Name :- Taha Mahmood

Reg No :- 2017472

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Take Home Assignment

First link

Kinetic Energy

$$k_1 = \frac{1}{2} m_1 V_{c_1}^T V_{c_1} + \frac{1}{2} {}^1\omega_1^T \, {}^{c_1}I_1 \, {}^1\omega_1$$

$$x_{c_1} = \frac{1}{2} l_1 c_1 \rightarrow \dot{x} = V_{xc_1} = -\frac{1}{2} l_1 s_1 \dot{\theta}_1$$

$$y_{c_1} = \frac{1}{2} l_1 s_1 \rightarrow \dot{y} = V_{yc_1} = \frac{1}{2} l_1 c_1 \dot{\theta}_1$$

$$V_{c_1} = \begin{bmatrix} -\frac{1}{2} l_1 s_1 \dot{\theta}_1 \\ \frac{1}{2} l_1 c_1 \dot{\theta}_1 \end{bmatrix}$$

$$V_{c_1}^T V_{c_1} = \frac{1}{2} (l_1 \dot{\theta}_1)^2$$

$$= {}^1\omega_1^T \, {}^{c_1}I_1 \, {}^1\omega_1$$

$$= I_{zz} (\dot{\theta}_1)^2 = \frac{m_1}{12} (l_1 \dot{\theta}_1)^2$$

Subsituting in orginal equation

$$k_1 = \frac{1}{4} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{12} m_1 l_1^2 \dot{\theta}_1^2 = \frac{1}{3} m_1 l_1^2 \dot{\theta}_1^2$$

Potential Energy

$$u_1 = -m_1 \, {}^0g^T \, {}^0P_{c_1}$$

$$= -m \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} l_1 c_1 \\ \frac{1}{2} l_1 s_1 \\ 0 \end{bmatrix} = \frac{1}{2} m g l_1 s_1$$

Second link

Kinetic Energy

$$X_{c_2} = l_1 c_1 + \frac{1}{2} l_2 c_{12} \rightarrow V_{xc_2} = -l_1 s_1 \dot{\theta}_1 - \frac{1}{2} l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$Y_{c_2} = l_1 s_1 + \frac{1}{2} l_2 s_{12} \rightarrow V_{yc_2} = l_1 c_1 \dot{\theta}_1 + \frac{1}{2} l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$V_{c_2} = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - \frac{1}{2} l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + \frac{1}{2} l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

$$V_{c_2}^T V_{c_2} = (-l_1 s_1 \dot{\theta}_1 - \frac{1}{2} l_2 s_{12} (\dot{\theta}_1 + \dot{\theta}_2))^2 + (l_1 c_1 \dot{\theta}_1 + \frac{1}{2} l_2 c_{12} (\dot{\theta}_1 + \dot{\theta}_2))^2$$

$$= l_1^2 \dot{\theta}_1^2 + l_1 l_2 c_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$${}^2\omega_2^{T}\, {}^{c_1}I_2\, {}^2\omega_2 = I_{zz_2}(\dot{\theta}_1)^2 = \frac{m_1 l_2^2}{12}(\dot{\theta}_1+\dot{\theta}_2)^2$$

$$k_2 = \frac{1}{2}m_2\left\{l_1^2\dot{\theta}_1^2 + \frac{1}{2}l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + l_1 l_2 c_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\right\}$$
$$+ \frac{1}{12}m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2$$

Potential Energy

$$u_2 = -m_2\, {}^0g^T\, {}^0P_{c2}$$

$$= -m_2\begin{bmatrix}0\\-g\\0\end{bmatrix}\begin{bmatrix}l_1c_1+\frac{1}{2}l_2c_{12}\\ l_1s_1+\frac{1}{2}l_2s_{12}\\ 0\end{bmatrix} = m_2 g\left(l_1 s_1 + \frac{1}{2}l_2 s_{12}\right)$$

Total Energy of manipulator

Kinetic Energy

$$k = k_1 + k_2$$

$$= \frac{1}{3}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left\{l_1^2\dot{\theta}_1^2 + \frac{2}{3}l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + l_1 l_2 c_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\right\}$$

Potential Energy

$$u = u_1 + u_2$$

$$= \frac{1}{2}m_1 g l_1 s_1 + m_2 g\left(l_1 s_1 + \frac{1}{2}l_2 s_{12}\right)$$

$$\frac{\partial k}{\partial \dot{\theta}} = \begin{vmatrix}\frac{2}{3}m_1 l_1^2\dot{\theta}_1 + \frac{1}{2}m_2\left\{2l_1^2\dot{\theta}_1 + \frac{4}{3}l_2^2(\dot{\theta}_1+\dot{\theta}_2) + 2l_1l_2c_2\dot{\theta}_1 + l_1l_2c_2\dot{\theta}_2\right\}\\ \frac{2}{3}m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2) + \frac{1}{2}m_2 l_1 l_2 c_2\dot{\theta}_1\end{vmatrix}$$

$$\frac{\partial k}{\partial \theta} = \begin{vmatrix}0\\ -\frac{1}{2}m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)s_2\end{vmatrix}$$

$$\frac{\partial u}{\partial \theta} = \begin{vmatrix}\frac{1}{2}m_1 g l_1 c_1 + m_2 g l_1 c_1 + \frac{1}{2}m_2 l_2 g c_{12}\\ \frac{1}{2}m_2 g l_2 c_{12}\end{vmatrix}$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}} = \begin{vmatrix}\frac{2}{3}m_1 l_1^2\ddot{\theta}_1 + m_2 l_1^2\ddot{\theta}_1 + \frac{2}{3}m_2 l_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2 l_1 l_2 c_2\ddot{\theta}_1 + \frac{1}{2}m_2 l_1 l_2 c_2\ddot{\theta}_2\\ - m_2 l_1 l_2 s_2\dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}m_2 l_1 l_2 s_2\dot{\theta}_2^2\\ \frac{2}{3}m_2 l_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + \frac{1}{2}m_2 l_1 l_2 c_2\ddot{\theta}_1 - \frac{1}{2}m_2 l_1 l_2 s_2\dot{\theta}_1\dot{\theta}_2\end{vmatrix}$$

Torque on first joint

$$\tau_1 = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_1} - \frac{\partial k}{\partial \theta_1} + \frac{\partial u}{\partial \theta_1}$$

$$= \frac{2}{3} m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + \frac{2}{3} m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 \ddot{\theta}_1$$
$$+ \frac{1}{2} m_2 l_1 l_2 c_2 \ddot{\theta}_2 - m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} m_2 l_1 l_2 c_2 \dot{\theta}_2^2 + \frac{1}{2} m_1 g l_1 c_1$$
$$+ m_2 g l_1 c_1 + \frac{1}{2} m_2 g l_2 c_{12}$$

Torque on second joint

$\cancel{\tau_2 =}$

$$\tau_2 = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_2} - \frac{\partial k}{\partial \theta_2} + \frac{\partial u}{\partial \dot{\theta}_2}$$
$$= \frac{2}{3} m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + \frac{1}{2} m_2 l_1 l_2 c_2 \ddot{\theta}_1 - \frac{1}{2} m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2$$
$$+ \frac{1}{2} m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) s_2 + \frac{1}{2} m_2 g l_2 c_{12}$$

$$= \frac{2}{3} m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + \frac{1}{2} m_2 l_1 l_2 c_2 \ddot{\theta}_1 + \frac{1}{2} m_2 l_1 l_2 \theta_1^2 s_2 + \frac{1}{2} m_2 g l_2 c_{12}$$

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

$$\begin{vmatrix} \tau_1 \\ \tau_2 \end{vmatrix} = \begin{vmatrix} \frac{2}{3} m_1 l_1^2 + m_2 l_1^2 + \frac{2}{3} m_2 l_2^2 + m_2 l_1 l_2 c_2 & \frac{2}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 c_2 \\ \frac{2}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 c_2 & \frac{2}{3} m_2 l_2^2 \end{vmatrix}$$

$$\times \begin{vmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{vmatrix} + \begin{vmatrix} -\frac{1}{2} m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ \frac{1}{2} m_2 g l_2 c_{12} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{1}{2} m_1 g l_1 c_1 + m_2 g l_1 c_1 + \frac{1}{2} m_2 g l_2 c_{12} \\ \frac{1}{2} m_2 g l_2 c_{12} \end{vmatrix}$$

Mass/Inertia matrix

$$M(\theta) = \begin{bmatrix} \frac{2}{3}m_1 l_1^2 + m_2 l_1^2 + \frac{2}{3}m_2 l_2^2 + m_2 l_1 l_2 c_2 & \frac{2}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 c_2 \\ \frac{2}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 c_2 & \frac{2}{3}m_2 l_2^2 \end{bmatrix}$$

Centrifugal and Coriolis vector

$$V(\theta, \dot{\theta}) = \begin{bmatrix} -\frac{1}{2}m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ \frac{1}{2}m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

Gravity Term

$$G(\theta) = \begin{bmatrix} \frac{1}{2}m_1 g l_1 c_1 + m_2 g l_1 c_1 + \frac{1}{2}m_2 g l_2 c_{12} \\ \frac{1}{2}m_2 g l_2 c_{12} \end{bmatrix}$$