“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 8

Any Reminders? Any Questions?

- I will be away on
  - Monday 2/16 --- no office hours that day
  - Tuesday 2/17 --- I will send videos to watch instead of class

- I will be back teaching in-person on Thursday 2/19
- I will do some review for the midterm during Thursday's class
- I will have regular office hours 2/19 – 3:30-4:30
- I will have additional office hours 2/19 – 4:30-5:30
- Calc 3 Calc Night: MONT 104 at 6:30-8:30pm on Thursdays!

- Exam 1 is on Friday, Feb 20th

EXAM 1 -- Friday, February 20th

Exam Covers:

- **Chapter 12**
  - Sections 12.1 – 12.6
- **Chapter 14**
  - Sections 14.1, 14.3 – 14.8

(NEW) Exam Study Guide and Practice Problems in HuskyCT

ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

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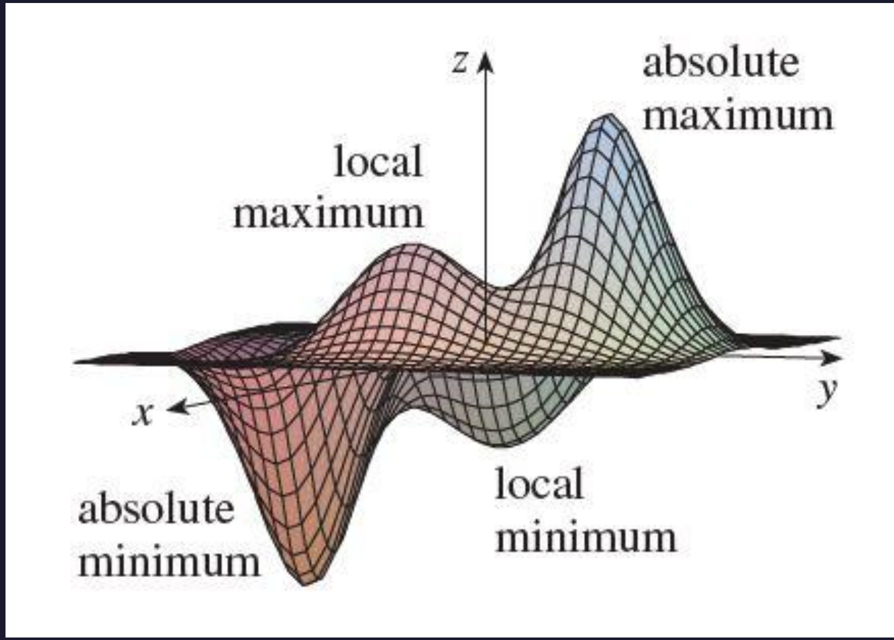
More on Maximum and Minimum Values

Today – Maximum and Minimum Values!

- Local Max and Min Values
- Second Derivative Test
- Absolute Max and Min Values



Local Max and Min Values

2 Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x\,(a, b) = 0$ and $f_y\,(a, b) = 0$.

A **critical point** for a function $f(x, y)$ is a point (a, b) where

$$\nabla f(a, b) = \vec{0},$$

that is $f_x(a, b) = 0, f_y(a, b) = 0$.

Example: Find all the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

Local Max and Min Values: Second Derivative Test

3 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x\left(a, b\right) = 0$ and $f_y\left(a, b\right) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D\left(a, b\right) = f_{xx}\left(a, b\right) f_{yy}\left(a, b\right) - \left[f_{xy}\left(a, b\right)\right]^2$$

(a) If $D > 0$ and $f_{xx}\left(a, b\right) > 0$, then $f\left(a, b\right)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}\left(a, b\right) < 0$, then $f\left(a, b\right)$ is a local maximum.

(c) If $D < 0$, then $f\left(a, b\right)$ is a saddle point of f.

WARNING! IF D = 0 , THE TEST IS INCONCLUSIVE.

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14$$

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Example: Find and classify the critical points for the function

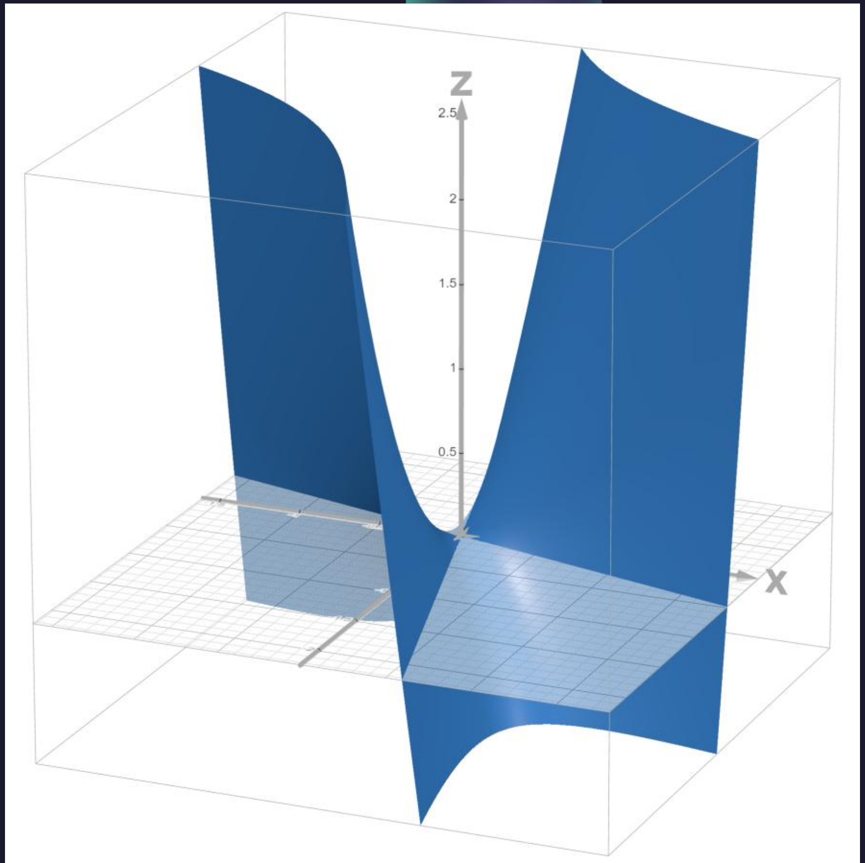
$$f(x,y) = x^2 + 4xy + y^2$$

[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find and classify the critical points for the function

$$f(x,y) = x^2 + 4xy + y^2$$



Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

8 Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Absolute Max and Min Values

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

$$D = D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Example: Find the absolute maximum and minimum values of

$$f(x,y) = xy^2$$

in the region $D = \{(x,y): x^2 + y^2 \leq 3\}$.

[Extra]

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

Example: Find the absolute maximum and minimum values of

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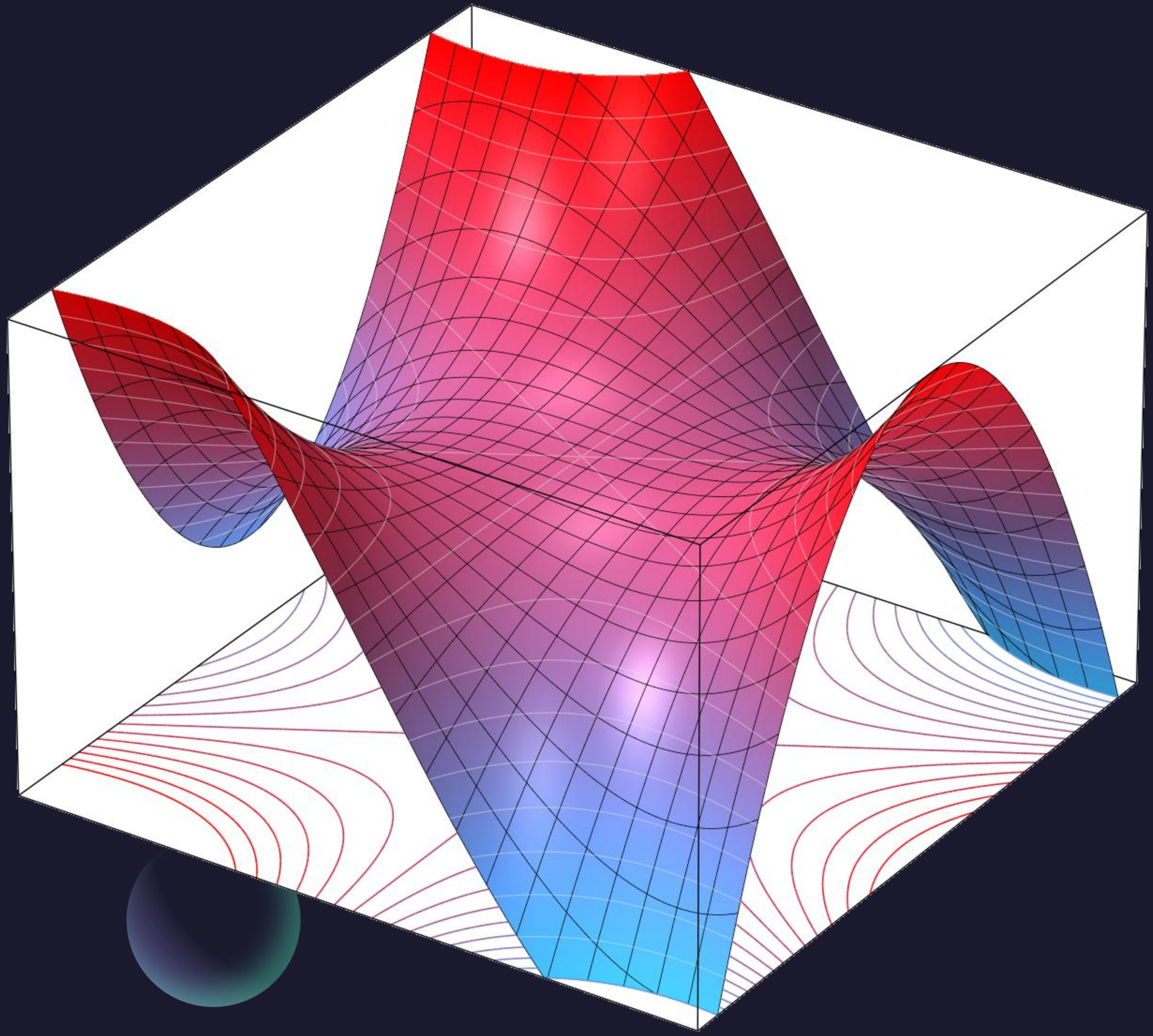
Questions?

Thank you

Until next time.



ALVARO: Start the recording!

“Calculus 3”

Multi-Variable Calculus

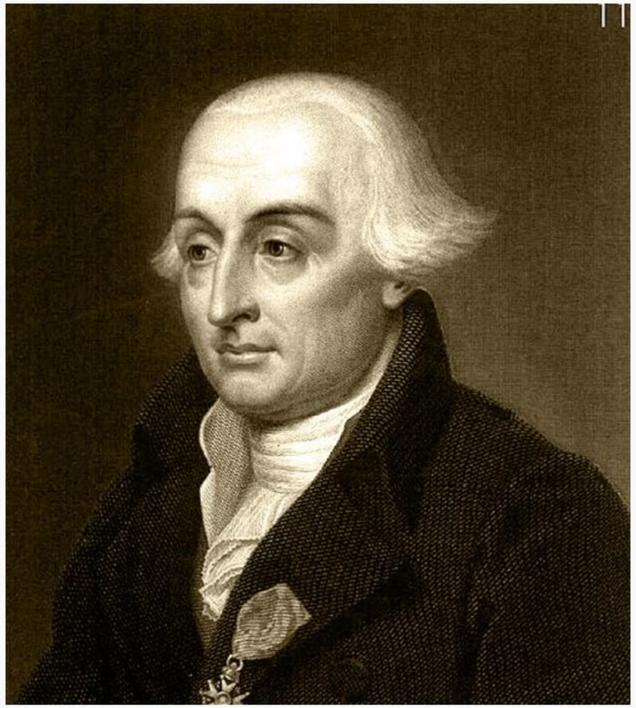
Instructor: Álvaro Lozano-Robledo

Lagrange Multipliers

Today – "Lagrange Multipliers!"

- The Method
- One Constraint
- Examples

Joseph-Louis Lagrange



Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin, Kingdom of Sardinia
Died	10 April 1813 (aged 77) Paris, First French Empire

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Recall: Properties of the Gradient Vector

Thus, the gradient vector for a surface $z = f(x, y)$ in three dimensions, $\nabla F = (f_x, f_y, -1)$ is normal to a surface at any point.

The gradient vector in two dimensions, $\nabla F = (f_x, f_y)$ is normal to any level curves of $f(x,y)$ at any point, indicating the maximum rate of change.

Example: Let $f(x,y) = 4 - x^2 - y^2$

(a) Find the normal vector to the graph of $f(x,y)$ at $(1,1,2)$

(b) Find the tangent plane to the graph of $f(x,y)$ at $(1,1,2)$

(c) Find the normal vector to the cross section $z = 0$ at $(1,1)$

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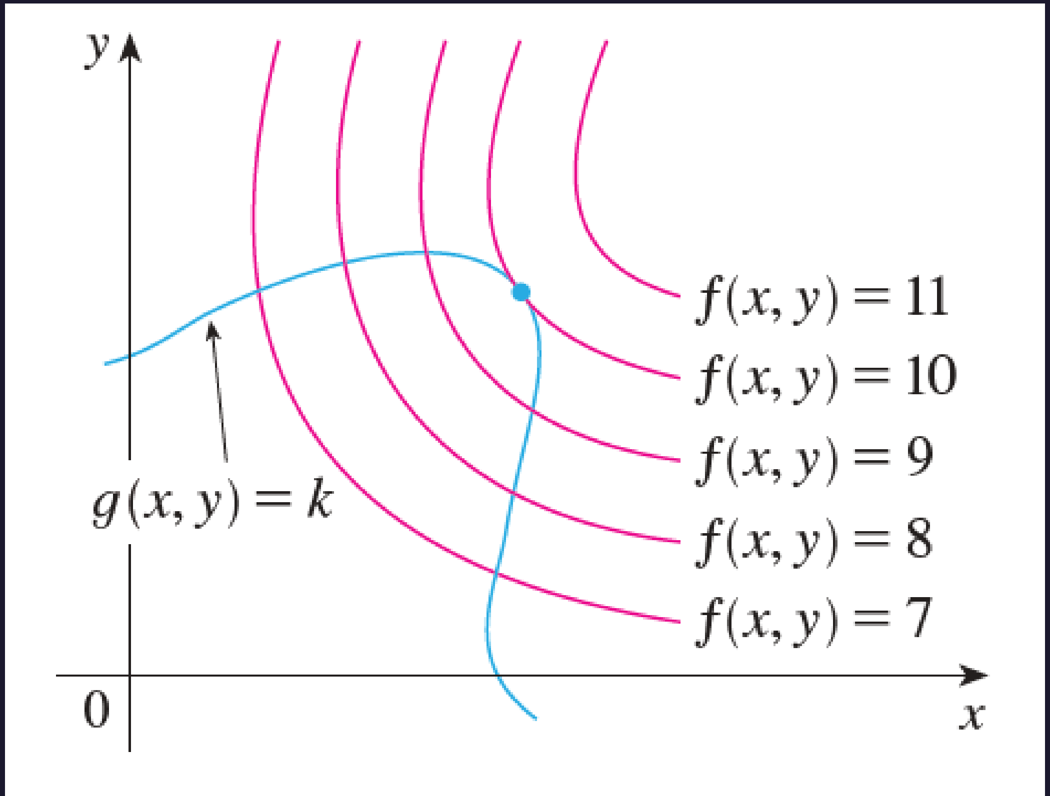
The "Lagrange Multipliers" Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.



The “Lagrange Multipliers” Method

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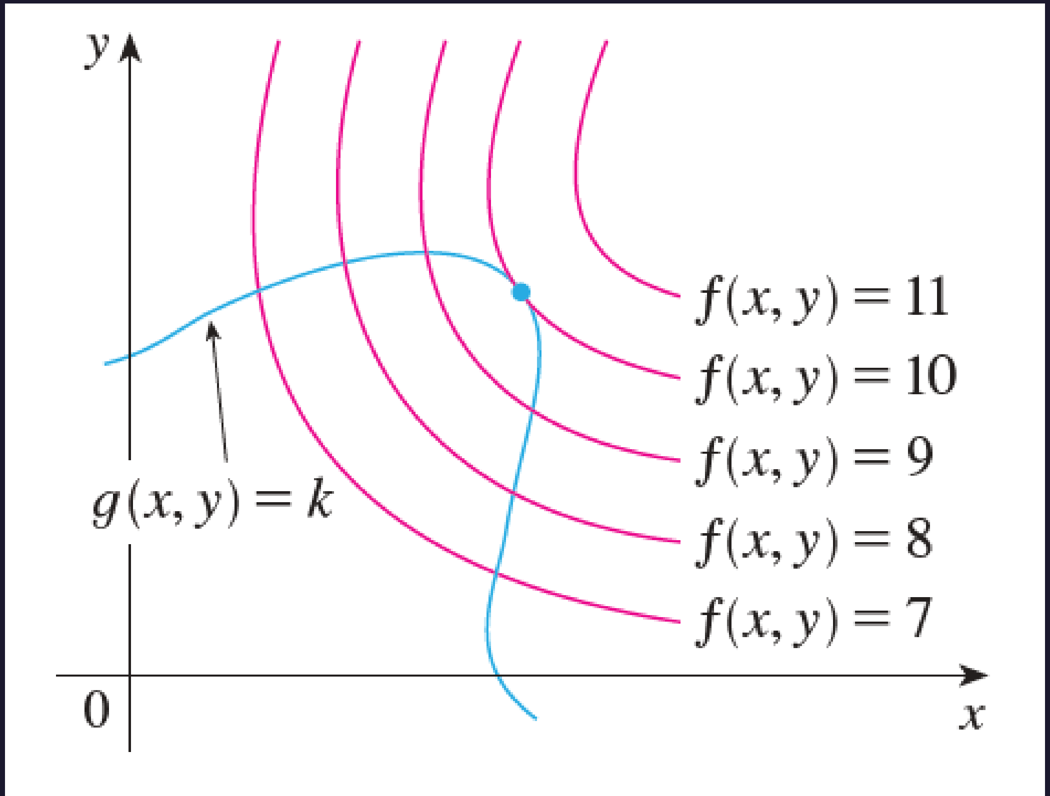


The value of f(x,y) on the curve g(x,y)=k will be maximized at some point (x_0,y_0) such that

$\nabla f\,(x_0,y_0)$ is parallel to $\nabla g\,(x_0,y_0)$

or equivalently a point (x_0,y_0) such that there is a constant λ with

$\nabla f\,(x_0,y_0) = \lambda \cdot \nabla g\,(x_0,y_0)$

and $g(x_0,y_0) = k$.

The “Lagrange Multipliers” Method

GOAL : Maximize $z = f(x,y)$ on the curve $g(x,y) = k$.

SOLVE:

$$\nabla f\,(x_0,y_0) = \lambda \cdot \nabla g\,(x_0,y_0)$$

$$g(x_0,y_0) = k\,.$$

$\nabla f(x_0, y_0) = \lambda \cdot \nabla g(x_0, y_0)$ and $g(x_0, y_0) = k$.

Example: Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

$\nabla f\,(x_0,y_0) = \lambda \cdot \nabla g\,(x_0,y_0)$ and $g(x_0,y_0) = k$.

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

$\nabla f\,(x_0,y_0) = \lambda \cdot \nabla g\,(x_0,y_0)$ and $g(x_0,y_0) = k$.

[Extra]

Example: Find the extreme values of $f(x,y) = x^2 + y^2$ on the curve $xy = 1$.

$$\nabla f(x_0,y_0) = \lambda \cdot \nabla g(x_0,y_0) \quad \text{and} \quad g(x_0,y_0) = k\,.$$

Example: Find the largest area of a rectangle with fixed perimeter equal to p.

The "Lagrange Multipliers" Method

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \, \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

$$\nabla f(x_0,y_0,z_0) = \lambda \cdot \nabla g(x_0,y_0,z_0) \quad \text{and} \quad g(x_0,y_0,z_0) = k\,.$$

Example: Find the dimensions of the closed box with the largest volume and fixed surface area S.

$$\nabla f(x_0,y_0,z_0) = \lambda \cdot \nabla g(x_0,y_0,z_0) \quad \text{and} \quad g(x_0,y_0,z_0) = k\ .$$

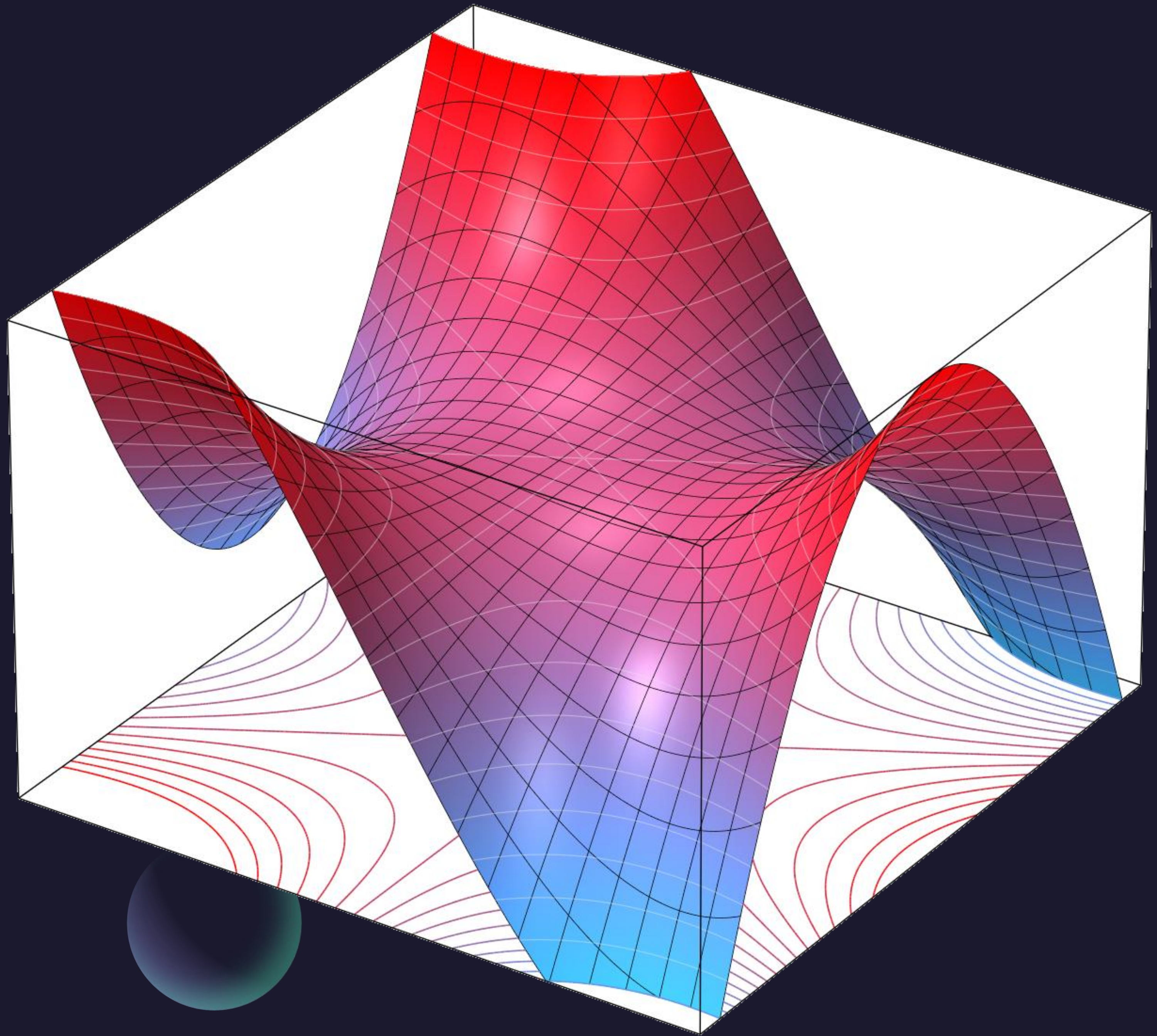
[Extra]

Example: Find the dimensions of the closed box with the largest volume and fixed surface area S.

Questions?

Thank you

Until next time.



ALVARO: Start the recording!

"Calculus 3"

Multi-Variable Calculus

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Double Integrals over Rectangles

Today – Double Integrals!

- The Definite Integral
- The Riemann Integral
- Iterated Integrals
- Fubini’s Theorem

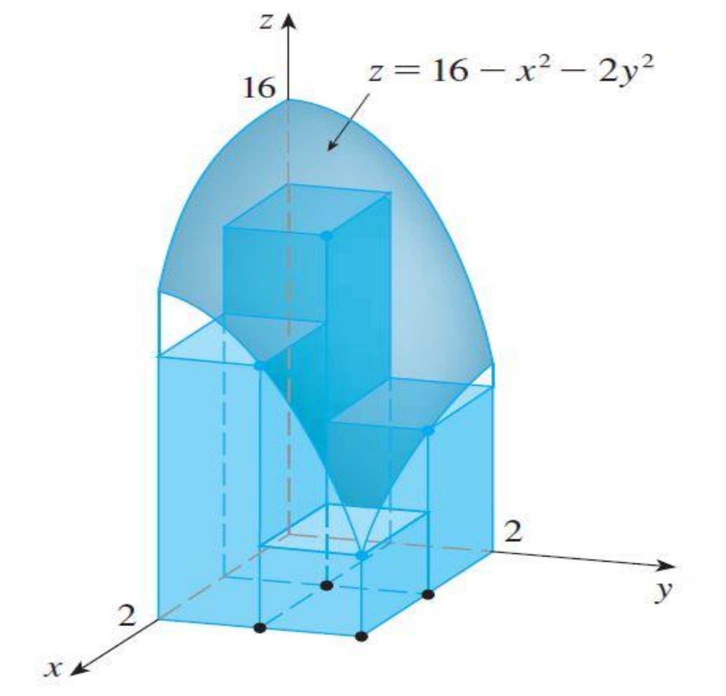
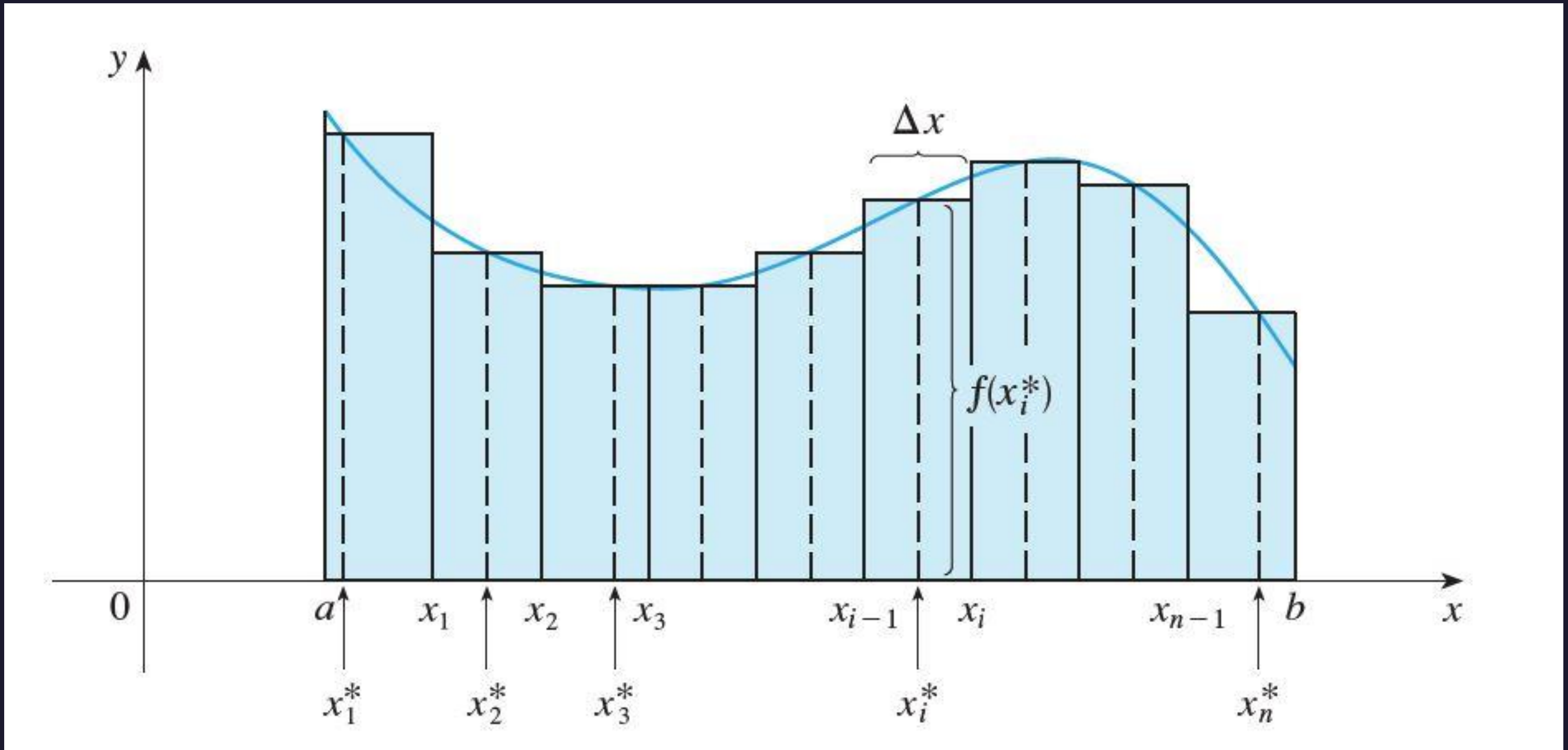
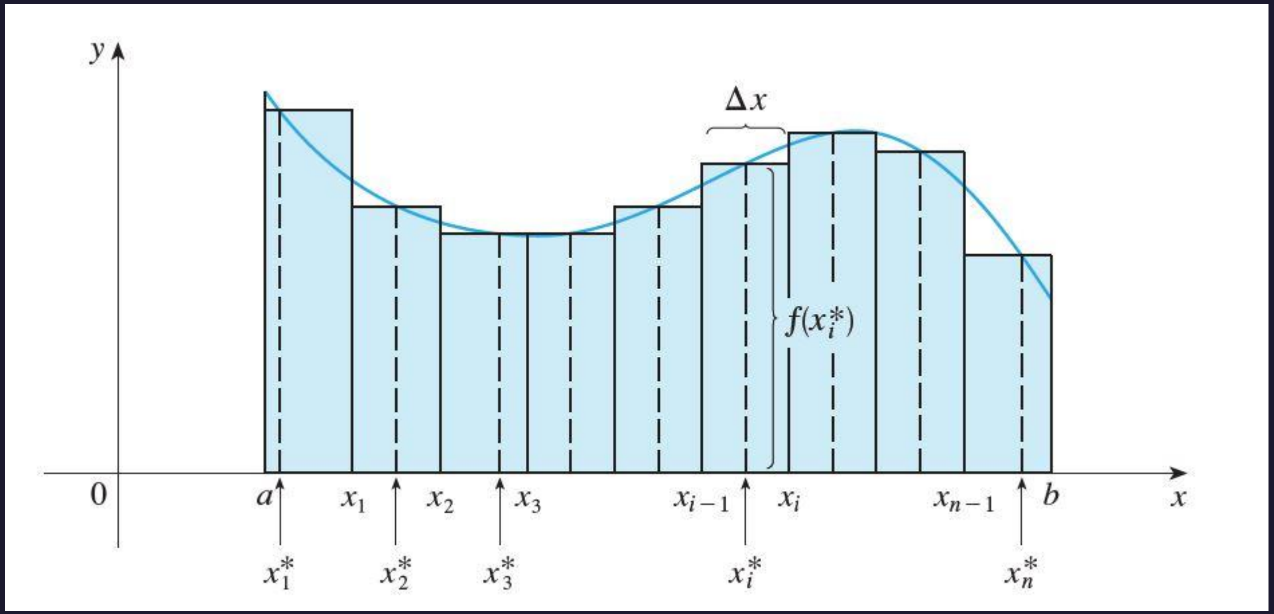


FIGURE 7

The Definite (Riemann) Integral



The Definite (Riemann) Integral



The Definite (Riemann) Integral

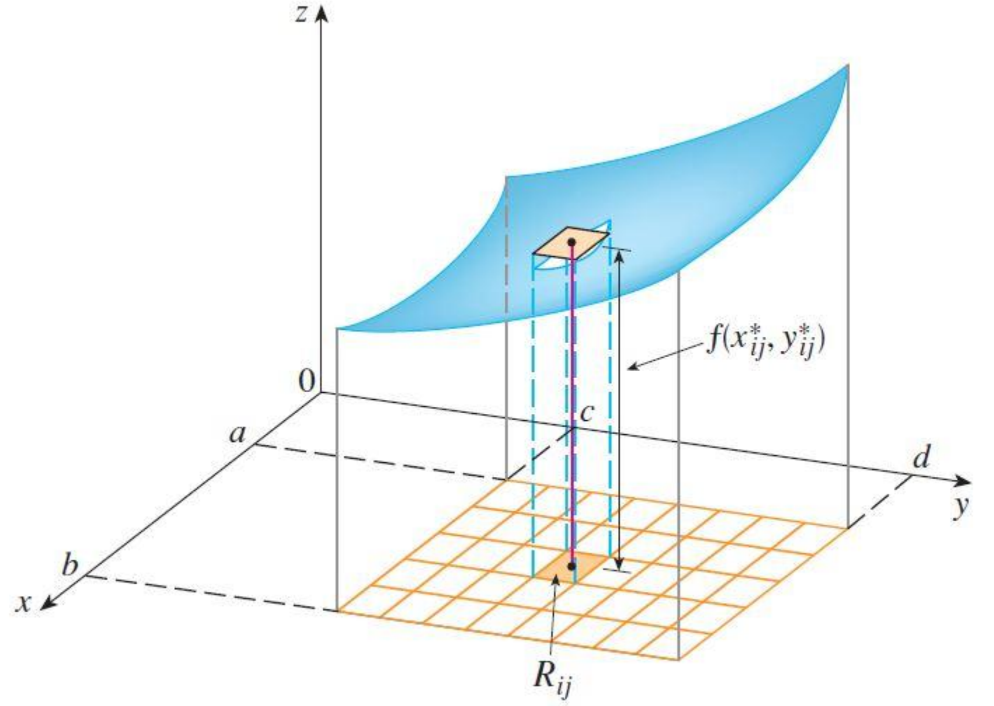


FIGURE 4

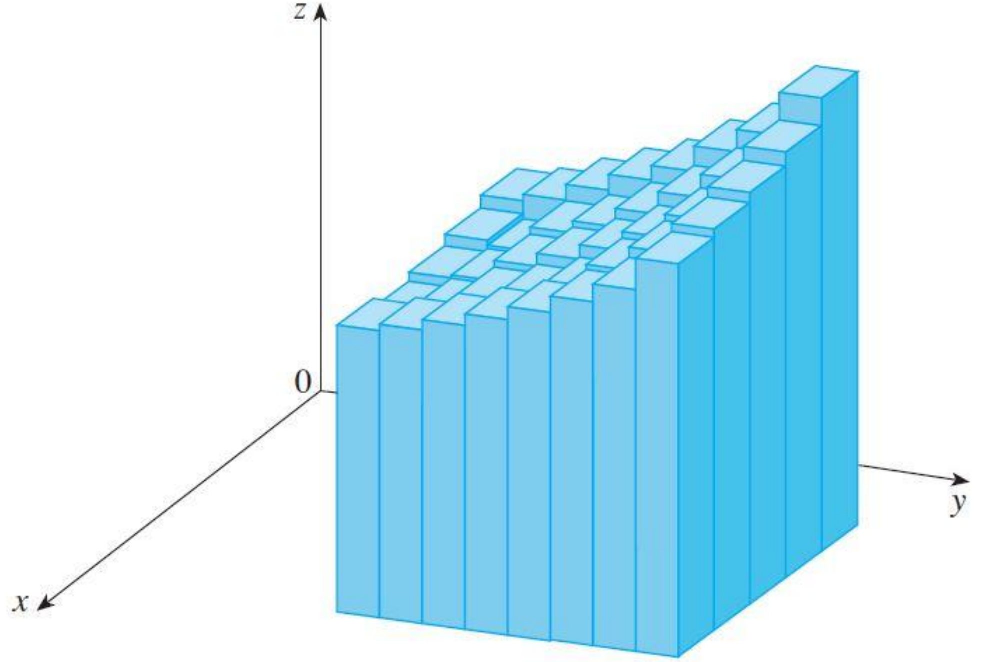


FIGURE 5

The Definite (Riemann) Integral

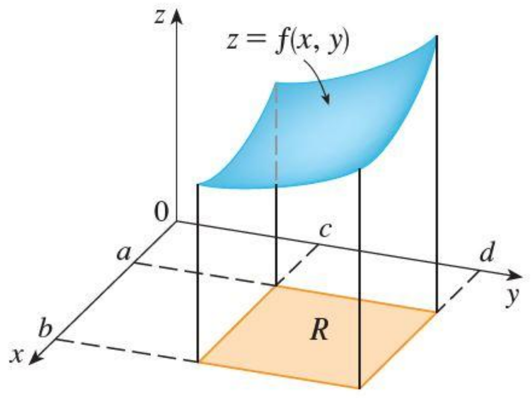


FIGURE 2

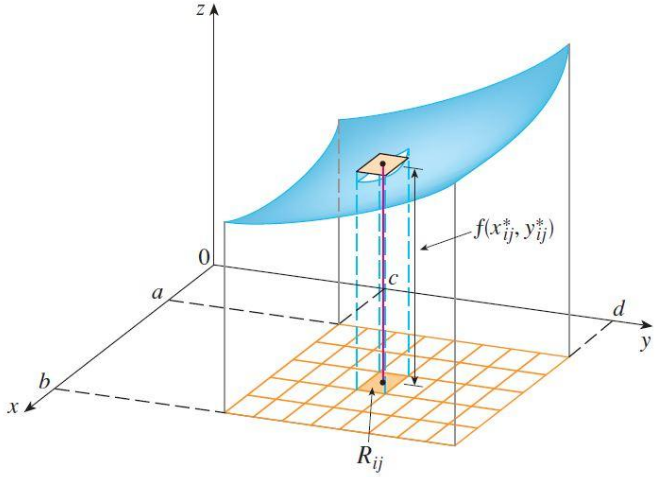


FIGURE 4

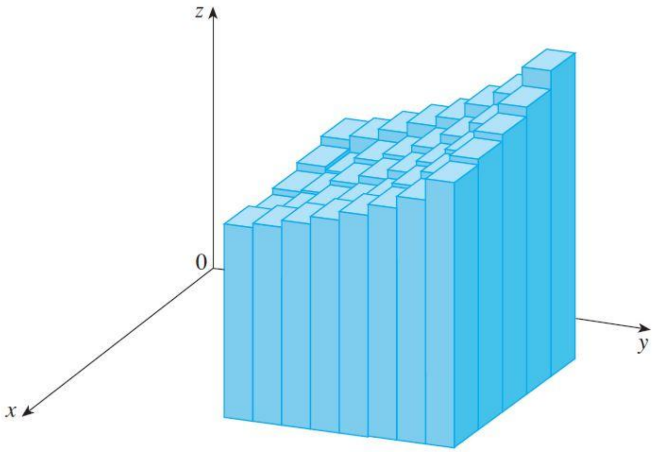


FIGURE 5

The Definite (Riemann) Integral

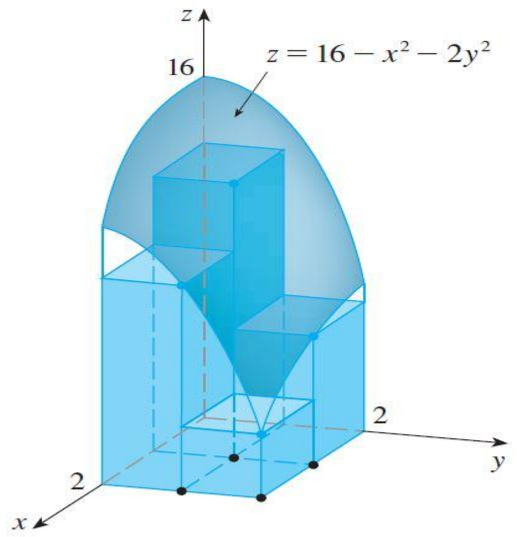
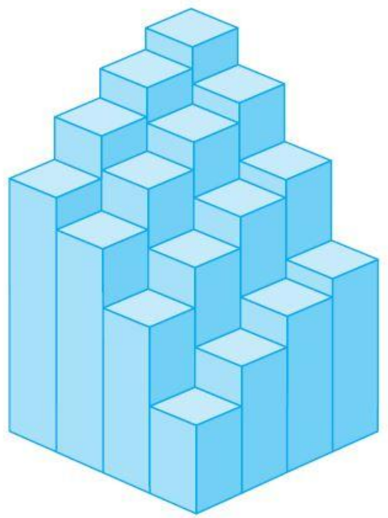


FIGURE 7



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

The usual properties of integration still hold for double integrals:

- $\displaystyle\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA.$
- For any constant c,

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA.$$

- If $f(x,y) \geq g(x,y)$ on the rectangle R, then

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA.$$

And when letting $m, n \to \infty$, we have $\Delta A \to dA = dx \cdot dy$. Then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy,$$

this is called an **iterated integral**, and we evaluate its value by computing the innermost integral first and then working the way out. Again, in the case this value represents a volume only if $f(x,y) \geq 0$ on R.

Example: Find the volume under the graph of $f(x, y) = 16 - x^2 - 2y^2$ above the square R = [0,2] x [0,2].

[Extra]

Example: Find the volume under the graph of $f(x,y) = 16 - x^2 - 2y^2$ above the square R = [0,2] x [0,2].

Example: Calculate the following iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

[Extra]

Example: Calculate the following iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

Fubini's Theorem

If f is continuous on the rectangle

$$R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$$

then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Guido Fubini

Born	19 January 1879 Venice
Died	6 June 1943 (aged 64) New York

Example: Evaluate the double integral

$$\iint_R \left(x - 3y^2\right) dA$$

where $R = \{(x, y): 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Example: Evaluate the double integral

$$\iint_R \left(x - 3y^2\right) dA$$

where $R = \{(x, y): 0 \le x \le 2, 1 \le y \le 2\}$.

Example: Evaluate the double integral

$$\iint_R y \sin(xy)\, dA$$

where $R = [1,2] \times [0, \pi]$.

[Extra]

Example: Evaluate the double integral

$$\iint_R y \sin(xy)\, dA$$

where $R = [1,2] \times [0, \pi]$.

When $f(x,y) = g(x) \cdot h(y)$, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b g(x)h(y)dxdy = \int_a^b g(x)dx \cdot \int_c^d h(y)dy$$

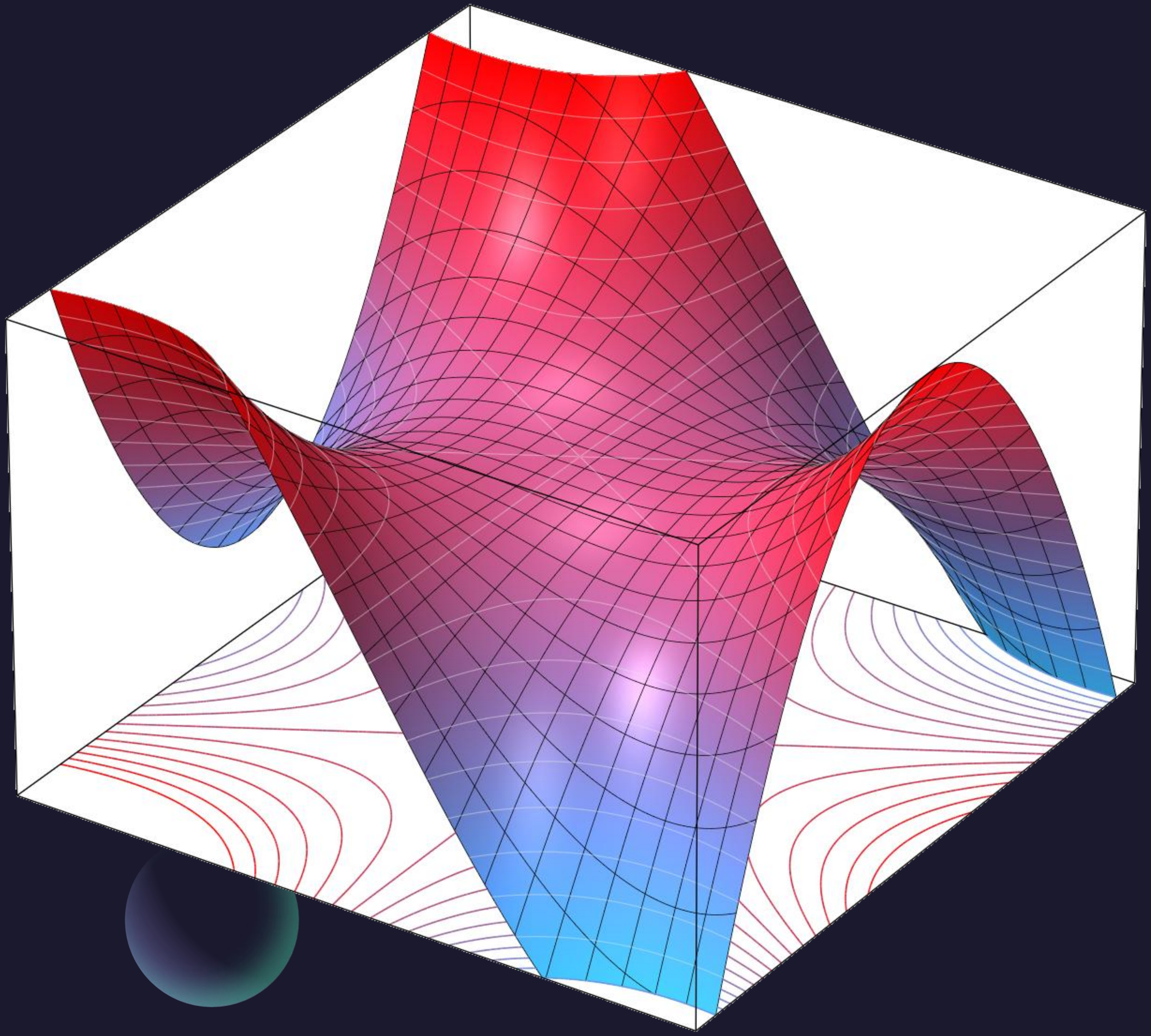
- Evaluate the iterated integral

$$\int_1^3 \int_1^5 \frac{\ln(y)}{xy} dxdy$$

Questions?

Thank you

Until next time.



"Calculus 3"

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Double Integrals over Regions

Today – Double Integrals in Regions!

- General Regions
- Regions of Type I and II
- Changing the Order of Integration
- Properties of Double Integrals

Regions of Type I and II

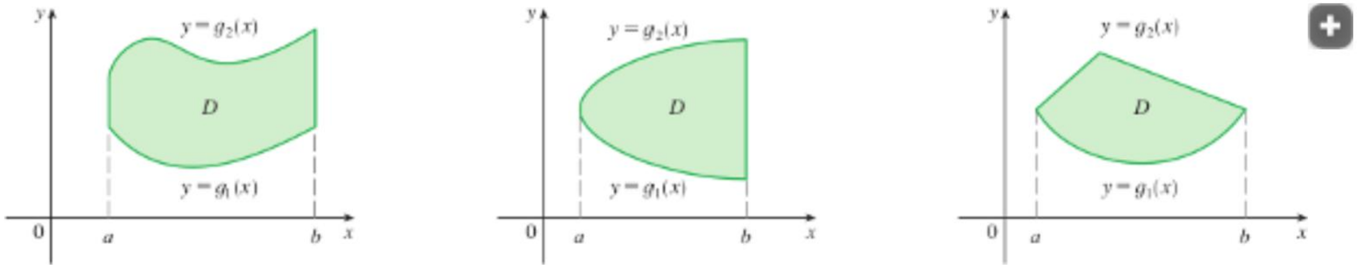
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

Figure 5

Some type I regions



Regions of Type I and II

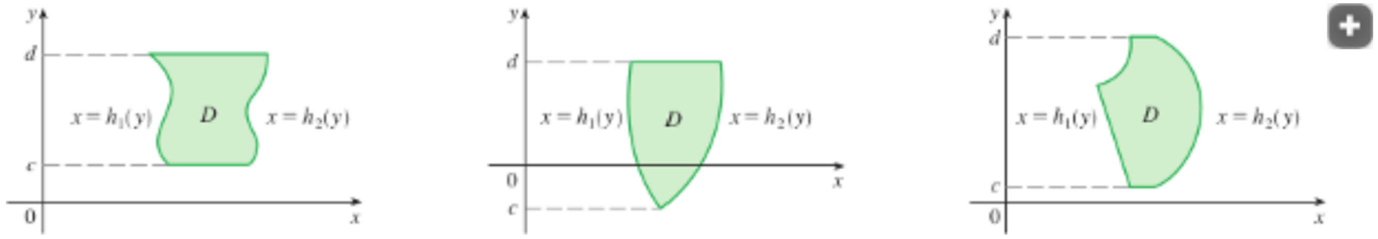
We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

Figure 7

Some type II regions



Integrals over Regions of Type I

3 If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Integrals over Regions of Type II

$\boxed{4}$ If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \text{ and } y = 1 + x^2 .$$

Example: Evaluate the double integral

$$\iint_R (x + 2y)\, dA$$

where R is the region bounded by the parabolas

$$y = 2x^2 \text{ and } y = 1 + x^2 .$$

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$. (As a Type I integral.)

[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

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[Extra]

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

[Extra]

Example: Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Properties of Double Integrals

$$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

$$\iint_D cf(x,y)\, dA = c\iint_D f(x,y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

7

$$\iint_D f(x,y)\, dA \geqslant \iint_D g(x,y)\, dA$$

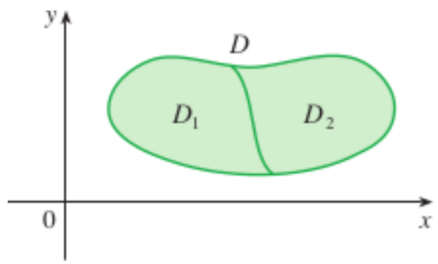
Properties of Double Integrals

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

$\boxed{8}$

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

Figure 17



Properties of Double Integrals

$$\iint_D 1 \, dA = A(D)$$

10 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y) \, dA \leqslant M \cdot A(D)$$

Example: Estimate the value of the double integral

$$\iint_R e^{-(x^2+y^2)}\,dA$$

where $R = \{(x, y): x^2 + y^2 \leq 1\}$ is the circle of radius 1.

Questions?

Thank you

Until next time.