Homework 2

梁昱桐 2100013116

Peking University

1. **Satisfaction from consumption is maximized when**

A. marginal cost equals zero.
B. marginal benefit equals zero.
C. marginal benefit equals marginal cost.
D. marginal benefit is maximum.

Answer: C

$$\text{maximize} \quad \pi = R - C \tag{1}$$

2. **Dr. J gives 4 homework assignments for the Intermediate Microeconomics course. Suppose he drops each student's two lower scores and uses the average score of the other two assignments to determine the final homework score. Naughty Brown is taking this class and gets a 60 and a 70 in the first two assignments. Let x be his score on the third assignment and y be his score on the fourth assignment. If we draw his indifference curves for the scores on the third and fourth assignment with x on the horizontal axis and y on the vertical axis, then his indifference curve that crosses the point of $(x, y) = (50, 70)$ is**

A. a line segment between $(0, 120)$ and $(120, 0)$.
B. two line segments with one being vertical and the other one being horizontal.
C. three line segments with one being vertical, one being horizontal, and the other one that links $(70, 50)$ and $(50, 70)$.
D. three line segments with one being vertical, one being horizontal, and the other one that links $(70, 60)$ and $(60, 70)$.

Answer: B

$$(0, 70) - (70, 70) - (70, 0) \tag{2}$$

3. **Pencils sell for 10 cents and pens sell for 50 cents. Suppose Fiori, whose preferences satisfy all the basic assumptions, buys 5 pens and one pencil each semester. With this consumption bundle, $|MRS_{Pencil,Pen}| = 3$. Which of the following is true?**

A. Fiori could optimize his utility by buying more pens and fewer pencils.
B. Fiori could optimize his utility by buying more pencils and fewer pens.
C. Fiori could optimize his utility by buying more pencils and more pens.
D. Fiori could optimize his utility by buying fewer pencils and fewer pens.

Answer: B

It can be seen through the graph.

4. **If $P_x = P_y$, then when the consumer maximizes utility,**

A. X must equal Y.
B. MU_x must equal MU_y.
C. $\mathrm{MRS}_{XY} = 0$.
D. X and Y must be substitutes.

Answer: B

There's only one restriction:

$$|\mathrm{MRS}| = \frac{P_y}{P_x} = 1 \tag{3}$$

$$\Rightarrow \frac{MU_x}{MU_y} = 1 \tag{4}$$

D: Only when we know the specific form of the utility function can we know how the demand for y changes (compared with 0) with the price change of x .

5. **Fiori derives utility from consuming iced tea and lemonade. For the bundle he currently consumes, the marginal utility he receives from iced tea is 16 , and the marginal utility he receives from lemonade is 8 . Instead of consuming this bundle, Fiori should**

A. buy more iced tea and less lemonade.
B. buy more lemonade and less iced tea.
C. buy more iced tea and lemonade.
D. buy less iced tea and lemonade.
E. None of the above is necessarily correct.

Answer: E

From the known information, we can only calculate that the current point with an MRS of -2 is on an indifference curve. However, since we do not know the prices of both goods, it is impossible to determine whether the point of utility maximization is on the left or right side.

6. **Fiori is a candy-loving kid. X units of candy provides him a utility of $\mathrm{u(X)} = 4\log(1+\mathrm{X})$ units, while holding $1 gives him 1 unit of utility. X does not have to be an integer.**

   (a) **If Fiori has $10 and the unit price of candy is $2, how many units of candy will Fiori purchase to maximize his utility? What is the maximized utility?**

   Let x be the units of candy Fiori buys.

   Total utility function:

$$U = 10 - 2x + 4\log(1+x) \tag{5}$$

   Maximize U ($x \in [0, 5]$) :

$$\begin{aligned} &\frac{\partial U}{\partial x} = -2 + \frac{4}{1+x} \\ \Rightarrow &x = 1 \\ &\frac{\partial^2 U}{\partial x^2} = -\frac{4}{(x+1)^2} = -1 < 0 \end{aligned} \tag{6}$$

   And:

$$\begin{gathered} U(0) = 10 \\ U(1) = 8 + 4\log 2 \approx 10.77 \\ U(5) = 4\log 6 \approx 7.17 \end{gathered} \tag{7}$$

   Therefore, Fiori will purchase 1 unit of candy, the maximized utility is $8 + 4\log 2 \approx 10.77$

   (b) **How about if Fiori has $1 and the unit price of candy is $2 ?**

   Let x be the units of candy Fiori buys.

   Total utility function:

$$U = 1 - 2x + 4\log(1+x) \tag{8}$$

   Maximize U ($x \in [0, \frac{1}{2}]$) :

$$\begin{aligned} &\frac{\partial U}{\partial x} = -2 + \frac{4}{1+x} \\ \Rightarrow &x = 1 \end{aligned} \tag{9}$$

   And:

$$\begin{gathered} U(0) = 1 \\ U(\frac{1}{2}) = 4\log\frac{3}{2} \approx 1.62 \end{gathered} \tag{10}$$

   Therefore, Fiori will purchase $\frac{1}{2}$ unit of candy, the maximized utility is $4\log\frac{3}{2} \approx 1.62$

   (c) **Derive Fiori's demand for candy when he has $3.**

   Let q be the units of candy Fiori buys, p be the price of candy.

   Total utility function:

$$U = 3 - pq + 4\log(1+q) \tag{11}$$

   Maximize U:

$$\begin{aligned} \frac{\partial U}{\partial q} &= -p + \frac{4}{1+q} = 0 \\ \Rightarrow q^* &= \frac{4}{p} - 1 \end{aligned} \tag{12}$$

i. $q^* \in [0, \frac{3}{p}]$

$$0 \leq \frac{4}{p} - 1 \leq \frac{3}{p} \Rightarrow 1 \leq p \leq 4 \tag{13}$$

ii. $q^* \in (\frac{3}{p}, +\infty)$

$$\frac{3}{p} < \frac{4}{p} - 1 \Rightarrow p < 1 \tag{14}$$

Because:

$$\frac{\partial^2 U}{\partial q^2} = -\frac{4}{(1+q)^2} < 0 \tag{15}$$

U increases as q increases

Thus, the optimal q would be $\frac{3}{p}$

iii. $q^* \in (-\infty, 0)$

$$\frac{4}{p} - 1 < 0 \Rightarrow p > 4 \tag{16}$$

Because:

$$\frac{\partial^2 U}{\partial q^2} = -\frac{4}{(1+q)^2} < 0 \tag{17}$$

U decreases as q increases

Thus, the optimal q would be 0

Above all, the demand function is:

$$q = \begin{cases} \frac{3}{p} \quad p \in (0,1) \\ \frac{4}{p} - 1 \quad p \in [1,4] \\ 0 \quad p \in (4, +\infty) \end{cases} \tag{18}$$

7. **Fiori is a utility-maximizing consumer. He has an income $m > 0$ to allocate between two goods (1 and 2). For each good, Fiori faces a constant price of p_1 and p_2. For each of the following utility function, derive Fiori's optimal demand $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.**

   (a) $u(x_1, x_2) = \sqrt{x_1} + 2\sqrt{x_2}$

   According to Cauchy's inequality:

$$(p_1 x_1 + p_2 x_2)\left(\frac{1}{p_1} + \frac{4}{p_2}\right) \geq \left(\sqrt{x_1} + 2\sqrt{x_2}\right)^2 = \mathrm{U}^2 \tag{19}$$

   Only when $\frac{x_2}{x_1} = \frac{4p_1^2}{p_2^2}$, the equation holds

   Do the calculation:

$$\begin{cases} x_1 = \frac{p_2}{4p_1^2 + p_1 p_2} m \\ x_2 = \frac{4p_1}{p_2^2 + 4p_1 p_2} m \end{cases} \tag{20}$$

(b) $u(x_1, x_2) = \int_0^{x_1^a x_2^b} f(x)dx$, **where** $a, b > 0$ **and** $f(x)$ **is positive**

$F(x) = \int_0^x f(t)dt$ is an increasing function when f is greater than zero

Therefore, $\max u(x_1, x_2) \Leftrightarrow \max x_1^a x_2^b$, the right side is exactly the Cobb-Douglas Utility Function

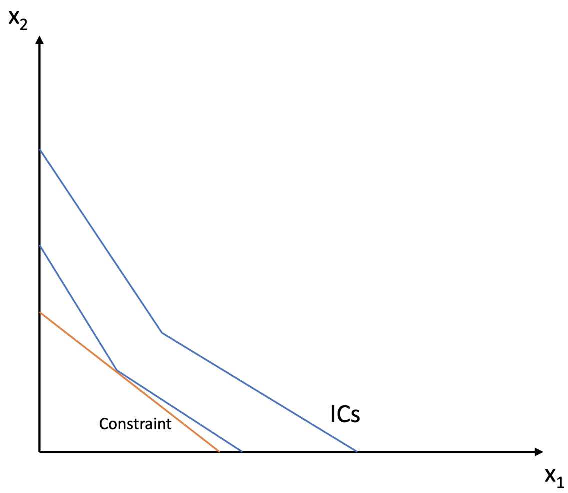
$$\begin{aligned} \frac{m}{a+b} &= \frac{p_1 x_1 + p_2 x_2}{a+b} \\ &= \frac{a \cdot (\frac{p_1}{a} x_1) + b \cdot (\frac{p_2}{b} x_2)}{a+b} \\ &\geqslant (\frac{p_1}{a} x_1)^{\frac{a}{a+b}} (\frac{p_2}{b} x_2)^{\frac{b}{a+b}} \\ &= \left(\frac{p_1^a p_2^b}{a^a b^b} x_1^a x_2^b \right)^{\frac{1}{a+b}} \\ &= \left(\frac{p_1^a p_2^b}{a^a b^b} \cdot \mathrm{U} \right)^{\frac{1}{a+b}} \end{aligned} \tag{21}$$

Only when $\frac{p_1 x_1}{a} = \frac{p_2 x_2}{b}$, the equation holds

Do the calculation:

$$\begin{cases} x_1 = \frac{a}{a+b} \cdot \frac{m}{p_1} \\ x_2 = \frac{b}{a+b} \cdot \frac{m}{p_2} \end{cases} \tag{22}$$

(c) $u(x_1, x_2) = \min\{x_1 + 2x_2, 2x_1 + x_2\}$



From the picture, it can be seen that:

i. $0 < \frac{p_1}{p_2} < \frac{1}{2}$

$$\begin{cases} x_1 = \frac{m}{p_1} \\ x_2 = 0 \end{cases} \tag{23}$$

ii. $\frac{p_1}{p_2} = \frac{1}{2}$

$$\begin{cases} x_1 = t \\ x_2 = \frac{m}{2p_1} - \frac{t}{2} \end{cases} \tag{24}$$

Where $t \in [\frac{m}{3p_1}, \frac{m}{p_1}]$

iii. $\frac{1}{2} < \frac{p_1}{p_2} < 2$

$$\begin{cases} x_1 = \frac{m}{p_1+p_2} \\ x_2 = \frac{m}{p_1+p_2} \end{cases} \tag{25}$$

iv. $\frac{p_1}{p_2} = 2$

$$\begin{cases} x_1 = t \\ x_2 = \frac{2m}{p_1} - 2t \end{cases} \tag{26}$$

Where $t \in [0, \frac{2m}{3p_1}]$

v. $2 < \frac{p_1}{p_2}$

$$\begin{cases} x_1 = 0 \\ x_2 = \frac{m}{p_2} \end{cases} \tag{27}$$

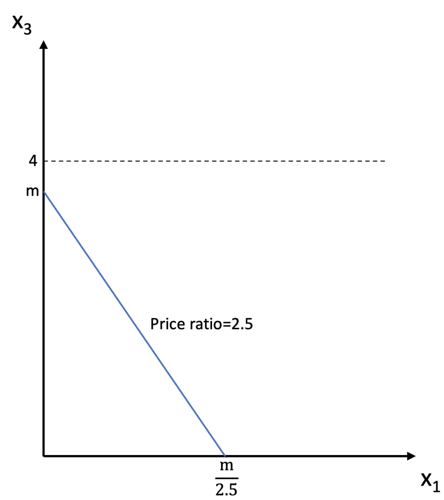
8. **Naughty Brown spends his monthly TA income m on food (x_1), clothing (x_2) and KTV (x_3) . The price of food is constant at $p_1 = 0.5$, and the price of clothing is constant at $p_2 = 2$, while the KTV currently offers a deal: every month, $p_3 = 1$ per hour for the first four hours and $p_4 = 4$ per hour for the following hours. Naughty Brown's utility function is given by $\mathrm{U}(x_1, x_2, x_3) = \sqrt{x_1 x_2} \cdot x_3$.**

   (a) **Assume that Naughty Brown buys the same amount of food and clothing this month. Derive his budget constraint and plot it with $x_1 (= x_2)$ on the horizontal axis and x_3 on the vertical axis. Label everything clearly.**

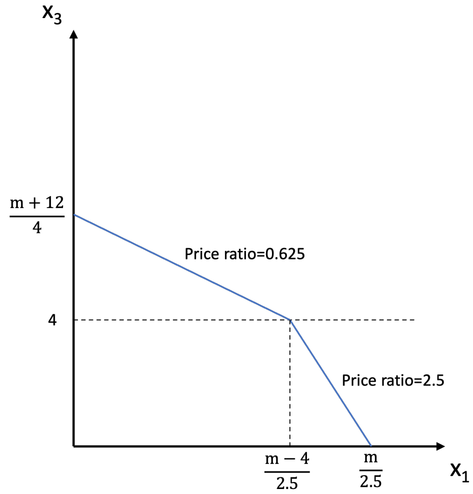
$$\begin{aligned} m &= \begin{cases} x_1p_1 + x_2p_2 + x_3p_3 & x_3 < 4 \\ x_1p_1 + x_2p_2 + 4p_3 + (x_3-4)p_4 & x_3 \geqslant 4 \end{cases} \\ &= \begin{cases} 2.5x_1 + x_3 & x_3 < 4 \\ 2.5x_1 + 4x_3 - 12 & x_3 \geqslant 4 \end{cases} \end{aligned} \tag{28}$$

Consider the range of x_3 , if m is small enough, there won't be a situation that $x_3 \geqslant 4$, so there are two types of curves classified by the existence of this situation.

i. $m < 4$



ii. $m \geqslant 4$



(b) **Derive Naughty Brown's demand function of $x_1(m)$, $x_2(m)$ and $x_3(m)$.**

The formal representation of this optimization problem:

$$\begin{aligned} \max \ \ \log \mathrm{U}(x_1, x_2, x_3) = \frac{1}{2}\log x_1 + \frac{1}{2}\log x_2 + \log x_3 \\ s.t. \ m = \begin{cases} 0.5x_1 + 2x_2 + x_3 & x_3 < 4 \\ 0.5x_1 + 2x_2 + 4x_3 - 12 & x_3 \geqslant 4 \end{cases} \\ x_1, x_2, x_3 \geqslant 0 \end{aligned} \tag{29}$$

There are only three possible optimal solutions, which can also be visually imagined through two-dimensional images.

And we first use the Lagrange multiplier method to calculate the solutions corresponding to these three cases.

i. The objective function is tangent to the plane represented by the first equation

$$\begin{aligned} \max \ \ \log \mathrm{U}(x_1, x_2, x_3) \\ s.t. \ m = 0.5x_1 + 2x_2 + x_3 \quad x_3 < 4 \\ x_1, x_2, x_3 \geqslant 0 \end{aligned} \tag{30}$$

The Lagrange function:

$$\mathrm{L} = \frac{1}{2}\log x_1 + \frac{1}{2}\log x_2 + \log x_3 - \lambda(0.5x_1 + 2x_2 + x_3 - m) \tag{31}$$

Do the calculation:

$$\begin{cases} \frac{\partial \mathrm{L}}{\partial x_1} = \frac{1}{2x_1} - \frac{\lambda}{2} = 0 \\ \frac{\partial \mathrm{L}}{\partial x_2} = \frac{1}{2x_2} - 2\lambda = 0 \\ \frac{\partial \mathrm{L}}{\partial x_3} = \frac{1}{x_3} - \lambda = 0 \\ 0.5x_1 + 2x_2 + x_3 - m = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{m}{2} \\ x_2 = \frac{m}{8} \\ x_3 = \frac{m}{2} \end{cases} \tag{32}$$

The restriction of x_3 :

$$x_3 = \frac{m}{2} < 4 \Rightarrow m < 8 \tag{33}$$

The utility:

$$\mathrm{U}_1 = \frac{m^2}{8} \tag{34}$$

ii. The objective function is tangent to the plane represented by the second equation

$$\begin{gathered} \max \ \ \log \mathrm{U}(x_1, x_2, x_3) \\ s.t.\ 0.5x_1 + 2x_2 + 4x_3 - 12 \quad x_3 \geqslant 4 \\ x_1, x_2, x_3 \geqslant 0 \end{gathered} \tag{35}$$

The Lagrange function:

$$\mathrm{L} = \frac{1}{2}\log x_1 + \frac{1}{2}\log x_2 + \log x_3 - \lambda(0.5x_1 + 2x_2 + 4x_3 - 12 - m) \tag{36}$$

Do the calculation:

$$\begin{cases} \frac{\partial \mathrm{L}}{\partial x_1} = \frac{1}{2x_1} - \frac{\lambda}{2} = 0 \\ \frac{\partial \mathrm{L}}{\partial x_2} = \frac{1}{2x_2} - 2\lambda = 0 \\ \frac{\partial \mathrm{L}}{\partial x_3} = \frac{1}{x_3} - 4\lambda = 0 \\ 0.5x_1 + 2x_2 + 4x_3 - 12 - m = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{m+12}{2} \\ x_2 = \frac{m+12}{8} \\ x_3 = \frac{m+12}{8} \end{cases} \tag{37}$$

The restriction of x_3 :

$$x_3 = \frac{m+12}{8} \geqslant 4 \Rightarrow m \geqslant 20 \tag{38}$$

The utility:

$$\mathrm{U}_2 = \frac{(m+12)^2}{32} \tag{39}$$

iii. The objective function passes through the intersection of two planes

$$\begin{gathered} \max \ \ \log \mathrm{U}(x_1, x_2) \\ s.t.\ 0.5x_1 + 2x_2 + 4 \\ x_1, x_2 \geqslant 0 \end{gathered} \tag{40}$$

The Lagrange function:

$$\mathrm{L} = \frac{1}{2}\log x_1 + \frac{1}{2}\log x_2 + \log 4 - \lambda(0.5x_1 + 2x_2 + 4) \tag{41}$$

Do the calculation:

$$\begin{cases} \frac{\partial \mathrm{L}}{\partial x_1} = \frac{1}{2x_1} - \frac{\lambda}{2} = 0 \\ \frac{\partial \mathrm{L}}{\partial x_2} = \frac{1}{2x_2} - 2\lambda = 0 \\ 0.5x_1 + 2x_2 + 4 - m = 0 \end{cases} \Rightarrow \begin{cases} x_1 = m - 4 \\ x_2 = \frac{m-4}{4} \\ x_3 = 4 \end{cases} \tag{42}$$

The restriction of x_1 and x_2 :

$$x_1 = m - 4 \geqslant 0 \Rightarrow m \geqslant 4 \tag{43}$$

The utility:

$$\mathrm{U}_3 = 2|m - 4| \tag{44}$$

Compare the range of m and U :

i. $0 \leqslant m < 4$

Only U_1 is valid

ii. $4 \leqslant m < 8$

$$\mathrm{U}_1 \geqslant \mathrm{U}_3 \Rightarrow \frac{m^2}{8} \geqslant 2|m-4| \Rightarrow m \in [4,8) \tag{45}$$

iii. $8 \leqslant m < 20$

Only $\mathbf{U}_3$ is valid

iv. $20 \leqslant m$

$$\mathrm{U}_2 \geqslant \mathrm{U}_3 \Rightarrow \frac{(m+12)^2}{32} \geqslant 2|m-4| \Rightarrow m \in [20,+\infty) \tag{46}$$

Above all, the demand functions are:

$$x_1 = \begin{cases} \frac{m}{2} & m \in [0,8) \\ m-4 & m \in [8,20) \\ \frac{m+12}{2} & m \in [20,+\infty) \end{cases} \tag{47}$$

$$x_2 = \begin{cases} \frac{m}{8} & m \in [0,8) \\ \frac{m-4}{4} & m \in [8,20) \\ \frac{m+12}{8} & m \in [20,+\infty) \end{cases} \tag{50}$$

$$x_3 = \begin{cases} \frac{m}{2} & m \in [0,8) \\ 4 & m \in [8,20) \\ \frac{m+12}{8} & m \in [20,+\infty) \end{cases} \tag{51}$$