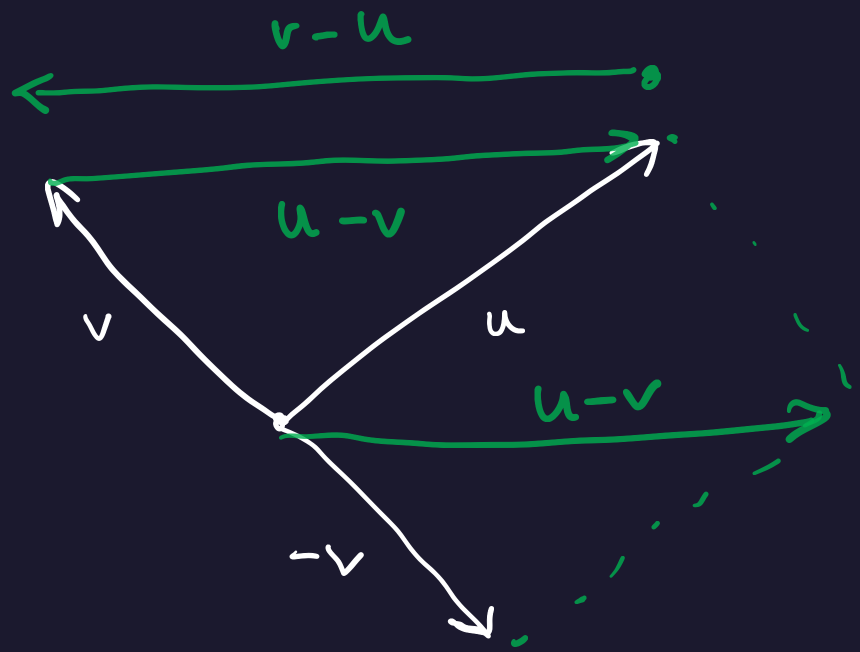
Vectors

A **vector** is a mathematical object with both magnitude (size) and direction, represented as a directed line segment (arrow).



Vector Addition and Scalar Multiplication





$$v - u = -(u - v)$$

Example: Let u = (2,3) and v = (-1,1). Find u+2v and u-2v.

$$2v = 2\cdot(-1,1) = (-2,2)$$

$$u + 2v = (2,3) + (-2,2)$$

$$= (0,5)$$

$$u - 2v = (2,3) + (2,-2) = (4,1)$$

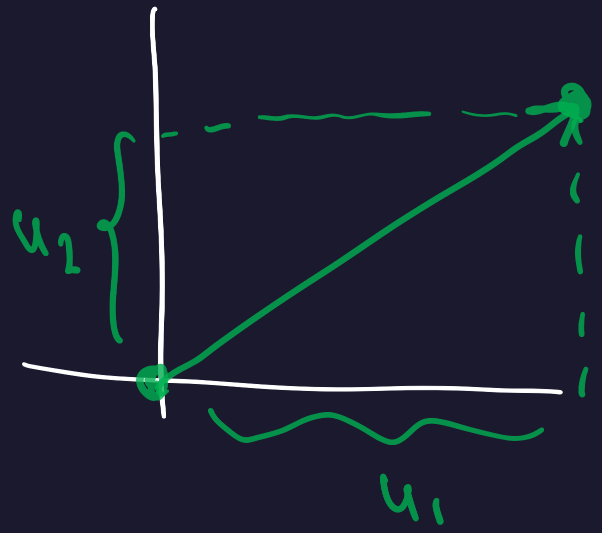
Example: Let u = (2,3,0) and v = (-1,1,2).
Find u+v and u-v.

$$u + v = (2, 3, 0) + (-1, 1, 2)$$

$$= (1, 4, 2)$$

Length of a Vector

$$|u| = \|u\| = \sqrt{u_1^2 + u_2^2}$$



$u = (u_1, u_2)$

$u = (u_1, u_2, u_3)$

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Example: Let a = (4, 0, 3) and b = (-2, 1, 5). Find the lengths of a+b and a-b.

$$a+b = (4,0,3) + (-2,1,5)$$

$$= (2,1,8)$$

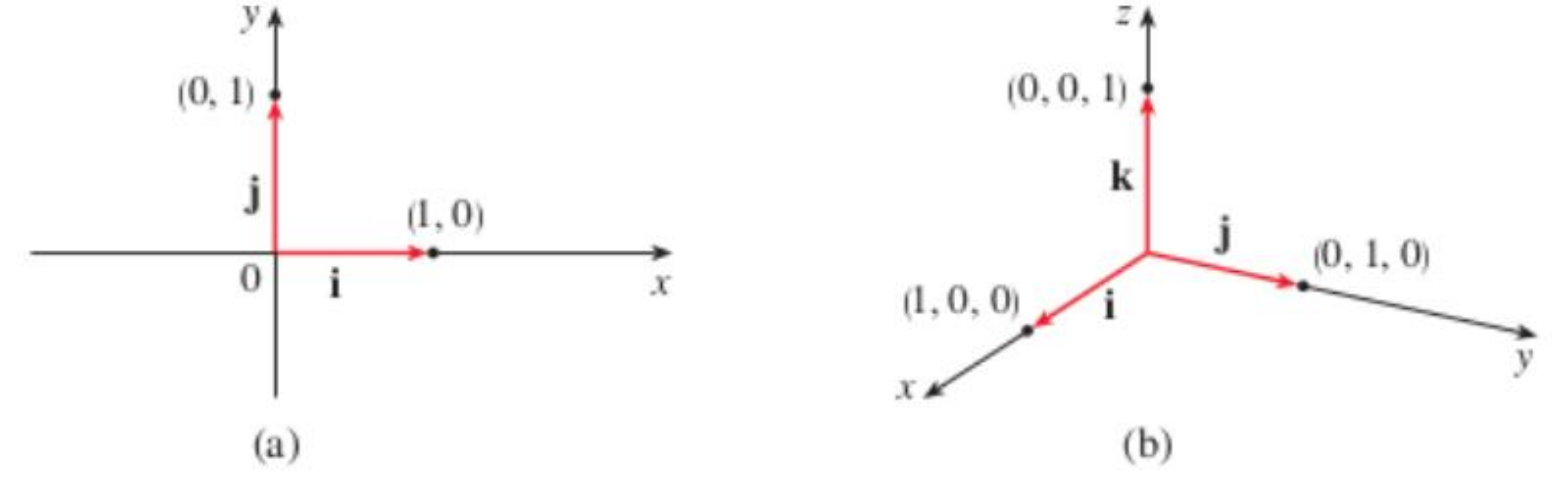
$$\|a+b\| = \sqrt{2^2 + 1^2 + 8^2} = \sqrt{69}$$

Properties of Vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c\,(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\,\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\,\mathbf{a} = c\,(d\mathbf{a})$
8. $\mathbf{1a} = \mathbf{a}$

$u = (2, 3, 5) = 2i + 3j + 5k$



Standard Basis Vectors

- $i = (1, 0, 0)$
- $j = (0, 1, 0)$
- $k = (0, 0, 1)$

Example: Let a = 4i+3k and b = -2i + j + 5k.
Find the length of c = 2a - b in terms of i, j, and k.

$$2a = 2\cdot(4i+3k) = 8i+6k$$

$$-b = -(-2i+j+5k) = 2i-j-5k$$

$$2a-b = (8i+6k)+(2i-j-5k)$$

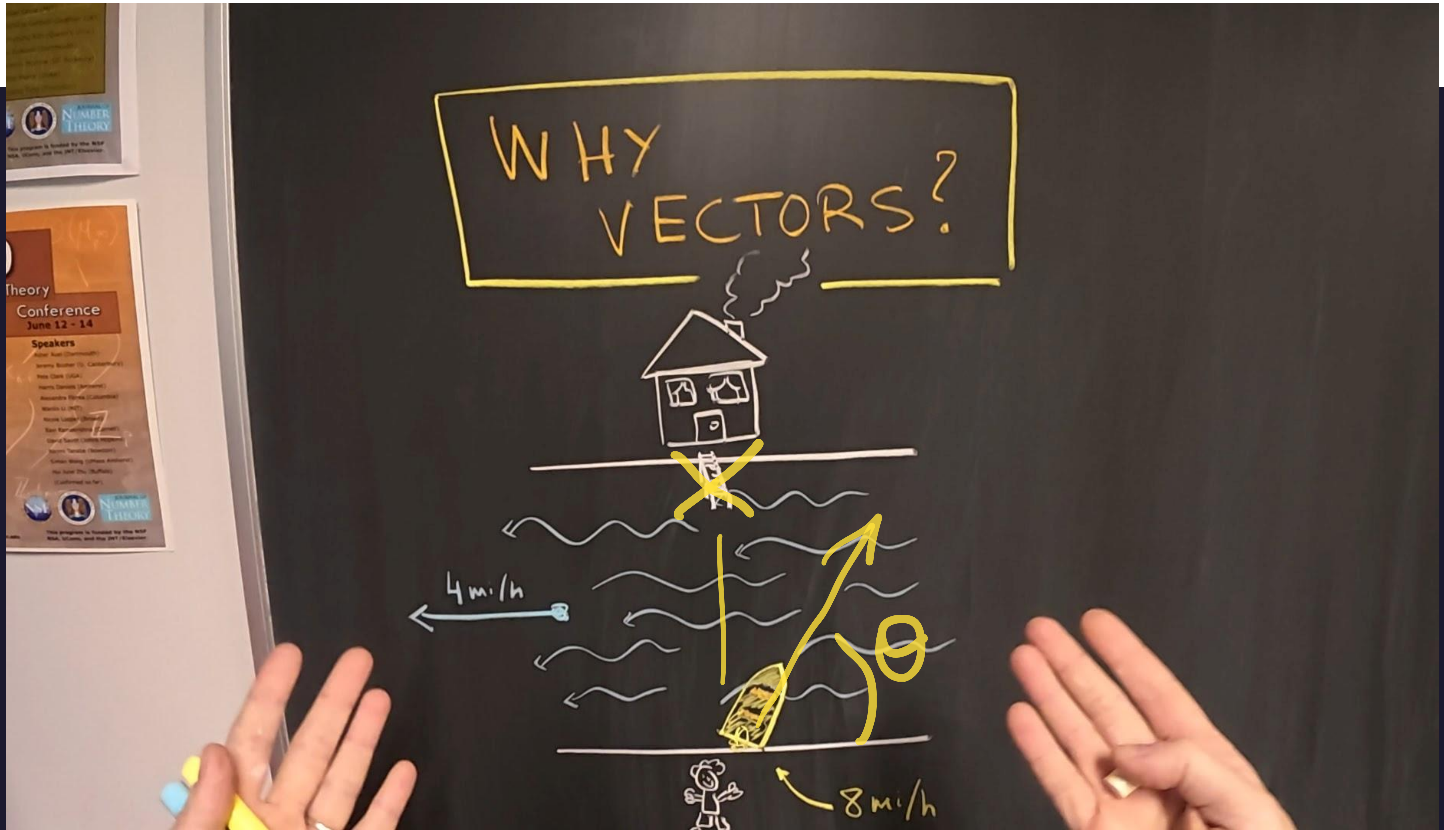
$$= \boxed{10i - j + k}$$

$$|2a-b| = \sqrt{10^2+(-1)^2+1^2}$$

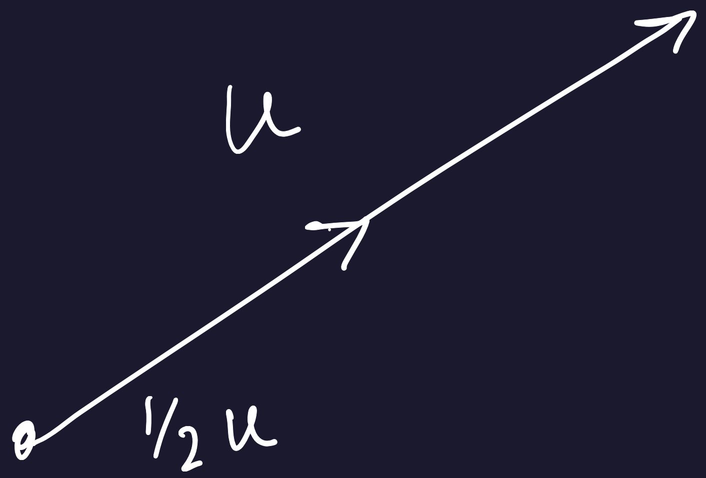
$$= \boxed{\sqrt{102}}$$

Example (an application of vectors):

A woman launches a boat from the south shore of a straight river that flows directly west at 4 mi/h. She wants to land at the point directly across on the opposite shore. If the speed of the boat (relative to the water) is 8 mi/h, in what direction should she steer the boat in order to arrive at the desired landing point?



Questions?



ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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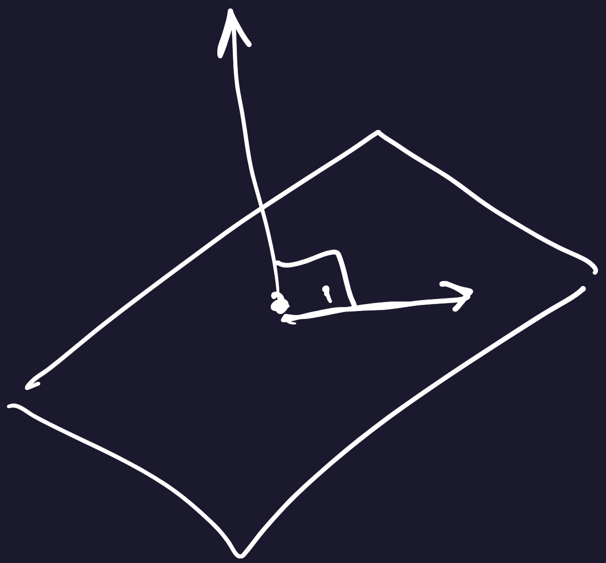
The Dot Product

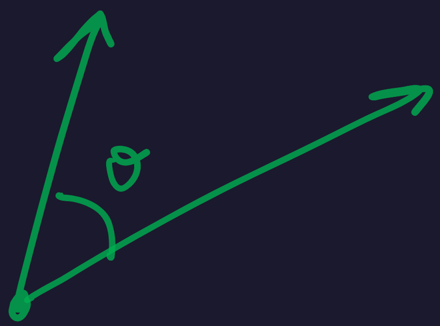
Today!

- The Dot Product
  - Definition and Properties
  - Direction Angles
  - Projections

What are we trying to do?



Angles?



The Dot Product

$(m, 1) \perp (-1, m)$

$m \cdot (-1) + 1 \cdot m = 0$

1 Definition of the Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The Dot Product

Example: Find the dot product of the vectors

$a = (1,0)$ and $b = (2,3)$.

$$a \cdot b = (1,0) \cdot (2,3)$$

$$= 1 \cdot 2 + 0 \cdot 3 = 2 + 0 = 2$$

The Dot Product

Example: Find the dot product of the vectors

a = (1,0,-1) and b = (2,5,2).

$$a \cdot b = (1,0,-1) \cdot (2,5,2)$$

$$= 1 \cdot 2 + 0 \cdot 5 + (-1) \cdot 2$$

$$= 2 + 0 - 2 = 0$$

Properties of the Dot Product

$$a \cdot b = 0$$

$$\Longleftrightarrow a \perp b$$

perpendicular

or <u>orthogonal</u>.

Properties of the Dot Product

Theorem. Two vectors u and v are perpendicular **if and only if** their dot product $u \cdot v = 0$.

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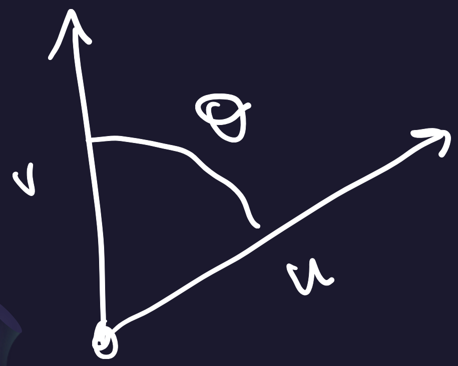
Theorem. If the angle between the vectors u and v is θ then



$$u \cdot v = |u||v|\cos(\theta)$$

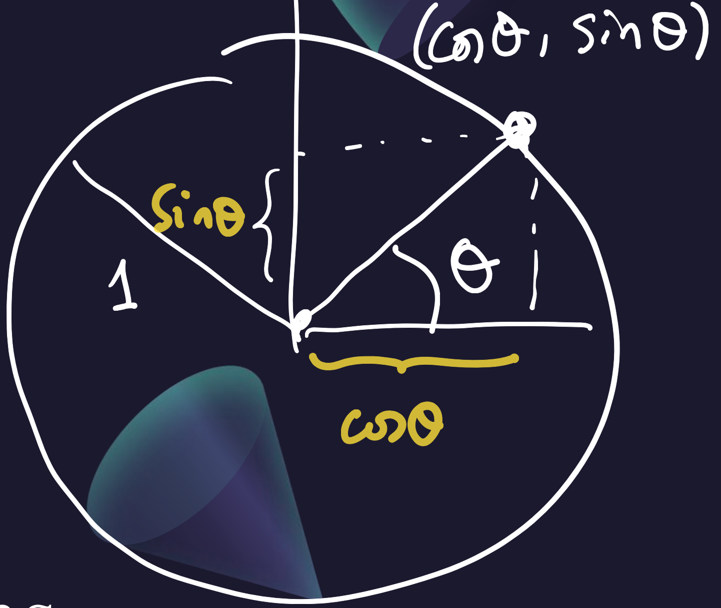
where $|u|$ and $|v|$ are their lengths.

Properties of the Dot Product

Theorem. Two vectors u and v are perpendicular **if and only if** their dot product $u \cdot v = 0$.

Theorem. If the angle between the vectors u and v is θ then

$$\cos(\theta) = \frac{u \cdot v}{|u||v|}$$



where $|u|$ and $|v|$ are their lengths.

The Dot Product

$$\cos(\theta) = \frac{a \cdot b}{|a| \cdot |b|}$$

Example: Find the angle between the two vectors a = (1,0,-1) and b = (2,5,2).

$$|a| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2} \qquad |b| = \sqrt{2^2 + 5^2 + 2^2} = \sqrt{33}$$

$$a \cdot b = (1, 0, -1) \cdot (2, 5, 2)$$

$$= 1 \cdot 2 + 0 \cdot 5 + (-1) \cdot 2 = 0$$

$$\Rightarrow \cos(\theta) = \frac{0}{\sqrt{2} \cdot \sqrt{33}} = 0 \Rightarrow \boxed{\theta = \pi/2} \quad a \perp b$$

The Dot Product

$$\cos\theta = \frac{a \cdot b}{|a| \cdot |b|}$$

Example: The angle between the two vectors
a = (1,1,0) and b = (√(3/2), √(3/2), 1).

$$a \cdot b = (1,1,0) \cdot \left(\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}}, 1\right)$$

$$= \sqrt{\tfrac{3}{2}} + \sqrt{\tfrac{3}{2}} + 0 = 2 \cdot \sqrt{\tfrac{3}{2}}$$

$$|a| = \sqrt{2}$$

$$|b| = \sqrt{\tfrac{3}{2} + \tfrac{3}{2} + 1} = 2$$

$$\cos\theta = \frac{\cancel{2} \cdot \sqrt{\tfrac{3}{2}}}{\sqrt{2} \cdot \cancel{2}} = \frac{\sqrt{\tfrac{3}{2}}}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\boxed{\theta = \frac{\pi}{6}}$$

$(= 30^\circ)$

Are the vectors 2i+2j-k and 5i-4j+2k perpendicular?

The Dot Product

Example: Are the vectors 2i+2j-k and 5i-4j+2k perpendicular?

$$(2i + 2j - k) \cdot (5i - 4j + 2k)$$

$$= 2 \cdot 5 + 2 \cdot (-4) + (-1) \cdot 2$$

$$= 10 - 8 - 2 = 0$$

Yes, they are orthogonal!

Questions?

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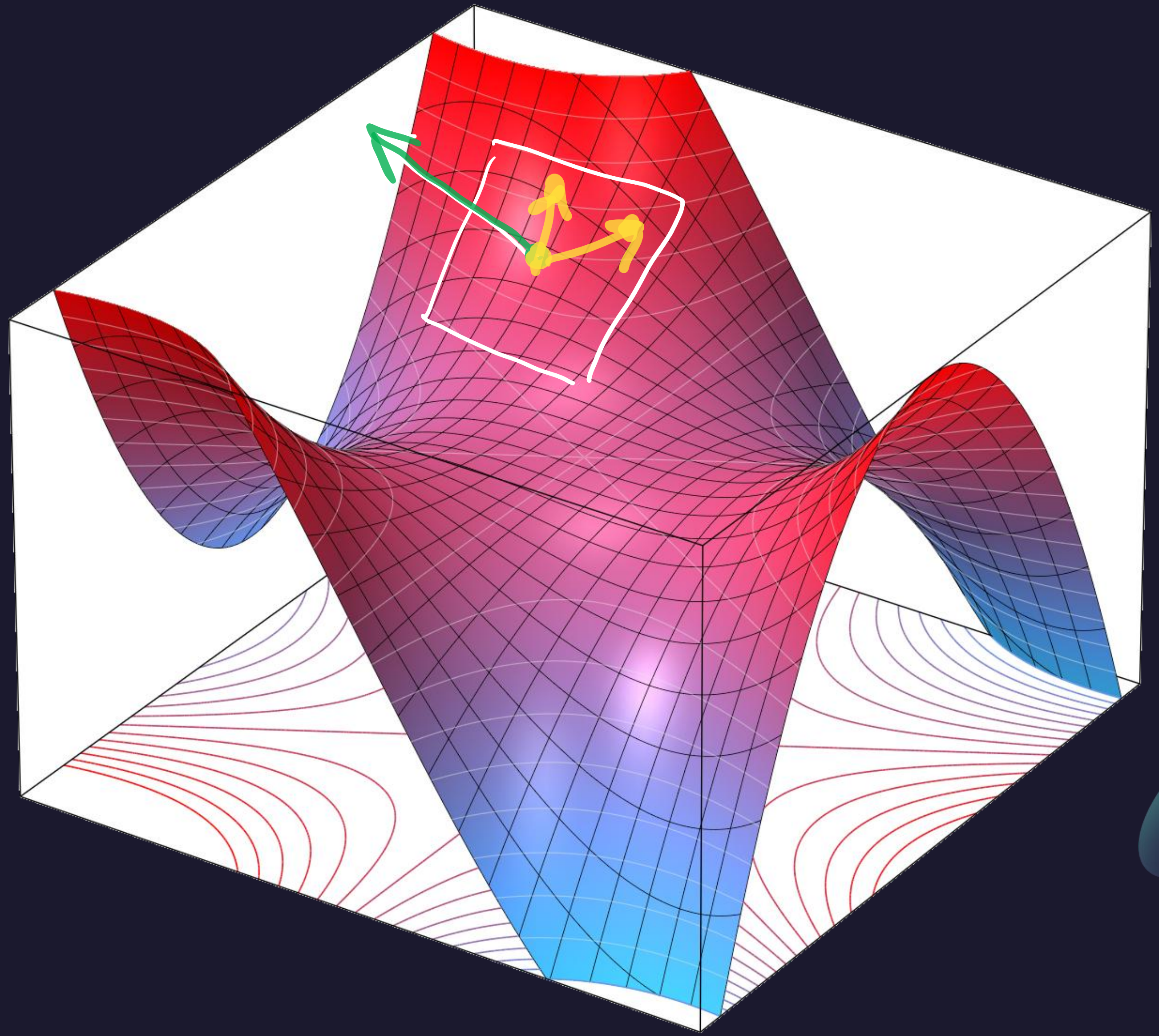
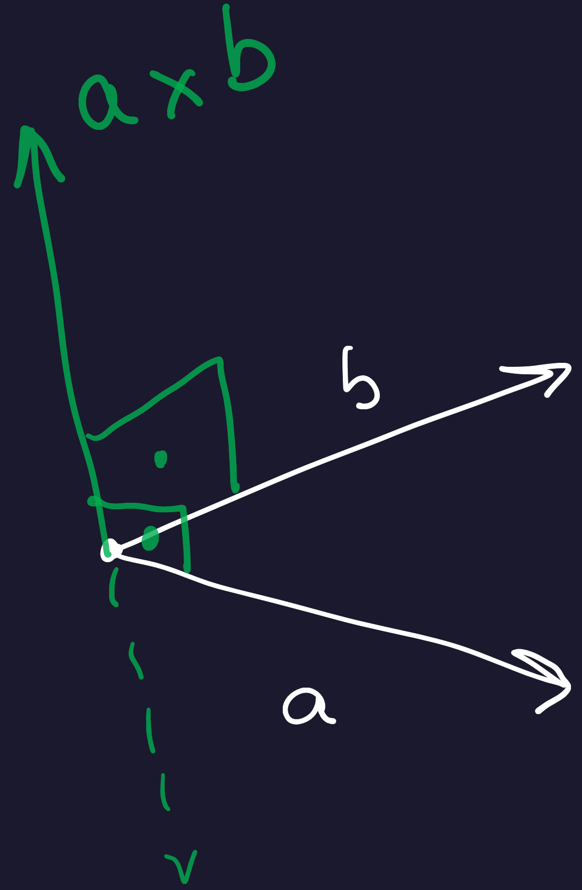
The Cross Product

Why Vectors?



Why Vectors?



Given two vectors, find a perpendicular one

$$a = (1, 1, 0)$$

$$b = \left(\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}}, 1\right)$$

$$c = (c_1, c_2, c_3)$$

$$a \cdot c = 0$$

$$b \cdot c = 0$$

$$\begin{cases} c_1 + c_2 = 0 \\ \sqrt{\tfrac{3}{2}}\, c_1 + \sqrt{\tfrac{3}{2}}\, c_2 + c_3 = 0 \end{cases}$$

Given two vectors, find a perpendicular one

4 Definition of the Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

A Slight Digression: Determinants

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\det} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$$

$$\left| \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Given two vectors, find a perpendicular one

4 Definition of the Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

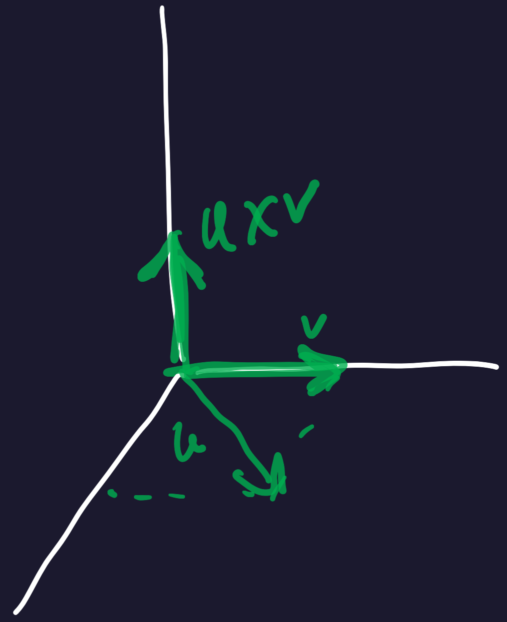
$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \cdot \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example: Find the cross product of

$u=$ (1, 1, 0) and $v=$ (0, 1, 0).



$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i\cdot\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} - j\cdot\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$+\,k\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= i\cdot\underbrace{(1\cdot 0 - 1\cdot 0)}_{0} - j\cdot\underbrace{(1\cdot 0 - 0\cdot 0)}_{0} + k\cdot(1\cdot 1 - 1\cdot 0)$$

$$= k\cdot(1) = k$$

Example: Find the cross product of

$u=$ (1, 1, 1) and (1, 0, -1).

$v=$

$1\cdot 0 - 1\cdot 1$

$''$

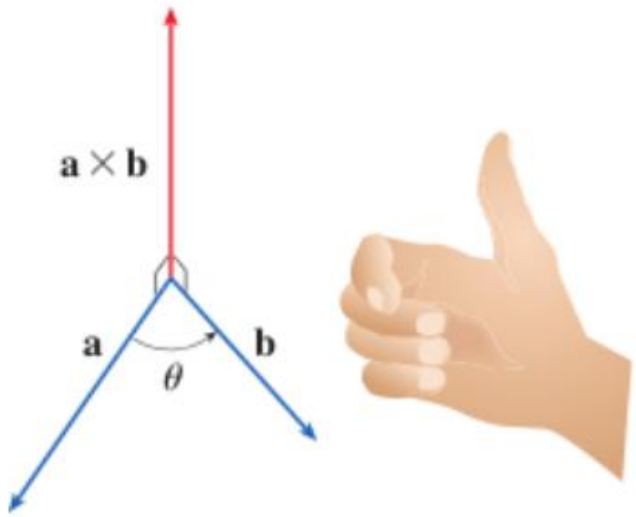
$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = i \cdot \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} - j \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + k \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= i \cdot (-1) - j \cdot (-2) + k \cdot (-1)$$

$$= -i + 2j - k = (-1, 2, -1)$$

Properties of the Cross Product

The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.



Properties of the Cross Product

9 Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

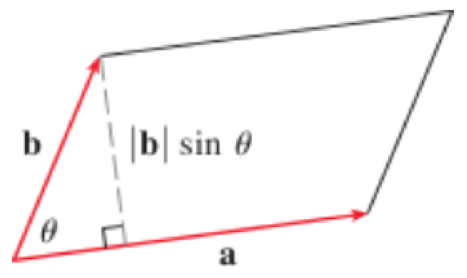
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \theta$$

9 Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

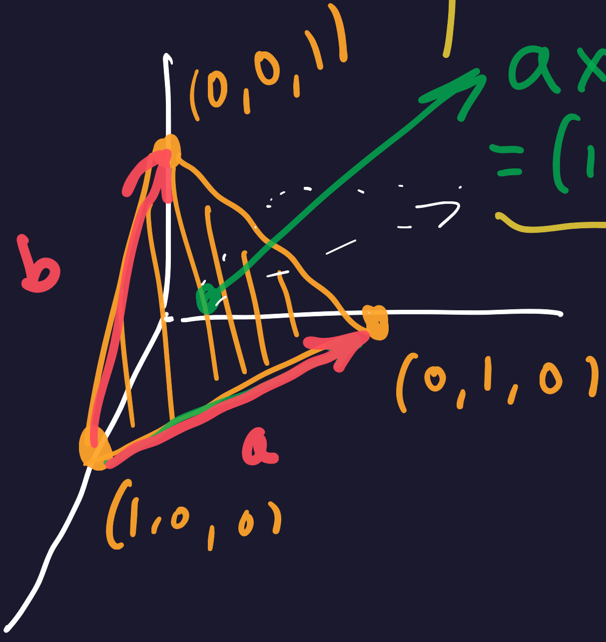
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$

Figure 2



The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Example: Find a vector perpendicular to the plane that passes through the points (1,0,0), (0,1,0), and (0,0,1).



$$\boxed{a \times b = (1,1,1)}$$

$$a = (0,1,0) - (1,0,0)$$

$$= (-1, 1, 0)$$

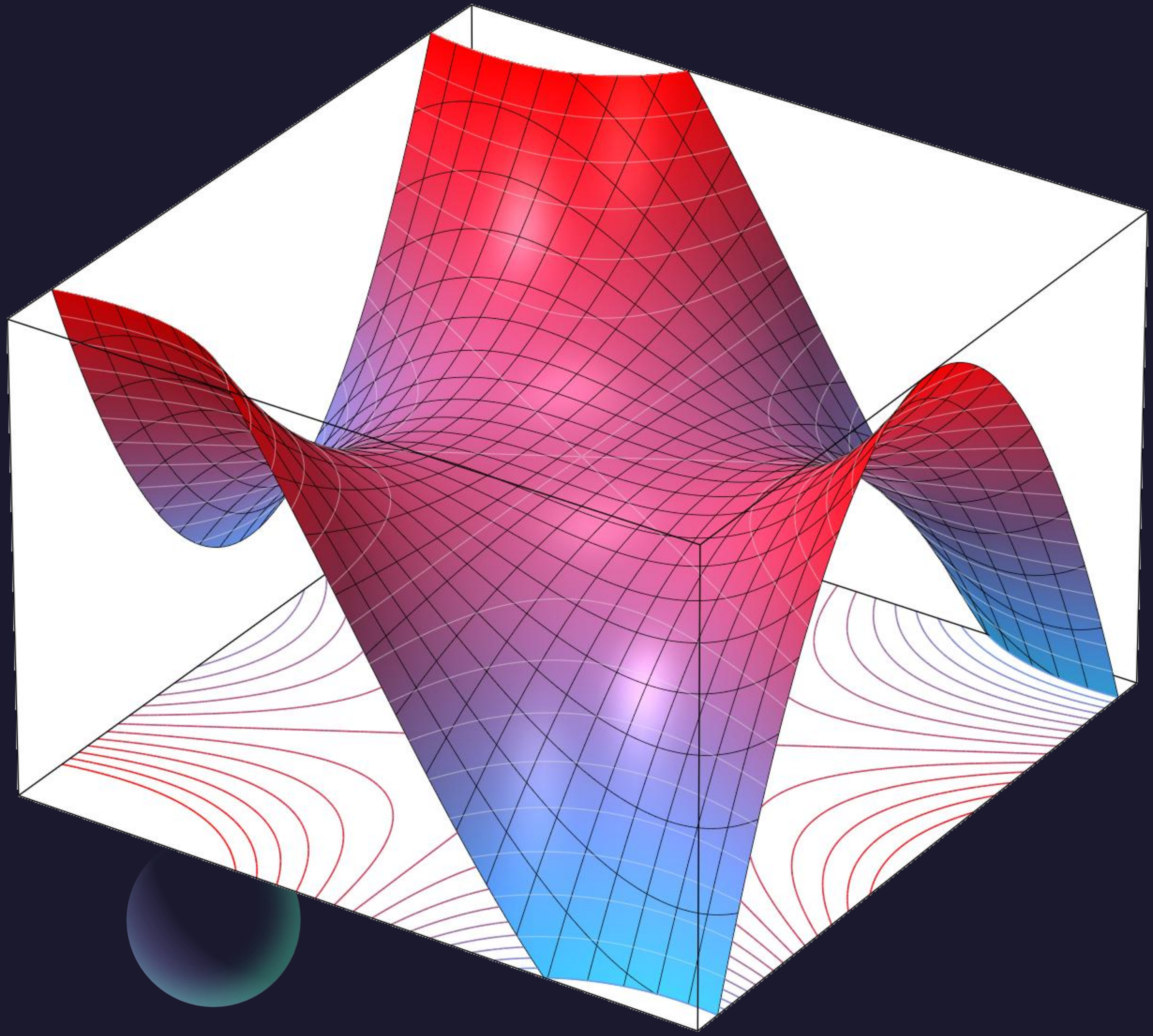
$$b = (0,0,1) - (1,0,0) = (-1,0,1)$$

$$a \times b = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = i \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}$$

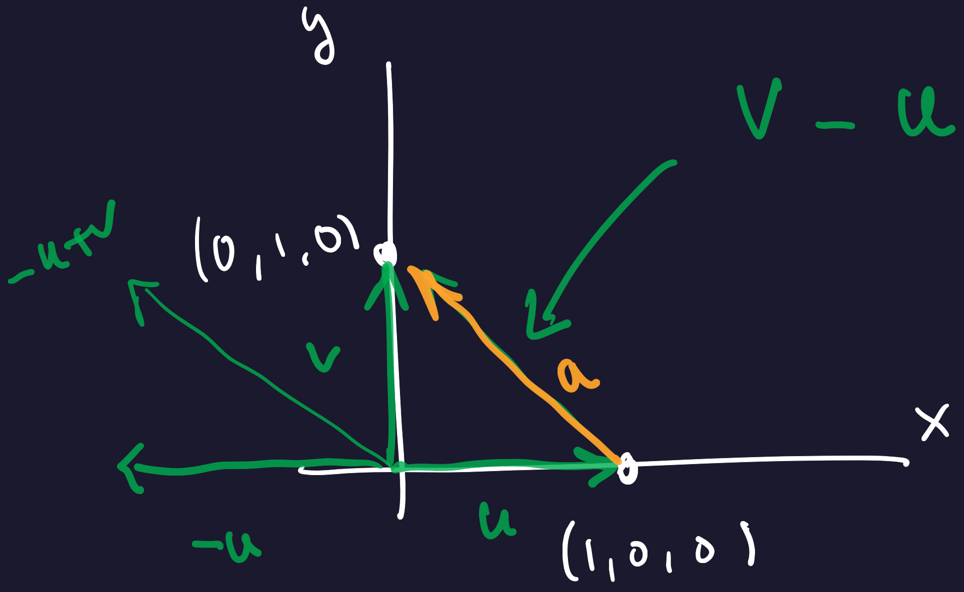
$$+ k \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = i + j + k$$

Thank you

Until next time.



Questions?



$$v - u = (0,1,0) - (1,0,0)$$
$$= (-1, 1, 0)$$