Computational Nonlinear Dynamics (MTH3039) Coursework 1
SIR model of infectious disease

This coursework consists of four questions. It is worth 35% of the credits for this unit. The maximum number of marks for this coursework is 100.

Deadline

3 November 2020 (Tuesday), 11:59am

This is an **individual coursework** and your attention is drawn to the College and University guidelines on collaboration and plagiarism, which are available from the College website.

Submitting your work

The coursework requires both paper (via BART) and electronic (via `http://empslocal.ex.ac.uk/submit/`) submission.

Material to be handed in

- (**Paper**) Document containing graphs and solutions with necessary headings, equations, annotations, and comments (no essay!),
- (**Paper**) Printout of program codes with comments.
- (**Electronic**) Upload four scripts `cw11.m`, `cw12.m`, `cw13.m` and `cw14.m` and all functions and scripts (also `m`-files) necessary to run the scripts `cw11.m`–`cw14.m` for this assignment. When I run the scripts they should recover all graphs and numerical outputs of your document. You may also include `mat`-files with pre-computed results. (This would make it possible for me, for example to check Q4b without having to run the code for Q4a first.) However, your scripts have to recover *all* results and graphs.

Detailed rules for electronic submission You should submit all files for this coursework via the electronic submission system at `http://empslocal.ex.ac.uk/submit/`. Make sure that your code is in files with the names specified in the questions and that all `m`-files that you need are in the current folder. On the `Matlab` commandline you can then type (for the example of coursework 1)

```
>>zip('cw1.zip',{'*.m','*.mat'});
```

(see help page for `zip`). This creates a zip file `cw1.zip` which contains all `m`-files and `mat`-files in the current folder. Then upload the `zip` file using the submit system.
You will be sent an email by the submit system asking you to confirm your submission by following a link. Your submission is not confirmed until you do this. It is best to do it straightaway, but there is a few hours leeway after the deadline has passed. It is possible to unsubmit and resubmit electronic coursework — follow the instructions on the submission website.

Marking criteria for the coding parts

Credit for the coding part of each question (see list of possible reasons for additional point deductions below):

100% Code performs computation correctly and efficiently and has comments that make it easy to understand;

80%–100% Code performs all computations correctly but has minor problems with its structure (see below);

60%–80% Code performs most computations correctly;

40%–60% Code does not perform computations correctly but could be made to work with minor corrections;

$\leq$ 40% Code does not perform computations correctly, and requires substantial changes to fix.

For example, points will be deducted (for bad structure) for:

- missing semicolons in functions (the function echoes its calculations);
- poorly structured code, for example;
  - some statements are unreachable,
  - variables are introduced and assigned but never used,
  - stray brackets, misleading variable names, variable usages (say, using `x(:)` if `x` is scalar), or function usages (for example, using `MySolve` for a system with too few or too many variables, or to find a root that has a singular Jacobian).
  - hard-coded "magic" numbers spread throughout the code,
  - one part of the code is a repetition of another part.
- Misspelled main file names (see each question) incur a 10% penalty.
- Not submitting hardcopies of your programs incurs a 10% penalty.

Note that your main script files for each question and other function files are permitted to call other functions that you have written yourself (for example, if you need to perform the same computation in different places). You need to upload these additional functions, too. You can freely choose their names.

You may also use functions that you did not write yourself and were not provided by me (e.g., from another student, or from the internet)

> **In this case you have to declare at the top of the paper submission and at the top of the relevant file(s) that you borrowed this file, and state the source. You will not get points for code that you borrowed.**

Some utility functions may be provided on ELE to help with certain aspects of the coursework. If you use them, you have to include them into your electronic submission. For questions, clarifications and further help contact (see office hours on ELE):

Dr James Rankin (`j.a.rankin@exeter.ac.uk`).

SIR infectious disease model

The following set of nonlinear differential equations is a simplified *SIR*-model of infectious disease spread, which describe the proportion of a population that are susceptible to (S), infected by (I) and recovered from (R) a particular infectious disease. You don't need to understand the biology behind the model to complete the coursework, apart from the fact that depending on a free parameter β the equations exhibit various steady state and periodic behaviours. The number of coexisting steady states and periodic orbits, and their stability, will change as β varies. A pair of nonlinear equations describe the time evolution of the proportion of the population that are infected I and a recovered R:

$$\begin{aligned}\dot{I} &= \beta I S(I,R) - (\mu+\sigma)I - \eta F(I,\gamma),\\ \dot{R} &= \mu I - (\nu+\sigma)R + \eta F(I,\gamma),\end{aligned} \tag{1}$$

where the susceptible population $S \in [0,1]$ is the proportion that are not infectious and not recovered $S(I,R) = 1 - I - R$. The infection rate is β, equal birth/death rate σ, recovery rate μ, decay of resistance rate ν, treatment rate η and the treatment capacity γ. The treatment saturation is controlled by the following function

$$F(I,\gamma) = \gamma\left(\frac{2}{1+\exp(-2I/\gamma)} - 1\right), \tag{2}$$

a smooth approximation of $F_{min}(I,\gamma) = \min(I,\gamma)$.

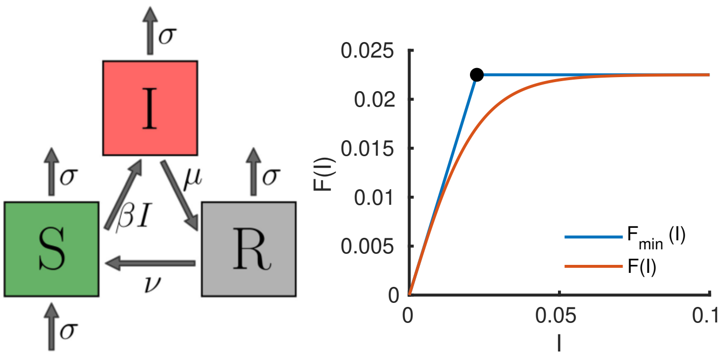

Figure 1: **(left)** Schematic of interacting S, I and R populations. **(right)** Smooth approximation $F(I)$ (red) of the nonlinear treatment saturation function (blue). The min-function used in the original study is discontinuous in its first derivative at (γ,γ) (black dot), see Vyska & Gilligan (2016) *Complex Dynamical Behaviour in an Epidemic Model with Control*

The nullclines for (1) can be written as the R-values expressed in terms of I:

$$\begin{aligned}R(I) &= (1-I) - \frac{(\mu+\sigma)}{\beta} - \frac{\eta F(I,\gamma)}{\beta I} \quad (I\text{-nullcline}),\\ R(I) &= \frac{\mu I + \eta F(I,\gamma)}{\nu+\sigma} \quad (R\text{-nullcline}).\end{aligned} \tag{3}$$

The model is implemented for you in the provided `Matlab` script `sir_model.m`, which contains the right-hand side for the ODE (1) as a function `rhs`. Most parameters of the system are kept fixed. **You obtain your personal value** for σ by entering your Student ID No. (see your student card) into the variable `SNumber` in `sir_model.m`. You should include this script into your solution files and upload it with your submission. The parameter β will be varied in the range $[3.5, 6]$ in this study.

Question 1: Location of equilibria and their stability [submit as `cw11.m`]

(a) Implement the functions `MyJacobian`, `MySolve` and `MyTrackCurve`, which are needed for this and several following questions. Detailed instructions for their interface and hints for their implementation are given in separate instruction sheets on ELE. [20 marks]

(b) **(Equilibria)** Find all equilibria with $I > 0$ of the system (1) for parameters $\beta \in [3.5, 6]$. Plot the curves of equilibria (the *bifurcation diagram*) in the (β, I)-plane. [5 marks]

Hint: find an equilibrium at $\beta = 6$ and follow the branch down in β. Find the starting equilibrium with `MySolve` and an initial guess for (I, R) such that $0 < S(I, R) < 1$.

(c) **(Stability)** Indicate the type of each equilibrium along the curves you obtained in part (b). For example, use a dot for equilibria that are stable (sinks), a cross for equilibria that have one unstable eigenvalue (saddles), a square for equilibria that have two unstable eigenvalues (sources). You may also use different colours instead of/as well as different symbols. [5 marks]

Total for Question 1: 30 marks

Hints:

- On ELE under `Useful Functions` you will find Matlab functions that are potentially useful, and that you can call as part of your own scripts and functions after downloading them. Beware that they are written by a colleague and only provided for your convenience. This means that they may not give meaningful error messages if you call them with inconsistent arguments. Report difficulties to me.
- In order for your answer to each question to be self-contained (`cw11.m`, `cw12.m`, …, etc) you may need to save specific variables and data structures (e.g. you might save the curve of equilibria `ylist` from Q1 and re-use it in later questions). You can include these with your submission, but your scripts must be able to recover all results and graphs from scratch.

Question 2: Phase portraits [submit as `cw12.m`]

(a) Implement the general-purpose function `MyIVP`, which is needed for this and several following questions. Detailed instructions for its interface and hints for its implementation are given separately on ELE. [5 marks]

(b) Plot all qualitatively different robust phase portraits. For each phase portrait:

- plot nullclines (requires definition of new functions based on (3))
- plot all equilibria, marking them differently, depending on their type (sink, source or saddle);
- plot any periodic orbits;
- for each sink, source and periodic orbit include a trajectory that approaches the sink/source/periodic orbit forward or backward in time;
- for each saddle include all four separatrices.

Based on the stability computations from Q1, find five parameter regions for β with qualitatively different phase portraits (further distinct regions can be identified after progress with Q4). Restrict your phase portraits to an appropriate rectangle (e.g., $I \in [0, 0.3]$, $R \in [0, 0.7]$). [15 marks]

Total for Question 2: 20 marks

Question 3: Bifurcations of equilibria and the parameter plane [submit as `cw13.m`]

(a) Calculate the following special equilibria and corresponding values of (β, I, R) to 5 significant digits. For each bifurcation, code a function `res=fhopf(y)` or `res=ffold(y)` that has the bifurcation point as a regular root, which you can then solve with `MySolve`.
(Hopf) As I_0 is varied, changes in stability of the equilibrium branch occurs at a Hopf bifurcation (two eigenvalues of the linearisation cross the imaginary axis). Locate two Hopf points at β_{H1} and β_{H2} ($\beta_{H1} < \beta_{H2}$) along with the corresponding Hopf equilibria (I_{H1}, R_{H1}) and (I_{H2}, R_{H2}). Insert these points into the bifurcation diagram from Question 1c.
(Fold) For some specific range of β within the interval $\beta \in [3.5, 6]$ the system possesses more than one equilibrium. This range in β is bounded by fold equilibria (I_{F1}, R_{F1}) and (I_{F2}, R_{F2}) at their corresponding parameter values β_{F1} and β_{F2}. Compute these points and add them to the bifurcation diagram from Question 1c. In order to obtain full marks you must solve the system of 5 equations in 5 unknowns given in the lecture notes. [10 marks]

(b) Explore how the model's possible dynamics change in the (β, γ)-plane by tracking the fold and Hopf bifurcations in two parameters. Extend the defining systems used in part (a) to include one extra unknown (γ) such that they can be used with `MyTrackCurve`. You can make use of the pre-defined function `rhs2P` (which includes γ as a second free parameter) given in `sir_model.m` to create the defining system. Investigate these curves in the box $(\beta, \gamma) \in [0, 12] \times [0, .1]$. Include a plot of the computed Fold and Hopf curves in the (β, γ)-plane. [10 marks]

Total for Question 3: 20 marks

Hints:

- In order to obtain the complete curves you will need to initiate the continuation for γ increasing, and for γ decreasing, in separate runs.
- A curve initialised at β_{H1} may also pass through β_{H2} (similarly for β_{F1} and β_{F2}).
- The defining system for a Hopf bifurcation given in lectures is also consistent with a neutral saddle with two real eigenvalues $\lambda_1 + \lambda_2 = 0$. You will need to check the eigenvalues along the Hopf curve and only plot the segment with imaginary eigenvalues.

Question 4: Periodic orbits and their bifurcations [submit as `cw14.m`]

(a) **(Periodic orbits)** Compute the family of periodic orbits that branches off from the Hopf bifurcation using your functions `MyTrackCurve` and `MyIVP`. Choose the Hopf point with larger β-value to start your computation (β_{H2}). Compute the eigenvalues of the linearised one-period map to determine stability of the periodic orbits.

Insert into the bifurcation diagram in the (β, I)-plane the I-maxima and I-minima of the periodic orbits for each parameter value β, indicating their stability either by using colours or symbols.

Plot the phase portraits at least 10 periodic orbits evenly spread along the branch, again using colours or symbols to distinguish between stable and unstable periodic orbits.

Plot the period T of the periodic orbits along the family in the (β, T)-plane. [15 marks]

(b) **(Fold of periodic orbits)** The family of periodic orbits makes a fold at some parameter value β_{PF}. Code a function `res=pfold(u)` that has the fold point as a regular root, which you then solve with `MySolve` (report β_{PF} to 5 significant digits). Include the fold of periodic orbits into the bifurcation diagram, the plot showing the periodic orbits from part (a) and the plot in the (β, T)-plane from part (a). You should now be able to identify at least one extra phase portrait as in Question 3, include this as a plot with the corresponding β-value indicated.

Extend the defining system for the fold of periodic orbits to include one extra unknown γ such that it can be used with `MyTrackCurve` to follow the locus of the bifurcation in the (β, γ)-plane. Include the curve (computed for both increasing and decreasing γ in your (β, γ)-plane plot from Question 3. [15 marks]

Total for Question 4: 30 marks

Hints:

- Whilst it is possible to track a branch of periodic orbits from the Hopf bifurcation with lower β-value (β_{H1}), this proves difficult to compute and requires a more advanced method that the one presented on this course. The branch you will compute, starting at β_{H2}, will also become difficult to compute as it approaches a fold for $\beta < \beta_{H1}$ and there is no need to attempt to follow the branch beyond this point.