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**EXAMINATION IN MATHEMATICS**
MAA153 Linear Algebra
*Date:* 2017-01-09 *Write time:* 5 hours
*Aid:* Writing materials, ruler

This examination consists of eight RANDOMLY ORDERED problems each of which is worth at maximum 5 points. The maximum sum of points is thus 40. The PASS-marks 3, 4 and 5 require a minimum of 18, 26 and 34 points respectively. The minimum points for the ECTS-marks E, D, C, B and A are 18, 20, 26, 33 and 38 respectively. Solutions are supposed to include rigorous justifications and clear answers. All sheets with solutions must be sorted in the order the problems are given in. Especially, avoid to write on back pages of solution sheets.

**1.** Let $\mathcal{M}$ denote the vector space of all $2 \times 2$ matrices with real-valued entries and with the trace (the sum of the diagonal entries) equal to zero. Find, relative to the standard ordered basis for $\mathbb{R}^3$ and the ordered basis $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ for $\mathcal{M}$, the matrix of the linear transformation $F : \mathbb{R}^3 \to \mathcal{M}$ defined as

$$F((x_1, x_2, x_3)) = \begin{pmatrix} x_1 + x_2 + x_3 & 2x_1 + x_2 + 3x_3 \\ x_1 - 2x_2 + 4x_3 & -x_1 - x_2 - x_3 \end{pmatrix}.$$

Also, find out whether $F$ is bijective or not.

**2.** Let $\mathbb{E}$ be the vector space $\mathbb{R}^3$ equipped with the inner product

$$\langle \mathbf{u} | \mathbf{v} \rangle = x_1 y_1 + 2x_2 y_2 + x_3 y_3 - (x_2 y_3 + x_3 y_2),$$

where $x_1, x_2, x_3$ and $y_1, y_2, y_3$ are the coordinates of $\mathbf{u}$ and $\mathbf{v}$ respectively in the (ordered) basis $(1,0,0), (0,1,0), (0,0,1)$. Find the orthogonal projection of the vector $(1,2,1)$ on the subspace $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\} \subset \mathbb{E}$.

**3.** The linear operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ has relative to the standard basis the matrix

$$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 0 \\ \beta & 2 & \beta \end{pmatrix}$$

where $\beta \in \mathbb{R}$. Find the numbers $\beta$ for which the operator är diagonalizable, and state a basis of eigenvectors for each of these $\beta$.

**4.** Find a basis for the subspace $Q = \left\{p \in \mathcal{P}_3 : p(1) = p(2) = \frac{1}{3}(p(3) - p(0))\right\}$ of the vector space $\mathcal{P}_3 = \operatorname{span}\{p_0, p_1, p_2, p_3\}$ where $p_0(x) = 1$ and $p_n(x) = x^n$, $n \geq 1$. Then find, with respect to the chosen basis, the coordinates of the polynomial function $18p_0 - (a+10)p_1 - 3p_2 + (2a-1)p_3$ for those $a$ for which the polynomial function belongs to $Q$.

**5.** The linear transformation $F : \mathbb{R}^5 \to \mathbb{R}^3$ is defined by

$$F(u) = (3a - 6b + c + 2d - 6e, -2a + 4b + 3c - 5d - 7e, 4a - 8b - c + 5d - e)$$

where $u = (a, b, c, d, e)$. Find the kernel and the image of $F$, and express them as linear spans generated by a set of basis vectors for each space respectively. Also, specify explicitly the dimensions of the two vector spaces.

**6.** Let $e_1, e_2, e_3$ be a basis for the vector space $L$, and introduce the vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ according to $\tilde{e}_1 = e_1 + 2e_2 - e_3$, $\tilde{e}_2 = 3e_2 + 2e_3$, $\tilde{e}_3 = e_1 + 6e_2 + 2e_3$. Prove that also $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a basis for $L$, and find the coordinates of the vector $e_1 + 2e_2 - 3e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

**7.** The intersection between the surface $5x^2 - y^2 + 5z^2 + 10xy + 10yz - 2zx = 1$ and a plane including the origin is a curve. Of all such curves, find the half-axes lengths of the ellipse whose enclosed plane region has the smallest possible area. It is assumed that $x, y, z$ denote the coordinates of a point in an orthonormal system.

**8.** Let $M = \{(a, b, c) \in \mathbb{E}^3 : 2a + 5b + 6c = 0, 3a + 4b + 2c = 0, -a + 3b + 8c = 0\}$. Find an orthonormal basis for the orthogonal complement to $M$ in $\mathbb{E}^3$.

**Om du föredrar uppgifterna formulerade på svenska, var god vänd på bladet.**

Denna tentamen består av åtta stycken om varannat SLUMPMÄSSIGT ORDNADE uppgifter som vardera kan ge maximalt 5 poäng. Den maximalt möjliga poängsumman är således 40. För GODKÄND-betygen 3, 4 och 5 krävs minst 18, 26 respektive 34 poäng. För ECTS-betygen E, D, C, B och A krävs 18, 20, 26, 33 respektive 38 poäng. Lösningar förutsätts innefatta ordentliga motiveringar och tydliga svar. Samtliga lösningsblad skall vid inlämning vara sorterade i den ordning som uppgifterna är givna i. Undvik speciellt att skriva på baksidor av lösningsblad.

**1.** Låt $\mathcal{M}$ beteckna det linjära rummet av alla matriser av typ $2\times 2$ med reellvärda element och med spåret (summan av diagonalelementen) lika med noll. Bestäm, med avseende på den ordnade standardbasen för $\mathbb{R}^3$ och den ordnade basen $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ för $\mathcal{M}$, matrisen för den linjära avbildningen $F:\mathbb{R}^3\to\mathcal{M}$ definierad enligt

$$F((x_1,x_2,x_3)) = \begin{pmatrix} x_1+x_2+x_3 & 2x_1+x_2+3x_3 \\ x_1-2x_2+4x_3 & -x_1-x_2-x_3 \end{pmatrix}.$$

Utred även om $F$ är bijektiv eller inte?

**2.** Låt $\mathbb{E}$ vara det det linjära rummet $\mathbb{R}^3$ utrustat med skalärprodukten

$$\langle \mathbf{u}|\mathbf{v}\rangle = x_1y_1 + 2x_2y_2 + x_3y_3 - (x_2y_3 + x_3y_2)\,,$$

där $x_1,x_2,x_3$ och $y_1,y_2,y_3$ är koordinaterna för $\mathbf{u}$ respektive $\mathbf{v}$ i (den ordnade) basen $(1,0,0),(0,1,0),(0,0,1)$. Bestäm den ortogonala projektionen av vektorn $(1,2,1)$ på underrummet $M=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1+x_2+x_3=0\}\subset\mathbb{E}$.

**3.** Den linjära operatorn $F:\mathbb{R}^3\to\mathbb{R}^3$ har relativt standardbasen matrisen

$$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 0 \\ \beta & 2 & \beta \end{pmatrix}$$

där $\beta\in\mathbb{R}$. Bestäm de tal $\beta$ för vilka operatorn är diagonaliserbar, och ange en bas av egenvektorer till $F$ för var och en av dessa $\beta$.

**4.** Bestäm en bas för underrummet $Q=\left\{p\in\mathcal{P}_3 : p(1)=p(2)=\frac{1}{3}(p(3)-p(0))\right\}$ till det linjära rummet $\mathcal{P}_3=\operatorname{span}\{p_0,p_1,p_2,p_3\}$ där $p_0(x)=1$ och $p_n(x)=x^n$, $n\geq 1$. Bestäm sedan, med avseende på den valda basen, koordinaterna för polynomfunktionen $18p_0-(a+10)p_1-3p_2+(2a-1)p_3$ för de $a$ för vilka polynomfunktionen tillhör $Q$.

**5.** Den linjära avbildningen $F:\mathbb{R}^5\to\mathbb{R}^3$ är definierad genom

$$F(u) = (3a-6b+c+2d-6e, -2a+4b+3c-5d-7e, 4a-8b-c+5d-e)$$

där $u=(a,b,c,d,e)$. Bestäm $F$:s nollrum och värderum, och uttryck dem som linjära höljen genererade av en uppsättning basvektorer för respektive rum. Ange även explicit dimensionerna av de två linjära rummen.

**6.** Låt $e_1,e_2,e_3$ vara en bas för det linjära rummet $L$, och introducera vektorerna $\tilde{e}_1,\tilde{e}_2,\tilde{e}_3$ enligt $\tilde{e}_1=e_1+2e_2-e_3$, $\tilde{e}_2=3e_2+2e_3$, $\tilde{e}_3=e_1+6e_2+2e_3$. Bevisa att även $\tilde{e}_1,\tilde{e}_2,\tilde{e}_3$ är en bas för $L$, och bestäm koordinaterna för vektorn $e_1+2e_2-3e_3$ relativt basen $\tilde{e}_1,\tilde{e}_2,\tilde{e}_3$.

**7.** Skärningen mellan ytan $5x^2-y^2+5z^2+10xy+10yz-2zx=1$ och ett plan inkluderande origo är en kurva. Av alla sådana skärningskurvor, bestäm halvaxellängderna hos den ellips vars inneslutna plana område har minsta möjliga area. Det antages att $x,y,z$ betecknar en punkts koordinater i ett ON-system.

**8.** Låt $M=\{(a,b,c)\in\mathbb{E}^3 : 2a+5b+6c=0, 3a+4b+2c=0, -a+3b+8c=0\}$. Bestäm en ON-bas för det ortogonala komplementet till $M$ i $\mathbb{E}^3$.

**If you prefer the problems formulated in English, please turn the page.**

# MAA153 / Solutions to the final exam 2017-01-09

**(1)** $F: \mathbb{R}^3 \to \mathcal{M}$ is given by $F((x_1,x_2,x_3)) = \begin{pmatrix} x_1+x_2+x_3 & 2x_1+x_2+3x_3 \\ x_1-2x_2+4x_3 & -x_1-x_2-x_3 \end{pmatrix}$

Let $\begin{cases} (1,0,0) = e_1 \\ (0,1,0) = e_2 \\ (0,0,1) = e_3 \end{cases}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = m_1$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = m_2$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = m_3$

Then $\begin{cases} F(e_1) = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = m_1 + m_2 + 2m_3 \\ F(e_2) = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = -2m_1 + m_2 + m_3 \\ F(e_3) = \begin{pmatrix} 1 & 3 \\ 4 & -1 \end{pmatrix} = 4m_1 + m_2 + 3m_3 \end{cases}$

and thus $\begin{pmatrix} F(e_1) & F(e_2) & F(e_3) \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

i.e. $F$ has the matrix $\begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} = A$ relative to the bases $e_1, e_2, e_3$ and $m_1, m_2, m_3$ of $\mathbb{R}^3$ and $\mathcal{M}$ respectively.

Since $A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & -3 \\ 0 & 5 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, we get that

$\operatorname{rank}(A) = 2 < \dim(\mathcal{M})$, implying that $\operatorname{im}(F)$ is a proper subset of $\mathcal{M}$ whereby $F$ is not bijective.

**(2)** $\langle u|v \rangle = x_1y_1 + 2x_2y_2 + x_3y_3 - (x_2y_3 + x_3y_2) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

where $x_1, x_2, x_3$ and $y_1, y_2, y_3$ are the coordinates of $u$ and $v$ respectively with respect to the (ordered) basis $(1,0,0), (0,1,0), (0,0,1)$.

$M = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1+x_2+x_3 = 0\} = \operatorname{span}\{\underbrace{(1,0,-1)}_{u_1}, \underbrace{(0,1,-1)}_{u_2}\} \subset E$

In order to find $\operatorname{proj}_M((1,2,1))$, we begin by constructing an ON-basis $e_1, e_2$ for $M$. The Gram-Schmidt orthog. procedure gives

$\|u_1\|^2 = \|(1,0,-1)\|^2 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2$

i.e. $e_1 = \frac{1}{\sqrt{2}} u_1 = \frac{1}{\sqrt{2}}(1,0,-1)$ ← reused result

$f_2 = u_2 - \langle u_2|e_1 \rangle e_1 = (0,1,-1) - \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} (1,0,-1) = (0,1,-1) - (1,0,-1) = (-1,1,0)$

where $\|f_2\|^2 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = 3$ i.e. $e_2 = \frac{1}{\sqrt{3}}(-1,1,0)$

Then $\operatorname{proj}_M((1,2,1)) = \langle (1,2,1)|e_1 \rangle e_1 + \langle (1,2,1)|e_2 \rangle e_2$

$= \frac{1}{2}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} (1,0,-1) + \frac{1}{3}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} (-1,1,0)$

$= (1,0,-1) + \frac{2}{3}(-1,1,0) = \frac{1}{3}(1,2,-3)$

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(3) $F: \mathbb{R}^3 \to \mathbb{R}^3$ has the matrix $A = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 0 \\ \beta & 2 & \beta \end{pmatrix}$, $\beta \in \mathbb{R}$, relative to the standard basis for $\mathbb{R}^3$.

Eigenvalues: $0 = \det(A-\lambda I) = \det \begin{pmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & 0 \\ \beta & 2 & \beta-\lambda \end{pmatrix} = (3-\lambda)\det \begin{pmatrix} 5+\beta-\lambda & 0 & 5+\beta-\lambda \\ 0 & 1 & 0 \\ \beta & 2 & \beta-\lambda \end{pmatrix}$

$$= (3-\lambda)(5+\beta-\lambda)\det\begin{pmatrix} 1 & 1 \\ \beta & \beta-\lambda \end{pmatrix} = -\lambda(\lambda-3)(\lambda-5-\beta)$$

F is definitely diagonalizable if all eigenvalues are distinct, i.e. if $\beta \neq -5, -2$. For those $\beta$, we get

$\lambda_1 = 0$: $A-\lambda_1 I = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & 0 \\ \beta & 2 & \beta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Eigenvectors are $t_1(1,0,-1)$, $t_1 \neq 0$

$\lambda_2 = 3$: $A-\lambda_2 I = \begin{pmatrix} 2 & -2 & 5 \\ 0 & 0 & 0 \\ \beta & 2 & \beta-3 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 5 \\ \beta+2 & 0 & \beta+2 \\ 0 & 0 & 0 \end{pmatrix} \overset{\beta\neq-2}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ Eigenvectors are $t_2(-2,3,2)$, $t_2 \neq 0$

$\lambda_3 = 5+\beta$: $A-\lambda_3 I = \begin{pmatrix} -\beta & -2 & 5 \\ 0 & -2-\beta & 0 \\ \beta & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} \beta & 2 & -5 \\ 0 & \beta+2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{\beta\neq-2}{\sim} \begin{pmatrix} \beta & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Eigenvectors are $t_3(5,0,\beta)$, $t_3 \neq 0$

If $\beta = -5$ then $\lambda_1 = \lambda_3$ but $\operatorname{rank}(A-\lambda_{1,3}I) = 2$ which means that $\dim(\text{Eigenspace}_0) = 1 < 2 =$ 'multiplicity of eigenvalue 0', i.e. F is not diagonalizable. If $\beta = -2$ then $\lambda_2 = \lambda_3$ and $\operatorname{rank}(A-\lambda_{2,3}I) = 1$ which means that $\dim(\text{Eigenspace}_3) = 2 =$ 'multiplicity of eigenvalue 3', i.e. F is diagonalizable.

Answer: F is diagonalizable iff $\beta \neq -5$, and a basis of eigenvectors is then e.g. $(1,0,-1)$, $(-2,3,2)$, $(5,0,\beta)$.

(4) $Q = \{p \in P_3 : p(1) = p(2) = \frac{1}{3}(p(3) - p(0))\}$

If $p \in P_3$ then $p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ where $c_0, c_1, c_2, c_3 \in \mathbb{R}$

If $p \in Q$ then $p \in P_3$ and $\begin{cases} c_0+c_1+c_2+c_3 = c_0+2c_1+4c_2+8c_3 \\ c_0+c_1+c_2+c_3 = \frac{1}{3}(c_0+3c_1+9c_2+27c_3-c_0) \end{cases}$

i.e. $p \in P_3$ and $\begin{cases} c_1+3c_2+7c_3 = 0 \\ c_0-2c_2-8c_3 = 0 \end{cases}$

i.e. $p_Q = (2c_2+8c_3)p_0 + (-3c_2-7c_3)p_1 + c_2 p_2 + c_3 p_3 = c_2(2p_0-3p_1+p_2) + c_3(8p_0-7p_1+p_3)$

i.e. a basis for Q is e.g. $\overbrace{2p_0-3p_1+p_2}^{q_1}$, $\overbrace{8p_0-7p_1+p_3}^{q_2}$

Furthermore, the polynomial function $18p_0-(a+10)p_1-3p_2+(2a-1)p_3$ iff it is a linear combination of $q_1$ and $q_2$, i.e. iff

$18p_0-(a+10)p_1-3p_2+(2a-1)p_3 = \alpha_1 q_1 + \alpha_2 q_2 = \alpha_1(2p_0-3p_1+p_2) + \alpha_2(8p_0-7p_1+p_3)$

$\Leftrightarrow \begin{pmatrix} 2 & 8 \\ -3 & -7 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 18 \\ -a-10 \\ -3 \\ 2a-1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2a-1 \\ 13a-26 \\ 32-16a \end{pmatrix}$ which has a root $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ iff $a = 2$

i.e. $18p_0-(a+10)p_1-3p_2+(2a-1)p_3 \in Q$ iff $a = 2$ and the coordinates relative to $q_1, q_2$ are then $-3, 3$.

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(5) $F: \mathbb{R}^5 \to \mathbb{R}^3$ where

$$F(a,b,c,d,e) = (3a-6b+c+2d-6e,\ -2a+4b+3c-5d-7e,\ 4a-8b-c+5d-e)$$

The matrix $A$ of $F$ relative to the standard bases is

$$A = \begin{pmatrix} 3 & -6 & 1 & 2 & -6 \\ -2 & 4 & 3 & -5 & -7 \\ 4 & -8 & -1 & 5 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & -3 & -13 \\ -2 & 4 & 3 & -5 & -7 \\ 0 & 0 & 5 & -5 & -15 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 4 & -3 & -13 \\ 0 & 0 & 11 & -11 & -33 \\ 0 & 0 & 1 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\begin{cases} \ker(F) = \operatorname{span}\{(2,1,0,0,0), (-1,0,1,1,0), (1,0,3,0,1)\} \\ \operatorname{im}(F) = \operatorname{span}\{(3,-2,4), (1,3,-1)\} \end{cases}$

and $\begin{cases} \dim(\ker(F)) = 5 - \operatorname{rank}(A) = 5-2 = 3 \\ \dim(\operatorname{im}(F)) = \operatorname{rank}(A) = 2 \end{cases}$

(6) $e_1, e_2, e_3$ is a basis for $L$ and $\begin{cases} \tilde{e}_1 = e_1 + 2e_2 - e_3 \\ \tilde{e}_2 = 3e_2 + 2e_3 \\ \tilde{e}_3 = e_1 + 6e_2 + 2e_3 \end{cases}$

We have $(\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3) = (e_1\ e_2\ e_3)\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 6 \\ -1 & 2 & 2 \end{pmatrix}$ i.e. $\tilde{e} = eS$

where $S \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

i.e. $\operatorname{rank}(S) = 3$ i.e. $S$ is invertible

i.e. $S$ is a change-of-basis matrix

i.e. $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a basis q.e.d.

Moreover $e_1 + 2e_2 - 3e_3 = (e_1\ e_2\ e_3)\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3) S^{-1}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$$= (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 6 \\ -1 & 2 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$= (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\frac{1}{(6+0+4)-(-3+12+0)}\begin{pmatrix} -6 & 2 & -3 \\ -10 & 3 & -4 \\ 7 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$= (\tilde{e}_1\ \tilde{e}_2\ \tilde{e}_3)\frac{1}{10-9}\begin{pmatrix} -6+4+9 \\ -10+6+12 \\ 7-4-9 \end{pmatrix} = 7\tilde{e}_1 + 8\tilde{e}_2 - 6\tilde{e}_3$$

i.e. $\operatorname{coord}_{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3}(e_1 + 2e_2 - 3e_3) = (7, 8, -6)$

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**7** $1 = 5x^2 - y^2 + 5z^2 + 10xy + 10yz - 2zx = (x\ y\ z)\begin{pmatrix}5 & 5 & -1\\ 5 & -1 & 5\\ -1 & 5 & 5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

$= X^T G X$ where $x, y, z$ are coordinates in an ON-system.

The eigenvalues of the symmetric operator which has the matrix $G$ are given by $0 = \det(G - \lambda I) = \det\begin{pmatrix}5-\lambda & 5 & -1\\ 5 & -1-\lambda & 5\\ -1 & 5 & 5-\lambda\end{pmatrix}$

$= \det\begin{pmatrix}6-\lambda & 0 & \lambda-6\\ 5 & -1-\lambda & 5\\ -1 & 5 & 5-\lambda\end{pmatrix} = \det\begin{pmatrix}6-\lambda & 0 & 0\\ 5 & -1-\lambda & 10\\ -1 & 5 & 4-\lambda\end{pmatrix} = (6-\lambda)\left[(\lambda+1)(\lambda-4) - 50\right]$

$= -(\lambda-6)(\lambda^2 - 3\lambda - 54) = -(\lambda-6)(\lambda-9)(\lambda+6) = -(\lambda+6)(\lambda-6)(\lambda-9)$

$\underline{\lambda_1 = -6}$: $A - \lambda_1 I = \begin{pmatrix}11 & 5 & -1\\ 5 & 5 & 5\\ -1 & 5 & 11\end{pmatrix} \sim \begin{pmatrix}0 & 0 & 0\\ 1 & 1 & 1\\ 0 & 6 & 12\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$ Eigenvectors are $t_1(1,-2,1)$, $t_1 \neq 0$

$\underline{\lambda_2 = 6}$: $A - \lambda_2 I = \begin{pmatrix}-1 & 5 & -1\\ 5 & -7 & 5\\ -1 & 5 & -1\end{pmatrix} \sim \begin{pmatrix}1 & -5 & 1\\ 0 & 18 & 0\\ 0 & 0 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$ Eigenvectors are $t_2(1,0,-1)$, $t_2 \neq 0$

$\underline{\lambda_3 = 9}$: $A - \lambda_3 I = \begin{pmatrix}-4 & 5 & -1\\ 5 & -10 & 5\\ -1 & 5 & -4\end{pmatrix} \sim \begin{pmatrix}0 & -15 & 15\\ 0 & 15 & -15\\ 1 & -5 & 4\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}$ Eigenvectors are $t_3(1,1,1)$, $t_3 \neq 0$

An ON-basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of eigenvectors is then e.g. $\begin{cases}\tilde{e}_1 = \frac{1}{\sqrt{6}}(e_1 - 2e_2 + e_3)\\ \tilde{e}_2 = \frac{1}{\sqrt{2}}(e_1 - e_3)\\ \tilde{e}_3 = \frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)\end{cases}$
where $e_1, e_2, e_3$ is the original ON-basis.

$\underline{\text{Thus}}$ $1 = X^T G X = (S\tilde{X})^T (S\tilde{G}S^{-1})(S\tilde{X}) = \tilde{X}^T \tilde{G}\tilde{X} = -6\tilde{x}^2 + 6\tilde{y}^2 + 9\tilde{z}^2$

$\underline{\text{where}}$ $S = \begin{pmatrix}1/\sqrt{6} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ -2/\sqrt{6} & 0 & \frac{1}{\sqrt{3}}\\ 1/\sqrt{6} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\end{pmatrix}$ is the orthogonal change-of-basis matrix from $e_1, e_2, e_3$ to $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$.

The intersection between the surface (a one-sheeted hyperboloid) and a plane including the origin is either a hyperbola (if the plane is parallel with the $\tilde{x}$-axis) or an ellipse. The ellipse which encloses the plane region with the smallest possible area is given by the intersection of the surface with the plane $\tilde{x} = 0$, and is thus given by the equations $1 = 6\tilde{y}^2 + 9\tilde{z}^2 = \left(\frac{\tilde{y}}{1/\sqrt{6}}\right)^2 + \left(\frac{\tilde{z}}{1/3}\right)^2$, $\tilde{x} = 0$.

The half-axes lengths are $\frac{1}{\sqrt{6}}$ and $\frac{1}{3}$.

**8** $M = \left\{(a,b,c) \in E^3 : \begin{matrix}2a + 5b + 6c = 0\\ 3a + 4b + 2c = 0\\ -a + 3b + 8c = 0\end{matrix}\right\} = \left\{(a,b,c) \in E^3 : \begin{matrix}a - 3b - 8c = 0\\ 11b + 22c = 0\\ 13b + 26c = 0\end{matrix}\right\}$

$= \left\{(a,b,c) \in E^3 : a - 2c = 0,\ b + 2c = 0\right\} = \left\{(a,b,c) \in E^3 : \begin{matrix}\langle (a,b,c) \mid (1,0,-2)\rangle = 0\\ \langle (a,b,c) \mid (0,1,2)\rangle = 0\end{matrix}\right\}$

$\underline{\text{i.e.}}$ $M^\perp$ is spanned by $v_1 = (1,0,-2)$ and $v_2 = (0,1,2)$.

The Gram-Schmidt orthogonalization procedure gives

$e_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{5}}(1,0,-2)$

$f_2 = v_2 - \langle v_2 \mid e_1\rangle e_1 = (0,1,2) - \frac{1}{5}(0+0-4)(1,0,-2) = \frac{1}{5}\left[(0,5,10) + (4,0,-8)\right]$

$= \frac{1}{5}(4,5,2)$ and $e_2 = \frac{1}{\sqrt{45}}(4,5,2) = \frac{1}{3\sqrt{5}}(4,5,2)$

Thus an ON-basis for the orthogonal complement to $M$ in $E^3$ is e.g. $\frac{1}{\sqrt{5}}(1,0,-2)$, $\frac{1}{3\sqrt{5}}(4,5,2)$

MÄLARDALEN UNIVERSITY
School of Education, Culture and Communication
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**EXAMINATION IN MATHEMATICS**
MAA153 Linear algebra
EVALUATION PRINCIPLES with POINT RANGES
*Academic year*: 2016/17

| **Examination 2017-01-09** | | Maximum points for subparts of the problems in the final examination |
|---|---|---|
| **1.** | The matrix of $F$ relative to the ordered bases for the domain and the codomain is equal to $\begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$. $F$ is not bijective. | **4p**: Correctly identified the matrix of $F$ relative to the ordered bases for the domain $R^3$ and the codomain $\mathcal{M}$<br>**1p**: Correctly concluded that $F$ is not bijective (due to the fact that the rank of its matrix is less than the dimension of the codomain $\mathcal{M}$) |
| **2.** | $\text{proj}_M((1,2,1)) = \frac{1}{3}(1,2,-3)$ | **1p**: Correctly found a basis for the subspace $M$<br>**2p**: Correctly found an orthogonal basis for $M$ (with the purpose of finding the orthogonal projection $\text{proj}_M((1,2,1))$)<br>**2p**: Correctly found the orthogonal projection of the vector $(1,2,1)$ on $M$ |
| **3.** | The linear operator $F$ is diagonalizable iff $\beta \neq -5$, and a basis of eigenvectors is then e.g. $(1,0,-1), (-2,3,2), (5,0,\beta)$ | **1p**: Correctly found that the linear operator (LO) is definitely diagonalizable if $\beta \neq -5,-2$<br>**1p**: Correctly found that the LO is diagonalizable if $\beta = -2$<br>**1p**: Correctly found that the LO is not diagonaliz. if $\beta = -5$<br>**2p**: Correctly for $\beta \neq -5$ found a basis of eigenvectors |
| **4.** | A basis for $Q$ is e.g. $q_1, q_2$ where $\begin{cases} q_1 = 2p_0 - 3p_1 + p_2 \\ q_2 = 8p_0 - 7p_1 + p_3 \end{cases}$ The polynomial function belong to $Q$ iff $a = 2$. The coordinates relative to the basis $q_1, q_2$ are $-3, 3$ | **3p**: Correctly found a basis for $Q$<br>**1p**: Correctly found that polynomial function $18p_0 - (a+10)p_1 - 3p_2 + (2a-1)p_3 \in Q$ iff $a = 2$<br>**1p**: Correctly found the coordinates of polynomial function $18p_0 - (a+10)p_1 - 3p_2 + (2a-1)p_3$ relative to a chosen basis for $Q$ |
| **5.** | $\ker(F) = \text{span}\{(2,1,0,0,0), (-1,0,1,1,0), (1,0,3,0,1)\}$<br>$\text{im}(F) = \text{span}\{(3,-2,4), (1,3,-1)\}$<br>$\dim(\ker(F)) = 3$, $\dim(\text{im}(F)) = 2$ | **3p**: Correctly found the kernel of $F$ expressed as a linear span<br>**1p**: Correctly found the image of $F$ expressed as a linear span<br>**1p**: Correctly found the dimensions of $\ker(F)$ and $\text{im}(F)$ |
| **6.** | Proof.<br>The coordinates of $e_1 + 2e_2 - 3e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are $7, 8, -6$ | **1p**: Correctly found the matrix relating $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ to $e_1, e_2, e_3$<br>**1p**: Correctly proved that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a basis for $L$<br>**3p**: Correctly found the coordinates of $e_1 + 2e_2 - 3e_3$ relative to the basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ |
| **7.** | The half-axes lengths of the intersection ellipse enclosing a plane region of smallest possible area are $\frac{1}{\sqrt{6}}$ and $\frac{1}{3}$ | **2p**: Correctly found that the equation is $1 = -6\tilde{x}^2 + 6\tilde{y}^2 + 9\tilde{z}^2$ if the coordinates are given relative to an ON-basis of eigenvectors of an implicit symmetric operator<br>**2p**: Correctly found that the intersection between the surface (a one-sheeted hyperboloid) and the plane $\tilde{x} = 0$ is the ellipse which encloses a plane region with the smallest possible area<br>**1p**: Correctly found the half-axes lengths of the intersection |
| **8.** | An ON-basis for the orthogonal complement to $M$ in $E^3$ is e.g. $\frac{1}{\sqrt{5}}(1,0,-2)$, $\frac{1}{3\sqrt{5}}(4,5,2)$ | **3p**: Correctly found a basis for the orthogonal complement to $M$ in $E^3$<br>**2p**: Correctly found an ON-basis for the orthogonal complement to $M$ in $E^3$ |