Intermediate Microeconomics Exercise Class 1

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Content

Thanks to Rui Ai

Derivative

- Definition of derivative: $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.
- Differential and derivative: $\Delta y = A\Delta x + o(\Delta x),\ \Delta x \to 0$. We call Δy and Δx differentials. $A = \frac{dy}{dx}|_{x=x_0}$ is called differential quotient.
- Derivatives of common functions:$\sin x, \cos x, \ln x, x^a, a^x \cdots$.
- Inverse function: suppose that $y = f(x)$is a bijective function,then we can define $x = g(y)$. It holds that $g'(y) = \frac{1}{f'(g(y))}$.
- The derivative of implicit functions.

Example

- Derive the derivative of $f(x) = x^{x^x}$.
- Suppose that $h(x, y) = y^2 - 2xy - x^2 + 2x - 4 = 0$, prove that $y'(x) = \frac{y(x)+x-1}{y(x)-x}$.

Derivative of Composition Functions

- Suppose that $h(x) = f \circ g(x)$, then it holds that $h'(x) = g'(f(x)) \cdot f'(x)$.
- One intuitionistic explanation is $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$.

Example

Define $g(x) = f(\frac{x-1}{x+1})$, where $f(x) = \arctan x$. Derive $g'(x)$.

Partial Derivative

- Definition: fix other variates. $\partial_x f(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$.
- Partial derivative of composition functions: suppose $z = f(u,v)$, $u = u(x,y)$ and $v = v(x,y)$, then it holds that

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}.$$
$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}.$$

Example

Define $f(x,y,z) = (\frac{2y}{z})^x$. Calculate partial derivatives at point $(1,2,1)$.

Total Differential

- Defintion: $df = \partial_x f(x_0, y_0)dx + \partial_y f(x_0, y_0)dy$.
- Gradient: $(\partial_x f, \partial_y f)$.

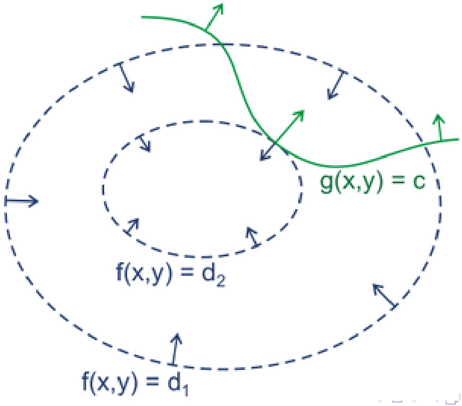
Example

For $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, Calculate the gradient of $x^\top Ax$.

Lagrange-Method (Simplified)

- $\min f(x,y)$ *subject to* $g(x,y)=0$. Construct $F(x,y,\lambda)=f(x,y)+\lambda g(x,y)$. Then, F.O.C. is

$$\begin{cases} \partial_x F(x,y,\lambda)=0, \\ \partial_y F(x,y,\lambda)=0, \\ \partial_\lambda F(x,y,\lambda)=0. \end{cases}$$



Lagrange-Method (Simplified)

Example

- Suppose that $x^2 + y^2 + z^2 = 10$, try to maximize $xy + 2yz$.
- Suppose that $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$, try to minimize $\frac{1}{x^2y^2z^2}$.
- Find the difference between the height of the highest and lowest points of the curve $\begin{cases} x - y + 4z = 1 \\ 2x^2 + 4y^2 = 3 \end{cases}$ in three dimensions.

AM-GM Inequality

- $a_1, a_2, ...a_n \geq 0$, $\frac{a_1+...+a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$
- More generally, we have
  $a_1, a_2, ...a_n \geq 0$, $\frac{a_1 b_1+...+a_n b_n}{b_1+...b_n} \geq \sqrt[b_1+...+b_n]{a_1^{b_1} a_2^{b_2} ... a_n^{b_n}}$

Example

- (Cobb-Douglas Utility) Try to find the pair (x, y) that maximizes $U = x^c y^d$ under the constraint of $ax + by = 1$

- The equation holds only when $a_1 = a_2 = ... = a_n$. Make sure you find the tightest inequality to solve the problem!

Review of Basic Economics

- Why Economics? Scarcity!
- Opportunity Cost: highest valued alternative
- Nominal variables VS real variables: USD/RMB=7
- Quantity demanded VS demand so as supply
- Inverse function

Elasticity

- Own-Price Elasticity of Demand

$$E_{Q_x^D,P_x} = \frac{\%\Delta Q_x^D}{\%\Delta P_x} = \frac{\partial Q_x^D}{\partial P_x}\frac{P_x}{Q_x^D}$$

Example

- (Double-Log Demand Function) $\log Q_x^D = \beta_0 + \beta_1 \log P_x$

Thanks!