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Tomasz Radożycki

Solving Problems in Mathematical Analysis, Part II

Definite, Improper and Multidimensional Integrals, Functions of Several Variables and Differential Equations

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Preface

The second part of this book series covers, roughly speaking, the material of the second semester course of mathematical analysis held by university departments of sciences. I tried to preserve the character of the first part, presenting very detailed solutions of all problems, without abbreviations or understatements. In contrast to authors of typical problem sets, my goal was not to provide the reader many examples for his own work in which only final results would be given (although such problems are also provided at the ends of chapters), but to clarify all steps of the solution to a given problem: from choosing the method, through its technical implementation to the final result.

A priority for me was the simplicity of argumentation although I am aware that the price for this is sometimes the loss of full precision. Concerned about the volume of the book and knowing that it is intended for students who have already passed the first semester of mathematical analysis, I let myself sometimes—in contrast to the first part—skip the details of very elementary transformations. The inspiration that guided me to prepare this particular set of problems is given in the preface of the first part.

The theoretical summaries placed at the beginning of each chapter only serve to collect the theorems and formulas that will be used. Theory of the mathematical analysis should be learned from other textbooks or lectures. The problems contained in this book can also be studied without reading these theoretical summaries since all necessary notions are repeated informally in the solution where applicable.

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Contents

Definitions and Notation

In this book, the conventions and definitions adopted in the first part are used. Several additional designations are listed below.

- The symbol $\int\limits_{[a,b]} f(x)dx$ denotes the Riemann integral on the interval $[a, b]$, and $\int\limits_a^b f(x)dx$ refers to the definite integral.
- The iterated limits are written in the form $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x, y)$ or $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x, y)$, while the full (i.e., non-iterated) limit is denoted as $\lim\limits_{(x,y)\to(a,b)} f(x, y)$.
- For typographic reasons, a column vector will be written in the text as $[v_1, v_2, \ldots, v_N]$ without adding any additional transposition index.
- The symbol $\mathbb{1}$ is used for the unit matrix, the dimension of which varies depending on the context.
- The symbol f_h' denotes the directional derivative along a given vector h.
- For the function of several variables, f' refers to the whole Jacobian matrix, and f'' to the matrix of second derivatives.
- The symbol $\|v\|$ denotes the norm of a vector v. In the case of $\vec{x} \in \mathbb{R}^N$, the notation $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$ is interchangeably used. Sometimes, the arrow over x is omitted if it is obvious from the context that $x \in \mathbb{R}^N$.

Chapter 1
Exploring the Riemann and Definite Integral

In this chapter, we explain in detail the idea of the Riemann integrability of functions and its connection to the definite integral obtained by the "antidifferentiation." We also learn how to apply these notions in practical calculations and how to avoid some hidden traps.

The **definite integral** is defined as

$$\int_a^b f(x)dx = F(b) - F(a), \tag{1.0.1}$$

where $F(x)$ is the primitive function for $f(x)$ (see Chap. 14 in Part I).

The **Riemann integral** on the interval $[a, b]$ is denoted as

$$\int_{[a,b]} f(x)dx. \tag{1.0.2}$$

For the precise definition see the first problem of this chapter where the entire construction is explained in detail. Unfortunately, no compact definition is possible. If Riemann's integral for a given function exists, the function is called **Riemann-integrable** or **R-integrable**. In particular, the continuous functions are Riemann-integrable.

If the Riemann integral exists, it equals the definite integral. It stems from the following **fundamental theorem of calculus**:

Given the continuous function $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. Then the function

$$F(x) = \int_{[a,x]} f(x')dx' \tag{1.0.3}$$

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is uniformly continuous on the interval $[a, b]$ and differentiable inside it, and for any x

$$\frac{d}{dx} F(x) = f(x). \tag{1.0.4}$$

This also means that

$$\int\limits_{[a,x]} f(x')dx' = \int\limits_a^x f(x')dx', \tag{1.0.5}$$

i.e., the Riemann integral equals the definite integral on the same interval.

The **first mean value theorem for definite integrals** has the following form:

Let the function $f : [a, b] \to \mathbb{R}$ be continuous on the interval $[a, b]$ and the function $g : [a, b] \to \mathbb{R}$ be integrable and not changing its sign in the considered interval. Then there exists $c \in]a, b[$ such that

$$\int\limits_a^b f(x)g(x)dx = f(c) \int\limits_a^b g(x)dx. \tag{1.0.6}$$

The **second mean value theorem for definite integrals** states the following:

If the function $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$ and the function $g : [a, b] \to \mathbb{R}$ is monotonic with continuous derivative (this purely technical assumption is sometimes omitted), then there exists $c \in]a, b[$ such that

$$\int\limits_a^b f(x)g(x)dx = g(a) \int\limits_a^c f(x)dx + g(b) \int\limits_c^b f(x)dx. \tag{1.0.7}$$

1.1 Examining the Integrability of Functions

Problem 1

The integrability of the function $f(x) = x^2$ on the interval [0, 1] will be examined.

Solution

Before starting to solve this task, one must clearly understand the components of the Riemann construction of an integral. In Fig. 1.1, an exemplary function $f(x)$ (not

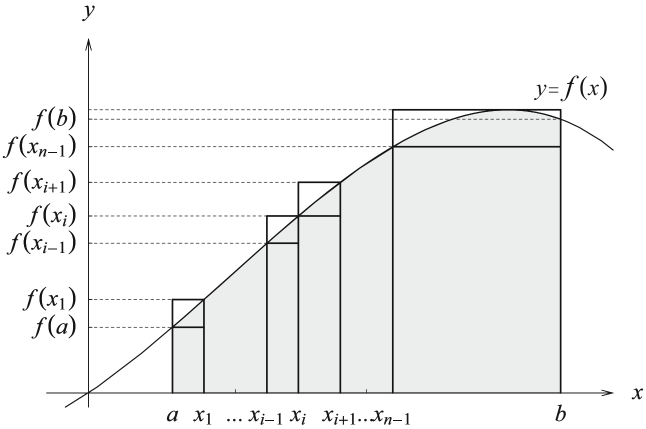

Fig. 1.1 The construction of the Riemann's sum and integral

necessarily that of the present exercise) is shown. We face the issue of calculating the area of the figure limited from the top by $f(x)$, from the bottom by the x-axis, and from the left and right by the lines $x = a$ and $x = b$ (naturally it is assumed that $a < b$). This figure is marked in gray.

It is not easy to directly calculate the surface area of the curvilinear figure, but is very easy to do it for a rectangle. The idea is, therefore, to replace the surface area in question by the sum of areas of certain rectangles. Some of them are drawn in the figure with thick lines. The procedure for the calculation of the area will be composed of the following steps:

1. The interval $[a, b]$ is divided into n parts (intervals) not necessarily equally spaced (but of course nonvanishing). The partition points are marked with symbols $x_0, x_1, \ldots, x_n$, where $x_0 = a$ and $x_n = b$.
2. In each interval $P_i := [x_{i-1}, x_i]$, into which the domain $[a, b]$ has been divided, one needs to find the maximum and minimum values of the function. (More precisely, we find bounds—upper and lower—of the sets $f(P_i)$, as the largest and the smallest value in a given interval may not exist if the function is not continuous.) As for the function f, one assumes that it is bounded on $[a, b]$, so it should not pose any problem. Let us denote them with symbols f_i^{max} and f_i^{min}. In the particular case of a constant function, both these values are equal, but this is not relevant for further discussion. As one can see in the figure, f_i^{max} or f_i^{min} can appear at the ends of a given interval, but also inside.
3. Each pair of successive points on the x-axis (i.e., x_{i-1} and x_i, for $i = 1.2 \ldots, n$) defines two rectangles: a greater one with vertices at points

$$(x_{i-1}, 0),\ (x_i, 0),\ (x_i, f_i^{max}),\ (x_{i-1}, f_i^{max})$$

and a smaller one with

$$(x_{i-1}, 0), (x_i, 0), (x_i, f_i^{min}), (x_{i-1}, f_i^{min}).$$

The sum of areas of all smaller rectangles (the so-called lower Riemann sum) when the interval $[a, b]$ is fragmented into n subintervals will be denoted with L_n, and the sum of larger ones (upper Riemann sum)—U_n. These values can be written as

$$L_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f_i^{min}, \quad U_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f_i^{max}. \tag{1.1.1}$$

Naturally, the following inequalities are satisfied:

$$L_n \le S \le U_n, \tag{1.1.2}$$

where S denotes the area below the curve.

4. The next step consists of making the partition of the interval $[a, b]$ still more dense by taking an increasingly large n. We expect that the sum of areas of the rectangles (both U_n and L_n) will then still more closely approximate the value of S. If, however, as a result of this procedure one wishes to obtain the exact value of the area of the curvilinear figure, the division cannot be done arbitrarily, but only in such a way that when $n \to \infty$ the lengths of *all* intervals P_i tend to zero, i.e.,

$$\max\{|x_1 - x_0|, |x_2 - x_1|, \ldots, |x_n - x_{n-1}|\} \underset{n\to\infty}{\longrightarrow} 0.$$

If at the limit any of the lengths remained finite, one would only get an *approximate* value of the area.

To facilitate our reasoning, it is assumed that the points $x_0, x_1, \ldots, x_n$ once selected on the x-axis are not changed, and the densification only consists of inserting some additional points in between (of course one needs to renumerate all points to have them in order).

Now, let us consider what happens to the ith larger rectangle if, due to this procedure, it is divided into two (or more) new ones. Is the sum of their areas still as that of the initial rectangle? Of course it may be the case, but in general this area decreases, because in each of the new subintervals, on which P_i has been divided, one needs to determine separately new maxima of the function (or upper bounds of its values), and these, at least for certain subintervals, will be smaller than f_i^{max} (we know for sure that they cannot increase). Thus one has

$$U_{n+1} \le U_n, \tag{1.1.3}$$

i.e., the sequence of the upper sums is not increasing. Similarly,

$$L_{n+1} \ge L_n, \tag{1.1.4}$$

and hence the sequence of the lower sums is not decreasing. Thanks to the condition (1.1.2), both these sequences are bounded, and therefore, they are convergent. Their limits are called the upper and the lower integrals. It remains only to establish whether they are equal to each other. If so, this means that we have simply found the integral, i.e., the value of the area S, and the function itself will be called *Riemann-integrable*. The answer to this question of equality between the upper and lower integrals is the subject of the present exercise.

Based on the above reasoning, one can see that in order to demonstrate the integrability of the function given in this exercise, it should be proved that

$$\lim_{n\to\infty}(U_n - L_n) = 0. \tag{1.1.5}$$

This will mean that sequences U_n and L_n converge to the common limit.

Let us, therefore, split the interval [0, 1] into n subintervals by the set of numbers:

$$0 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{n-1} < x_n = 1. \tag{1.1.6}$$

With the use of (1.1.1), one can construct now two sequences. It should be noted that the function $f(x) = x^2$ is increasing for nonnegative x's, and therefore, for each interval $[x_{i-1}, x_i]$ the minimal value appears on the left end (i.e., $f_i^{min} = f(x_{i-1}) = x_{i-1}^2$), and the maximal one on the right end (i.e., $f_i^{max} = f(x_i) = x_i^2$). As a result, one gets

$$U_n = \sum_{i=1}^{n}(x_i - x_{i-1})f_i^{max} = \sum_{i=1}^{n}(x_i - x_{i-1})x_i^2, \tag{1.1.7}$$

$$L_n = \sum_{i=1}^{n}(x_i - x_{i-1})f_i^{min} = \sum_{i=1}^{n}(x_i - x_{i-1})x_{i-1}^2. \tag{1.1.8}$$

Taking the difference of these two values, one obtains

$$U_n - L_n = \sum_{i=1}^{n}(x_i - x_{i-1})x_i^2 - \sum_{i=1}^{n}(x_i - x_{i-1})x_{i-1}^2 = \sum_{i=1}^{n}(x_i - x_{i-1})(x_i^2 - x_{i-1}^2). \tag{1.1.9}$$

We wish to show below that this difference can be made arbitrarily small (i.e., smaller than any chosen $\epsilon > 0$) if the points x_i are thickened with the request that for each i: $x_i - x_{i-1} < \delta$, and δ is the value at our disposal (it will be adjusted to ϵ). It simply corresponds to the choice of an appropriately large n. As a result, one has the estimate:

$$0 \le U_n - L_n < \delta \sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) = \delta \sum_{i=1}^{n}(x_i - x_{i-1})(x_i + x_{i-1}). \tag{1.1.10}$$

As the highest value which may be taken by x_i's is 1—ultimately, the integrability on the interval [0, 1] is being investigated—one can write

$$0 \leq U_n - L_n < \delta \sum_{i=1}^{n} (x_i - x_{i-1})(1+1) = 2\delta \sum_{i=1}^{n} (x_i - x_{i-1}) = 2\delta \cdot 1 = 2\delta. \tag{1.1.11}$$

We have also made use of the fact that the sum of lengths of all intervals P_i is always equal to 1. Now, if one wishes to make the difference $U_n - L_n$ smaller than a certain tiny ϵ, one can simply choose $\delta = \epsilon/2$, and this requirement will be met. The conclusion is U_n and L_n have a common limit, and as a consequence, the function $f(x) = x^2$ is integrable on the specified interval.

It is worth adding that apart from the lower and upper sums defined by the equations (1.1.1), one could also create an "intermediate" sum:

$$S_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f(\xi_i), \tag{1.1.12}$$

i.e., such that the points ξ_i are freely chosen within the intervals $[x_{i-1}, x_i]$. In view of the obvious inequalities:

$$L_n \leq S_n \leq U_n \tag{1.1.13}$$

all these three sums will coincide in the limit of infinitely dense partition (for an integrable function).

Problem 2

The integrability of the function $f(x) = \sin x$ on the interval $[-\pi/2, \pi/2]$ will be examined.

Solution

The sine function is bounded, so the Riemann construction described in the previous example can be applied. To this end, the interval $[-\pi/2, \pi/2]$ should be divided into n subintervals $[x_{i-1}, x_i]$, with the use of a sequence of numbers

$$-\pi/2 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{n-1} < x_n = \pi/2.$$

On the entire interval, the sine function is increasing, so one has

$$f_i^{max} = \sin x_i \quad f_i^{min} = \sin x_{i-1}. \tag{1.1.14}$$

It is clear that if we were to examine the Riemann integrability of a non-monotonic function, for example the sine function on the interval $[0, \pi]$, the most convenient method of proceeding would be to split the interval $[0, \pi]$ into the sum of intervals $[0, \pi/2]$ and $[\pi/2, \pi]$. On each of them, the sine function is monotonic: increasing on the former, so our reasoning would proceed exactly as in the present problem, and decreasing on the latter, where f_i^{max} and f_i^{min} would simply swap their roles. Naturally, there are also functions for which it is impossible to indicate the intervals of monotonicity (such a function will be dealt with in Problem 4) and then one has to proceed with a different strategy.

Now, the upper sum can be written in the form:

$$U_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f_i^{max} = \sum_{i=1}^{n} (x_i - x_{i-1}) \sin x_i \tag{1.1.15}$$

and the lower one as:

$$L_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f_i^{min} = \sum_{i=1}^{n} (x_i - x_{i-1}) \sin x_{i-1}. \tag{1.1.16}$$

Similarly as in Problem 1, it should be proved that

$$\lim_{n\to\infty} (U_n - L_n) = 0. \tag{1.1.17}$$

Let us write

$$\begin{aligned} U_n - L_n &= \sum_{i=1}^{n} (x_i - x_{i-1}) \sin x_i - \sum_{i=1}^{n} (x_i - x_{i-1}) \sin x_{i-1} \\ &= \sum_{i=1}^{n} (x_i - x_{i-1})(\sin x_i - \sin x_{i-1}) \end{aligned} \tag{1.1.18}$$

and use the well-known formula for the difference of two sines:

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}. \tag{1.1.19}$$

In this way, one gets

$$U_n - L_n = 2\sum_{i=1}^{n} (x_i - x_{i-1}) \cos\frac{x_i + x_{i-1}}{2} \sin\frac{x_i - x_{i-1}}{2}. \tag{1.1.20}$$

We are going to prove below that the value of the limit of this quantity equals zero, which simultaneously entails the equality:

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n \tag{1.1.21}$$

(due to the fact that both limits do exist, which is known from the introduction to the previous example). This also indicates the integrability of the considered function. For this purpose, the well-known estimate is used: $\sin\phi \leq \phi$ for positive values of ϕ (see Exercise 5 in Sect. 5.1 of Part I), thanks to which one can write

$$0 \leq U_n - L_n \leq \sum_{i=1}^{n} (x_i - x_{i-1})^2 \cos\frac{x_i + x_{i-1}}{2} \leq \sum_{i=1}^{n} (x_i - x_{i-1})^2. \tag{1.1.22}$$

Additionally, we have made use of the fact that cosine is bounded by 1.

If one now makes the partition points x_i more dense, so that for each i, $x_i - x_{i-1} < \delta$ where the value of δ is fixed below, the following estimate will be true:

$$0 \leq U_n - L_n = \sum_{i=1}^{n} (x_i - x_{i-1})(x_i - x_{i-1}) < \delta \sum_{i=1}^{n} (x_i - x_{i-1}) = \delta \cdot \pi. \tag{1.1.23}$$

The number of π which appears in the above result is simply the length of the interval $[-\pi/2, \pi/2]$. Since one requires the difference $U_n - L_n$ to be smaller than a certain tiny ϵ, it is sufficient to choose $\delta = \epsilon/\pi$ (i.e., to make n large enough) and this condition will be met. Therefore, U_n and L_n do have a common limit, and as a result, the function $\sin x$ is integrable on the specified interval.

Problem 3

The integrability of the function $f(x) = \log x$ on the interval $[1, 2]$ will be examined.

Solution

The logarithmic function is bounded on $[1, 2]$, so we know how to handle it. The interval is divided into n subintervals $[x_{i-1}, x_i]$ with the use of partition numbers x_i with the usual assumption:

$$1 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{n-1} < x_n = 2.$$

The natural logarithm on its entire domain is increasing, and therefore,

$$f_i^{max} = \log x_i, \quad f_i^{min} = \log x_{i-1}. \tag{1.1.24}$$

The upper sum has then the form:

$$U_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f_i^{max} = \sum_{i=1}^{n} (x_i - x_{i-1}) \log x_i \tag{1.1.25}$$

and the lower one:

$$L_n = \sum_{i=1}^{n} (x_i - x_{i-1}) f_i^{min} = \sum_{i=1}^{n} (x_i - x_{i-1}) \log x_{i-1}. \tag{1.1.26}$$

In order to demonstrate that

$$\lim_{n\to\infty} (U_n - L_n) = 0, \tag{1.1.27}$$

similarly as in the previous examples, we are going to study the difference:

$$\begin{aligned} U_n - L_n &= \sum_{i=1}^{n} (x_i - x_{i-1}) \log x_i - \sum_{i=1}^{n} (x_i - x_{i-1}) \log x_{i-1} \\ &= \sum_{i=1}^{n} (x_i - x_{i-1}) \log \frac{x_i}{x_{i-1}}. \end{aligned} \tag{1.1.28}$$

Now suppose that, after the appropriate densification of the partition, one has $x_i - x_{i-1} < \delta$, i.e., $x_i < x_{i-1} + \delta$ for some $\delta > 0$. Again making use of the monotonicity of the log function, one gets the estimate:

$$\begin{aligned} 0 \le U_n - L_n &< \sum_{i=1}^{n} (x_i - x_{i-1}) \log \frac{x_{i-1} + \delta}{x_{i-1}} = \sum_{i=1}^{n} (x_i - x_{i-1}) \log \left(1 + \frac{\delta}{x_{i-1}}\right) \\ &\le \sum_{i=1}^{n} (x_i - x_{i-1}) \log \left(1 + \frac{\delta}{1}\right) = \log(1+\delta) \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= 1 \cdot \log(1+\delta) = \log(1+\delta), \end{aligned} \tag{1.1.29}$$

where, while passing to the second line, we used the inequality: $x_{i-1} \ge 1$.

From the obtained result, it can be seen that for any small $\epsilon > 0$ a certain $\delta > 0$ can be chosen in such a way that $\epsilon = \log(1+\delta)$ (i.e., $\delta = e^{\epsilon} - 1 > 0$), and the condition

$$U_n - L_n < \epsilon \tag{1.1.30}$$

will be met. This proves that the function is integrable.

Problem 4

Let a set A be defined as follows:

$$A = \{1/n \mid n \in \mathbb{N}\}. \tag{1.1.31}$$

The integrability of the characteristic function of this set $\chi_A(x)$ on the interval $P = [0, 1]$ will be examined.

Solution

In the previous problems, the integrability of functions could be demonstrated quite easily. As one can clearly see, a common feature that united these examples was the *continuity* of considered functions. Thereby the question arises whether, by any chance, the integrability of arbitrary continuous (and therefore, bounded) function on $[a, b]$ can be demonstrated. As the reader probably knows from the lecture of analysis, the answer to this question is affirmative. In this exercise, we are going to examine the less obvious case: the integrability of a function with discontinuities (and even an infinite number of them). However, one must start by revisiting the definition of the characteristic function for a given set C:

$$\chi_C(x) = \begin{cases} 1 \text{ for } x \in C, \\ 0 \text{ for } x \notin C. \end{cases} \tag{1.1.32}$$

From this definition, it follows that the largest and smallest values of the function on its entire domain are

$$f^{max} = 1 \quad \text{and} \quad f^{min} = 0. \tag{1.1.33}$$

Imagine now that P is partitioned into subintervals P_i, where $i = 1, 2, \ldots, N$. The split points are denoted, as previously, with symbols x_i and it is also assumed that

$$0 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{N-1} < x_N = 1.$$

If in a given interval P_i there appears a number $1/n$ for $n \in \mathbb{N}$, naturally, one will have $f_i^{max} = 1$. Otherwise, $f_i^{max} = 0$. The minimal value for any P_i always equals zero ($f_I^{min} = 0$) since in each interval there exists a number unequal to $1/n$.

Let us now form the upper and the lower sum:

$$U_N = \sum_{i=1}^{N} (x_i - x_{i-1}) f_i^{max}, \quad L_N = \sum_{i=1}^{N} (x_i - x_{i-1}) f_i^{min} = 0. \tag{1.1.34}$$

It must be checked if $|U_N - L_N| = |U_N| < \epsilon$ for any $\epsilon > 0$, as long as sufficiently large N has been chosen, thereby shortening the length of all subintervals. The construction of the Riemann integral requires that the length of the largest one (its length will be denoted with δ) tends to zero. At a certain moment, it must then become smaller than the (positive) expression:

$$\frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{2} \cdot \frac{1}{k(k-1)}, \tag{1.1.35}$$

where k is a (large) natural number chosen by us. Expression (1.1.35) is simply a half of the distance between numbers $1/k$ and $1/(k-1)$. This guarantees that they cannot be simultaneously found in any of the intervals P_i, nor can it happen for numbers $1/(k-2), 1/(k-3), \ldots, 1$, for which "distances" are still greater. This situation is shown in Fig. 1.2. The marked intervals lying to the right of $1/k$ and giving nonzero contribution each contain exactly one number of the form $1/n$. On the contrary, the interval $[0, 1/k]$ (or $[0, 1/(k-1)[$) contains an infinite number of them. It breaks down to the sum of intervals of lengths smaller than δ too, but for the estimate (1.1.36) it does not matter.

The upper sum in our case is the sum of the lengths of all intervals that contain at least one number $1/n$, and the lower sum, as we know, is zero. Thus, one has the following estimate as δ is smaller than expression (1.1.35):

$$\begin{aligned} |U_N - L_N| &< \frac{1}{k-1} + (k-1)\delta < \frac{1}{k-1} + \frac{k-1}{2k(k-1)} \\ &= \frac{1}{k-1} + \frac{1}{2k} < \frac{1}{k-1} + \frac{1}{k-1} < \frac{2}{k-1}. \end{aligned} \tag{1.1.36}$$

Now, if it is required that $|U_N - L_N| < \epsilon$, one can easily meet this condition by selecting $k > (2+\epsilon)/\epsilon$, and then

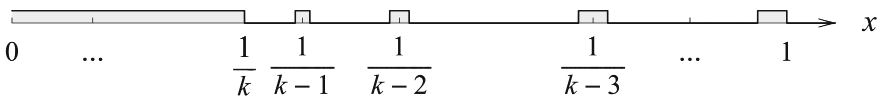

Fig. 1.2 Sample intervals containing numbers of the form $1/n$, where $n \in \mathbb{N}$

$$\delta < \frac{1}{2} \cdot \frac{1}{k(k-1)} < \frac{\epsilon^2}{4(2+\epsilon)}.$$

Thus, we deduce that $|U_N - L_N| \underset{N\to\infty}{\longrightarrow} 0$, and the function in the text of the exercise is integrable.

At the end, it is worth making some digression. If one was dealing with the characteristic function of the set of rational numbers, then for each subinterval and for any partition of [0, 1] one would have

$$f_i^{max} = 1, \quad f_i^{min} = 0, \tag{1.1.37}$$

for, in any of them, both rational and irrational numbers could be found. This means that the upper sum would be constantly equal to unity and the lower one to zero. Therefore,

$$|U_n - L_N| = 1 \underset{N\to\infty}{\longrightarrow} 1 \neq 0, \tag{1.1.38}$$

and the function would turn out to be non-integrable. It bears the name of the Dirichlet function.

1.2 Finding Riemann Integrals by Definition

Problem 1

The Riemann integral

$$\int\limits_{[1,2]} x^\alpha \, dx, \tag{1.2.1}$$

where $\alpha \in \mathbb{R}$ will be found.

Solution

When solving the examples in the previous section, we learned how to show that the function is (or is not) integrable in the Riemann sense on an interval $[a, b]$. While performing proofs, however, nowhere had any specific form of the partition of $[a, b]$ been assumed, except the points x_i must not overlap (which, in any case, is not important) and the largest contract to zero as the partition points become more dense (which is important). This means that the outcome of the procedure, for an integrable function, will be identical for each partition meeting the above conditions. This fact opens the possibility of relatively simple calculation of the values of integrals (and not only demonstrating that they exist) if functions are not too complicated. It is sufficient to carry out the procedure for such a partition, for which it is the easiest to do, and the result of the calculation will be the same as for any other. This method will be followed when calculating the integral (1.2.1).

The function in question is continuous and, therefore, integrable. Thus, one is allowed to choose such a partition which is convenient for our goal. We have of course

$$1 = x_0 < x_1 < \ldots < x_{i-1} < x_i < \ldots < x_{N-1} < x_N = 2, \tag{1.2.2}$$

but still specific values x_i must be chosen. There appears a question of what should guide us. Naturally no rigid rule can be given here, and certainly the specificity of a function should be taken into account. In the present case, some hint may constitute its power-law character. If the interval $[1, 2]$ was divided in a "power-law" way as well (which in this exercise means the "geometrical" way), there would be a good chance that the Riemann sum will turn out to be simply the sum of N terms of a geometric sequence, and this could be calculated without difficulty. Let us assume, therefore, $x_i = q^i$, and the quotient (i.e., the number q) will be fixed below. For $i = 0$, one automatically gets $x_0 = q^0 = 1$. In addition, there must be $x_N = q^N = 2$, and hence $q = \sqrt[N]{2}$. The following partition points are then chosen:

$$x_i = (\sqrt[N]{2})^i, \quad \text{for } \ i = 0, 1, \ldots, N. \tag{1.2.3}$$

Let us now form the Riemann sum: upper, lower, or any intermediate one (see Problem 1 in Sect. 1.1). For all integrable functions they must lead to the same limit. To calculate it, any values taken by the function in the interval $[x_{i-1}, x_i]$ can be used. In particular, we choose for this goal $f(x_i)$'s, i.e., the values $\left[(\sqrt[n]{2})^i\right]^\alpha$. We have then

$$\begin{aligned} I_N &:= \sum_{i=1}^{N} (x_i - x_{i-1}) f(x_i) = \sum_{i=1}^{N} \left[(\sqrt[N]{2})^i - (\sqrt[N]{2})^{i-1}\right]\left[(\sqrt[N]{2})^i\right]^\alpha \\ &= (\sqrt[N]{2} - 1) \sum_{i=1}^{N} (\sqrt[N]{2})^{i\alpha+i-1} = \frac{\sqrt[N]{2} - 1}{\sqrt[N]{2}} \sum_{i=1}^{N} \left[(\sqrt[N]{2})^{\alpha+1}\right]^i. \end{aligned} \tag{1.2.4}$$

In the obtained expression, the sum of N first terms of the geometric sequence with the quotient $(\sqrt[N]{2})^{\alpha+1}$ appears. Let us temporarily assume that $\alpha \neq -1$, and the case $\alpha = -1$ will be dealt with later. The sum (1.2.4) can be easily performed:

$$I_N = \frac{\sqrt[N]{2}-1}{\sqrt[N]{2}} (\sqrt[N]{2})^{\alpha+1} \frac{1-\left[(\sqrt[N]{2})^{\alpha+1}\right]^N}{1-(\sqrt[N]{2})^{\alpha+1}} = \frac{\sqrt[N]{2}-1}{\sqrt[N]{2}} (\sqrt[N]{2})^{\alpha+1} \frac{1-2^{\alpha+1}}{1-(\sqrt[N]{2})^{\alpha+1}}. \tag{1.2.5}$$

In order to find the integral, one has to let N go to infinity: $I = \lim_{N\to\infty} I_N$. For this purpose let us rewrite (1.2.5) in the form:

$$I_N = \frac{\sqrt[N]{2}-1}{\sqrt[N]{2^{\alpha+1}}-1} (\sqrt[N]{2})^{\alpha} (2^{\alpha+1}-1). \tag{1.2.6}$$

The limit of the expression $\sqrt[N]{2}$ equals 1, and fractions of the kind:

$$\frac{\sqrt[N]{a}-1}{\sqrt[N]{b}-1}$$

were dealt with in the first part of this book (see Problem 1 in Sect. 14.3). We then know that

$$\lim_{N\to\infty} \frac{\sqrt[N]{a}-1}{\sqrt[N]{b}-1} = \lim_{N\to\infty} \frac{N(\sqrt[N]{a}-1)}{N(\sqrt[N]{b}-1)} = \frac{\log a}{\log b}. \tag{1.2.7}$$

In our case $a = 2$ and $b = 2^{\alpha+1}$. Collecting the results, one finds the value of the integral as:

$$I = \lim_{N\to\infty} I_N = \frac{\log 2}{\log 2^{\alpha+1}} \cdot 1 \cdot (2^{\alpha+1}-1) = \frac{2^{\alpha+1}-1}{\alpha+1}. \tag{1.2.8}$$

The case $\alpha = -1$ still has to be considered. For this value, formula (1.2.4) takes the form:

$$I_N = \frac{\sqrt[N]{2}-1}{\sqrt[N]{2}} \sum_{i=1}^{N} 1 = \frac{N(\sqrt[N]{2}-1)}{\sqrt[N]{2}}. \tag{1.2.9}$$

When calculating the limit of this expression, one finally finds

$$I = \lim_{N\to\infty} I_N = \frac{\log 2}{1} = \log 2. \tag{1.2.10}$$

Problem 2

The Riemann integral

$$\int\limits_{[0,\pi]} \sin x \, dx \tag{1.2.11}$$

will be found.

Solution

The sine function is continuous, and therefore, integrable on any interval $[a, b]$. Each properly made partition of this interval (spoken of in Sect. 1.1) will lead to the identical value of the integral. When selecting this partition for the segment $[0, \pi]$, the following formula, to be proved relatively easily by mathematical induction or by polar decomposition of complex numbers, turns out to be helpful:

$$\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha = \frac{\sin(n\alpha/2)\sin((n+1)\alpha/2)}{\sin(\alpha/2)}, \tag{1.2.12}$$

for $\alpha \neq 2k\pi$ $(k \in \mathbb{Z})$. It will be demonstrated with the use of the former method.

For $n = 1$, the formula is obviously true, since its left-hand side is simply equal to $\sin\alpha$ and the right-hand one to $\sin(\alpha/2)\sin(2\alpha/2)/\sin(\alpha/2) = \sin\alpha$. In the second step, one must show that the inductive hypothesis, which has the form identical to (1.2.12), implies the inductive thesis:

$$\sin\alpha + \sin 2\alpha + \ldots + \sin(n+1)\alpha = \frac{\sin((n+1)\alpha/2)\sin((n+2)\alpha/2)}{\sin(\alpha/2)}. \tag{1.2.13}$$

Left-hand sides of (1.2.12) and (1.2.13) differ only by $\sin(n+1)\alpha$, so we simply add it to both sides of (1.2.12). If one manages to show that

$$\frac{\sin(n\alpha/2)\sin((n+1)\alpha/2)}{\sin(\alpha/2)} + \sin(n+1)\alpha$$
$$= \frac{\sin((n+1)\alpha/2)\sin((n+2)\alpha/2)}{\sin(\alpha/2)}, \tag{1.2.14}$$

this will mean that the right-hand sides are also equal and formula (1.2.12) for any natural n will be demonstrated. Therefore, let us transform the left-hand side of (1.2.14) in the following way:

$$\frac{\sin(n\alpha/2)\sin((n+1)\alpha/2)}{\sin(\alpha/2)} + \sin(n+1)\alpha$$
$$= \frac{\sin(n\alpha/2)\sin((n+1)\alpha/2) + \sin(n+1)\alpha\sin(\alpha/2)}{\sin(\alpha/2)}$$

$$
\begin{aligned}
&= \frac{\sin(n\alpha/2)\sin((n+1)\alpha/2) + 2\sin((n+1)\alpha/2)\cos((n+1)\alpha/2)\sin(\alpha/2)}{\sin(\alpha/2)} \\
&= \frac{\sin((n+1)\alpha/2)}{\sin(\alpha/2)}\left[\sin\left(\frac{(n+1)\alpha}{2} - \frac{\alpha}{2}\right) + 2\cos\frac{(n+1)\alpha}{2}\sin\frac{\alpha}{2}\right] \\
&= \frac{\sin((n+1)\alpha/2)}{\sin(\alpha/2)}\left[\sin\frac{(n+1)\alpha}{2}\cos\frac{\alpha}{2} - \cos\frac{(n+1)\alpha}{2}\sin\frac{\alpha}{2}\right. \\
&\quad \left. + 2\cos\frac{(n+1)\alpha}{2}\sin\frac{\alpha}{2}\right] = \frac{\sin((n+1)\alpha/2)}{\sin(\alpha/2)}\left[\sin\frac{(n+1)\alpha}{2}\cos\frac{\alpha}{2}\right. \\
&\quad \left. + \cos\frac{(n+1)\alpha}{2}\sin\frac{\alpha}{2}\right] = \frac{\sin((n+1)\alpha/2)\sin((n+2)\alpha/2)}{\sin(\alpha/2)}. \qquad (1.2.15)
\end{aligned}
$$

The right-hand side of (1.2.14) has actually be obtained, which means that the proof is complete.

Using formula (1.2.12), one can divide the interval $[0, \pi]$ into intervals of equal lengths (e.g., with points $x_i = \pi i/N$). Now, one knows that the appropriate Riemann sum in this case can be performed; it has the form:

$$
I_N = \sum_{i=1}^{N}(x_i - x_{i-1})\sin x_i = \sum_{i=1}^{N}\left(\frac{\pi i}{N} - \frac{\pi(i-1)}{N}\right)\sin\frac{\pi i}{N} = \frac{\pi}{N}\sum_{i=1}^{N}\sin\frac{\pi i}{N}. \qquad (1.2.16)
$$

As the reader knows from the previous exercise and probably from the lecture of analysis, in each subinterval $[x_{i-1}, x_i]$, any function value can be chosen to create the sum (we choose $f(x_i)$). Thanks to this, the expression obtained has the necessary form. Using formula (1.2.12) for $\alpha = \pi/N$, one gets

$$
\begin{aligned}
I_N &= \frac{\pi}{N}\cdot\frac{\sin(N\pi/(2N))\sin((N+1)\pi/(2N))}{\sin(\pi/(2N))} \\
&= 2\,\frac{\pi/(2N)}{\sin(\pi/(2N))}\,\sin((N+1)\pi/(2N)), \qquad (1.2.17)
\end{aligned}
$$

where it has been put $\sin(N\pi/(2N)) = \sin(\pi/2) = 1$. When $N \to \infty$, the last factor in (1.2.17) becomes equal to 1, since the argument of the sine function also goes to $\pi/2$ (and the function is continuous), while the limit of the fraction is identical (see Part I, Problem 5 in Sect. 5.1). One gets, therefore, the following value of the integral (1.2.11):

$$
I = \lim_{N\to\infty} I_N = 2\cdot 1\cdot 1 = 2. \qquad (1.2.18)
$$

Problem 3

The moment of inertia of a uniform cylinder of mass M and radius R with respect to its axis of symmetry will be found.

Solution

To calculate the moment of inertia, we will make use of Riemann's idea. As we know from classical mechanics, the moment of inertia of a solid with respect to a certain rotation axis is defined as follows:

$$I = \sum_{i=0}^{N-1} m_i r_i^2, \tag{1.2.19}$$

where the summation is performed over all N "pieces" of the solid, into which it has been contractually divided and which can be considered as points. The symbol m_i stands for the mass of the i-th "point" and r_i—its distance from the rotation axis. Strictly speaking, it is not necessary for masses to be point-like, to which we return later.

This expression is reminiscent of a Riemann sum, and this similarity will become fully visible while solving this exercise. This time, the physically existing solid with the well-defined moment of inertia is divided into parts. By adding together the contributions coming from "point-like" masses, its value certainly will be recovered regardless of what partition was used as long as it will be sufficiently dense so that for each mass m_i, its distance r_i from the axis is unambiguous. This means—similarly as in the calculation of Riemann's integral for a continuous function—that one can choose a partition that is the most convenient for us. One for which it is the easiest to carry out the sum (1.2.19) is similar to what is shown in Fig. 1.3.

Due to the axial symmetry, one chooses as elementary masses m_i the cylindrical surfaces with radii r_i, altitude equal to the cylinder height (be it h) and the thicknesses $\triangle r_i$. As for the latter ones, it is assumed that $\triangle r_i \ll R$ (i.e., walls are thin). The reader might question that choice: is it legitimate to treat a cylinder with even the thinnest wall as point-like? However, the proposed partition is correct and it can be easily justified. From the formal point of view, these cylindrical surfaces should actually be subdivided into still smaller parts, but each of these fragments is equally distant from the axis of rotation (has the same r_i). Therefore, they can be summed up in the first instance and just the mass of the i-th layer multiplied by r_i^2 is obtained.

Now, let us agree that

$$0 = r_0 < r_1 < \ldots < r_{i-1} < r_i < \ldots < r_{N-1} < r_N = R. \tag{1.2.20}$$

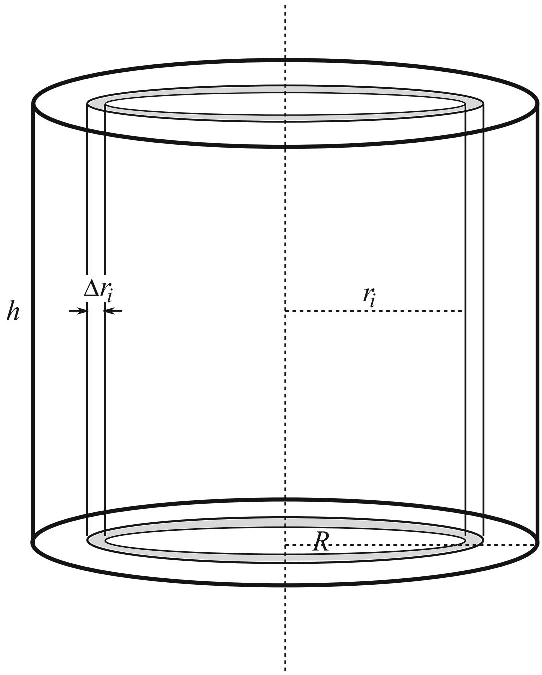

Fig. 1.3 The partition of the cylinder into fragments of well-defined distances from the axis of rotation

The cylinder is homogeneous, so its density is obtained from the formula:

$$\rho = \frac{M}{\pi R^2 h}. \tag{1.2.21}$$

On the other hand, m_i arises as the difference of masses of the cylinder of the radius $r_i + \triangle r_i$ and that of r_i:

$$m_i = [\pi(r_i + \triangle r_i)^2 h - \pi r_i^2 h]\rho = \pi h \,\triangle r_i (2r_i + \triangle r_i)\rho = \frac{M}{R^2}\,\triangle r_i (2r_i + \triangle r_i). \tag{1.2.22}$$

If one omits the expression $\triangle r_i$ in brackets (which will be found to be completely negligible), it is apparent that (1.2.19), after inserting m_i in the form (1.2.22), becomes the Riemann sum for the function $f(r) = 2Mr^3/R^2$:

$$\sum_{i=0}^{N-1} f(r_i)\,\triangle r_i = 2\,\frac{M}{R^2}\sum_{i=0}^{N-1} r_i^3\,\triangle r_i.$$

In accordance with previous observations, the partition points r_i of the interval $[0, R]$ can now be selected in the form of

$$r_i = R\,\frac{i}{N}, \qquad i = 0, 1, 2, \ldots, N, \tag{1.2.23}$$

which implies that $\triangle r_i = R/N$. For $N \to \infty$ one actually has $\triangle r_i \ll R$. With this choice, the sum (1.2.19) will take the form:

$$I = \sum_{i=0}^{N-1} \frac{M}{R^2}\,\triangle r_i (2r_i + \triangle r_i) r_i^2 = \frac{M}{R^2}\left(\frac{R}{N}\right)^4 \sum_{i=0}^{N-1} (2i+1)i^2 = \alpha M R^2, \tag{1.2.24}$$

where α is a dimensionless coefficient:

$$\begin{aligned}\alpha &= \frac{1}{N^4}\sum_{i=0}^{N-1}(2i+1)i^2 = \frac{1}{N^4}\sum_{i=0}^{N-1}(2i^3+i^2)\\ &= \frac{1}{N^4}\left(2\,\frac{(N-1)^2N^2}{4} + \frac{(N-1)N(2N-1)}{6}\right).\end{aligned} \tag{1.2.25}$$

Here we have made use of the known formulas, easy to demonstrate by mathematical induction:

$$\begin{aligned}1^3 + 2^3 + 3^3 + \ldots + n^3 &= \frac{n^2(n+1)^2}{4},\\ 1^2 + 2^2 + 3^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6}.\end{aligned} \tag{1.2.26}$$

The exact value of the moment of inertia is obtained for $N \to \infty$, when layers become infinitely thin and r_i's are actually well defined. Now from Exercise 1 in Sect. 5.2 of the first part of this book, we know that the limit of a rational function with two polynomials of the same degree in variable N is the ratio of coefficients at the highest powers. From the formula for α, it is easily seen that one is dealing with fourth degree polynomials and the ratio of the coefficients at N^4 is $1/2$. It comes from the first term in brackets in (1.2.25). The second term is a polynomial of the third degree, so it does not contribute to the limit. It has already been mentioned that the term $\triangle r_i$ in brackets in (1.2.22) can be omitted as a small quantity of higher order. The moment of inertia in question is, therefore, equal to

$$I = \frac{1}{2} M R^2. \tag{1.2.27}$$

1.3 Finding Limits of Certain Sequences

Problem 1

The limit of the sequence:

$$a_n = \frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \ldots + \frac{n}{n^2+n^2} \tag{1.3.1}$$

will be found.

Solution

The calculation of limits of sequences was dealt with in the first part of this book series. However, there exists a certain class of sequences for which the limits can be easily found with the use of the Riemann sum and integral, and it is why their study was postponed until now. The idea consists of writing down the Riemann integral in two ways: as the Riemann sum and as the appropriate definite integral. As we know from the fundamental theorem of calculus, the Riemann integral of an integrable function is simply equal to the corresponding definite integral:

$$\int\limits_{[a,b]} f(x)dx = \int\limits_a^b f(x)dx = F(b) - F(a), \tag{1.3.2}$$

where $F(x)$ denotes the primitive function for $f(x)$ and thus satisfies the relation $F'(x) = f(x)$. Now we will learn how to apply this method.

First of all, one should consider whether any Riemann sum can be recognized in expression (1.3.1). If so, it should be possible to give it the form:

$$\sum_{i=1}^{n} (x_i - x_{i-1}) f(\tilde{x}_i), \tag{1.3.3}$$

in which x_i's are points of a certain interval $[a, b]$ (a and b still have to be determined), and $\tilde{x}_i \in [x_{i-1}, x_i]$. Let us now rewrite the formula for a_n as

$$\begin{aligned} a_n &= \frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \ldots + \frac{n}{n^2+n^2} \\ &= \frac{1}{n^2}\left(\frac{n}{1+1^2/n^2} + \frac{n}{1+2^2/n^2} + \ldots + \frac{n}{1+n^2/n^2}\right) = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(i/n)^2}. \end{aligned} \tag{1.3.4}$$

Comparing this expression with (1.3.3), one sees that if we assume $x_0 = 0$ and

$$x_i - x_{i-1} = \frac{1}{n}, \quad \text{so} \quad x_i = \frac{i}{n}, \quad \text{for } i = 1, \ldots, n \tag{1.3.5}$$

with $\tilde{x}_i = x_i$, then we actually have to do with the Riemann sum for the function defined by the formula $f(x) = 1/(1+x^2)$ with the partition of a certain interval into n subintervals of equal lengths $1/n$. What is more, since $x_0 = 0$ and $x_n = 1$, it is clear that the partitioned interval is simply [0, 1].

The function f is continuous on this interval and, therefore, is Riemann integrable. Therefore, each partition and each choice of $\tilde{x}_i$'s (if only $\tilde{x}_i \in [x_{i-1}, x_i]$)—in particular, that given by (1.3.5)—will lead to the same value of the integral. As a result, one has

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(i/n)^2} = \int\limits_{[0,1]} \frac{1}{1+x^2} dx. \tag{1.3.6}$$

This formula contains the essence of the applied method: instead of calculating the limit, the Riemann integral on the right-hand side can be computed, and this is simply the definite integral of the function f:

$$\lim_{n\to\infty} a_n = \int_0^1 \frac{1}{1+x^2}\, dx = \arctan x \Big|_0^1 = \frac{\pi}{4}. \tag{1.3.7}$$

Problem 2

The limit of the sequence:

$$a_n = \frac{1}{\sqrt{2n^2-1^2}} + \frac{1}{\sqrt{2n^2-2^2}} + \ldots + \frac{1}{\sqrt{2n^2-(n-1)^2}} + \frac{1}{\sqrt{2n^2-n^2}} \tag{1.3.8}$$

will be found.

Solution

The expression for a_n has, as in the previous problem, the form of a sum, which offers the opportunity of using the same method. In order to give (1.3.8) the desirable shape (1.3.3), let us extract the factor $1/n$, or even better $1/(n\sqrt{2})$ which—as we expect—will constitute the length of the required subintervals:

$1/(n\sqrt{2}) = x_i - x_{i-1}$. We have then

$$a_n = \frac{1}{n\sqrt{2}} \sum_{i=1}^{n} \frac{1}{\sqrt{1-(i/n)^2/2}} = \frac{1}{n\sqrt{2}} \sum_{i=1}^{n} f(\frac{i}{n\sqrt{2}}), \tag{1.3.9}$$

where $f(x) = 1/\sqrt{1-x^2}$. Comparing the above expression to that of the Riemann sum, one can see that both conform to each other if one makes the following identification:

$$x_0 = 0, \quad x_i = \frac{i}{n\sqrt{2}} \quad \text{and} \quad \tilde{x}_i = x_i = \frac{i}{n\sqrt{2}}, \quad \text{for } i = 1, \ldots, n. \tag{1.3.10}$$

Then, one has to do with the uniform partition of the interval $[a, b]$ with the ends

$$a = x_0 = 0, \quad b = x_n = \frac{1}{\sqrt{2}}. \tag{1.3.11}$$

The function $f(x)$ is continuous on $[0, 1/\sqrt{2}]$ and, therefore, integrable: Riemann's procedure leads to the identical result for each (correct) partition and arbitrary choice of $\tilde{x}_i \in [x_{i-1}, x_i]$. Because the Riemann integral is equal to the definite integral, one can write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n\sqrt{2}} \sum_{i=1}^{n} f(\frac{i}{n\sqrt{2}}) = \int\limits_{[0,1/\sqrt{2}]} \frac{1}{\sqrt{1-x^2}}\, dx$$
$$= \int\limits_{0}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x \Big|_0^{1/\sqrt{2}} = \frac{\pi}{4}. \tag{1.3.12}$$

Problem 3

The limit of the sequence:

$$a_n = \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}} \tag{1.3.13}$$

will be found.

Solution

As we will see below, using the method of the Riemann sum and interval, we will be able to find limits of sequences whose general term has not only the form of a sum but also a product of factors. This possibility is owed to the properties of the logarithmic function which converts products into sums in accordance with the formula:

$$\log(x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_k) = \log x_1 + \log x_2 + \log x_3 + \ldots + \log x_k, \tag{1.3.14}$$

for positive numbers $x_1, x_2, \ldots, x_k$.

The idea is, instead of calculating the limit of a_n, to find the limit of $\log a_n$. If so, one needs to check whether the expression $\log a_n$ has the form of a Riemann sum. It is easy to see that it is actually the case:

$$\begin{aligned}\log a_n &= \log\left(\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}}\right) = \log\left(\frac{(2n)!}{n!n^n}\right)^{1/n} = \frac{1}{n}\left(\log\frac{(2n)!}{n!} - \log n^n\right)\\ &= \frac{1}{n}\big[\log(n+1) + \log(n+2) + \log(n+3) + \ldots + \log(2n) - n\log n\big]\\ &= \frac{1}{n}\sum_{i=1}^{n}\big[\log(n+i) - \log n\big] = \frac{1}{n}\sum_{i=1}^{n}\log\left(1+\frac{i}{n}\right).\end{aligned} \tag{1.3.15}$$

By selecting $x_i = i/n$, one has $x_i - x_{i-1} = 1/n$ and the coefficient in front of the sum in (1.3.15) becomes simply the length of the subinterval $[x_{i-1}, x_i]$. Next, one can take $\tilde{x}_i = x_i = i/n \in [x_{i-1}, x_i]$ and $f(x) = \log(1+x)$. As a result, it can be seen that

$$\log a_n = \sum_{i=1}^{n}(x_i - x_{i-1})f(\tilde{x}_i). \tag{1.3.16}$$

In the limit $n \to \infty$, the integral over the interval $[a, b]$ is obtained, where $a = x_0 = 0$ and $b = x_n = 1$ ($\log(1+x)$ is continuous on this interval, i.e., integrable), easy to be calculated by parts:

$$\begin{aligned}\lim_{n\to\infty}\log a_n &= \int\limits_{[0,1]}\log(1+x)dx = \int\limits_0^1\log(1+x)dx\\ &= [(1+x)\log(1+x) - x]\Big|_0^1 = 2\log 2 - 1 = \log\frac{4}{e}.\end{aligned} \tag{1.3.17}$$

Again, using the continuity of the logarithm this time in the neighborhood of $4/e$, one can write

$$\lim_{n\to\infty} \log a_n = \log\left(\lim_{n\to\infty} a_n\right) = \log\frac{4}{e}. \tag{1.3.18}$$

Finally, thanks to the injectiveness of the logarithm, one finds

$$\lim_{n\to\infty} a_n = \frac{4}{e}. \tag{1.3.19}$$

Problem 4

The limit of the sequence:

$$a_n = \sqrt[n^2]{\left(1+\frac{1^2}{n^2}\right)^1 \cdot \left(1+\frac{3^2}{n^2}\right)^3 \cdot \ldots \cdot \left(1+\frac{(2n-1)^2}{n^2}\right)^{2n-1}} \tag{1.3.20}$$

will be found.

Solution

As in the previous example, considering $\log a_n$, in place of a_n, will allow us to achieve two goals: to get rid of the troublesome root and to replace the product of factors with the appropriate sum. Since one has

$$\begin{aligned}\log a_n &= \log \sqrt[n^2]{\left(1+\frac{1^2}{n^2}\right)^1 \cdot \left(1+\frac{3^2}{n^2}\right)^3 \cdot \ldots \cdot \left(1+\frac{(2n-1)^2}{n^2}\right)^{2n-1}} \\ &= \frac{1}{n^2}\left[\log\left(1+\frac{1^2}{n^2}\right) + \log\left(1+\frac{3^2}{n^2}\right)^3 + \ldots + \log\left(1+\frac{(2n-1)^2}{n^2}\right)^{2n-1}\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{2i-1}{n} \log\left(1+\frac{(2i-1)^2}{n^2}\right),\end{aligned} \tag{1.3.21}$$

this expression becomes (up to the factor $1/2$) the Riemann sum:

$$\sum_{i=1}^{n} (x_i - x_{i-1}) f(\tilde{x}_i) \tag{1.3.22}$$

if it is assumed

$$x_i = \frac{2i}{n}, \quad \tilde{x}_i = \frac{2i-1}{n} \in [x_{i-1}, x_i], \quad f(x) = x\log(1+x^2). \tag{1.3.23}$$

If so, then $x_i - x_{i-1} = 2/n$ and we have

$$\lim_{n\to\infty} \log a_n = \frac{1}{2}\int\limits_{[a,b]} f(x)dx, \tag{1.3.24}$$

with $a = x_0 = 0$ and $b = x_n = 2$. The Riemann integral is, in turn, equal to the definite integral and that can be calculated with the use of the simple substitution $t = x^2$ and the primitive function specified in the previous exercise (see formula (1.3.17)):

$$\begin{aligned}\int\limits_{[0,2]} f(x)\,dx &= \int\limits_0^2 x\log(1+x^2)\,dx = \frac{1}{2}\int\limits_0^4 \log(1+t)\,dt \\ &= \frac{1}{2}[(1+t)\log(1+t) - t]\Big|_0^4 = \frac{5}{2}\log 5 - 2 = \log\left[\sqrt{5}\left(\frac{5}{e}\right)^2\right].\end{aligned} \tag{1.3.25}$$

As a consequence, taking into account the coefficient $1/2$ in (1.3.24) and the properties of continuity and injectivity of the logarithm, one gets

$$\lim_{n\to\infty} a_n = \frac{5^{5/4}}{e}. \tag{1.3.26}$$

1.4 Calculating Various Interesting Definite Integrals

Problem 1

The definite integral:

$$I_{n,m} = \int\limits_0^{\pi/2} \sin^{2n} x \cos^{2m} x\,dx, \tag{1.4.1}$$

where $n, m \in \mathbb{N}$, will be calculated.

Solution

In the first volume, among various methods of finding indefinite integrals, we met the so-called recursive method (see Sect. 14.2). Since it could be applied then to more difficult problems, as finding the primitive function, we hope that, all the more, it can turn out to be useful also for simpler ones, i.e., calculating the definite integrals which are numbers. At first glance, some difficulty in this exercise can come from the fact that in (1.4.1) there are two parameters (n and m) over which the recursion is to be led. One should not, however, worry about it, but derive the recursive formula for either of them. To this end, we will use the fact that functions $\sin x$ and $\cos x$ change into each other after differentiation or integration. One has

$$\begin{aligned}
I_{n,m} &= \int\limits_0^{\pi/2} \sin^{2n} x \cos^{2m} x\, dx = \int\limits_0^{\pi/2} \sin^{2n} x\, \cos x\, \cos^{2m-1} x\, dx \\
&= \frac{1}{2n+1} \int\limits_0^{\pi/2} \left[\sin^{2n+1} x\right]' \cos^{2m-1} x\, dx = \frac{1}{2n+1} \sin^{2n+1} x \cos^{2m-1} x \Big|_0^{\pi/2} \\
&- \frac{1}{2n+1} \int\limits_0^{\pi/2} \sin^{2n+1} x \left[\cos^{2m-1} x\right]' dx \\
&= 0 + \frac{2m-1}{2n+1} \int\limits_0^{\pi/2} \sin^{2(n+1)} x \cos^{2(m-1)} x\, dx = \frac{2m-1}{2n+1} I_{n+1,m-1}. \qquad (1.4.2)
\end{aligned}$$

The formula for the integration by parts and also the fact that $\sin 0 = \cos(\pi/2) = 0$ have been used here.

We have succeeded to relate by an equation the index m with $m-1$, similarly as in the examples from the first part of the book, but for the price of increasing the second index: $n \mapsto n+1$. It is then not a recurrence that can be unraveled in one step. However, formula (1.4.2) enables us to transform the two-index expression $I_{n,m}$ into one bearing one index if the following iteration is made:

$$\begin{aligned}
I_{n,m} &= \frac{2m-1}{2n+1} I_{n+1,m-1} = \frac{(2m-1)(2m-3)}{(2n+1)(2n+3)} I_{n+2,m-2} \\
&= \ldots = \frac{(2m-1)!!(2n-1)!!}{(2(n+m)-1)!!} I_{n+m,0}. \qquad (1.4.3)
\end{aligned}$$

Let us denote now

$$\tilde{I}_k := I_{k,0} = \int\limits_0^{\pi/2} \sin^{2k} x\, dx, \tag{1.4.4}$$

and thus, the question of calculating the integral (1.4.1) boils down to determining and resolving the recursive equation for $\tilde{I}_k$. The method applied for this type of integration is already well known—one just uses the Pythagorean trigonometric identity as follows:

$$\begin{aligned}
\tilde{I}_k &= I_{k,0} = \int\limits_0^{\pi/2} \sin^{2k} x\, dx = \int\limits_0^{\pi/2} \sin^{2k-2} \sin^2 x\, dx \\
&= \int\limits_0^{\pi/2} \sin^{2k-2}(1-\cos^2 x)\, dx = \int\limits_0^{\pi/2} \sin^{2k-2}\, dx - \int\limits_0^{\pi/2} \sin^{2k-2} \cos^2 x\, dx \\
&= \tilde{I}_{k-1} - \frac{1}{2k-1} \int\limits_0^{\pi/2} \left[\sin^{2k-1} x\right]' \cos x\, dx \\
&= \tilde{I}_{k-1} - \underbrace{\frac{1}{2k-1} \sin 2k-1x \cos x \Big|_0^{\pi/2}}_{=0} + \frac{1}{2k-1} \int\limits_0^{\pi/2} \sin^{2k-1} x\, [\cos x]'\, dx \\
&= \tilde{I}_{k-1} - \frac{1}{2k-1} \int\limits_0^{\pi/2} \sin^{2k}\, dx = \tilde{I}_{k-1} - \frac{1}{2k-1} \tilde{I}_k.
\end{aligned} \tag{1.4.5}$$

Solving the equation for $\tilde{I}_k$, one gets the relation

$$\tilde{I}_k = \frac{2k-1}{2k}\, \tilde{I}_{k-1}. \tag{1.4.6}$$

For $k = 0$, the integral is easy to calculate because the integrand function becomes a unity:

$$\tilde{I}_0 = \int\limits_0^{\pi/2} dx = \frac{\pi}{2}, \tag{1.4.7}$$

and so, after the iteration (1.4.6), one has

$$\tilde{I}_k = \frac{(2k-1)!!}{(2k)!!}\,\tilde{I}_0 = \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}. \tag{1.4.8}$$

Using now (1.4.3), the final result for the integral $I_{n,m}$ can be written:

$$\begin{aligned} I_{n,m} &= \frac{(2m-1)!!(2n-1)!!}{(2(n+m)-1)!!} I_{n+m,0} = \frac{(2m-1)!!(2n-1)!!}{(2(n+m)-1)!!}\tilde{I}_{n+m} \\ &= \frac{(2m-1)!!(2n-1)!!}{(2(n+m)-1)!!}\cdot\frac{(2(n+m)-1)!!}{(2(m+m))!!}\cdot\frac{\pi}{2} = \frac{(2m-1)!!(2n-1)!!}{(2(m+m))!!}\cdot\frac{\pi}{2}. \end{aligned} \tag{1.4.9}$$

The reader more familiar with mathematics could, in formula (1.4.1), easily recognize the so-called Euler's Beta function, defined by the formula:

$$B(u,v) = 2\int_0^{\pi/2} \sin^{2u-1} x \cos^{2v-1} x\, dx, \tag{1.4.10}$$

for $u, v > 0$. Thereby, the expression in the text of this exercise is simply

$$I_{n,m} = \frac{1}{2}\, B\left(n+\frac{1}{2}, m+\frac{1}{2}\right), \tag{1.4.11}$$

and the result (1.4.9) can be found in manuals of the so-called special functions.

Problem 2

The definite integrals:

$$\begin{aligned} &\text{(a)} \quad I_{nm} = \int_0^{2\pi} \sin nx \sin mx\, dx, \\ &\text{(b)} \quad J_{nm} = \int_0^{2\pi} \cos nx \cos mx\, dx, \\ &\text{(c)} \quad K_{nm} = \int_0^{\pi} \sin nx \cos mx\, dx, \end{aligned} \tag{1.4.12}$$

where $n, m \in \mathbb{N}$, will be calculated.

Solution

The definite integrals to be found in this exercise are not particularly difficult. However, it is useful to calculate them in this section since they will find the application to Fourier series examined in the third part of this book series. Notably, the results obtained in (a) prove that—having properly defined the scalar product which will be referred to later—the functions $\sin nx$ and $\sin mx$ are orthogonal, for $m \neq n$. The same result is got in (b) for the functions $\cos nx$ and $\cos mx$. The calculation of the integral in (c) is, in turn, nothing else than the proof of orthogonality of $\sin nx$ and $\cos mx$ for any $n, m \in \mathbb{N}$.

Integral (a)

As we remember from the first volume, it is much easier to integrate a single trigonometric function than a product of them. For this reason the expression $\sin nx \sin mx$ will be replaced by the following sum of terms, using the known trigonometric formula:

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}. \tag{1.4.13}$$

In the first integrand expression in (1.4.12), one can recognize (up to a factor) the right-hand side of (1.4.13) if one makes the identification:

$$\alpha = (n+m)x, \quad \beta = (n-m)x. \tag{1.4.14}$$

Applying the relations (1.4.13) and (1.4.14), one obtains

$$I_{nm} = \int_0^{2\pi} \sin nx \sin mx\, dx = -\frac{1}{2}\int_0^{2\pi} [\cos(n+m)x - \cos(n-m)x]\, dx$$
$$= -\frac{1}{2}\int_0^{2\pi} \cos(n+m)x\, dx + \frac{1}{2}\int_0^{2\pi} \cos(n-m)x\, dx. \tag{1.4.15}$$

The first integral naturally vanishes since it is that of the cosine function over its entire period (or even $n+m$ periods because the function $\cos kx$ has a period $2\pi/k$). The second integral again equals zero for the same reason, except for one particular situation: when $n = m$. In this special case, the integrand function equals $\cos 0 = 1$, and the integration gives simply the length of the interval (i.e., 2π). As a result, we obtain

$$
\begin{aligned}
I_{nm} &= 0 + \frac{1}{2}\, 2\pi = \pi, \quad \text{for } \; n = m, \\
I_{nm} &= 0 + 0 = 0, \quad \text{for } \; n \neq m.
\end{aligned} \tag{1.4.16}
$$

Integral (b)

Here, we are going to proceed in the same way as above with the difference that instead of (1.4.13) we will use the formula:

$$
\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}. \tag{1.4.17}
$$

Thus, one has

$$
\begin{aligned}
J_{nm} &= \int_0^{2\pi} \cos nx \cos mx\, dx = \frac{1}{2}\int_0^{2\pi} (\cos(n+m)x + \cos(n-m)x)\, dx \\
&= \frac{1}{2}\int_0^{2\pi} \cos(n+m)x\, dx + \frac{1}{2}\int_0^{2\pi} \cos(n-m)x\, dx.
\end{aligned} \tag{1.4.18}
$$

Thanks to the identical arguments as above, one gets the result:

$$
\begin{aligned}
J_{nm} &= 0 + \frac{1}{2}\, 2\pi = \pi, \quad \text{for } \; n = m, \\
J_{nm} &= 0 + 0 = 0, \quad \text{for } \; n \neq m.
\end{aligned} \tag{1.4.19}
$$

Integral (c)

This time we will use the formula:

$$
\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}, \tag{1.4.20}
$$

and as a result one can write

$$
K_{nm} = \int_0^{2\pi} \sin nx \cos mx\, dx = \frac{1}{2}\int_0^{2\pi} (\sin(n+m)x + \sin(n-m)x)\, dx
$$

$$= \frac{1}{2}\int_0^{2\pi} \sin(n+m)x\,dx + \frac{1}{2}\int_0^{2\pi} \sin(n-m)x\,dx = 0 + 0 = 0. \tag{1.4.21}$$

For $n \neq m$, one gets zero exactly as for the cosine function, and in the case $n = m$ again zero, since $\sin 0 = 0$. These results should be kept in mind since they will become useful in the third volume.

Problem 3

The definite integral:

$$I = \int_0^{\pi} \frac{\sin x}{\sqrt{1 - 2a\cos x + a^2}}\,dx, \tag{1.4.22}$$

for $a \neq \pm 1$, will be calculated.

Solution

The expressions containing denominators in the form:

$$\sqrt{1 - 2a\cos x + a^2}$$

appear often in physics, in particular when one calculates the gravitational or electrical potential originating from a certain distribution of mass or charge. Since the potential of the point charge (or point mass) is proportional to $|\vec{R} - \vec{r}|^{-1}$, $\vec{R}$ being the vector of the observation point and $\vec{r}$ that of the charge position, such an expression inevitably emerges under the appropriate volume integral (such integrals are dealt with in Chaps. 12 and 13). Usually, it is transformed as follows:

$$\frac{1}{|\vec{R} - \vec{r}|} = \frac{1}{\sqrt{\vec{R}^2 - 2\vec{R}\cdot\vec{r} + \vec{r}^2}} = \frac{1}{R}\cdot\frac{1}{\sqrt{1 - 2\vec{R}\cdot\vec{r}/R^2 + r^2/R^2}}. \tag{1.4.23}$$

The symbols R and r denote the lengths of the vectors $\vec{R}$ and $\vec{r}$. Now, using spherical variables (rather than Cartesian $\vec{r}$) which are defined in Fig. 6.1 on page 150 and selecting the z-axis along the vector $\vec{R}$, one comes to the following integral over the angle θ (this is the angle between vectors $\vec{R}$ and $\vec{r}$ as shown in Fig. 1.4):

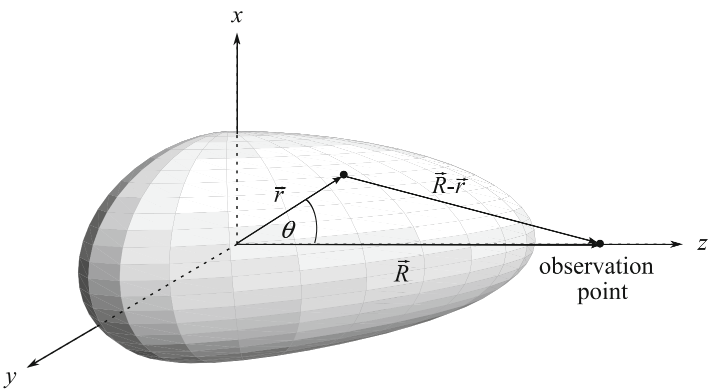

Fig. 1.4 Definition of the vectors $\vec{R}$ and $\vec{r}$ and of the angle θ

$$\int_0^{\pi} \frac{f(\theta)}{\sqrt{1 - 2r/R\,\cos\theta + r^2/R^2}}\sin\theta\, d\theta. \tag{1.4.24}$$

The function $f(\theta)$ contains the details of the particular charge distribution (it may also depend on the azimuthal angle φ, but this is not relevant for this exercise), and in the simplest case, when this distribution is spherically symmetric, $f(\theta)$ is simply equal to 1. The additional $\sin\theta$ comes from the Jacobian determinant due to the change of variables (see Chap. 13). These issues will be examined in detail in the final chapters of this book. Keep in mind that this example is no purely academic and will be useful later. For now, it is sufficient to observe that the integral in the text of the problem and that given by (1.4.24) are the same upon the designation $a = r/R$ (or $a = -r/R$, for $a < 0$).

Going ahead to the calculation of the integral I, one sees that it depends only on the cosine and the sine of the variable x, the latter entering into the integrand expression linearly. As we know from Sect. 14.4 of the first volume, in such situations the substitution $t = \cos x$ gets imposed. Since $t' = -\sin x$, then one gets

$$I = \int_1^{-1} \frac{1}{\sqrt{1 - 2at + a^2}}\,(-1)\, dt = \int_{-1}^{1} \frac{1}{\sqrt{1 - 2at + a^2}}\, dt. \tag{1.4.25}$$

For $a = 0$ the value of this integral equals 2, and for $a \neq 0$ the root under the integral can be rewritten in the form of a derivative:

$$
\begin{aligned}
\frac{1}{\sqrt{1-2at+a^2}} &= -\frac{1}{2a}\,2\frac{d}{dt}\left[\sqrt{1-2at+a^2}\right] \\
&= -\frac{1}{a}\cdot\frac{d}{dt}\left[\sqrt{1-2at+a^2}\right].
\end{aligned} \tag{1.4.26}
$$

As a result the integral can be found immediately:

$$
\begin{aligned}
I &= -\frac{1}{a}\int_{-1}^{1}\frac{d}{dt}\left[\sqrt{1-2at+a^2}\right]dt = -\frac{1}{a}\,\sqrt{1-2at+a^2}\Big|_{-1}^{1} \\
&= -\frac{1}{a}\left[\sqrt{(1-a)^2}-\sqrt{(1+a)^2}\right] = \frac{|1+a|-|1-a|}{a}.
\end{aligned} \tag{1.4.27}
$$

According to the exercise, $a \neq \pm 1$, so this gives (including also the case $a = 0$)

$$
I = \begin{cases} 2/|a|, & \text{when } a \in]-\infty,-1[\,\cup\,]1,\infty[, \\ 2, & \text{when } \quad a \in]-1,1[. \end{cases} \tag{1.4.28}
$$

The case $|a| < 1$ in our physical picture means that we measure the potential from outside of the domain of the charge (or mass) distribution ($R > r$), and $|a| > 1$ conversely. These situations will be dealt with in detail in Exercises 1 and 2 in Sect. 13.5.

The particular situation $a = \pm 1$ (i.e., when $r = R$) has been omitted here. The integral I still does exist then, and its value can be found by letting in the upper or lower formula of (1.4.28) a pass to ± 1. However, in this case, one has to deal with an improper integral because the integrand expression diverges on the integration interval. These kinds of integrals will constitute the subject of the next chapter.

Problem 4

The definite integral:

$$
I_n = \int_{-z}^{z}(x^2-z^2)^n\,dx, \tag{1.4.29}
$$

where $z > 0$ and $n \in \mathbb{N}$, will be calculated.

Solution

In order to find the integral (1.4.29), the recursive method will be used already known from the first part of this book (see Sect. 14.2). In the current exercise, it should be easier to apply, since one does not deal here with indefinite integrals (i.e., functions) but definite ones (i.e., numbers). It was already spoken of in the first problem of this section.

The simplest way to derive a recursive formula is to first insert the unity, written as $1 = [x]'$, into the integrand function and next integrate by parts. In this way, the relation between I_n and I_{n-1} will be obtained:

$$\begin{aligned}
I_n &= \int_{-z}^{z} (x^2 - z^2)^n \, dx = \int_{-z}^{z} [x]'(x^2 - z^2)^n \, dx = x(x^2 - z^2)^n \Big|_{-z}^{z} \\
&- \int_{-z}^{z} x \left[(x^2 - z^2)^n\right]' dx = 0 - 2n \int_{-z}^{z} x^2(x^2 - z^2)^{n-1} \, dx \\
&= -2nz^2 \int_{-z}^{z} (x^2 - z^2)^{n-1} \, dx - 2n \int_{-z}^{z} (x^2 - z^2)^n \, dx = -2nz^2 I_{n-1} - 2n I_n.
\end{aligned} \tag{1.4.30}$$

Solving this equation for I_n, one finds

$$I_n = -\frac{2nz^2}{2n+1} I_{n-1}. \tag{1.4.31}$$

This recursion can be resolved by iterations:

$$\begin{aligned}
I_n &= -\frac{2nz^2}{2n+1} I_{n-1} = (-1)^2 \frac{2nz^2}{2n+1} \cdot \frac{(2n-2)z^2}{2n-1} I_{n-2} \\
&= \ldots = \frac{(-1)^n z^{2n} (2n)!!}{(2n+1)!!} I_0.
\end{aligned} \tag{1.4.32}$$

Calculating explicitly the value of I_0 which is simply equal to the length of the interval (that is just $2z$), one gets

$$I_n = \int_{-z}^{z} (x^2 - z^2)^n \, dx = 2(-1)^n z^{2n+1} \frac{(2n)!!}{(2n+1)!!}. \tag{1.4.33}$$

Problem 5

The definite integral:

$$I = \int_0^{\pi/4} \log(1 + \tan x)\, dx \tag{1.4.34}$$

will be calculated.

Solution

A natural method of calculating definite integrals is to first find the primitive function (i.e., indefinite integral) for integrand expression, and then substitute the values for the interval ends: $\int_a^b f(x)\, dx = F(b) - F(a)$. Unfortunately, attempting to apply this approach to the integral (1.4.34), one would face a serious problem. The primitive function cannot be expressed by elementary functions and one is unable to give it an explicit form. If under the symbol of integration quite natural change of variables is implemented using $t = \tan x$, the obtained indefinite integral would have the form:

$$\int \frac{\log(1+t)}{1+t^2}\, dt, \tag{1.4.35}$$

which leads to the appearance of a special function called "polylogarithm."

However, one should not give up at this point, but rather realize that our job is in fact much simpler: to find (1.4.34), one does not need to know the whole function $F(x)$ but only its values at $x = 0$ and $x = \pi/4$ (strictly speaking one needs to know even less: only the difference $F(\pi/4) - F(0)$). This raises the question of whether one is able to find a definite integral, i.e., a number, without determining first the whole function $F(x)$. In some cases—as in the current problem—it is possible, in others not. An elegant method of finding values of definite integrals for certain types of integrand expressions will be introduced in the third part of this book. It is the method of complex variables and the so-called residue theorem. In this exercise, however, we must deal with some "cunning" way. It consists of the following change of variables: $x' = \pi/4 - x$. It should be noted that the limits of integration do not vary: the integral still runs from 0 to $\pi/4$. Now, if one uses the known formula for $\tan(\alpha - \beta)$:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}, \tag{1.4.36}$$

the expression under the logarithm will change in the following way:

$$1+\tan x \longmapsto 1+\tan\left(\frac{\pi}{4}-x'\right) = 1+\frac{\tan(\pi/4)-\tan x'}{1+\tan(\pi/4)\tan x'}$$
$$= \; 1+\frac{1-\tan x'}{1+\tan x'} = \frac{2}{1+\tan x'}. \tag{1.4.37}$$

Omitting the factor 2, one can see that the inverse of the initial expression has been obtained. Since the logarithm possesses a well-known property: $\log 1/a = -\log a$, there is a good chance that our unknown integral (1.4.34), denoted with symbol I, will be expressed by itself. For, one also has $dx \longmapsto -dx'$, which leads to

$$I = \int_0^{\pi/4} \log(1+\tan x)\,dx = \int_{\pi/4}^{0} \log\left(\frac{2}{1+\tan x'}\right)(-dx')$$
$$= \int_0^{\pi/4} \left[\log 2 - \log(1+\tan x')\right] dx' = \frac{\pi}{4}\log 2 - I. \tag{1.4.38}$$

Solving this equation for I, we finally find

$$I = \int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2. \tag{1.4.39}$$

Problem 6

The definite integral:

$$I = \int_0^{\pi/2} \frac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x}\,dx, \tag{1.4.40}$$

where $\alpha \in \mathbb{R}$, will be calculated.

Solution

Similarly as in the previous exercise, the definite integral (1.4.40) would be difficult to find by calculating first the primitive function, especially because the parameter α can take any real value, even an irrational one. We will try then the similar trick as in the transition from (1.4.34) to (1.4.38). This time, we will make use of the well-known relations:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x. \tag{1.4.41}$$

If a new variable $x' = \pi/2 - x$ is introduced into the integral, we will obtain

$$\begin{aligned}\int\limits_0^{\pi/2} \frac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x}\, dx &= \int\limits_{\pi/2}^{0} \frac{\sin^\alpha(\pi/2 - x')}{\sin^\alpha(\pi/2 - x') + \cos^\alpha(\pi/2 - x')}\,(-dx') \\ &= \int\limits_0^{\pi/2} \frac{\cos^\alpha x'}{\cos^\alpha x' + \sin^\alpha x'}\, dx'.\end{aligned} \tag{1.4.42}$$

The name of the integration variable (x or x') is inessential, so one can write

$$\begin{aligned}\int\limits_0^{\pi/2} \frac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x}\, dx &= \frac{1}{2}\left(\int\limits_0^{\pi/2} \frac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x}\, dx + \int\limits_0^{\pi/2} \frac{\cos^\alpha x}{\sin^\alpha x + \cos^\alpha x}\, dx\right) \\ &= \frac{1}{2}\int\limits_0^{\pi/2} \frac{\sin^\alpha x + \cos^\alpha x}{\sin^\alpha x + \cos^\alpha x}\, dx = \frac{1}{2}\int\limits_0^{\pi/2} 1\, dx = \frac{\pi}{4}.\end{aligned} \tag{1.4.43}$$

Interestingly, the result turned out to be independent of the parameter α.

1.5 Explaining Several Apparent Paradoxes

Problem 1

The definite integral:

$$\int\limits_{-1}^{1} \frac{1}{x^2}\, dx \tag{1.5.1}$$

will be "calculated."

Solution

In this problem, the word "calculated" is used in quotes, since, as we will see in a moment, the integral (1.5.1) actually does not exist. To start with, let us assume that we are ignorant of this and so we try to calculate the integral, first deriving the primitive function and then substituting the limits of integration:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{1} \left[\frac{-1}{x}\right]' dx = -\left.\frac{1}{x}\right|_{-1}^{1} = -1 - 1 = -2 < 0. \tag{1.5.2}$$

At this time, one should already realize that our calculation must be incorrect because the definite integral of a positive function which undoubtedly is $1/x^2$ cannot be negative.

To determine what has gone wrong, let us look at the integrand function. One can clearly see that it is unbounded, when $x \to 0$. If so, one should rather write:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx \tag{1.5.3}$$

and face the problem of calculating the integral in the form

$$\int_{-1}^{0} \frac{1}{x^2}\, dx, \tag{1.5.4}$$

i.e., the improper integral (such integrals are dealt with in the following section). It should be determined according to the rule:

$$\int_{-1}^{0} \frac{1}{x^2}\, dx = \lim_{\epsilon \to 0^+} \int_{-1}^{-\epsilon} \frac{1}{x^2}\, dx = \lim_{\epsilon \to 0^+} \left.\frac{-1}{x}\right|_{-1}^{-\epsilon} = \lim_{\epsilon \to 0^+} \left(\frac{1}{\epsilon} - 1\right). \tag{1.5.5}$$

However, this limit does not exist, which means that the integral is divergent. The same applies to the expression:

$$\int_{0}^{1} \frac{1}{x^2}\, dx = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} \frac{1}{x^2}\, dx = \lim_{\epsilon \to 0^+} \left.\frac{-1}{x}\right|_{\epsilon}^{1} = \lim_{\epsilon \to 0^+} \left(-1 + \frac{1}{\epsilon}\right). \tag{1.5.6}$$

We conclude, therefore, that the initial integral does not exist. This example should warn us against the "automatic" application of the rule:

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{1.5.7}$$

without considering whether given expressions make sense.

Problem 2

The definite integral:

$$\int_0^{2\pi} \frac{1}{3+\sin x}\,dx \tag{1.5.8}$$

will be calculated.

Solution

As in the previous problem, we will first try to find the indefinite integral using the change of variables discussed in detail in the first volume (see Problem 2 in Sect. 14.4): $t = \tan(x/2)$. As we know in this case, the following substitutions should be done under the integral:

$$\sin x \longmapsto \frac{2t}{1+t^2}, \quad dx \longmapsto \frac{2}{1+t^2}\,dt. \tag{1.5.9}$$

In this way, one obtains

$$\begin{aligned} I := \int \frac{1}{3+\sin x}\,dx &= \int \frac{1}{3+2t/(1+t^2)} \cdot \frac{2}{1+t^2}\,dt \\ &= \frac{2}{3}\int \frac{1}{t^2+2t/3+1}\,dt = \frac{2}{3}\int \frac{1}{(t+1/3)^2+8/9}\,dt. \end{aligned} \tag{1.5.10}$$

By introducing the subsequent variable:

$$t + \frac{1}{3} = \frac{2\sqrt{2}}{3}\,u, \tag{1.5.11}$$

one is able to extract the factor of $8/9$ from the denominator and get an expression easy to integrate:

$$
\begin{aligned}
I &= \frac{\sqrt{2}}{2}\int \frac{1}{u^2+1}\,du = \frac{\sqrt{2}}{2}\arctan u + C \\
&= \frac{\sqrt{2}}{2}\arctan\frac{3t+1}{2\sqrt{2}} + C = \frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} + C. \qquad (1.5.12)
\end{aligned}
$$

It would seem that it is sufficient to substitute the limits of integration to get the desired result:

$$
\begin{aligned}
&\int_0^{2\pi} \frac{1}{3+\sin x}\,dx = \left(\frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} + C\right)\Bigg|_0^{2\pi} \\
&= \frac{\sqrt{2}}{2}\arctan\frac{3\tan\pi+1}{2\sqrt{2}} + C - \frac{\sqrt{2}}{2}\arctan\frac{3\tan 0+1}{2\sqrt{2}} - C = 0, \qquad (1.5.13)
\end{aligned}
$$

since $\tan 0 = \tan\pi = 0$. Once again we have come to a paradox: the integral of a positive function has turned out to be equal to zero! An error must have been committed. This time to find it, one needs a more sophisticated eye. Well, the expression that was considered to be a primitive function, i.e.,

$$
\frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} + C, \qquad (1.5.14)
$$

is not continuous in the whole integration interval $[0, 2\pi]$. To confirm this, let us calculate

$$
\begin{aligned}
&\lim_{x\to\pi^-}\frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\cdot\frac{\pi}{2} = \frac{\pi\sqrt{2}}{4}, \\
&\lim_{x\to\pi^+}\frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\cdot\frac{-\pi}{2} = -\frac{\pi\sqrt{2}}{4}, \qquad (1.5.15)
\end{aligned}
$$

where the known limits have been used:

$$
\lim_{x\to\pi/2^-}\tan x = \infty, \qquad \lim_{x\to\pi/2^+}\tan x = -\infty \qquad (1.5.16)
$$

and

$$
\lim_{x\to\pm\infty}\arctan x = \pm\frac{\pi}{2}. \qquad (1.5.17)
$$

The one-sided limits (1.5.15) are then different, and at the point $x = \pi$ itself the function is not defined at all. Even if the definition was supplemented by assigning any value at this point, one would not be in a position to do this in a continuous manner, as the limit of the function at this point does not exist (see Fig. 1.5).

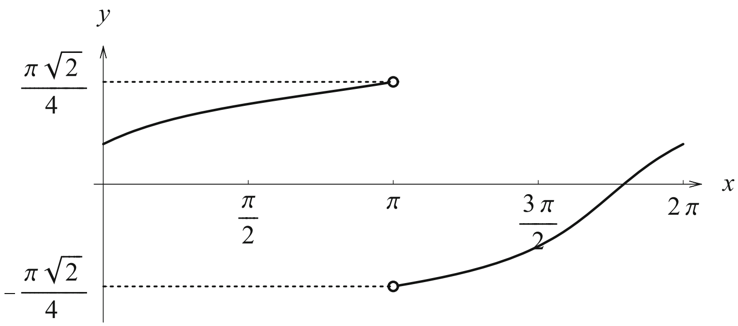

Fig. 1.5 The plot of the function given by (1.5.14) for the constant $C = 0$. For other values of C the graph will be shifted in the vertical direction

Meanwhile, the primitive function in formula (1.5.7) must ultimately be continuous (since it is differentiable!). Thus one has two possible next steps from this situation: either to consider the function found above only on disjoint intervals, on which it is continuous, or to look for a *true* primitive function which would be continuous on the entire interval $[0, 2\pi]$. We are going to deal with these two possibilities in turn.

The arctangent function is continuous everywhere as well as the tangent function on each of the intervals $]-\pi/2 + k\pi, \pi/2 + k\pi[$, where $k \in \mathbb{Z}$. Thereby, the composite function (1.5.14) is certainly continuous in the domains where $\tan(x/2)$ is: $[0, \pi[$ and $]\pi, 2\pi]$. Therefore, one gets:

$$\int_0^{2\pi} \frac{1}{3+\sin x}\,dx = \int_0^{\pi} \frac{1}{3+\sin x}\,dx + \int_{\pi}^{2\pi} \frac{1}{3+\sin x}\,dx$$
$$= \left(\frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} + C\right)\Bigg|_0^{\pi} + \left(\frac{\sqrt{2}}{2}\arctan\frac{3\tan(x/2)+1}{2\sqrt{2}} + C\right)\Bigg|_{\pi}^{2\pi}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\pi}{2} - \frac{\sqrt{2}}{2}\arctan\frac{1}{2\sqrt{2}} + \frac{\sqrt{2}}{2}\arctan\frac{1}{2\sqrt{2}} - \frac{\sqrt{2}}{2}\cdot\frac{-\pi}{2} = \frac{\pi\sqrt{2}}{2}. \tag{1.5.18}$$

This time, the positive value has been obtained in accordance to our expectations.

Now the second method will be used: define the *true* primitive function which is continuous on the interval $[0, 2\pi]$. To this end, it should be noted that our expression (1.5.14) has the correct derivative, the integrand function in (1.5.8), so the primitive function may differ from it by at most a constant. But—and this is a key point here—the constants may vary between the intervals $[0, \pi[$ and $]\pi, 2\pi]$. Therefore, let us write

$$F(x) = \begin{cases} \dfrac{\sqrt{2}}{2}\arctan\dfrac{3\tan(x/2)+1}{2\sqrt{2}} + C_1 & \text{for } \ 0 < x < \pi, \\ \dfrac{\sqrt{2}}{2}\arctan\dfrac{3\tan(x/2)+1}{2\sqrt{2}} + C_2 & \text{for } \pi < x < 2\pi, \\ C_3 & \text{for } \quad x = \pi. \end{cases} \tag{1.5.19}$$

The inserted constants do not affect the continuity of the function, neither on $[0, \pi[$ nor on $]\pi, 2\pi]$. Additionally we want $F(x)$ to be continuous at $x = \pi$. Therefore, the following condition is required:

$$\lim_{x\to\pi^-} F(x) = \lim_{x\to\pi^+} F(x) = F(\pi), \tag{1.5.20}$$

meaning

$$\frac{\pi\sqrt{2}}{4} + C_1 = -\frac{\pi\sqrt{2}}{4} + C_2 = C_3. \tag{1.5.21}$$

For the constants C_2 and C_3, one gets

$$C_2 = C_1 + \frac{\pi\sqrt{2}}{2}, \quad C_3 = C_1 + \frac{\pi\sqrt{2}}{4}. \tag{1.5.22}$$

Now, one is allowed to use the formula:

$$\int_0^{2\pi} \frac{1}{3+\sin x}\,dx = F(2\pi) - F(0) = \frac{\sqrt{2}}{2}\arctan\frac{1}{2\sqrt{2}} + C_2 - \frac{\sqrt{2}}{2}\arctan\frac{1}{2\sqrt{2}} - C_1$$

$$= C_2 - C_1 = \frac{\pi\sqrt{2}}{2}. \tag{1.5.23}$$

As one can see, we have obtained again the result given by (1.5.18). It should be added that the value of the constant C_3 does not play any role in calculating the definite integral because it is taken by the function at a single point (as they say: “on a set of zero measure”).

Problem 3

The definite integral:

$$\int_{-1}^{1} \left[\arctan \frac{1}{x}\right]' dx \tag{1.5.24}$$

will be calculated.

Solution

At first glance, this problem seems trivial. After all, one has immediately the primitive function:

$$F(x) = \arctan \frac{1}{x}. \tag{1.5.25}$$

So it remains only to write

$$\int_{-1}^{1} \left[\arctan \frac{1}{x}\right]' dx = \arctan \frac{1}{x}\bigg|_{-1}^{1} = \arctan 1 - \arctan(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \frac{\pi}{2}. \tag{1.5.26}$$

Apparently, everything seems to be in order. However, more attentive analysis reveals a certain paradox: the function $\arctan(1/x)$ is decreasing, and therefore, its derivative (i.e., the integrand in (1.5.24)) should be negative. This derivative, after all, can be calculated explicitly:

$$\left[\arctan \frac{1}{x}\right]' = \frac{1}{1 + 1/x^2} \cdot \frac{-1}{x^2} = \frac{-1}{x^2 + 1} < 0. \tag{1.5.27}$$

How is it possible that when integrating a negative function, one attains a positive result (1.5.26)? The answer is analogous to that of the previous exercise: the function $F(x)$ which has been acknowledged to be the primitive one is in fact not because it is discontinuous at zero. Since one has

$$\begin{aligned} \lim_{x\to 0^-} F(x) &= \lim_{x\to 0^-} \arctan \frac{1}{x} = -\frac{\pi}{2}, \\ \lim_{x\to 0^+} F(x) &= \lim_{x\to 0^+} \arctan \frac{1}{x} = \frac{\pi}{2}, \end{aligned} \tag{1.5.28}$$

the one-sided limits are different. The function considered is indeterminate at zero, and no matter what value was assigned to it at this point—in this way supplementing its definition—it would never be a continuous function (see Fig. 1.6).

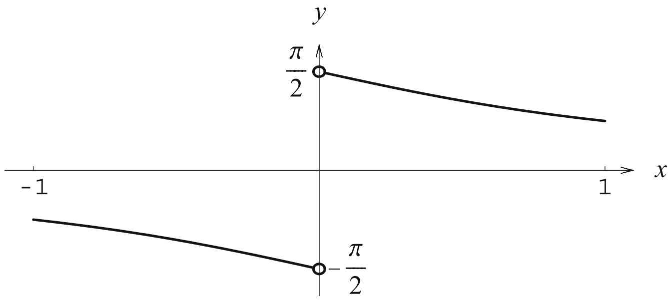

Fig. 1.6 The graph of the function (1.5.25)

As we know from the previous example, one can proceed by splitting the integration interval into such subintervals for which the function $F(x)$ would be continuous. In the present case, these subintervals are $[-1, 0[$ and $]0, 1]$. Thereby, one obtains

$$\int_{-1}^{1} \left[\arctan\frac{1}{x}\right]' dx = \int_{-1}^{0} \left[\arctan\frac{1}{x}\right]' dx + \int_{0}^{1} \left[\arctan\frac{1}{x}\right]' dx$$
$$= \arctan\frac{1}{x}\bigg|_{-1}^{0} + \arctan\frac{1}{x}\bigg|_{0}^{1} = -\frac{\pi}{2} - \frac{-\pi}{4} + \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{2}. \tag{1.5.29}$$

In contrast to (1.5.26), the obtained value is nonnegative.

The second possible way is to derive an expression for the true primitive function on the entire interval $[-1, 1]$. It has the form:

$$F(x) = \begin{cases} \arctan\frac{1}{x} + C_1 & \text{for } -1 \le x < 0, \\ C_3 & \text{for } \quad x = 0, \\ \arctan\frac{1}{x} + C_2 & \text{for } \ 0 < x \le 1, \end{cases} \tag{1.5.30}$$

where the constants C_1, C_2, and C_3 are related via the conditions of continuity:

$$\lim_{x\to 0^-} F(x) = \lim_{x\to 0^+} F(x) = F(0). \tag{1.5.31}$$

This means that:

$$-\frac{\pi}{2} + C_1 = \frac{\pi}{2} + C_2 = C_3, \tag{1.5.32}$$

and consequently

$$C_2 = C_1 - \pi, \quad C_3 = C_1 - \frac{\pi}{2}. \tag{1.5.33}$$

It now remains only to apply the following formula for the definite integral:

$$\int_{-1}^{1}\left[\arctan\frac{1}{x}\right]' dx = F(1) - F(-1) = \frac{\pi}{4} + C_2 - \frac{-\pi}{4} - C_1 = \frac{\pi}{2} + C_2 - C_1$$
$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}. \tag{1.5.34}$$

1.6 Using the (Second) Mean Value Theorem

Problem 1

It will be proved that

$$\lim_{n\to\infty}\int_a^b \frac{\cos nx}{\sqrt{x}}dx = 0, \tag{1.6.1}$$

where $0 < a < b$.

Solution

In this problem, we will apply the second mean value theorem, which was formulated at the beginning of this chapter. Below, we briefly recall that it states that the integral

$$\int_a^b f(x)g(x)dx,$$

where f is a continuous function on the interval $[a, b]$, and g is a monotonic function with continuous derivative, can be given the form

$$\int_a^b f(x)g(x)dx = g(a)\int_a^c f(x)dx + g(b)\int_c^b f(x)dx, \tag{1.6.2}$$

c being a number belonging to the interval $]a, b[$.

If one looks back at the integral (1.6.1), it can be seen that it has the perfect form to apply this theorem. The function $\cos nx$ is obviously continuous irrespective of the value of n and it will play the role of the function f. Moreover, $1/\sqrt{x}$ satisfies the conditions that are required from the function g: monotonicity and continuity of its derivative (for $a, b > 0$). Thus, one can write

$$\int_a^b \frac{\cos nx}{\sqrt{x}} dx = \frac{1}{\sqrt{a}} \int_a^c \cos nx dx + \frac{1}{\sqrt{b}} \int_c^b \cos nx dx. \tag{1.6.3}$$

In this way, the troubling factor $\sqrt{x}$ has been removed from the integrand and the integral of the cosine function can trivially be calculated. One gets

$$\int_a^c \cos nx\, dx = \frac{1}{n} \sin nx \Big|_a^c = \frac{1}{n}(\sin nc - \sin na), \tag{1.6.4}$$

which leads to

$$\left| \int_a^c \cos nx\, dx \right| = \left| \frac{1}{n}(\sin nc - \sin na) \right| \leq \frac{1}{n}(1+1) = \frac{2}{n}. \tag{1.6.5}$$

Identically,

$$\left| \int_c^b \cos nx\, dx \right| \leq \frac{2}{n}. \tag{1.6.6}$$

For the integral in the current exercise, one gets the estimate:

$$\begin{aligned} 0 \leq \left| \int_a^b \frac{\cos nx}{\sqrt{x}} dx \right| &\leq \left| \frac{1}{\sqrt{a}} \int_a^c \cos nx\, dx \right| + \left| \frac{1}{\sqrt{b}} \int_c^b \cos nx\, dx \right| \\ &\leq \frac{2}{n} \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a}} \right) \underset{n\to\infty}{\longrightarrow} 0. \end{aligned} \tag{1.6.7}$$

Recalling now the squeeze theorem for sequences (see Sect. 5.1 in the first part of the book), one can see that

$$\lim_{n\to\infty} \int_a^b \frac{\cos nx}{\sqrt{x}} dx = 0, \tag{1.6.8}$$

which was our goal to demonstrate.

Problem 2

The sign of the definite integral:

$$I = \int_0^{2\pi} x^n \sin^{2m+1} x \, dx \tag{1.6.9}$$

will be determined, for $n, m \in \mathbb{N}$.

Solution

Because of the sine function, the integrand expression takes both positive and negative values within the integration interval. The subject of this exercise is to determine the sign of this integral, and to this end the theorem we became acquainted with in the previous example will be used.

The function $g(x) = x^n$ is increasing and has the continuous derivative, while the function $f(x) = \sin^{2m+1} x$ is continuous. This allows us to write the integral in the form:

$$I = \int_0^{2\pi} x^n \sin^{2m+1} x \, dx = 0^n \int_0^{c} \sin^{2m+1} x \, dx + (2\pi)^n \int_c^{2\pi} \sin^{2m+1} x \, dx, \tag{1.6.10}$$

where $0 < c < 2\pi$. The former of these expressions naturally equals zero and the integral in the latter can be written as

$$\int_c^{2\pi} \sin^{2m+1} x \, dx = \int_0^{2\pi} \sin^{2m+1} x \, dx - \int_0^{c} \sin^{2m+1} x \, dx = 0 - \int_0^{c} \sin^{2m+1} x. \tag{1.6.11}$$

Here we have made use of the fact that

$$\int_0^{2\pi} \sin^{2m+1} x \, dx \underset{x \mapsto 2\pi - x}{=} \int_0^{2\pi} \sin^{2m+1}(2\pi - x) \, dx = -\int_0^{2\pi} \sin^{2m+1} x \, dx$$
$$\Longrightarrow \int_0^{2\pi} \sin^{2m+1} x \, dx = 0. \tag{1.6.12}$$

Depending on what one needs, the integral I can be written in a twofold way. Either in the form of

$$I = (2\pi)^n \int_c^{2\pi} \sin^{2m+1} x \, dx, \tag{1.6.13}$$

or as

$$I = -(2\pi)^n \int_0^c \sin^{2m+1} x \, dx. \tag{1.6.14}$$

Admittedly, one does not know the value of c, but one can consider separate cases, $0 < c \leq \pi$ and $\pi < c < 2\pi$. In the former, we choose the second one of the above forms (i.e., (1.6.14)). Without any problem, one can then conclude that $I < 0$ because in $]0, c[$ the sine is positive as well as the entire integrand function.

If $\pi < c < 2\pi$ the first form of I is used. Equally straightforward, the integrand function can be evaluated as negative, since for $x \in]c, 2\pi[$ one has $\sin x < 0$. In conclusion, it can be said that

$$\int_0^{2\pi} x^n \sin^{2m+1} x \, dx < 0. \tag{1.6.15}$$

Problem 3

It will be proved that for $n \in \mathbb{N}$, the following inequalities hold:

$$0 \leq \int_0^1 \frac{\sin(2\pi n x)}{\sqrt{1+x^2}} \, dx \leq \frac{1}{\pi n}\left(1 - \frac{1}{\sqrt{2}}\right). \tag{1.6.16}$$

Solution

Let us define auxiliary functions $f(x) = \sin(2\pi n x)$ and $g(x) = 1/\sqrt{1+x^2}$. They comply with the assumptions of the mean value theorem dealt with in the preceding exercises, so one can use formula (1.6.2) for $a = 0$ and $b = 1$ and write:

$$\int_0^1 \frac{\sin(2\pi nx)}{\sqrt{1+x^2}}\,dx = \frac{1}{\sqrt{1+0^2}} \cdot \int_0^c \sin(2\pi nx)\,dx + \frac{1}{\sqrt{1+1^2}} \int_c^1 \sin(2\pi nx)\,dx. \tag{1.6.17}$$

The parameter c fulfills the condition $0 < c < 1$. The obtained integrals are very simple and can be easily found, since

$$\int \sin(2\pi nx)\,dx = -\frac{1}{2\pi n}\cos(2\pi nx). \tag{1.6.18}$$

One then gets

$$\begin{aligned}\int_0^1 \frac{\sin(2\pi nx)}{\sqrt{1+x^2}}\,dx &= -\frac{1}{2\pi n}\cos(2\pi nx)\Big|_0^c - \frac{1}{\sqrt{2}}\cdot\frac{1}{2\pi n}\cos(2\pi nx)\Big|_c^1 \\ &= \frac{1}{2\pi n}\left(1-\frac{1}{\sqrt{2}}\right)(1-\cos(2\pi nc)),\end{aligned} \tag{1.6.19}$$

as $\cos(2\pi n) = \cos 0 = 1$.

The parameter c is unknown, so one cannot determine the exact value of the integral. However, the cosine function takes its values from the interval $[-1, 1]$. Therefore, one can use the estimate:

$$0 \le 1 - \cos(2\pi nc) \le 2, \tag{1.6.20}$$

which immediately gives the result (1.6.16) that we were hoping to reach.

1.7 Exercises for Independent Work

Exercise 1 Using Riemann's definition of the integral, examine the integrability of functions:

(a) $f(x) = x^3$, on the interval $[1, 2]$,
(b) $f(x) = \tan x$, on the interval $[-\pi/4, \pi/4]$,
(c) $f(x) = \chi_A(x)$, where $A = \{n2^{-m} \mid n, m \in \mathbb{N}\}$, on the interval $[0, 1]$.

Answers
(a) Integrable,
(b) Integrable,
(c) Non-integrable.

Exercise 2 Calculating Riemann sums, find the values of the integrals:

(a) 1°. $\int\limits_{[0,1]} x^3 dx$, 2°. $\int\limits_{[0,\pi]} \sin x\, dx$ (hint: make use of formula (1.2.12)),
(b) 1°. $\int\limits_{[0,2]} 2^x dx$, 2°. $\int\limits_{[1,2]} \frac{1}{x}\, dx$, 3°. $\int\limits_{[0,1]} \cosh x\, dx$.

Answers
(a) 1°. $1/4$, 2°. 2,
(b) 1°. $3/\log 2$, 2°. $\log 2$, 3°. $\sinh 1$.

Exercise 3 Using the Riemann integral, find $\lim\limits_{n\to\infty} a_n$ for:

(a) $a_n = \left[\sqrt[n]{e} + 2(\sqrt[n]{e})^2 + \ldots + n(\sqrt[n]{e})^n\right]/n^2$,
(b) $b_n = [\sin(\pi/n) + \sin(2\pi/n) + \ldots + \sin(n\pi/n)]/n$.

Answers
(a) $\lim_{n\to\infty} a_n = 1$,
(b) $\lim_{n\to\infty} b_n = 2$.

Exercise 4 Find definite integrals:

(a) 1°. $\int\limits_0^{\pi} \log \sin x\, \cos 2x\, dx$, 2°. $\int\limits_0^{2\pi} \frac{1}{5+\cos x}\, dx$, 3°. $\int\limits_0^{\sqrt{3}} x \arctan^2 x\, dx$,
(b) 1°. $\int\limits_0^{\pi/2} \frac{\sin x}{\sqrt{1+\sin^2 x}}\, dx$, 2°. $\int\limits_0^{\pi/2} \frac{\sqrt{\tan x}}{\sin x(\cos x + \sin x)}\, dx$,
(c) 1°. $\int\limits_0^{\pi/4} \frac{1}{\left(\sqrt{\cos x} + \sqrt{\sin x}\right)^8}\, dx$, 2°. $\int\limits_0^{\pi/2} \log \tan x\, dx$.

Answers
(a) 1°. $-\pi/2$, 2°. $\pi/\sqrt{6}$, 3°. $\log 2 - \pi/\sqrt{3} + 2\pi^2/9$,
(b) 1°. $\pi/4$, 2°. π,
(c) 1°. $1/21$, 2°. 0.

Exercise 5 Using the mean value theorem, demonstrate that

(a) $$-\frac{200}{101} \cdot \frac{1}{\pi} \le \int_{\pi}^{101\pi} \frac{\sin x}{x}\, dx \le 0,$$

(b) $$\frac{3}{\log 2} \le \int_{2}^{8} \frac{x}{\log^2 x}\, dx \le \frac{15}{\log 2}.$$

Chapter 2
Examining Improper Integrals

This chapter is concerned with the so-called improper integrals. They constitute the generalization of the definite integral when either the integration interval is infinite or the integrand function is not bounded. We will learn how to examine the convergence of such integrals and apply these results to certain infinite series.

The **improper integrals** of the first kind are defined as follows:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx, \quad \text{or} \quad \int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx. \tag{2.0.1}$$

The integral over the entire real axis has the form

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b f(x)dx. \tag{2.0.2}$$

In the case of improper integrals of the second kind, the function $f(x)$ diverges in either a or b. Then, one respectively has

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx, \quad \int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx. \tag{2.0.3}$$

If the function diverges to infinity at either end the integral is defined as a double limit:

$$\int_a^b f(x)dx = \lim_{c\to a^+}\lim_{d\to b^-}\int_c^d f(x)dx. \tag{2.0.4}$$

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Various tests for checking the convergence of improper integrals are introduced and discussed when solving specific problems in Sect. 2.2.

The behavior of improper intervals of the first kind can be used to test the convergence of series. It is the so-called **integral test** or **Maclaurin–Cauchy test**. It has the following form:

Let N be a natural number. Assume that a certain function $f : [N, \infty[\to \mathbb{R}$ is nonnegative, continuous, and monotonically decreasing. Then the infinite series $\sum\limits_{N}^{\infty} f(n)$ is convergent if and only if the improper integral $\int\limits_{N}^{\infty} f(x)dx$ exists.

2.1 Investigating the Convergence of Integrals by Definition

Problem 1

The existence of integrals:

$$I_1 = \int\limits_{1}^{\infty} \frac{dx}{x^\alpha}, \quad I_2 = \int\limits_{0}^{1} \frac{dx}{x^\alpha} \tag{2.1.1}$$

will be examined for $\alpha \in \mathbb{R}$. From this, it will be determined for what values of α the following integral is convergent:

$$I_3 = \int\limits_{0}^{\infty} \frac{dx}{x^\alpha}. \tag{2.1.2}$$

Solution

While setting about solving similar problems, it is convenient to determine points that are "dangerous" for the convergence of a given integral. If the integration extends to infinity—as in the case of I_1—this "point" is always suspect and the appropriate limiting procedure must be carefully executed. Such a "dangerous" point can also happen at a finite limit of the integration interval. If the function is unbounded, the "dangerous" point can appear while approaching it from the interior of the integration interval. This case will be encountered in I_2.

First, consider the integral denoted by I_1. The common method involves replacing the upper (infinite) limit of integration with a finite one (called Λ), calculating the definite integral—in so far as this is possible—and then examining the transition $\Lambda \to \infty$. Thus, one has

$$I_1^\Lambda := \int\limits_1^\Lambda \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\, x^{1-\alpha}\Big|_1^\Lambda = \frac{1}{1-\alpha}(\Lambda^{1-\alpha} - 1). \tag{2.1.3}$$

It has been assumed for a moment that $\alpha \neq 1$ (this case will be dealt with later). Now, one has to examine whether there exists a limit:

$$\lim_{\Lambda\to\infty} I_1^\Lambda. \tag{2.1.4}$$

For $\alpha > 1$, the power of Λ on the right-hand side of (2.1.3) is negative. This means that $\Lambda^{1-\alpha} \underset{\Lambda\to\infty}{\longrightarrow} 0$, and therefore, the limit (2.1.4) does exist and

$$I_1 = \lim_{\Lambda\to\infty} I_1^\Lambda = -\frac{1}{1-\alpha}. \tag{2.1.5}$$

The situation is different when $\alpha < 1$. Then the exponent of Λ is positive and the expression is divergent. The improper integral does not exist.

There is still the case $\alpha = 1$ remaining. Then one gets

$$I_1^\Lambda := \int\limits_1^\Lambda \frac{dx}{x} = \log|x|\,\Big|_1^\Lambda = \log\Lambda. \tag{2.1.6}$$

Since $\log\Lambda \underset{\Lambda\to\infty}{\longrightarrow} \infty$, in this case the integral is again divergent.

The integral I_2 has the same integrand function, and only the integration limits are changed, so one can easily use the above results. Because this function diverges at the lower limit (i.e., at zero), we will replace 0 with ϵ and calculate the ordinary definite integral:

$$I_2^\epsilon := \int\limits_\epsilon^1 \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\, x^{1-\alpha}\Big|_\epsilon^1 = \frac{1}{1-\alpha}(1 - \epsilon^{1-\alpha}), \tag{2.1.7}$$

where again it has been assumed that $\alpha \neq 1$. Then the limit,

$$\lim_{\epsilon\to 0^+} I_2^\epsilon, \tag{2.1.8}$$

has to be examined. It is clear that when $\alpha < 1$ the expression $\epsilon^{1-\alpha} \underset{\epsilon\to 0^+}{\longrightarrow} 0$ and one gets

$$I_2 = \lim_{\epsilon \to 0^+} I_2^\epsilon = \frac{1}{1-\alpha}. \tag{2.1.9}$$

The opposite is true for $\alpha > 1$. In this case the exponent of ϵ is negative and the limit (2.1.8) does not exist. The integral I_2 is then divergent.

In the case, when $\alpha = 1$, the integral is again divergent, since

$$\int_\epsilon^1 \frac{dx}{x} = \log|x| \Big|_\epsilon^1 = \log\frac{1}{\epsilon} \underset{\epsilon \to 0^+}{\longrightarrow} \infty. \tag{2.1.10}$$

A conclusion can be made for the integral I_3: the integral does not exist for any value of α. If it is convergent at one integration limit, it becomes divergent at the other. No such value of α exists for which it would converge at both ends.

The results of this exercise should be kept in mind as they constitute a point of reference for other, more complex integrations.

Problem 2

The existence of the integral:

$$I = \int_0^1 x^p \log^q x \, dx, \tag{2.1.11}$$

will be examined for $p, q \in \mathbb{R}$.

Solution

It should be noted that—depending on the parameters p and q—the problem in the above integral can appear both at the lower and at the upper integration limits. When $p < 0$, the integrand function diverges at zero regardless of the value of the second parameter (q). However, if $q < 0$, the divergence (of the integrand) appears at the upper limit too. For this reason, it is convenient to split the integration interval $]0, 1[$ into the sum of intervals $]0, a[\cup [a, 1[$, where for a any number between zero and one may be chosen. Fixing this value as $a = 1/2$, one can write

$$\int_0^1 x^p \log^q x\, dx = \underbrace{\int_0^{1/2} x^p \log^q x\, dx}_{I_1} + \underbrace{\int_{1/2}^1 x^p \log^q x\, dx}_{I_2}, \tag{2.1.12}$$

and separately examine the convergence at 0 and at 1. Of course, in order for the integral (2.1.11) to exist, both sub-integrals I_1 and I_2 must exist.

Let us start with the former integration, i.e., I_1. The mean value theorem will be used that we became acquainted with in the previous chapter. To this end, let us rewrite the integral in the form of

$$I_1 = \int_0^{1/2} x^p \log^q x\, dx = \int_0^{1/2} x^{p+1} \frac{1}{x} \log^q x\, dx \tag{2.1.13}$$

and define

$$I_1^\epsilon := \int_\epsilon^{1/2} x^{p+1} \frac{1}{x} \log^q x\, dx. \tag{2.1.14}$$

A question can arise at this point, why have we wished to get an additional factor $1/x$ under the integral and why have the expression x^p has been rewritten as x^{p+1}/x? This will become clear below while discussing formula (2.1.15).

It should be noted that the function $1/x \log^q x$ is continuous on the interval $[\epsilon, 1/2]$, so it can serve as a function f in formula (1.6.2). And, the function x^{p+1} is monotonic in this interval (the case $p = -1$ will be dealt with separately) and has a continuous derivative. The theorem assumptions are, therefore, fulfilled by the function g. Pursuant to the mean value theorem, there exists a certain number c (satisfying $\epsilon < c < 1/2$) that

$$I_1^\epsilon := \epsilon^{p+1} \int_\epsilon^{c} \frac{1}{x} \log^q x\, dx + \left(\frac{1}{2}\right)^{p+1} \int_c^{1/2} \frac{1}{x} \log^q x\, dx. \tag{2.1.15}$$

At this point, it becomes clear why the additional factor $1/x$ under the integral was desirable: this expression is the derivative of the natural logarithm, so both integrals in (2.1.15) can be easily calculated. For one has (if $q \neq -1$)

$$\int \frac{1}{x} \log^q x\, dx = \frac{1}{q+1} \int \left[\log^{q+1} x\right]' dx = \frac{1}{q+1} \log^{q+1} x, \tag{2.1.16}$$

where the integration constant has been omitted, since it will disappear in a moment in definite integrals. In the case when $q = -1$, the result of the integration is slightly different:

$$\int \frac{1}{x\log x}\,dx = \int \left[\log(\log x)\right]' dx = \log(\log x). \tag{2.1.17}$$

For $q \neq -1$, the mean value theorem leads then to the relation:

$$\begin{aligned} I_1^\epsilon &= \epsilon^{p+1}\,\frac{1}{q+1}\,\log^{q+1} x\Big|_\epsilon^c + \left(\frac{1}{2}\right)^{p+1}\frac{1}{q+1}\,\log^{q+1} x\Big|_c^{1/2} \\ &= \underline{\epsilon^{p+1}\,\frac{1}{q+1}(\log^{q+1} c - \log^{q+1}\epsilon)} + \left(\frac{1}{2}\right)^{p+1}\frac{1}{q+1}(\log^{q+1}\frac{1}{2} - \log^{q+1} c). \end{aligned} \tag{2.1.18}$$

Now, the question arises, whether in this expression one can go to the limit $\epsilon \to 0^+$. If so, the integral I_1 is convergent, and its value will be given by

$$I_1 = \lim_{\epsilon\to 0^+} I_1^\epsilon. \tag{2.1.19}$$

Looking at the underlined term on the right-hand side of (2.1.18) and at its dependence on ϵ, one sees that one has to do with the expression which is a product of a power function and logarithmic function. As we know, in this case the convergence is determined by the power factor (see Problem 2 in Sect. 11.4 of the first volume). For $p > -1$, the exponent $p+1$ is positive and the term dependent on ϵ goes to zero regardless of the value of q. In turn, for $p < -1$ the situation is reversed: the integral diverges regardless of q.

In the particular case when $q = -1$ instead of (2.1.16), formula (2.1.17) must be used. We obtain then

$$\begin{aligned} I_1^\epsilon &= \epsilon^{p+1}\log(\log x)\Big|_\epsilon^c + \left(\frac{1}{2}\right)^{p+1}\log(\log x)\Big|_c^{1/2} \\ &= \epsilon^{p+1}\,\log\frac{\log c}{\log\epsilon} + \left(\frac{1}{2}\right)^{p+1}\log\frac{\log(1/2)}{\log c}. \end{aligned} \tag{2.1.20}$$

Again the power factor ϵ^{p+1} rules on convergence, so the conclusions are the same as above.

There is still the case $p = -1$ remaining. It is not necessary to use here the mean value theorem because one is able to find the integral explicitly using formulas (2.1.16) and (2.1.17):

$$I_1^\epsilon = \int_\epsilon^{1/2} \frac{1}{x}\log^q x\,dx = \begin{cases} \dfrac{1}{q+1}\log^{q+1}\dfrac{1}{2\epsilon} & \text{for } q \neq -1, \\[2ex] \log\dfrac{\log(1/2)}{\log\epsilon} & \text{for } q = -1. \end{cases} \tag{2.1.21}$$

The divergence as $\epsilon \to 0^+$ appears for $q > -1$ (upper formula) and for $q = -1$ (lower formula). In turn, for $q < -1$ the (upper) expression tends to zero and the integral is convergent. Thus, we have a complete set of results for the integral I_1:

- if $p > -1$, the integral is convergent regardless of q,
- if $p < -1$, the integral is divergent regardless of q,
- if $p = -1$ and $q < -1$, the integral is convergent,
- if $p = -1$ and $q \geq -1$, the integral is divergent.

Below we turn to the second integration (I_2). Let us again define the auxiliary integral:

$$I_2^\epsilon := \int_{1/2}^{1-\epsilon} x^{p+1}\frac{1}{x}\log^q x\,dx. \tag{2.1.22}$$

Applying similarly as above the mean value theorem, one gets either

$$\begin{aligned} I_2^\epsilon &= \left(\frac{1}{2}\right)^{p+1}\frac{1}{q+1}(\log^{q+1} c - \log^{q+1}\frac{1}{2}) \\ &+ (1-\epsilon)^{p+1}\frac{1}{q+1}(\log^{q+1}(1-\epsilon) - \log^{q+1} c), \end{aligned} \tag{2.1.23}$$

for $q \neq -1$ or

$$I_2^\epsilon = \left(\frac{1}{2}\right)^{p+1}\log\frac{\log c}{\log 1/2} + (1-\epsilon)^{p+1}\log\frac{\log(1-\epsilon)}{\log c}, \tag{2.1.24}$$

for $q = -1$. This time, naturally, the parameter c satisfies the condition $1/2 < c < 1-\epsilon$.

The integral given by (2.1.23) is convergent for $\epsilon \to 0^+$ if the exponent of the logarithm of $(1-\epsilon)$ is positive. This happens only for $q > -1$. For $q < -1$, the divergence is obvious. In turn, when $q = -1$, the integral I_2 is divergent as can be seen from (2.1.24). The parameter p is not important for the convergence of I_2, which could have been predicted, as the factor x^p for $x \to 1$ does not behave in a singular way for any value of p.

For the existence of the integral I given in the text of the exercise, the simultaneous convergence of I_1 and I_2 has to take place. It is easy to see that it happens if both parameters $p, q > -1$.

2.2 Using Different Criteria

Problem 1

The convergence of the integral:

$$I = \int_0^\infty \frac{\tanh x}{x^\alpha + x^\beta}, \tag{2.2.1}$$

for $\alpha > \beta$, will be examined.

Solution

One often encounters the situation of not being able to explicitly calculate the integral, but the information about its convergence is necessary. Therefore, we must have at our disposal a method of determining this convergence other than directly using the definition. This opportunity is given by convergence criteria discussed in this section. One of the most commonly used criteria in this context is the so-called limit comparison test. The same criterion is used for testing the convergence of series of positive terms, dealt with in the first volume of this book (see Sect. 14.2).

For nonnegative functions f and g defined on the interval $[A, \infty[$, it can be formulated as follows. Let us assume that our goal is to investigate the convergence (at infinity) of the integral $\int_A^\infty f(x)dx$. To achieve this goal, a subsidiary function g is needed, for which the convergence of $\int_A^\infty g(x)dx$ can be easily established. Now suppose that there exists a limit:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = K < \infty. \tag{2.2.2}$$

The number K may be finite but may also equal zero. In simple words, this result indicates that at infinity the function f behaves "not worse" (i.e., it tends to zero similarly or more rapidly) than g. In both these cases, the convergence of $\int_A^\infty g(x)dx$ entails the convergence of $\int_A^\infty f(x)dx$. For, if the area under the graph of the function g is finite, then so is the area under the graph of the function f. On the other hand, if the limit (2.2.2) is infinite, this criterion does not adjudicate, as the convergence of $\int_A^\infty g(x)dx$ prejudges neither convergence nor divergence of $\int_A^\infty f(x)dx$.

If the integral $\int_A^\infty g(x)dx$ is divergent, this test still turns out to be useful. For $K > 0$ or if the limit (2.2.2) is infinite, one has an opposite situation: the function f behaves "worse" (or similarly) than the function g. The criterion guarantees then the divergence of $\int_A^\infty f(x)dx$. In the special case of $K = 0$, the convergence of the tested integral cannot be established in this way.

Comparing these two cases, one can conclude—as it was in the case of series—that the most favorable situation emerges when K does exist and is finite. Then the test unequivocally rules on the convergence of a given integral. Therefore, the question arises how to make the number of K finite. The answer is quite obvious: a subsidiary function g which, after all, is at our disposal, should be appropriately adjusted. The word "appropriately" means here "in such a way that f and g behave *identically* at infinity." In the first part of this book series we used the symbol $\simeq$ to denote this fact (see "Definitions and Notation"). The procedure of applying the limit comparison test starts then always with the preliminary analysis which allows one to assess the asymptotic behavior of function f.

An analogous test, which does not require detailed discussion, is also applicable for improper integrations of the type $\int_a^b f(x)dx$ if the integrand diverges on either end of the integration interval (e.g., at a). One must then choose the function g and examine the limit:

$$\lim_{x \to a^+} \frac{f(x)}{g(x)}. \tag{2.2.3}$$

If this limit exists, it is again denoted with K. The conclusions to be drawn as to the convergence of $\int_a^b f(x)dx$ on the basis of the knowledge of $\int_a^b g(x)dx$ are the same as above.

After this theoretical introduction one can set about examining the integral (2.2.1). First we are going to consider the upper limit of integration, and subsequently the lower one, as the problem of convergence may occur at both of them. In accordance with our considerations, we must first evaluate the behavior at infinity of the integrand function:

$$f(x) = \frac{\tanh x}{x^\alpha + x^\beta}. \tag{2.2.4}$$

One knows that as $x \to \infty$, the hyperbolic tangent function goes to unity, since

$$\lim_{x \to \infty} \tanh x = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1, \tag{2.2.5}$$

so the behavior of the function (2.2.4) is determined by its denominator. For $\alpha > \beta$, the leading expression is naturally x^α. Thus, one has

$$f(x) = \frac{\tanh x}{x^\alpha + x^\beta} = \frac{1}{x^\alpha} \cdot \frac{\tanh x}{1 + x^{\beta - \alpha}}. \tag{2.2.6}$$

Since $\beta - \alpha < 0$, the second fraction tends to a constant and the entire function behaves like $1/x^\alpha$. This is why we choose the test function in the form of

$$g(x) = \frac{1}{x^\alpha}. \tag{2.2.7}$$

In Problem 1 of the previous section, it was announced that the knowledge of convergence of integrals with power functions is very useful, as they are most often used as auxiliary functions. According to the presently applied test, one needs to calculate

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\tanh x/(x^\alpha + x^\beta)}{1/x^\alpha} = \lim_{x\to\infty} \frac{\tanh x}{1 + x^{\beta-\alpha}} = 1 =: K. \tag{2.2.8}$$

The number K has proved to be finite, and therefore, based on the criterion, we know the solution to the problem. The integral (2.2.1) is convergent at the upper limit if and only if the integral of the function $1/x^\alpha$ is. As we learned in the previous section, this happens for $\alpha > 1$.

Now one can move on to the lower limit. If $x \to 0^+$, the integrand function may be given the form:

$$f(x) = \frac{\tanh x}{x^\alpha + x^\beta} = \frac{\tanh x}{x} \cdot \frac{1}{x^{\alpha-\beta} + 1} \cdot \frac{1}{x^{\beta-1}}. \tag{2.2.9}$$

The known fact that

$$\lim_{x\to 0^+} \frac{\tanh x}{x} = \lim_{x\to 0^+} \frac{\sinh x}{x} \cdot \frac{1}{\cosh x} = 1 \tag{2.2.10}$$

shows that both the first and second fractions tend to constants (because $\alpha - \beta > 0$), so the behavior of the function is determined by the factor $1/x^{\beta-1}$. Therefore, in this case one should choose

$$g(x) = \frac{1}{x^{\beta-1}} \tag{2.2.11}$$

arriving at

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\tanh x/(x^\alpha + x^\beta)}{1/x^{\beta-1}} = \lim_{x\to 0^+} \frac{\tanh x}{x} \cdot \frac{1}{x^{\alpha-\beta} + 1} = 1 =: K. \tag{2.2.12}$$

The function g has been correctly chosen (since $K \neq 0$), so the test rules on the convergence of (2.2.1) at zero: this integral is convergent if and only if the integral of the function g is, and this takes the place for $\beta - 1 < 1$, or $\beta < 2$.

In conclusion, one can write that the integral given in the text of this exercise exists only for $\alpha > 1$ (due to the upper limit) and for $\beta < 2$ (due to the lower limit), on the assumption that $\alpha > \beta$ (in fact, our considerations are also valid for $\alpha = \beta$, with the inessential difference that the number K equals now $1/2$). If this last condition is not met and $\alpha < \beta$, these parameters simply swap their roles.

Problem 2

The convergence of the integral:

$$I = \int_0^1 \frac{x}{\log^n x}\, dx, \tag{2.2.13}$$

for $n \in \mathbb{Z}$, will be examined.

Solution

The potentially "dangerous" points for the convergence of the above integral are both the lower and the upper ends of the integration interval. However, as $x \to 0^+$, the integrand function goes to zero even if $n < 0$, since in the expression

$$\frac{x}{\log^n x} = x \log^{-n} x = x \log^{|n|} x \tag{2.2.14}$$

the power factor decides on the convergence and not the logarithm (see Problem 2 in Sect. 11.4 of the first volume).

On the other hand, at the upper end, the function is indeed unbounded for $n > 0$ due to the fact that the denominator goes to zero (the numerator becomes a nonzero constant). Therefore, when examining the convergence of the integral (2.2.13), it is sufficient to handle only this point. For this purpose, the limit comparison test formulated in the previous example will be used. However, one needs to have some intuition as to the particular behavior of the integrand function close to 1. From Taylor's formula (see Problem 3 in Sect. 12.2 of Part I), one knows that

$$\log x = \log(1 + x - 1) = (x - 1) + o(x - 1), \tag{2.2.15}$$

where the symbol $o(x - 1)$ means a negligible of higher order than $(x - 1)$, i.e., satisfying

$$\lim_{x \to 1} \frac{o(x - 1)}{x - 1} = 0. \tag{2.2.16}$$

In such a case (note that here $n > 0$),

$$\log^n x = [(x - 1) + o(x - 1)]^n = (x - 1)^n + o((x - 1)^n) \tag{2.2.17}$$

and it appears that the integrand function has the following behavior close to unity: $f(x) \sim \text{const}/(x - 1)^n$. This knowledge allows us to select the auxiliary (comparative) function in the form of

$$g(x) = \frac{1}{(x-1)^n}. \tag{2.2.18}$$

According to the limit comparison test, we calculate now

$$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} \frac{x/\log^n x}{1/(x-1)^n} = \lim_{x\to 1^-} x \left(\frac{x-1}{\log x}\right)^n = 1\cdot (1)^n = 1 =: K, \tag{2.2.19}$$

where the limit

$$\begin{aligned} \lim_{x\to 1^-} \frac{x-1}{\log x} \underset{t:=1-x}{=} -\lim_{t\to 0^+} \frac{t}{\log(1-t)} &= -\lim_{t\to 0^+} \frac{1}{\log(1-t)^{1/t}} \\ = -\frac{1}{\log(1/e)} &= 1 \end{aligned} \tag{2.2.20}$$

has been used (see Problem 2 in Sect. 7.2 of the first volume).

When calculating (2.2.19), a finite value of K was obtained, and therefore, the limit comparison test unequivocally ruled on convergence. The integral (2.2.13) exists if and only if the integral

$$\int_0^1 \frac{1}{(x-1)^n}\, dx$$

exists, and this takes place for $n < 1$, i.e., $n = 0, -1, -2, \ldots$

Problem 3

The convergence of the integral:

$$I = \int_0^\infty \cos x^2\, dx, \tag{2.2.21}$$

called the Fresnel integral, will be examined.

Solution

The integrand function approaches a constant at the lower end, so this point does not constitute any danger for the convergence of the integral. Therefore, we focus only on the upper end. It is clear that as $x \to \infty$, the function $\cos x^2$ strongly oscillates and does not approach zero. Thus, the limit comparison test used in the previous exercises cannot be applied here. The convergence of the integral—in so far as it takes place at all—is owed not to the quick approach of the integrand to zero (since it is not!), but to the fact that areas below and above the x-axis contribute with opposite signs and cancel each other. As we remember, the limit comparison test was applicable only for functions with a fixed sign.

Before the appropriate criterion is selected for this example, let us change the variables, which will enable us to get rid of the composite argument from the cosine function: we substitute $t = x^2$. The integral will now take the form:

$$I = \int_0^\infty \cos x^2 \, dx = \frac{1}{2} \int_0^\infty \frac{\cos t}{\sqrt{t}} \, dt. \tag{2.2.22}$$

The change of variables—as it often happens—made the new integrand function unbounded at the lower end of the integration interval, because the denominator approaches zero. Actually, one would not need to examine this point, because for $t \to 0^+$ one has also $x \to 0^+$, and we know that (2.2.21) is convergent at zero. However, for completeness, the limit comparison test will be applied by selecting the auxiliary function $g(t) = 1/\sqrt{t}$ and writing

$$\lim_{t \to 0^+} \frac{\cos t/\sqrt{t}}{1/\sqrt{t}} = \lim_{t \to 0^+} \cos t = 1 =: K. \tag{2.2.23}$$

Since the number K has proved to be finite and $1/\sqrt{t}$ is integrable at zero, the same can be said about (2.2.22).

The reader might have protested here, for it was written a little earlier that it was not possible to apply this test to the considered function (neither in the variable x nor t) because of the undefined sign of the integrand function. This is true, but we referred only to the upper end of the integration interval. There, the integral will really be found convergent thanks to the cancellation of positive and negative values which originate from the cosine, since the factor $1/\sqrt{t}$ approaches zero too slowly (from Problems 1 and 2, we know that the exponent of t would have to be greater than one and here it is equal to $1/2$ only). However, in the lower limit of integration, such a problem does not appear. On the interval $[0, \pi/2[$, the cosine function has a fixed sign and if the integral is to be convergent, one must have the appropriate powers of t (the exponent must be less than 1). Whatever happens outside that interval does not matter for the convergence at zero.

For the examination of (2.2.22) at infinity, we will use the so-called Dirichlet's test which seems to be ideally suited to this example. This criterion applies to integrations just of the form:

$$\int_a^\infty f(x)g(x)\,dx. \tag{2.2.24}$$

It states that if for any $b > a$ there exists the integral $\int_a^b f(x)\,dx$ and it is bounded with respect to b, i.e., a fixed number M can be chosen such that

$$\left|\int_a^b f(x)\,dx\right| \leq M, \tag{2.2.25}$$

and in addition the function $g(x)$ monotonically approaches zero for $x \to \infty$, then the integral (2.2.24) is convergent too. So, let us examine whether these assumptions are all satisfied by the functions:

$$f(t) = \cos t \quad \text{and} \quad g(t) = \frac{1}{\sqrt{t}}. \tag{2.2.26}$$

Surely

$$\left|\int_a^b f(t)\,dt\right| = \left|\int_a^b \cos t\,dt\right| = |-\sin b + \sin a| \leq 2, \tag{2.2.27}$$

and the function $g(t)$ has, for $t \to \infty$, the limit equal to zero and is monotonic. In consequence, Dirichlet's test assures that the integral (2.2.22) is convergent too. As will be seen in the third volume, one is even able to calculate the value of this integral—despite the fact that it is impossible to explicitly find the primitive function—and it turns out to be equal to $\sqrt{\pi/8}$.

Problem 4

The convergence of the integral:

$$I = \int_1^\infty \frac{\sin x \arctan x}{x}\,dx \tag{2.2.28}$$

will be examined.

Solution

In this example, only the upper limit of integration creates a problem. In order to examine, the results of the previous exercise will partly be used. According to Dirichlet's test, the following integral is convergent:

$$\int_1^\infty \frac{\sin x}{x}\, dx, \tag{2.2.29}$$

exactly due to the same arguments that were used to show the convergence of $\int_0^\infty \cos t/\sqrt{t}\, dt$. The integral of the sine function is bounded by two, and the factor $1/x$ monotonically approaches zero.

Let us now denote

$$f(x) = \frac{\sin x}{x}, \qquad g(x) = \arctan x \tag{2.2.30}$$

and refer to the so-called Abel's test, applicable to integrations of the type:

$$\int_a^\infty f(x)g(x)\, dx. \tag{2.2.31}$$

Pursuant to this test, if the function $f(x)$ is integrable on the interval $[a, \infty[$ and the function $g(x)$—monotonic and bounded, the integral (2.2.31) is convergent. The first condition is met—it was spoken of above—so one needs to check out the second one only. One has

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}, \tag{2.2.32}$$

which means that the function $g(x)$ is bounded. It is also monotonic, which is a well-known fact (it is the inverse function to the increasing function tan). One can easily convince oneself about it, by calculating the derivative:

$$g'(x) = \frac{1}{1+x^2} > 0 \tag{2.2.33}$$

and noting that it is constantly positive, i.e., the function g—increasing. The assumptions of Abel's test are, therefore, met and the integral (2.2.28) is convergent.

Problem 5

The convergence of the integral:

$$I = \int_{2}^{\infty} \frac{\sin x}{x^{\lambda} + \sin x}\, dx, \tag{2.2.34}$$

where $\lambda > 0$, will be examined.

Solution

Investigating the integral (2.2.22) at infinity, we relied on Dirichlet's test. We then made use of the fact that the factor $1/\sqrt{t}$ monotonically approaches zero. Exactly the same arguments would apply to the integral:

$$\int_2^{\infty} \frac{\sin x}{x^{\lambda}}, \tag{2.2.35}$$

since for $\lambda > 0$ the expression $1/x^{\lambda}$ goes to zero in a monotonic way. It is clear, therefore, that the integral (2.2.35) is convergent. It might apparently seem that the same can be said about the integral given in the exercise: after all, for large x (and only those values matter for the convergence of (2.2.34)), $\sin x$ in the denominator can be neglected as compared to x^{λ}, leading *de facto* to the integral (2.2.35). As it will turn out in a moment, this reasoning is incorrect and could constitute a warning against brushing aside the assumptions of mathematical theorems.

In order to examine the convergence of (2.2.34), we will use the identity (true if $a \neq -b$ and $b \neq 0$)

$$\frac{a}{a+b} = \frac{a}{b} - \frac{a^2}{b(b+a)}, \tag{2.2.36}$$

substituting $a = \sin x$ and $b = x^{\lambda}$ (for very large x one certainly has $\sin x \neq -x^{\lambda}$):

$$\frac{\sin x}{\sin x + x^{\lambda}} = \frac{\sin x}{x^{\lambda}} - \frac{\sin^2 x}{x^{\lambda}(x^{\lambda} + \sin x)}. \tag{2.2.37}$$

The former of the two obtained expressions leads to the convergent integral, spoken of above. As regards to the latter, it is nonnegative (for large value x), so the appropriate evaluations can be made. For $\lambda > 1/2$ one can write

$$0 \le \frac{\sin^2 x}{x^\lambda(x^\lambda + \sin x)} \le \frac{1}{x^\lambda(x^\lambda + \sin x)} \le \frac{1}{x^\lambda(x^\lambda - 1)}. \tag{2.2.38}$$

Upon the limit comparison test, it is evident that the integral

$$\int_2^\infty \frac{1}{x^\lambda(x^\lambda - 1)}\, dx \tag{2.2.39}$$

is convergent if the leading exponent in the denominator (equal to 2λ) is larger than 1. From lectures of analysis, we know that the following integral has to be convergent too:

$$\int_2^\infty \frac{\sin^2 x}{x^\lambda(x^\lambda + \sin x)}\, dx. \tag{2.2.40}$$

In fact, since the area below the graph of the function $1/(x^\lambda(x^\lambda - 1))$ is finite, then—because of the inequality (2.2.38)—even more must the area given by integral (2.2.40) be finite.

For $\lambda \le 1/2$, one has, however, the opposite estimate:

$$\begin{aligned}\frac{\sin^2 x}{x^\lambda(x^\lambda + \sin x)} &\ge \frac{\sin^2 x}{x^\lambda(x^\lambda + 1)} = \frac{1}{2} \cdot \frac{1 - \cos 2x}{x^\lambda(x^\lambda + 1)} \\ &= \frac{1}{2x^\lambda(x^\lambda + 1)} - \frac{\cos 2x}{x^\lambda(x^\lambda + 1)} \ge 0. \end{aligned} \tag{2.2.41}$$

Now, it should be noted that

- the integral $\int_2^\infty 1/(2x^\lambda(x^\lambda + 1))\, dx$ is divergent (infinite) by virtue of the limit comparison test,
- the integral $\int_2^\infty \cos 2x/(2x^\lambda(x^\lambda + 1))\, dx$ is convergent by virtue of Dirichlet's test.

The sum (or difference) of divergent and convergent integrals is, of course, divergent, so the first inequality of (2.2.41) implies the divergence (i.e., infiniteness) of the integral I. Let us summarize the results: the integral I is convergent for $\lambda > 1/2$ and divergent for $0 < \lambda \le 1/2$.

There still remains the question: why couldn't one directly use Dirichlet's test? At which point were the assumptions of the theorem violated inasmuch as the results appear to contradict them? Well, it turns out that the function $g(x) := 1/(x^\lambda + \sin x)$ in (2.2.34) does not meet the requirement of monotonicity even for very large x. To see this, let us calculate its derivative:

$$g'(x) = -\frac{1}{(x^\lambda + \sin x)^2}\,(\lambda x^{\lambda-1} + \cos x). \tag{2.2.42}$$

The denominator is always positive, so it does not affect the sign of the expression. Note, however, that in view of the negative exponent $\lambda - 1$ (for $\lambda < 1$), the first component in brackets for large values of x becomes very small and the sign of the derivative will be changing due to cosine oscillations. A variable sign of the derivative indicates that the function is not monotonic and it fails to comply with the requirements of Dirichlet's test.

One more question may appear here: why in this case did the integral prove to be convergent for all values of $\lambda > 1/2$ inasmuch as the assumptions are not met even in $[1/2, 1[$? Similar answers have already been given several times: a mathematical theorem tells us that a given thesis is true if the assumptions are satisfied; however, it does not rule on what happens when these assumptions are violated. In our case it simply means that for $\lambda \in [1/2, 1[$ one was not allowed to use Dirichlet's test and for the examination of the convergence one had to adopt a different criterion.

2.3 Using Integral Test for Convergence of Series

Problem 1

The convergence of the series:

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} \tag{2.3.1}$$

will be examined.

Solution

Since we have already learned how to check the convergence of improper integrals over infinite intervals, this knowledge can be now used to study the convergence of certain series. It becomes possible thanks to the so-called integral test which states that if $f(x)$ is positive and decreasing on the interval $[1, \infty[$, then the convergence of the integral $\int_1^\infty f(x)\,dx$ takes place if and only if the series $\sum_1^\infty f(n)$ is convergent. In fact, the requirements with respect to f can be somewhat mitigated: it is sufficient that f is decreasing starting from a certain point. The idea of this test becomes clear if one looks at Fig. 2.1.

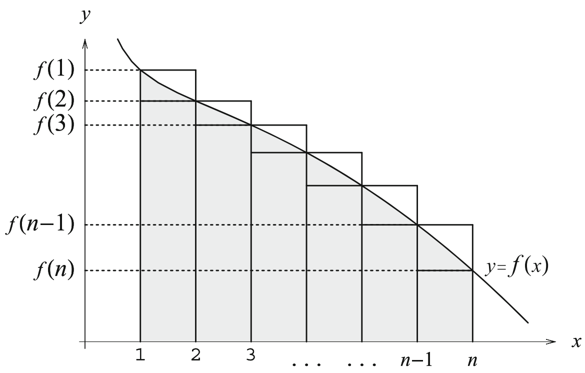

Fig. 2.1 The connection between an improper integral and a series

The value of the integral $P := \int_1^\infty f(x)\,dx$ is the gray area under the curve (let us assume that it extends to infinity), and the sum of areas of large rectangles equals nothing other than

$$P_u := f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + \ldots \sum_{n=1}^{\infty} f(n). \tag{2.3.2}$$

The sum of the areas of small rectangles equals, in turn,

$$P_l := f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + \ldots = \sum_{n=2}^{\infty} f(n) = P_u - f(1). \tag{2.3.3}$$

The obvious inequality holds:

$$P_l < P < P_u, \quad \text{i.e.,} \quad P_u - f(1) < P < P_u < P + f(1). \tag{2.3.4}$$

Since $f(1)$ is a finite and fixed number, it is clear that the existence of P_u entails the existence of P, and conversely the existence of P entails the existence of P_u. Of course, one can shift the lower end of the integral or that of the sum if necessary (in this exercise they both equal to 2).

To examine the convergence of (2.3.1), it is sufficient to determine whether the integral

$$I := \int_2^\infty \frac{1}{(\log x)^{\log x}}\,dx \tag{2.3.5}$$

exists. The natural logarithm is positive for $x > 1$ and increasing, so similarly positive and increasing is also the function $(\log x)^{\log x}$, and in consequence, the integrand function in (2.3.5) is decreasing. The assumptions of the integral test are satisfied. Now we will make the substitution $x = e^t$ which simplifies the integral, and one has

$$I = \int_{\log 2}^{\infty} \left(\frac{e}{t}\right)^t dt. \tag{2.3.6}$$

The lower limit of integration does not pose any problem. In regards to the upper one, one can easily handle it if it is noticed that for $t > e$, the fraction e/t becomes smaller than unity. To make use of this fact, one can split the interval of the integration $[\log 2, \infty[$ into the sum of intervals on $[\log 2, a]$ and $]a, \infty[$, where $a > e$ (for convenience $a = e^2$ is chosen). We then estimate

$$\begin{aligned} 0 < I &= \int_{\log 2}^{e^2} \left(\frac{e}{t}\right)^t dt + \int_{e^2}^{\infty} \left(\frac{e}{t}\right)^t dt < \int_{\log 2}^{e^2} \left(\frac{e}{t}\right)^t dt + \int_{e^2}^{\infty} \left(\frac{e}{e^2}\right)^t dt \\ &= \int_{\log 2}^{e^2} \left(\frac{e}{t}\right)^t dt + \int_{e^2}^{\infty} e^{-t}\, dt. \end{aligned} \tag{2.3.7}$$

In the former of these integrals, one has a continuous function on a bounded and closed interval, and therefore, a finite number is obtained. The latter can be computed explicitly:

$$\int_{e^2}^{\infty} e^{-t}\, dt = \lim_{A\to\infty} \int_{e^2}^{A} e^{-t}\, dt = -\lim_{A\to\infty} e^{-t}\Big|_{e^2}^{A} = \lim_{A\to\infty} (e^{-e^2} - a^{-A}) = e^{-e^2} < \infty. \tag{2.3.8}$$

Since the integrand function in (2.3.5) is positive, the obtained estimate proves the convergence of the integral which entails the convergence of the series (2.3.1).

Problem 2

The convergence of the series:

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha} \log^{\beta} n}, \tag{2.3.9}$$

where $\alpha, \beta \in \mathbb{R}$, will be examined.

Solution

For $x > 2$, the expression

$$\frac{1}{x^\alpha \log^\beta x} \tag{2.3.10}$$

is positive. Let us calculate its derivative:

$$\left[\frac{1}{x^\alpha \log^\beta x}\right]' = -\frac{1}{x^{\alpha+1} \log^\beta x}\left[\alpha + \frac{\beta}{\log x}\right]. \tag{2.3.11}$$

As long as $\alpha > 0$, this expression becomes negative for suitably large values of x (since $\beta/\log x \underset{x\to\infty}{\longrightarrow} 0$), which means that the function (2.3.10) is decreasing. One can, therefore, apply the integral test, and instead of directly studying the convergence of the series (2.3.9), examine the convergence of the integral:

$$I := \int\limits_2^\infty \frac{1}{x^\alpha \log^\beta x}\, dx. \tag{2.3.12}$$

The substitution simplifying the expression is certainly $x = e^t$, i.e., $t = \log x$. If done, the integral gains the following form, easy to examine:

$$I = \int\limits_{\log 2}^\infty e^{-\alpha t} t^{-\beta} e^t dt = \int\limits_{\log 2}^\infty e^{-(\alpha-1)t} t^{-\beta} dt. \tag{2.3.13}$$

For $\alpha > 1$ the integrand function decreases exponentially, and therefore, faster than any power of t. The integral is then clearly convergent and consequently the series (2.3.9) as well. For $\alpha < 1$, the situation is reversed: the integrand grows, so the divergence of the integral (and of the series) is out of the question. In the borderline case $\alpha = 1$, the exponential function becomes in fact a constant (the unity) and the convergence is dependent on the value of the parameter β. As one knows from the first exercise of this section, for the convergence of (2.3.13) and (2.3.9), the value $\beta > 1$ is required.

It is worth recalling at the end that the result in the latter case was obtained in the first part of this book with the use of the so-called Cauchy's condensation test (see Problem 5 in Sect. 13.2).

2.4 Exercises for Independent Work

Exercise 1 Examine the existence of improper integrals:

(a) $\displaystyle\int_1^2 \frac{\log^\alpha x}{x-1}\,dx$ (depending on the value of the parameter $\alpha \in \mathbb{R}$),

(b) 1°. $\displaystyle\int_0^{\pi/2} \tan x\,dx$, 2°. $\displaystyle\int_0^{\pi/2} \cot x\,dx$, 3°. $\displaystyle\int_0^{\pi/2} \sqrt{\cot x}\,dx$,

(c) 1°. $\displaystyle\int_0^1 \frac{\sin x - \sinh x}{x^3}\,dx$, 2°. $\displaystyle\int_1^\infty \frac{\sin x - \sinh x}{x^3}\,dx$,

(d) 1°. $\displaystyle\int_0^2 \frac{\sqrt{x}-\sqrt{x+1}}{x}\,dx$, 2°. $\displaystyle\int_2^\infty \frac{\sqrt{x}-\sqrt{x+1}}{x}\,dx$,

(e) 1°. $\displaystyle\int_0^\infty \sin(x\sqrt{x})\,dx$, 2°. $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x+1}\sin x\,dx$.

Answers

(a) The integral exists for $\alpha > 0$ and does not for $\alpha \le 0$,
(b) 1°. The integral does not exist, 2°. The integral does not exist, 3°. The integral exists,
(c) 1°. The integral exists, 2°. The integral does not exist,
(d) 1°. The integral does not exist, 2°. The integral exists,
(e) 1°. The integral exists, 2°. The integral exists.

Exercise 2 Using the integral test, investigate the convergence of series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan^{100} n}{n^2+1}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\log n\log(\log n)\log(\log(\log n))}$.

Answers

(a) The series is convergent,
(b) The series is divergent.

Chapter 3
Applying One-Dimensional Integrals to Geometry and Physics

The present chapter is devoted to the applications of one-dimensional integrals to the calculation of some geometrical quantities like lengths of curves or areas of surfaces as well as physical ones. The formulas for the typical geometrical quantities are collected below. They are discussed in detail when solving the appropriate problems.

The **length of a curve**, which constitutes the graph of a function $y(x)$ contained between $x = a$ and $x = b$, may be calculated from the expression

$$l_{ab} = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx. \tag{3.0.1}$$

Sometimes the curve is defined in the parametric way by specifying the formulas for the Cartesian coordinates $x(t)$ and $y(t)$, where t is a parameter. In such a case, the length contained between the points $(x(\alpha), y(\alpha))$ and $(x(\beta), y(\beta))$ can be found from the formula:

$$l_{\alpha\beta} = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\, dt. \tag{3.0.2}$$

The **surface area** below the graph of a positive function $y(x)$ contained between the vertical lines $x = a$ and $x = b$ obviously equals

$$S = \int_a^b y(x)dx, \tag{3.0.3}$$

which stems directly from Chap. 1. In the case of the polar coordinates r and φ, in which the curve is defined by specifying the dependence $r(\varphi)$, the area contained between the curve $r = r(\varphi)$ and the radial straight lines $\varphi = \varphi_1$ and $\varphi = \varphi_2$ is given by

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Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-36848-7_3

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r(\varphi)^2\, d\varphi. \tag{3.0.4}$$

The one-dimensional integrals sometimes allow us also to find certain volumes. For instance, the **volume** of a solid obtained by rotating the fragment of the graph of the function $y(x)$ contained between $x = a$ and $x = b$ around the x-axis can be found with the use of the formula:

$$V = \pi \int_a^b y(x)^2\, dx. \tag{3.0.5}$$

If the rotated curve is given in the parametric form by specifying $x(t)$ and $y(t)$, where $\alpha < t < \beta$, then the following formula holds:

$$V = \pi \int_\alpha^\beta y(t)^2 x'(t)\, dt. \tag{3.0.6}$$

In order to calculate the corresponding surface areas one can use the expressions

$$S = 2\pi \int_a^b |y(x)|\sqrt{1 + y'(x)^2}\, dx, \tag{3.0.7}$$

or in the parametric case:

$$S = 2\pi \int_\alpha^\beta |y(t)|\sqrt{x'(t)^2 + y'(t)^2}\, dt. \tag{3.0.8}$$

3.1 Finding Lengths of Curves

Problem 1

The length of the curve given by the formula:

$$y(x) = -\log\cos x, \tag{3.1.1}$$

for $x \in [\pi/6, \pi/3]$, will be found.

Solution

Suppose that the curve in the xy plane is a section of the graph of a certain function $y(x)$ contained between the points $(a, y(a))$ and $(b, y(b))$, and that this function is continuously differentiable. As we know from the theoretical introduction, to find its length the following formula may be used:

$$l_{ab} = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx. \tag{3.1.2}$$

In the present problem one deals exactly with such a situation. The function $y(x)$ is given in the text of the exercise, and the outermost points are equal to $a = \pi/6$, $b = \pi/3$. The appropriate graph is shown in Fig. 3.1.

The derivative of the function is given by the formula:

$$y'(x) = -\frac{1}{\cos x}(-\sin x) = \tan x, \tag{3.1.3}$$

so one has to calculate the integral:

$$l_{ab} = \int_a^b \sqrt{1 + \tan^2 x}\, dx = \int_a^b \sqrt{\frac{1}{\cos^2 x}}\, dx = \int_a^b \frac{1}{\cos x}\, dx. \tag{3.1.4}$$

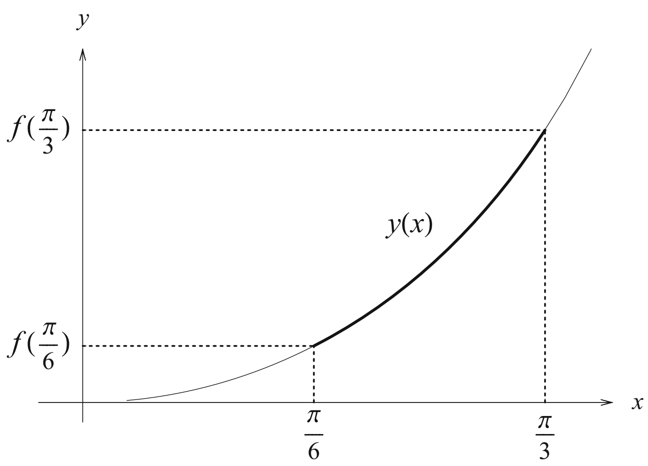


Fig. 3.1 The behavior of the function (3.1.1). The thick segment corresponds to the line, the length of which is to be found

On the whole interval $[\pi/6, \pi/3]$ the cosine function is positive, so one could write $\sqrt{\cos^2 x} = \cos x$.

The method of finding integrals containing rational functions of two arguments, $\sin x$ and $\cos x$, was studied in Part I (see Sect. 14.4). The best substitution, which will allow to calculate (3.1.4), is certainly $t = \sin x$, since the integral function changes its sign while replacing $\cos x \mapsto -\cos x$. Let us then first calculate the indefinite integral (the integration constant is omitted):

$$\begin{aligned}\int \frac{1}{\cos x}\,dx &= \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx\\ &= \int \frac{1}{1-t^2}\,dt = \frac{1}{2}\int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt\\ &= \frac{1}{2}(-\log|1-t| + \log|1+t|) = \frac{1}{2}\log\left|\frac{1+t}{1-t}\right|. \end{aligned} \tag{3.1.5}$$

Having regard to the fact that $\sin(\pi/6) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$, one easily gets the result for the length:

$$l_{ab} = \frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\,\Bigg|_{1/2}^{\sqrt{3}/2} = \frac{1}{2}\log\left|\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}\cdot\frac{1-1/2}{1+1/2}\right| = \log\left(1+\frac{2}{\sqrt{3}}\right). \tag{3.1.6}$$

Problem 2

The length of the nth coil of the spiral defined in polar coordinates as $r(\varphi) = e^{\varphi}$ will be bound.

Solution

The curve, the length of which we are supposed to calculate, is not always the graph of a function. It may be that of an equation described in a parametric way, i.e., by the dependences $x(t)$ and $y(t)$. As written at the beginning of this chapter, formula (3.1.2) should then be replaced with

$$l_{\alpha\beta} = \int\limits_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt, \tag{3.1.7}$$

where α and β are the initial and final values of the parameter t, respectively.

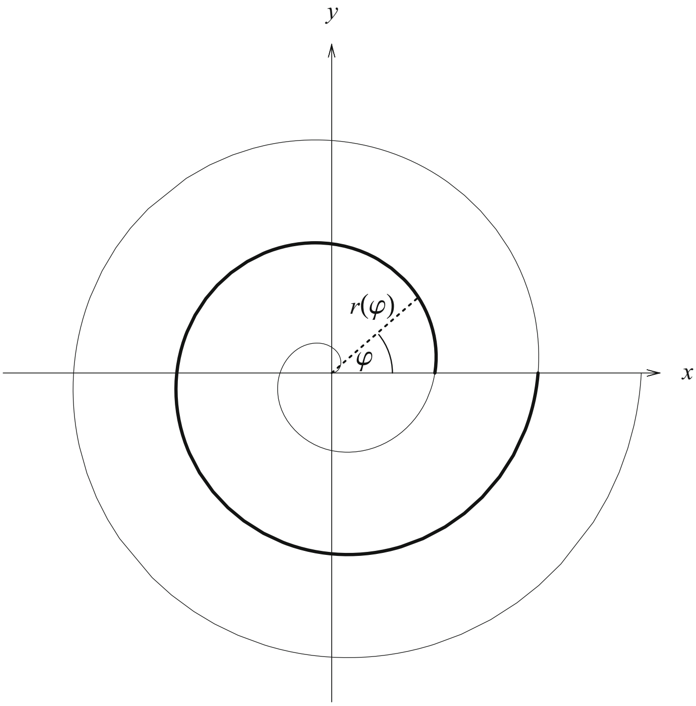


Fig. 3.2 The spiral $r(\varphi) = e^{\varphi}$ plotted in the logarithmic scale. The thick segment indicates one of the coils, the length of which is being calculated

In the case of the spiral of our current exercise, it is convenient to choose as a parameter the angle φ, sometimes called the azimuthal angle and sometimes the amplitude of the point (x, y). The considered spiral is shown in Fig. 3.2 in the logarithmic scale (i.e., by measuring the distance of a point on the graph from the origin, one finds $\log r$ instead of r).

We have the following parameterization:

$$x(\varphi) = r(\varphi)\cos\varphi = e^{\varphi}\cos\varphi,$$
$$y(\varphi) = r(\varphi)\sin\varphi = e^{\varphi}\sin\varphi, \tag{3.1.8}$$

and one can make use of formula (3.1.7), calculating first

$$x'(\varphi) = \left[e^{\varphi}\cos\varphi\right]' = e^{\varphi}(\cos\varphi - \sin\varphi),$$
$$y'(\varphi) = \left[e^{\varphi}\sin\varphi\right]' = e^{\varphi}(\sin\varphi + \cos\varphi) \tag{3.1.9}$$

and then

$$(x'(\varphi))^2 + (y'(\varphi))^2 = e^{2\varphi}\left[(\cos\varphi - \sin\varphi)^2 + (\sin\varphi + \cos\varphi)^2\right] = 2e^{2\varphi}, \tag{3.1.10}$$

where the Pythagorean trigonometric identity has been used. The values α and β corresponding to two ends of the curve still have to be determined. We are interested in the nth coil, which means that $2(n-1)\pi \le \varphi < 2n\pi$. Its length equals, therefore,

$$l_n = \int\limits_{2(n-1)\pi}^{2n\pi} \sqrt{2e^{2\varphi}}\, d\varphi = \sqrt{2} \int\limits_{2(n-1)\pi}^{2n\pi} e^{\varphi}\, d\varphi = \sqrt{2}\, e^{\varphi}\Big|_{2(n-1)\pi}^{2n\pi}$$
$$= \sqrt{2}\, e^{2\pi n}\left(1 - e^{-2\pi}\right). \tag{3.1.11}$$

It should be noted that the following coil would have the length:

$$l_{n+1} = \sqrt{2} e^{2\pi(n+1)}\left(1 - e^{-2\pi}\right) = e^{2\pi} l_n, \tag{3.1.12}$$

which means that the lengths of consecutive coils form a geometric sequence.

Since spirals are described quite often in polar variables by the equation $r = r(\varphi)$, it would be worthwhile at the end to derive the "polar counterpart" of formula (3.1.7) to be used in other problems:

$$l_{\alpha\beta} = \int\limits_{\alpha}^{\beta} \sqrt{[r(\varphi)\cos\varphi]'^2 + [r(\varphi)\sin\varphi]'^2}\, d\varphi$$
$$= \int\limits_{\alpha}^{\beta} \sqrt{[r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi]^2 + [r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi]^2}\, d\varphi$$
$$= \int\limits_{\alpha}^{\beta} \sqrt{r'(\varphi)^2 + r(\varphi)^2}\, d\varphi. \tag{3.1.13}$$

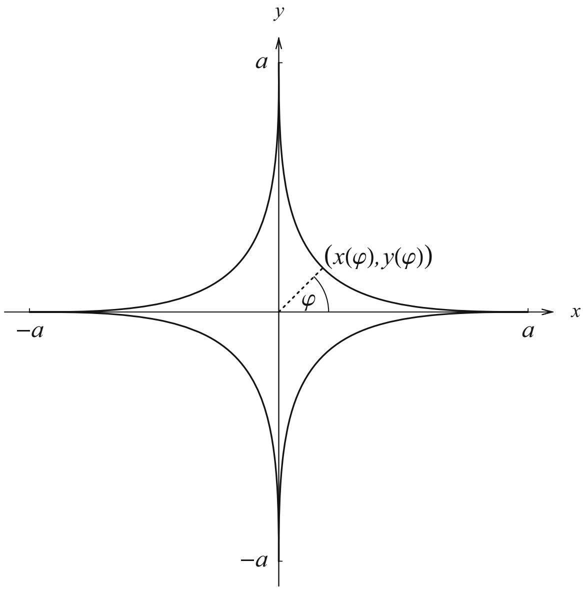
Problem 3

The circumference of the curve with the parametric equation:

$$x(\varphi) = a\cos^5\varphi, \quad y(\varphi) = a\sin^5\varphi, \tag{3.1.14}$$

where $a > 0$ and $\varphi \in [0, 2\pi[$, will be found.

Fig. 3.3 The curve (3.1.14)



Solution

The curve dealt with in this exercise is drawn in Fig. 3.3. It recalls an asteroid, for which, however, the exponents in formulas (3.1.14) would be equal to 3. In order to calculate its circumference, formula (3.1.7) can be used, taking the azimuthal angle φ as a parameter t.

In view of the symmetry of the figure, one can limit oneself to the first quarter only and multiply the result by 4. One finds

$$
\begin{aligned}
l &= \int\limits_0^{2\pi} \sqrt{x'(\varphi)^2 + y'(\varphi)^2}\, d\varphi = 4 \int\limits_0^{\pi/2} \sqrt{x'(\varphi)^2 + y'(\varphi)^2}\, d\varphi \\
&= 4 \int\limits_0^{\pi/2} \sqrt{\left(-5a\cos^4\varphi \sin\varphi\right)^2 + \left(5a\sin^4\varphi\cos\varphi\right)^2}\, d\varphi \\
&= 20a \int\limits_0^{\pi/2} \sqrt{\cos^8\varphi \sin^2\varphi + \sin^8\varphi\cos^2\varphi}\, d\varphi \\
&= 10a \int\limits_0^{\pi/2} |\sin 2\varphi| \sqrt{\cos^6\varphi + \sin^6\varphi}\, d\varphi.
\end{aligned}
\tag{3.1.15}
$$

To simplify the integrand expression, the Pythagorean trigonometric identity will be used, leading to

$$\begin{aligned}1 = &\ (\cos^2\varphi + \sin^2\varphi)^3 = \cos^6\varphi + \sin^6\varphi + 3\cos^4\varphi\sin^2\varphi \\ &+3\cos^2\varphi\sin^4\varphi = \cos^6\varphi + \sin^6\varphi + 3\cos^2\varphi\sin^2\varphi(\cos^2\varphi + \sin^2\varphi) \\ = &\ \cos^6\varphi + \sin^6\varphi + \frac{3}{4}\sin^2 2\varphi,\end{aligned} \tag{3.1.16}$$

from which it follows that

$$\cos^6\varphi + \sin^6\varphi = 1 - \frac{3}{4}\sin^2 2\varphi = \frac{1}{4} + \frac{3}{4}\cos^2 2\varphi. \tag{3.1.17}$$

The goal of these transformations will be clear if one notes that the above expression depends only on $\cos 2\varphi$. Under the integral (3.1.15), we also find a factor $\sin 2\varphi = -(\cos 2\varphi)'/2$ (the modulus may be omitted, as for the angle $\varphi \in [0, \pi/2]$, one has $\sin 2\varphi \geq 0$), which makes the substitution $t = \cos 2\varphi$ especially effective. In this way, one gets the integral:

$$l = \frac{5}{2}a\int\limits_{-1}^{1}\sqrt{1+3t^2}\,dt, \tag{3.1.18}$$

to be calculated with the use of the methods dealt with in the first part of this book (see Sect. 14.6). Now the best substitution seems to be $\sqrt{3}\,t = \sinh u$ and after having applied it, one obtains (upon neglecting the integration constant)

$$\begin{aligned}I &= \int\sqrt{1+3t^2}\,dt = \int\sqrt{1+\sinh^2 u}\,\frac{1}{\sqrt{3}}\cosh u\,du \\ &= \frac{1}{\sqrt{3}}\int\cosh^2 u\,du = \frac{1}{2\sqrt{3}}\int(\cosh 2u + 1)\,du \\ &= \frac{1}{4\sqrt{3}}(\sinh 2u + 2u) = \frac{1}{4\sqrt{3}}(2\sinh u\cosh u + 2u).\end{aligned} \tag{3.1.19}$$

Performing these transformations, we have used the known formulas:

$$\cosh 2u = \cosh^2 u + \sinh^2 u$$

and

$$\sinh 2u = 2\cosh u\sinh u,$$

as well as the hyperbolic version of the Pythagorean trigonometric identity. Now one must return to the variable t, and for this goal, the value of the hyperbolic cosine has to be found:

$$\cosh u = \sqrt{1 + \sinh^2 u} = \sqrt{1 + 3t^3}.$$

The result is

$$I = \frac{1}{2\sqrt{3}}(\sqrt{3}\,t\sqrt{1 + 3t^2} + \operatorname{arsinh}\sqrt{3}\,t). \tag{3.1.20}$$

To calculate the length of the considered curve, one has to incorporate the integration limits from (3.1.18):

$$\begin{aligned} l &= \frac{a}{2\sqrt{3}}(\sqrt{3}\,t\sqrt{1 + 3t^2} + \operatorname{arsinh}\sqrt{3}\,t)\Big|_{-1}^{1} \\ &= \frac{a}{2\sqrt{3}}(4\sqrt{3} + 2\operatorname{arsinh}\sqrt{3}) = \left(2 + \frac{\operatorname{arsinh}\sqrt{3}}{\sqrt{3}}\right)a \\ &= \left(2 + \frac{\log(2 + \sqrt{3})}{\sqrt{3}}\right)a, \end{aligned} \tag{3.1.21}$$

simultaneously applying the identity (see (15.1.54) in Part I):

$$\operatorname{arsinh} x = \log(x + \sqrt{1 + x^2}). \tag{3.1.22}$$

3.2 Calculating Areas of Surfaces

Problem 1

The area of the ellipse with axes a and b will be found, by calculating the integral in Cartesian coordinates.

Solution

As we know from the theoretical introduction and the lecture of analysis, the area below the graph of a continuous and positive function $y(x)$ contained between two straight lines $x = A$ and $x = B$ is given by the integral (it is assumed that $A < B$)

$S = \int_A^B y(x)\,dx$. This formula will be used below to find the area of the ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{3.2.1}$$

Naturally, the graph of this equation is not a graph of a function, but due to the mirror symmetry ($y \mapsto -y$) of the ellipse, one may concentrate on its upper half only and simply multiply the result by 2. Thereby the function $y(x)$ can be obtained by solving (3.2.1) with respect to y (and assuming $y > 0$). It is given by the formula:

$$y(x) = b\sqrt{1 - \frac{x^2}{a^2}}. \tag{3.2.2}$$

Now the following integral has to be found:

$$S = 2\int\limits_A^B b\sqrt{1 - \frac{x^2}{a^2}}\,dx, \quad \text{where} \quad A = -a,\ B = a. \tag{3.2.3}$$

There are a few ways to find it. For example, one can use the substitution $x = a\sin t$, to which the reader is encouraged to try independently, but we are going to calculate it integrating by parts. To accomplish this let us insert $1 = [x]'$ and write

$$\begin{aligned}
S &= 2\,\frac{b}{a}\int\limits_{-a}^{a}\sqrt{a^2 - x^2}\,dx = 2\,\frac{b}{a}\int\limits_{-a}^{a}[x]'\sqrt{a^2 - x^2}\,dx \\
&= 2\,\frac{b}{a}\,x\sqrt{a^2 - x^2}\,\bigg|_{-a}^{a} - 2\,\frac{b}{a}\int\limits_{-a}^{a} x\left[\sqrt{a^2 - x^2}\right]'\,dx = 0 + 2\,\frac{b}{a}\int\limits_{-a}^{a}\frac{x^2}{\sqrt{a^2 - x^2}}\,dx \\
&= 2\,\frac{b}{a}\int\limits_{-a}^{a}\frac{x^2 - a^2 + a^2}{\sqrt{a^2 - x^2}}\,dx = -2\,\frac{b}{a}\int\limits_{-a}^{a}\sqrt{a^2 - x^2}\,dx + 2ab\int\limits_{-a}^{a}\frac{1}{\sqrt{a^2 - x^2}}\,dx \\
&= -P + 2ab\arcsin\frac{x}{a}\bigg|_{-a}^{a} = -P + 2ab\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -P + 2\pi ab.
\end{aligned} \tag{3.2.4}$$

Solving the obtained equation for P, one finds the area as equal to πab.

Problem 2

The area of the folium of Descartes defined by the equation:

$$x^3 + y^3 = 6xy \tag{3.2.5}$$

will be found.

Solution

In Fig. 3.4, the so-called "folium of Descartes" is depicted. The purpose of this exercise is to calculate the area contained inside the loop. The easiest way to solve this problem is in polar variables r and φ associated with Cartesian coordinates via the equations: $x = r\cos\varphi$, $y = r\sin\varphi$. Eliminating x and y from (3.2.5), one gets the equation of the curve in the form of:

$$r^3(\cos^3\varphi + \sin^3\varphi) = 6r^2\cos\varphi\sin\varphi, \tag{3.2.6}$$

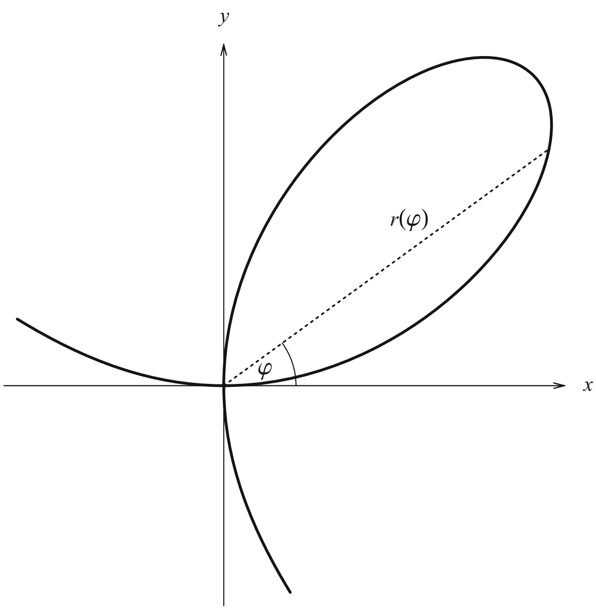


Fig. 3.4 Folium of Descartes

i.e.,

$$r^2[r(\cos^3\varphi + \sin^3\varphi) - 6\cos\varphi\sin\varphi] = 0. \tag{3.2.7}$$

When $\varphi \in]0, \pi/2[$, the expression in square brackets becomes zero for

$$r(\varphi) = \frac{6\cos\varphi\sin\varphi}{\cos^3\varphi + \sin^3\varphi} > 0 \tag{3.2.8}$$

and one gets the loop shown in the figure. It should be noted that the Eq. (3.2.7) in this domain has no other solutions apart from $r = 0$.

In order to calculate the area, we will make use of the formula, probably known to the reader from the lecture and recalled at the beginning of the chapter (see (3.2.9)), for the value of area bounded by a curve $r = r(\varphi)$ and radial straight lines $\varphi = \varphi_1$ and $\varphi = \varphi_2$:

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r(\varphi)^2\, d\varphi. \tag{3.2.9}$$

The sense of formula (3.2.9) is intuitively clear: as far as in the Cartesian variables the area is calculated as the sum of areas of certain rectangles (e.g., see the Riemann construction in Sect. 1.1), now one will have the sum of triangles, such as that presented in Fig. 3.5.

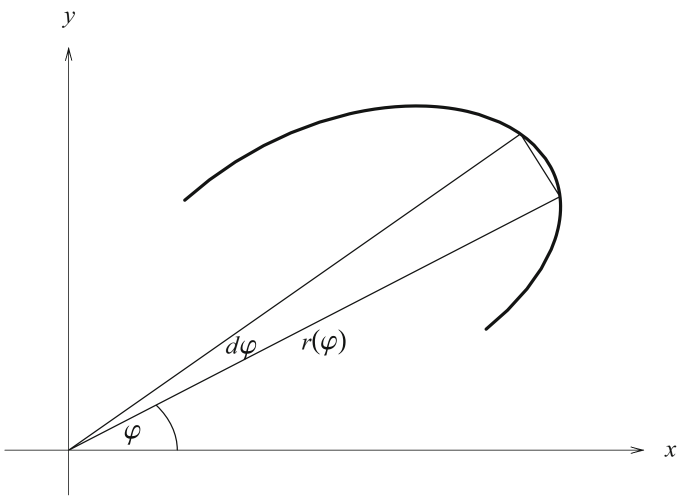


Fig. 3.5 The calculation of the area in polar coordinates

Since $d\varphi$ is small, the area of a single triangle can be calculated as

$$\frac{1}{2}\underbrace{r(\varphi)d\varphi}_{\text{base}}\cdot\underbrace{r(\varphi)}_{\text{altitude}}=\frac{1}{2}r(\varphi)^2d\varphi. \tag{3.2.10}$$

This formula requires some comment. In the first place, for a very small value $d\varphi$, the base of the triangle (i.e., the chord) and the corresponding arc length (i.e., $rd\varphi$) become actually identical. Second, in the same limit, the altitude of the triangle is simply $r(\varphi)$, since the side of the triangle is almost perpendicular to the base. And finally, the circular segment (i.e., the tiny region contained between the chord and the arc), neglected here, is proportional to $(d\varphi)^2$, and therefore, it is negligible of higher order not contributing to S.

Substituting (3.2.8) into (3.2.9), with $\varphi_1=0$, and $\varphi_2=\pi/2$ (note that $r(\varphi)>0$ for $\varphi_1<\varphi<\varphi_2$ and $r(\varphi)\underset{\varphi\to\varphi_{1,2}}{\longrightarrow}0$), we face the problem of calculating the integral:

$$S=\frac{6^2}{2}\int\limits_0^{\pi/2}\frac{\cos^2\varphi\sin^2\varphi}{(\cos^3\varphi+\sin^3\varphi)^2}\,d\varphi. \tag{3.2.11}$$

In the first part of this book we learned how to integrate the rational functions of trigonometric functions (see Sect. 14.4). Because the integrand does not change when simultaneously applying the replacements: $\sin\varphi\mapsto-\sin\varphi$ and $\cos\varphi\mapsto-\cos\varphi$, one knows that the most effective substitution is $t=\tan\varphi$. It can be easily perceived if one reassigns (3.2.11) in the form:

$$S=\frac{6^2}{2}\int\limits_0^{\pi/2}\frac{\cos^2\varphi\sin^2\varphi}{(\cos^3\varphi+\sin^3\varphi)^2}\cdot\frac{1/\cos^6\varphi}{1/\cos^6\varphi}\,d\varphi=\frac{6^2}{2}\int\limits_0^{\pi/2}\frac{\tan^2\varphi}{(1+\tan^3\varphi)^2}\cdot\frac{1}{\cos^2\varphi}\,d\varphi, \tag{3.2.12}$$

and recall that the term $1/\cos^2\varphi$ constitutes the derivative of the function $\tan\varphi$. In addition when $\varphi\to0$, one has $t=\tan\varphi\to0$, while for $\varphi\to\pi/2^-$ one gets $t=\tan\varphi\to\infty$. In the new variable t, the integrand function becomes simply a rational function, and upon performing the (improper) integral one obtains the value of the area as

$$S=18\int\limits_0^{\infty}\frac{t^2}{(1+t^3)^2}\,dt=18\int\limits_0^{\infty}\left[\frac{-1/3}{1+t^3}\right]'\,dt=\left.\frac{-6}{1+t^3}\right|_0^{\infty}=6. \tag{3.2.13}$$

Problem 3

The area enclosed with the curve:

$$(x^2 + y^2)^2 = a^2(x^2 + y^2 - 4xy), \tag{3.2.14}$$

where $a > 0$, will be found.

Solution

In polar coordinates, the curve (3.2.14) is defined by the equation:

$$r^2[r^2 - a^2(1 - 2\sin 2\varphi)] = 0. \tag{3.2.15}$$

In the case when $\pi/12 < \varphi < 5\pi/12$ and $13\pi/12 < \varphi < 17\pi/12$ the expression $a^2(1 - 2\sin 2\varphi)$ is negative. It is, therefore, impossible for the factor in square brackets to vanish and the only solution of the Eq. (3.2.15) is $r = 0$. For other angles of the interval $[0, 2\pi]$, one gets the nonzero solution that is needed

$$r(\varphi) = a\sqrt{1 - 2\sin 2\varphi}. \tag{3.2.16}$$

The obtained curve is shown in Fig. 3.6. In accordance with our results of acceptable values of the angle φ, it extends mainly in the second and fourth quadrants of the coordinate system. To calculate the area bounded by the curve, formula (3.2.9) will be used. Similarly as before, the symmetry of the Eq. (3.2.16), while substituting $\varphi \mapsto \varphi + \pi$, may be used: one finds the area of one loop and then multiplies it by 2.

Taking $\varphi_1 = 5\pi/12$ and $\varphi_2 = 13\pi/12$, one gets

$$\begin{aligned} S &= 2\,\frac{1}{2}\int\limits_{5\pi/12}^{13\pi/12} r(\varphi)^2\, d\varphi = a^2 \int\limits_{5\pi/12}^{13\pi/12} (1 - 2\sin 2\varphi)\, d\varphi = a^2(\varphi + \cos 2\varphi)\Big|_{5\pi/12}^{13\pi/12} \\ &= a^2\left(\frac{8\pi}{12} + \cos\frac{13\pi}{6} - \cos\frac{5\pi}{6}\right) \\ &= a^2\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{2}{3}\pi + \sqrt{3}\right). \end{aligned} \tag{3.2.17}$$

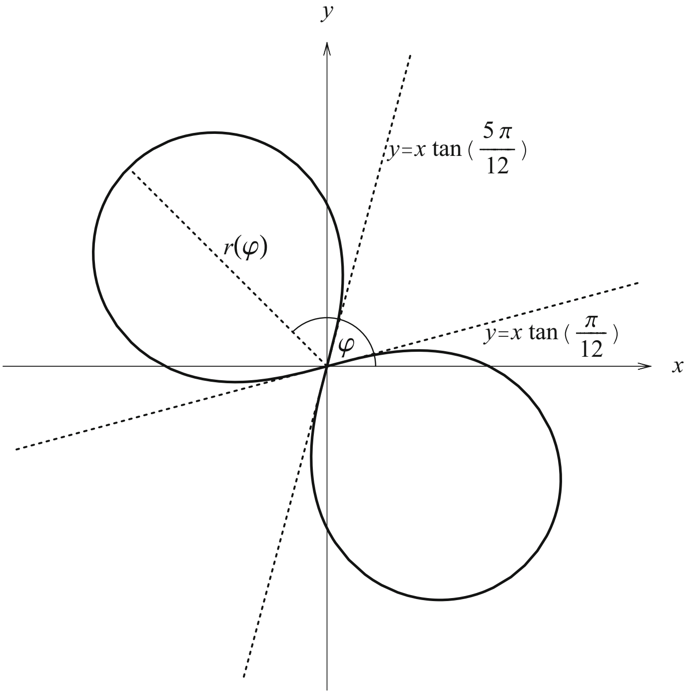


Fig. 3.6 The curve (3.2.14)

Problem 4

The area of one coil of the hyperbolic spiral:

$$r(\varphi)\,\varphi = a, \tag{3.2.18}$$

where $a > 0$, will be found.

Solution

By the area of one coil, we understand that it is bounded by two subsequent coils of the spiral, as it is shown in Fig. 3.7 (note that the angle φ is not restricted to the interval $]0, 2\pi]$, to which one is accustomed while using polar coordinates). With the increase of the angle φ, the spiral gets tighter and tighter. It is assumed that for an outer coil this angle is chosen as $2\pi(n-1) < \varphi \leq 2\pi n$ and for the inner one $2\pi n < \varphi \leq 2\pi(n+1)$, where n is a natural number (a winding number). The area to be found is marked in the figure with gray color.

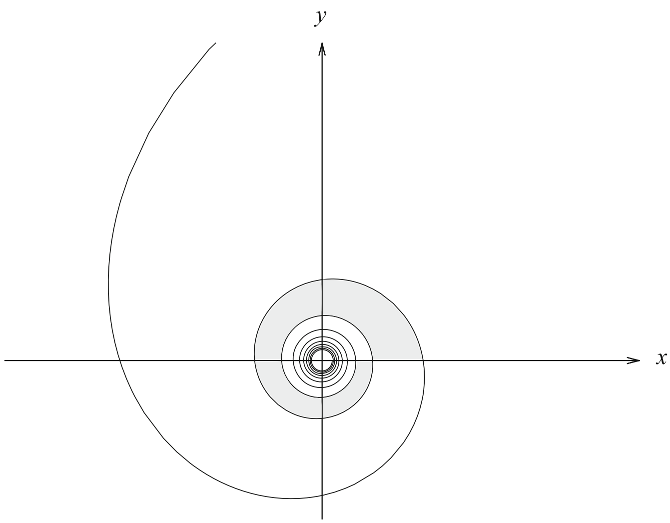


Fig. 3.7 An exemplary hyperbolic spiral (3.2.18). The area between the nth and $n+1$st coil is marked in gray

If now, based on formula (3.2.9), we simply write

$$S = \frac{1}{2} \int\limits_{2\pi(n-1)}^{2\pi n} r(\varphi)^2 \, d\varphi,$$

one will not find the area between coils but the area contained within the nth coil including all subsequent ones. For this reason, one must rather write

$$\begin{aligned} S_n &= \frac{1}{2} \int\limits_{2\pi(n-1)}^{2\pi n} r(\varphi)^2 \, d\varphi - \frac{1}{2} \int\limits_{2\pi n}^{2\pi(n+1)} r(\varphi)^2 \, d\varphi \\ &= \frac{1}{2} \int\limits_{2\pi(n-1)}^{2\pi n} \frac{a^2}{\varphi^2} \, d\varphi - \frac{1}{2} \int\limits_{2\pi n}^{2\pi(n+1)} \frac{a^2}{\varphi^2} \, d\varphi = -\frac{a^2}{2} \cdot \frac{1}{\varphi}\bigg|_{2\pi(n-1)}^{2\pi n} + \frac{a^2}{2} \cdot \frac{1}{\varphi}\bigg|_{2\pi n}^{2\pi(n+1)} \\ &= \frac{a^2}{4\pi} \left(-\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n} \right) = \frac{a^2}{2\pi} \left(\frac{n}{n^2-1} - \frac{1}{n} \right) = \frac{a^2}{2\pi n(n^2-1)}. \end{aligned} \tag{3.2.19}$$

3.3 Finding Volumes and Surface Areas of Solids of Revolution

Problem 1

The volume of the ellipsoid of revolution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1, \tag{3.3.1}$$

where $a, b > 0$, will be found.

Solution

The ellipsoid (3.3.1) can be obtained by rotating, around the x-axis, the ellipse of the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{3.3.2}$$

or rather its upper "half," for which $y \geq 0$. This is shown in Fig. 3.8, for which the solid line represents the curve being rotated.

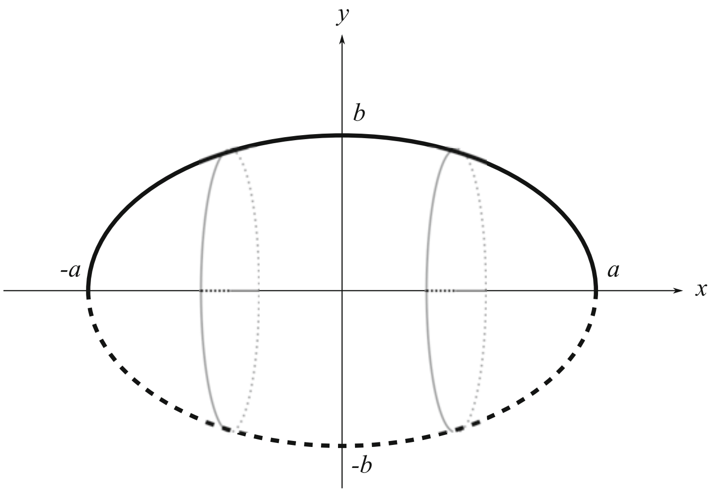


Fig. 3.8 Rotary ellipsoid (3.3.1)

If one solves the Eq. (3.3.2) with respect to y (under the condition that $y \geq 0$), it will become clear that the rotated curve will constitute the graph of function:

$$f(x) = y(x) = b\sqrt{1 - \frac{x^2}{a^2}}. \tag{3.3.3}$$

In such a case, as it was written at the beginning of this chapter and surely known from lectures of analysis, for $A \leq x \leq B$ the volume of the solid can be found with the use of the formula:

$$V = \pi \int_A^B f(x)^2\, dx. \tag{3.3.4}$$

It is intuitively clear, as can be seen from Fig. 3.9. The volume is equal to the sum of volumes of cylinders (such as that marked in gray) of infinitesimal height dx and the base of radius equal to $f(x)$. Of course, for this radius one could use any of the values assumed by the function f in the interval $[x, x + dx]$, but the result does not depend on this choice. The difference between the two values is in fact negligible of higher order, i.e., it is proportional to $(dx)^2$.

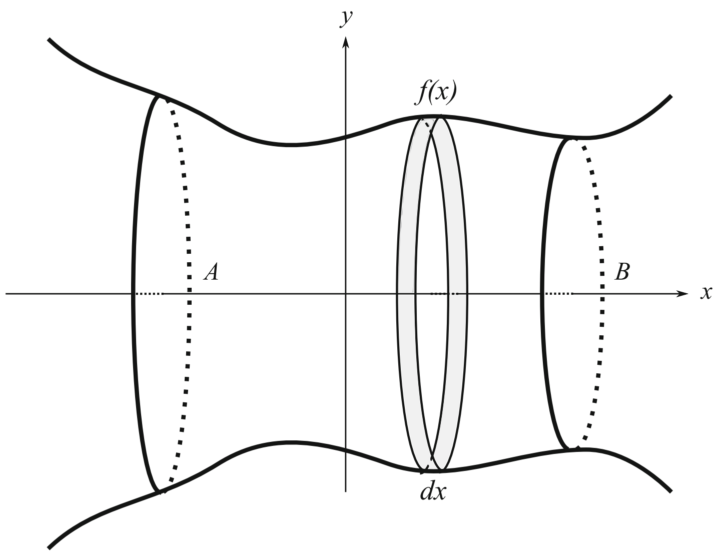


Fig. 3.9 The visualization of how the volume is calculated

Using formula (3.3.4) for $A = -a$ and $B = a$, one can write

$$V = \pi \int_{-a}^{a} y(x)^2\, dx = \pi b^2 \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx$$
$$= \pi b^2 \left(x - \frac{x^3}{3\,a^2}\right)\Bigg|_{-a}^{a} = 2\pi b^2 \left(a - \frac{a^3}{3\,a^2}\right) = \frac{4}{3}\,\pi a b^2. \qquad (3.3.5)$$

In the particular case when $a = b = r$, one obviously gets the expression for the volume of the sphere $V = 4/3\,\pi r^3$.

Problem 2

The volume and surface area of the torus defined by the equation:

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2, \qquad (3.3.6)$$

where $R > r$, will be found.

Solution

If a curve being rotated is given in the parametric form, i.e., it is defined by relations $x(t)$ and $y(t)$ for some parameter t satisfying the requirement $\alpha < t < \beta$, then in place of (3.3.4), one can make use of the formula:

$$V = \pi \int_{\alpha}^{\beta} y(t)^2 x'(t)\, dt. \qquad (3.3.7)$$

In the case of a torus, as that shown in Fig. 3.10, the role of the parameter t is played by θ, whereas the angle φ is related to the rotation of the curve

$$x(\theta) = R + r\cos\theta, \quad z(\theta) = r\sin\theta, \qquad (3.3.8)$$

i.e., the circle lying in the plane xz, around the z-axis. The role of $x(t)$ in formula (3.3.7) is now played by $z(\theta)$, and that of $y(t)$—by $x(\theta)$. As regards to the domain of the parameter θ, for a torus, one has $0 \leq \theta < 2\pi$. It is interesting to note that formula (3.3.7) can also be applied for closed curve which is the circle of Fig. 3.10. For $0 \leq \theta < \pi/2$ and $3\pi/2 < \theta < 2\pi$, the integral contributes with positive sign while for $\pi/2 \leq \theta < 3\pi/2$ with negative sign (for $z'(\theta)$ is found to be negative). And after the subtraction of both of them, one gets in fact the volume of the "tube" itself which is the interior of the torus. One has then

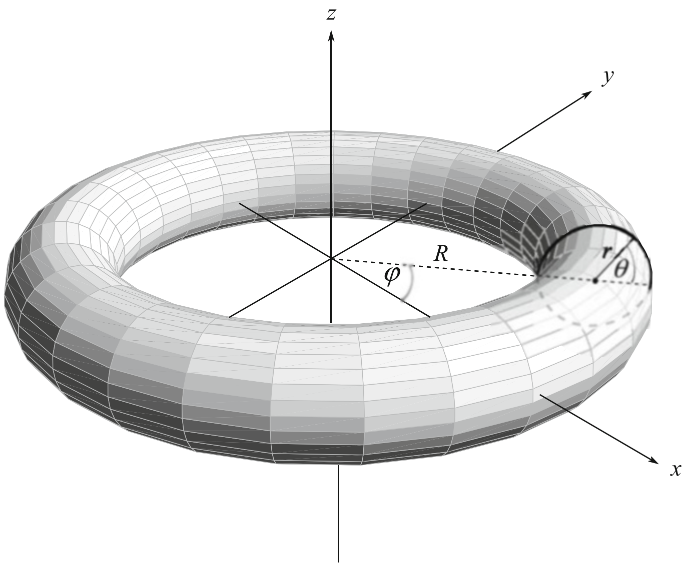


Fig. 3.10 The torus given by the Eq. (3.3.6)

$$V_{\text{torus}} = \pi \int_0^{2\pi} (R + r\cos\theta)^2 r\cos\theta \, d\theta$$
$$= \pi r \int_0^{2\pi} (R^2\cos\theta + 2Rr\cos^2\theta + r^2\cos^3\theta)\, d\theta. \tag{3.3.9}$$

The first integral (in the second line) is equal to zero since it is a definite integral of the cosine function over its entire period. Identically, any other odd power of cosine also gives zero. It is easy to find out by writing

$$I = \int_0^{2\pi} \cos^{2n+1}\theta \, d\theta = \int_0^{\pi} \cos^{2n+1}\theta \, d\theta + \int_{\pi}^{2\pi} \cos^{2n+1}\theta \, d\theta \tag{3.3.10}$$

and substituting $\theta' = \theta - \pi$ in the second integral. Thus one gets

$$I = \int_0^{\pi} \cos^{2n+1}\theta \, d\theta + \int_0^{\pi} \cos^{2n+1}(\theta' + \pi)\, d\theta'$$
$$= \int_0^{\pi} \cos^{2n+1}\theta \, d\theta - \int_0^{\pi} \cos^{2n+1}\theta \, d\theta = 0, \tag{3.3.11}$$

where the reduction formula $\cos(\theta+\pi) = -\cos\theta$ has been used, and in the second line, the prime at θ has been omitted. As a result, one has to calculate only

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} (\cos 2\theta + 1)\, d\theta = \frac{1}{2}(0 + 2\pi) = \pi \tag{3.3.12}$$

and the volume is found to be

$$V_{\text{torus}} = 2\pi^2 r^2 R. \tag{3.3.13}$$

The surface area of a solid of revolution emerging from the rotation of a curve around the x-axis can be calculated as the sum of gray stripes ("hoops") of Fig. 3.9. Each of them can be determined according to the formula:

$$\text{Area} \;=\; \text{circumference} \;\times\; \text{strip width}.$$

The result is $2\pi|y(x)| \cdot \sqrt{1 + y'(x)^2}\, dx$ (the factor $\sqrt{1 + y'(x)^2}$ is nothing else but the inverse of the cosine of the slope for the curve $y(x)$ at x), so one has

$$S = 2\pi \int_a^b |y(x)|\sqrt{1 + y'(x)^2}\, dx. \tag{3.3.14}$$

If a curve is given in the parametric form, then one obtains

$$S = 2\pi \int_\alpha^\beta |y(t)|\sqrt{x'(t)^2 + y'(t)^2}\, dt. \tag{3.3.15}$$

Applying this latter formula to our area, we find

$$\begin{aligned} S_{\text{torus}} &= 2\pi \int_0^{2\pi} (R + r\cos\theta)\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, d\theta \\ &= 2\pi r \int_0^{2\pi} (R + r\cos\theta)\, d\theta = 2\pi r(2\pi R + 0) = 4\pi^2 r R. \end{aligned} \tag{3.3.16}$$

Problem 3

The volume of the solid formed by rotating the curve:

$$y(x) = \frac{1}{x^2 + 1} \tag{3.3.17}$$

around the x-axis, will be found.

Solution

In accordance with formula (3.3.4), the volume of the solid shown in Fig. 3.11 can be calculated by the following integration:

$$V = \pi \int_{-\infty}^{\infty} y(x)^2 \, dx. \tag{3.3.18}$$

A new element to be noted here is the infinite extension of the solid. Contrary to previous examples, for which $a \le x \le b$, this time one has $-\infty < x < \infty$. In such a case, before calculating the integral (3.3.18), one has to make sure that it does exist (this is ultimately an improper integral). The study of such integrals was covered in

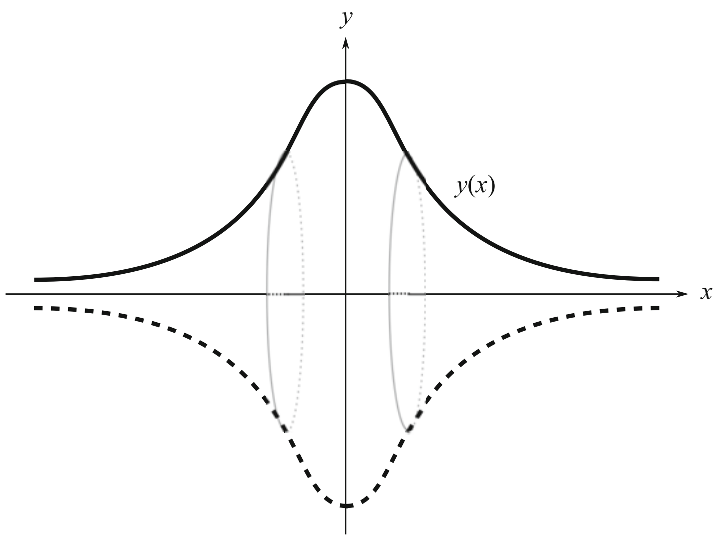


Fig. 3.11 Rotation of the curve (3.3.17) around x-axis

Chap. 2. We found out then that if the integrand function decreases at infinity as a certain power of argument and this power is larger than 1 (i.e., $f(x) \simeq 1/x^p$ and $p > 1$), the integral is convergent. It is exactly the case dealt with in (3.3.18) if one substitutes $y(x)$ in the form of (3.3.17). In our case it happens that $y(x)^2 \simeq 1/x^4$ and the integral is convergent at both ∞ and $-\infty$. No other "dangerous" points appear.

In order to find the volume, let us first calculate the indefinite integral, integrating by parts:

$$
\begin{aligned}
I &= \int \frac{1}{(x^2+1)^2}\, dx = \int \frac{1}{x^2+1} \cdot \frac{x^2+1-x^2}{x^2+1}\, dx \\
&= \int \frac{1}{x^2+1}\, dx - \int x \frac{x}{(x^2+1)^2}\, dx = \arctan x + \frac{1}{2} \int x \left[\frac{1}{x^2+1}\right]' dx \\
&= \arctan x + \frac{1}{2} \cdot \frac{x}{x^2+1} - \frac{1}{2} \int \frac{1}{x^2+1}\, dx = \frac{1}{2} \arctan x + \frac{1}{2} \cdot \frac{x}{x^2+1}.
\end{aligned} \tag{3.3.19}
$$

Substituting the integration limits one gets

$$
V = \pi I \Big|_{-\infty}^{\infty} = \frac{\pi}{2} \left(\arctan x + \frac{x}{x^2+1}\right) \Big|_{-\infty}^{\infty} = \frac{\pi^2}{2}. \tag{3.3.20}
$$

Problem 4

The area of the patch of the rotary paraboloid defined by the equations $z = x^2 + y^2$, where $x^2 + y^2 \leq 1$, will be found.

Solution

The patch in question shown in Fig. 3.12 will be calculated with the use of formulas (3.0.7) and (3.3.14), naturally with some obvious modifications. In the present case, the graph of the function $x(z) = \sqrt{z}$ is rotated around the z-axis, and therefore,

$$
S = 2\pi \int_0^1 |x(z)| \sqrt{1 + x'(z)^2}\, dz. \tag{3.3.21}
$$

Inserting the derivative $x'(z) = 1/(2\sqrt{z})$ into the above integral, one gets the required area:

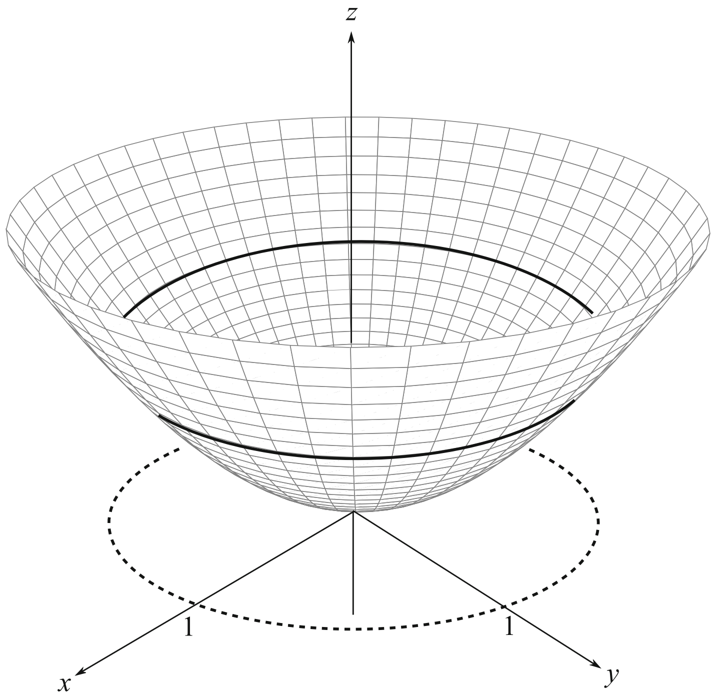


Fig. 3.12 The patch of the paraboloid

$$\begin{aligned} S &= 2\pi \int_0^1 \sqrt{z}\sqrt{1+\left(\frac{1}{2\sqrt{z}}\right)^2}\, dz = 2\pi \int_0^1 \sqrt{z+\frac{1}{4}}\, dz \\ &= 2\pi\, \frac{2}{3} \left(\sqrt{z+\frac{1}{4}}\right)^3 \Bigg|_0^1 = \frac{4\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - \frac{1}{8}\right] = \frac{\pi}{6}(5\sqrt{5}-1). \end{aligned} \tag{3.3.22}$$

3.4 Finding Various Physical Quantities

Problem 1

The moment of inertia of a uniform cone of mass M and radius R about its axis of symmetry will be calculated.

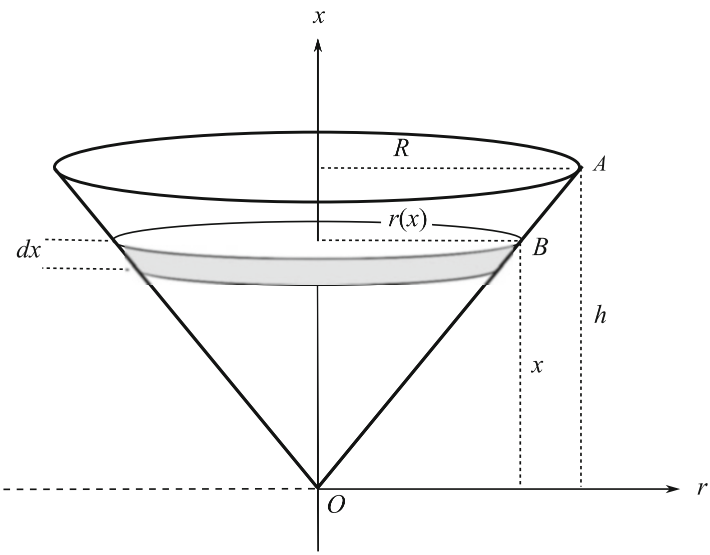


Fig. 3.13 The way of calculating the moment of inertia of a cone

Solution

Let us denote the height of the cone with the symbol h. The final result, naturally, shall not depend on this quantity because compressing and stretching out the cone vertically (by modifying h) do not change the mass distribution around the rotation axis.

The moment of inertia will be found with the use of the result obtained in Exercise 3 of Sect. 1.2, as shown in Fig. 3.13. Namely, let us divide the cone into slices, infinitely thin in the limit (so they may be easily treated as thin cylinders), and for each of them, formula (1.2.27) can be used. The total moment of inertia—in accordance with the additivity principle—equals the sum of the slices and will become the integral in the limit.

Denoting the (constant) mass density with the symbol ρ, one can write

$$I = \int\limits_0^h \frac{1}{2} r(x)^2 \underbrace{\pi r(x)^2 \rho \, dx}_{dM} = \frac{\pi \rho}{2} \int\limits_0^h r(x)^4 \, dx, \tag{3.4.1}$$

$r(x)$ being the radius of the slice. It can be found using the similarity of rectangular triangles seen in the figure. The bigger triangle has R and h as its sides (its hypotenuse OA is the generating line for the cone) and the smaller one $r(x)$ and x respectively (the hypotenuse is the segment of the generating line between points O and B). Creating the appropriate proportion, one gets

$$\frac{r(x)}{x} = \frac{R}{h} \implies r(x) = \frac{R}{h}\, x, \tag{3.4.2}$$

and after the substitution into (3.4.1) and the calculation of the integral, we come to

$$I = \frac{\pi\rho R^4}{2h^4} \int_0^h x^4\, dx = \frac{\pi\rho R^4}{10h^4}\, x^5 \Big|_0^h = \frac{\pi\rho R^4 h}{10}. \tag{3.4.3}$$

The mass density ρ can be eliminated if the known formula for cone volume is made use of:

$$\rho = \frac{M}{V_{\text{cone}}} = \frac{M}{\pi R^2 h/3}. \tag{3.4.4}$$

Finally, we find

$$I = \frac{3}{10} M R^2. \tag{3.4.5}$$

Problem 2

The moment of inertia of the uniform sphere of mass M and radius R about its axis of symmetry will be found.

Solution

In the preceding problem, we saw that the method of dividing a solid into slices is very convenient for calculating moments of inertia if all of them can be considered as infinite thin cylinders. The similar gimmick should, therefore, be effective also in the case of a sphere if it is done in a way shown in Fig. 3.14.

This time one has

$$I = \int_{-R}^{R} \frac{1}{2} r(x)^2 \underbrace{\pi r(x)^2 \rho\, dx}_{dM} = \frac{\pi\rho}{2} \int_{-R}^{R} r(x)^4\, dx, \tag{3.4.6}$$

$r(x)$ being the radius of a given slice to be found from the Pythagorean theorem applied to the rectangular triangle OAB:

$$r(x)^2 = R^2 - x^2. \tag{3.4.7}$$

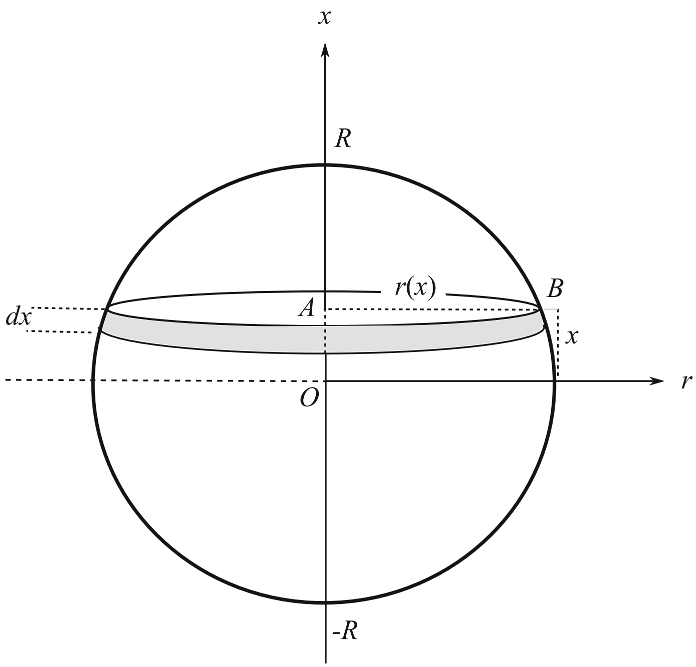


Fig. 3.14 A visual for calculating the moment of inertia of a sphere

Then one must carry out the integral in the form:

$$I = \frac{\pi\rho}{2}\int_{-R}^{R}(R^2 - x^2)^2\,dx = \frac{\pi\rho}{2}\int_{-R}^{R}(R^4 - 2R^2x^2 + x^4)\,dx$$
$$= \frac{\pi\rho}{2}\left(R^4x - \frac{2}{3}R^2x^3 + \frac{1}{5}x^5\right)\Bigg|_{-R}^{R} = \frac{8\pi\rho R^5}{15}. \tag{3.4.8}$$

The mass density of the uniform sphere can be easily expressed by its total mass:

$$\rho = \frac{M}{V_{\text{sphere}}} = \frac{M}{4\pi R^3/3}, \tag{3.4.9}$$

thanks to which we get from (3.4.8)

$$I = \frac{2}{5}MR^2. \tag{3.4.10}$$

Problem 3

The moment of inertia of a uniform trapezoid of mass M, bases a and b, height h about the axis parallel to the bases, and distance c between the axis and one of the bases will be found.

Solution

Only the situation where the axis of rotation lies outside the trapezoid is considered, as it is shown in Fig. 3.15. The reader is encouraged to solve the other case independently.

Let us denote the surface mass density with the symbol σ. The contribution to the moment of inertia coming from an infinitesimally thin "bar" (marked in the figure in gray) of length $l(x)$ and thickness dx, shifted by x from the axis of rotation, is

$$dI = \sigma l(x)dx \cdot x^2, \tag{3.4.11}$$

since it can be treated as a rectangle with the area of $l(x) \cdot dx$. The full moment of inertia is, therefore, given by the integral:

$$I = \int\limits_{c}^{c+h} dx\, x^2 l(x)\sigma. \tag{3.4.12}$$

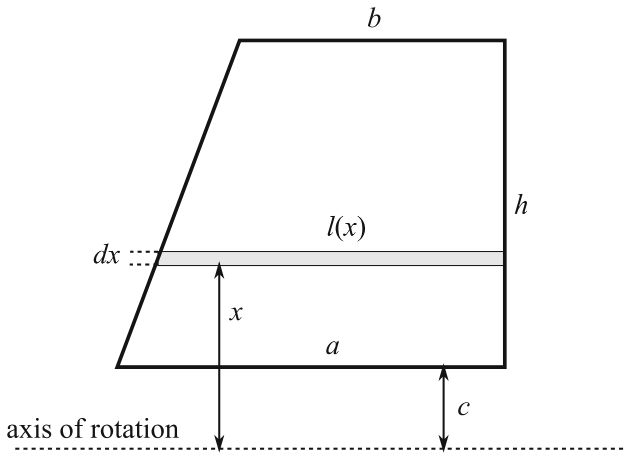


Fig. 3.15 A visual for calculating the moment of inertia of a trapezoid

The dependence $l(x)$ is linear. In addition, we know that $l(c) = a$ and $l(c+h) = b$. From the known formula for a straight line passing through two chosen points of coordinates (x_1, y_1) and (x_2, y_2) (naturally, it is assumed that $x_1 \neq x_2$), i.e.,

$$y(x) = \frac{y_2 - y_1}{x_2 - x_1}\,(x - x_1) + y_1, \tag{3.4.13}$$

one immediately gets for $y_1 = a$, $y_2 = b$, $x_1 = c$

$$l(x) = \frac{b-a}{h}\,(x - c) + a. \tag{3.4.14}$$

By plugging this expression into (3.4.12), one has to integrate the polynomial:

$$\begin{aligned}
I &= \sigma \int\limits_c^{c+h} dx\, x^2 \left(\frac{b-a}{h}(x-c) + a\right) \\
&= \sigma \int\limits_c^{c+h} dx \left[\frac{b-a}{h}\,x^3 + \left(a - \frac{(b-a)c}{h}\right) x^2\right] \\
&= \sigma \left[\frac{b-a}{4h}\,x^4 + \frac{1}{3}\left(a - \frac{(b-a)c}{h}\right) x^3\right]\Bigg|_c^{c+h} \\
&= \sigma \left[\frac{b-a}{4h}((c+h)^4 - c^4) + \frac{1}{3}\left(a - \frac{(b-a)c}{h}\right)((c+h)^3 - c^3)\right].
\end{aligned} \tag{3.4.15}$$

Now one can eliminate the surface mass density σ in favor of the total mass M. As we know, the surface area of a trapezoid is given by $S = (a+b)h/2$. Thus, we find

$$M = \sigma\,\frac{(a+b)h}{2} \quad\Longrightarrow\quad \sigma = \frac{2M}{(a+b)h}. \tag{3.4.16}$$

Upon inserting this result into (3.4.15), after some simplifications one arrives at:

$$I = M\left[c^2 + \frac{2}{3}\,ch + \frac{1}{6}\,h^2 + bh\,\frac{2c+h}{3(a+b)}\right]. \tag{3.4.17}$$

Let us check if this result leads to the well-known moment of inertia of a rod of length h and mass M rotating around an axis perpendicular to it and passing through one of its ends:

$$I_{\text{rod}} = \frac{1}{3}\,Mh^2.$$

To this end in formula (3.4.17), one should set $b = a$ (the trapezoid will then become a rectangle) and $c = 0$, and as a result, one actually gets

$$I = M\left[0 + 0 + \frac{1}{6}\,h^2 + ah\,\frac{h}{3(a+a)}\right] = \frac{1}{3}\,Mh^2. \tag{3.4.18}$$

Problem 4

The center of mass of half of the homogeneous cylinder described by the equations:

$$x^2 + y^2 \leq R^2, \quad x \geq 0, \quad 0 \leq z \leq H, \tag{3.4.19}$$

where $R, H > 0$, will be found.

Solution

As we know, the center of mass of a solid with mass M, which consists of (point-like) masses m_i ($i = 1, 2, \ldots, N$), can be found from the formula:

$$\vec{R}_{CM} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i. \tag{3.4.20}$$

The vector $\vec{r}_i$ denotes here the position of the ith mass.

In our case, $\vec{R}_{CM} = [x_{CM}, y_{CM}, z_{CM}]$, but of these three coordinates, the values of two can be immediately guessed. The symmetry of the solid, as shown in Fig. 3.16, immediately leads to

$$y_{CM} = 0 \quad \text{and} \quad z_{CM} = \frac{H}{2}. \tag{3.4.21}$$

It remains, therefore, to calculate x_{CM}, using the method of dividing the solid into slices and the formula which is a continuous (i.e., integral) version of (3.4.20). These slices are now parallel to the plane yz and perpendicular to the x-axis, as it is shown in the figure. Each slice has transverse dimensions $2\sqrt{R^2 - x^2} \times H$ and thickness dx. Therefore,

$$x_{CM} = \frac{1}{M}\int_0^R x \cdot \underbrace{2\sqrt{R^2 - x^2}H\rho\, dx}_{dM} = \frac{H\rho}{M}\int_0^R 2x\sqrt{R^2 - x^2}\, dx. \tag{3.4.22}$$

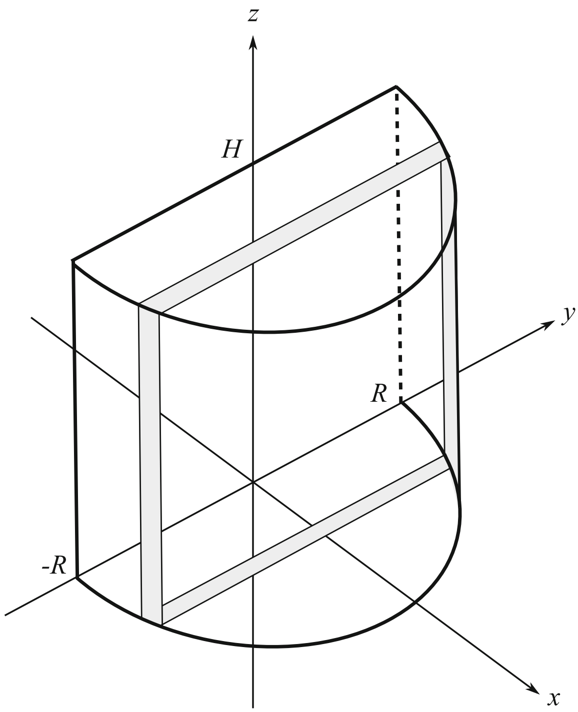


Fig. 3.16 The half of the cylinder described by the Eq. (3.4.19)

The easiest way to perform this integral is by substituting $t = x^2$. Then,

$$x_{CM} = \frac{H\rho}{M}\int\limits_0^{R^2}\sqrt{R^2 - t}\,dt = -\frac{2H\rho}{3M}\left(\sqrt{R^2 - t}\right)^3\Bigg|_0^{R^2} = \frac{2HR^3\rho}{3M}, \tag{3.4.23}$$

and having regard to the fact that the mass density is equal to

$$\rho = \frac{M}{V_{\mathrm{walec}}/2} = \frac{M}{\pi R^2 H/2}, \tag{3.4.24}$$

one gets

$$x_{CM} = \frac{4}{3\pi}R. \tag{3.4.25}$$

Notice that if one missed at the beginning that $y_{CM} = 0$ and tried to calculate it, then in place of (3.4.22), one would have to work with the following integral (slices would be now perpendicular to the y-axis):

$$y_{CM} = \frac{1}{M}\int_{-R}^{R} y \underbrace{\sqrt{R^2-y^2}H\rho\, dy}_{dM}, \tag{3.4.26}$$

which obviously equals zero because of the odd parity of the integrand function in the variable y. In turn, when calculating z_{CM}, the slices would be chosen as horizontal and each of them would have identical volume (and hence the mass too). Thus, one would obtain

$$z_{CM} = \frac{1}{M}\int_{0}^{H} z \underbrace{\frac{1}{2}\pi R^2\sigma dz}_{dM} = \frac{\pi R^2\sigma}{2M}\int_{0}^{H} z dz = \frac{H}{2}, \tag{3.4.27}$$

in accordance with our prediction.

Problem 5

The electrostatic potential at the center of the square frame of side l uniformly charged with total charge Q will be found.

Solution

Let us denote with the symbol λ the linear charge density $\lambda = Q/(4l)$. As we know, the electrostatic potential coming from charges q_i, $i = 1, \ldots, N$, measured at a certain point $\vec{R}$, is given by the formula:

$$V(\vec{R}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{R}-\vec{r}_i|}, \tag{3.4.28}$$

where $\vec{r}_i$ means the location of the ith charge and ε_0, the vacuum permittivity. For a continuous distribution, the sum should be replaced with an integral. In our case, the charge is distributed on the line (and not on a surface or in space), so one will have to work with a single integral over the circumference of the frame.

In order to calculate the potential, let us divide the frame into (infinitesimal) sections—such as that shown in Fig. 3.17—each of which can be practically treated as a point. The vector $[x, l/2]$ plays the role of $\vec{r}_i$ and its length is denoted by $r(x)$. Since we are interested in the field at the center of the frame, then $\vec{R} = [0, 0]$ and $|\vec{R} - \vec{r}_i|$ simply equals $r(x) = \sqrt{x^2 + l^2/4}$. Thus, formula (3.4.28) assumes the form:

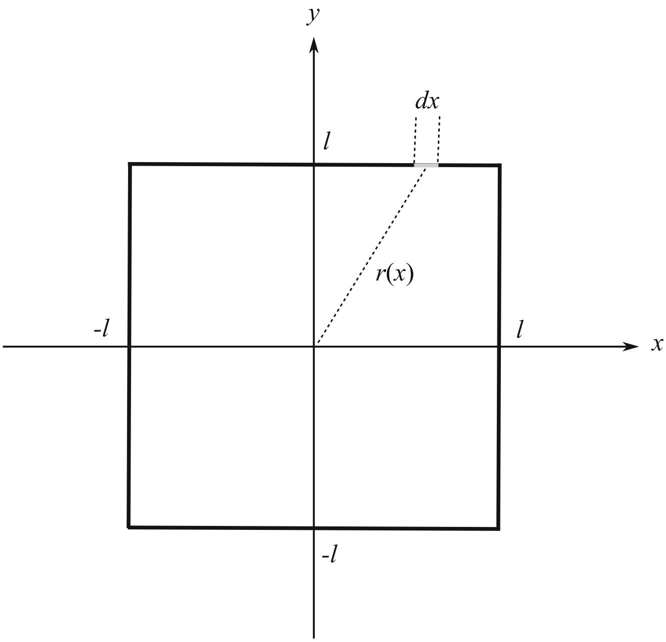


Fig. 3.17 Uniformly charged frame

$$V_1 = \frac{1}{4\pi\epsilon_0}\int\limits_{-l/2}^{l/2} \underbrace{\frac{1}{\sqrt{x^2+l^2/4}}}_{1/|\vec{R}-\vec{r}_i|}\underbrace{\lambda\, dx}_{q_i}, \tag{3.4.29}$$

the integration being performed over one side of the frame. The complete result, due to the symmetry of the square under rotation by the right angle and the superposition principle, will be obtained by multiplying (3.4.29) by 4. To calculate the indefinite integral, one substitutes $x = l/2 \cdot \sinh t$:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+l^2/4}}\, dx &= \int \frac{\cosh t}{\cosh t}\, dt = \int dt = t \\ &= \operatorname{arsinh}\frac{2x}{l} = \log\left(\frac{2x}{l} + \sqrt{\frac{4x^2}{l^2}+1}\right),\end{aligned} \tag{3.4.30}$$

where the hyperbolic counterpart of the Pythagorean trigonometric identity and formula (3.1.22) have been used and an irrelevant integration constant has been omitted. The potential can be now found by the substitution of the integration limits:

$$V = 4\,\frac{\lambda}{4\pi\epsilon_0}\,\log\left(\frac{2x}{l} + \sqrt{\frac{4x^2}{l^2} + 1}\right)\Bigg|_{-l/2}^{l/2} \tag{3.4.31}$$

$$= \frac{Q}{4\pi l\epsilon_0}\left[\log\left(1+\sqrt{2}\right) - \log\left(-1+\sqrt{2}\right)\right] = \frac{Q}{4\pi l\epsilon_0}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

3.5 Exercises for Independent Work

Exercise 1 Find the lengths of curves defined by the formulas:

(a) $y(x) = \sqrt{1-x^2}$, for $x \in [-1, 1]$,
(b) $y(x) = \cosh x$, for $x \in [0, 1]$,
(c) $y(x) = x^{3/2}$, for $x \in [0, 5]$.

Answers

(a) π (half of the circumference of the unit circle),
(b) $(e - 1/e)/2$,
(c) $335/27$.

Exercise 2 Find the areas of the flat surface:

(a) bounded by the x-axis and the curve given by $y(x) = 1 - x^4$,
(b) enclosed by the curve $(x^2 + y^2)^2 \le x^3$,
(c) defined (in polar coordinates) by the inequality $r(\varphi) \le \cos 2\varphi$.

Answers

(a) $8/5$,
(b) $5\pi/32$,
(c) $\pi/4$.

Exercise 3 Find the volume of the solid created by rotating around the x-axis

(a) the curve given by $y(x) = \arcsin x$, for $x \in [0, 1]$,
(b) the curve given by $y(x) = \log x$, for $x \in [1, e]$,
(c) the curve given by $y(x) = 1/(x + x^2)$, for $x \in [1, 3]$,
(d) the curve defined by the parametric equation $x(t) = 3t\sin t$, $y(t) = t\cos t$, for $t \in [0, \pi/2]$.

Answers

(a) $\pi(\pi^2 - 8)/4$,
(b) $\pi(e - 2)$,
(c) $\pi[11/12 + 2\log(2/3)]$,
(d) $\pi[\pi^3/4 - 6\pi + 12]$.

Exercise 4 Find the total surface area of the solid created by rotating around the x-axis:

(a) the curve given by $y(x) = \sqrt{x}$, for $x \in [0, 2]$,
(b) the curve defined by the parametric equation $x(t) = \cosh t$, $y(t) = t$, for $t \in [0, 1]$.

Answers

(a) $19\pi/3$,
(b) $\pi(3 - 2/e)$.

Exercise 5 Find the moments of inertia of

(a) the homogeneous trapezoid of mass M, bases a and b, and height h about the axis parallel to the bases passing at an equal distance between them,
(b) the rod of mass M, length l, and linear mass density $\rho(x) = \text{const} \cdot x^2$ (where x is the distance from one of the ends) about the axes perpendicularly passing through rod's ends.

Answers

(a) $Mh^2/12$,
(b) $3Ml^2/5$ for the lighter end, $Ml^2/10$ for the heavier one.

Chapter 4
Dealing with Functions of Several Variables

This chapter contains introductory notions referring to the functions of several variables. We learn how to find images and preimages of various sets and deal with the limits and continuity in more than one dimension.

The notions of images and preimages introduced in Part I (see Chap. 2) hold in many dimensions as well. They are recalled below.

Consider a function $f : X \to Y$. The **image** or **range** of a set $A \subset X$ denoted by $f(A)$ is defined in the following way:

$$f(A) = \{y \in Y \mid y = f(x) \wedge x \in A\}. \tag{4.0.1}$$

The **preimage** or **inverse image** of a set $A \subset f(X) \subset Y$ is

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}. \tag{4.0.2}$$

Below, only functions defined in $\mathbb{R}^N$ are studied. **Heine's definition of a limit** of a function, defined in certain domain $D \subset \mathbb{R}^N$ at a given point $x_0 \in \mathbb{R}^N$ that is the cluster point for D, has the following form:

$$\lim_{x\to x_0} f(x) = g \iff \big[\forall_{(x_n)} (x_n \in D \wedge x_n \neq x_0, n = 1, 2, 3, \ldots)$$
$$\wedge \lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = g\big]. \tag{4.0.3}$$

By $\lim\limits_{x\to x_0}$, we understand $\lim\limits_{|x-x_0|\to 0}$. Consequently, $\lim\limits_{n\to\infty} x_n = x_0$ denotes $\lim\limits_{n\to\infty} |x_n - x_0| = 0$. It should be noted that the obtained results must be identical for all sequences x_n converging to x_0, the variety of which is now infinitely richer than in the one-dimensional case. Therefore, the existence of the limit is a much stronger requirement than the existence of limits individually in each argument.

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The similar definition of a **continuous function** at a given point x_0 belonging to the domain $D \subset \mathbb{R}^N$ is the following:

$$\text{a function } f \text{ is continuous at the point } x_0 \in D \iff \tag{4.0.4}$$
$$\Big[\forall_{(x_n)}\ (x_n \in D, n = 1, 2, 3, \dots)\ \wedge\ \lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)\Big].$$

The function continuous with respect to each variable x_i does not have to be continuous according to the above definition.

The **Darboux property** can be stated as follows:

Let $f : \mathbb{R}^N \supset D \to \mathbb{R}$ be continuous. Then the image of a connected subset of D is an interval.

4.1 Finding Images and Preimages of Sets

Problem 1

Given the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by the formula:

$$f(x, y, z) = x^2 + y^2 + z^2. \tag{4.1.1}$$

The image of the set:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 6\} \tag{4.1.2}$$

under function f will be found.

Solution

While looking for images of sets under various functions, one cannot, unfortunately, rely on any algorithm and our method needs each time to be adapted to the specificity of a particular example. In the present problem, first of all, one should pay attention to the function f which can be interpreted as a square distance of the point (x, y, z) from the origin. If so, the text of the exercise can be restated in the following way: on the given plane $x + 2y + 3z = 6$, find the points of minimal and maximal distances from the point $(0, 0, 0)$. If such points existed, they could be marked as $d_{\min}$ and $d_{\max}$, and the image would be in this case the interval $[d_{\min}, d_{\max}]$. This stems from the fact that the function f is a polynomial (in three variables) i.e., the continuous function and the set A is connected. The image must, in this case, also be a connected set (it is the so-called Darboux property), and the only sets of that kind in $\mathbb{R}$ are intervals. Below we will try to examine whether such extremal points exist and possibly find them.

Since one is dealing with the plane, i.e., an unbounded set, it is obvious that the "farthest" point does not exist. For example, let us choose a point of coordinates $(6-2\xi, \xi, 0)$, where ξ is any real number. Regardless of its choice, the coordinates satisfy the equation of the plane, since

$$6-2\xi+2\xi+3\cdot 0=6. \tag{4.1.3}$$

The value assumed by the function f at this point, or in other words, its distance from the origin of the coordinate system, is

$$f(6-2\xi,\xi,0)=(6-2\xi)^2+\xi^2+0^2=5\xi^2-24\xi+36. \tag{4.1.4}$$

This expression describes the ordinary quadratic function and it is clear that it can be made arbitrarily large by the appropriate choice of ξ. Thus, the result indicates that $d_{\max}$ does not exist and $f(A)$ will probably have the form $[d_{\min}, \infty[$.

In order to find on the plane $Ax+By+Cz+D=0$, the point nearest to the origin, one can make use of the known fact that the vector $v=[A,B,C]$ is perpendicular to this plane. In our case it has the form $v=[1,2,3]$. This vector spans the whole straight line of the parametric equation ($t\in\mathbb{R}$):

$$[x,y,z]=[x_0,y_0,z_0]+v\cdot t, \quad \text{i.e.,} \quad \begin{cases} x=x_0+t,\\ y=y_0+2t,\\ z=z_0+3t, \end{cases} \tag{4.1.5}$$

where it has been set $x_0=y_0=z_0=0$, since we are interested in a line passing through the origin.

Now we have to ask ourselves: at which point does this straight line intersect the plane? Naturally, this takes place for that value of the parameter t, for which the coordinates (4.1.5) meet its equation:

$$\underbrace{t}_{x}+2\cdot\underbrace{2t}_{y}+3\cdot\underbrace{3t}_{z}=6 \implies t=\frac{3}{7}. \tag{4.1.6}$$

This value substituted into (4.1.5) gives $(3/7, 6/7, 9/7)$. Thus, one finally gets

$$d_{\min}=\sqrt{(3/7)^2+(6/7)^2+(9/7)^2}=\sqrt{18/7} \tag{4.1.7}$$

and the set one has been looking for, has the form

$$f(A)=\left[\sqrt{18/7},\infty\right[. \tag{4.1.8}$$

Problem 2

Given the mapping f defined on the plane $\mathbb{R}^2$ (beyond the straight lines $y = \pm x$) by the formula:

$$f(x, y) = \left[\frac{y}{x^2 - y^2}, \frac{x}{x^2 - y^2}\right]. \tag{4.1.9}$$

The image of the set:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\} \tag{4.1.10}$$

in this mapping will be found.

Solution

As in the previous example, at the beginning one should consider whether the function has any special properties. For this goal, let us introduce the quantities:

$$u = \frac{y}{x^2 - y^2}, \qquad v = \frac{x}{x^2 - y^2}. \tag{4.1.11}$$

After a careful look, it can be seen that the following equation can be obtained for them:

$$u^2 - v^2 = \left(\frac{y}{x^2 - y^2}\right)^2 - \left(\frac{x}{x^2 - y^2}\right)^2 = \frac{y^2 - x^2}{(x^2 - y^2)^2} = \frac{1}{y^2 - x^2}. \tag{4.1.12}$$

Thus, for all points belonging to A, one has $u^2 - v^2 = -1$. This equation defines a hyperbola (denoted below with H), which means that the image $f(A)$ is a subset of H. It remains to be settled if $f(A) = H$ or $f(A) \subset H$ only.

If the first possibility ($f(A) = H$) was realized, then each point $(u, v) \in H$ would have to be the image of a certain point $(x, y) \in A$. This means that for any numbers u, v meeting the condition $u^2 - v^2 = -1$ the set equations:

$$\begin{cases} \dfrac{y}{x^2 - y^2} = u, \\ \dfrac{x}{x^2 - y^2} = v \end{cases} \tag{4.1.13}$$

would have to have a solution for x and y satisfying $x^2 - y^2 = 1$. Let us then try to solve (4.1.13). First it should be noted that $v \neq 0$, since otherwise the condition $u^2 - v^2 = -1$ cannot be fulfilled. Consequently, the second equation of (4.1.13) means that $x \neq 0$ either. Dividing both equations by each other and subtracting them, one gets

$$\begin{cases} \dfrac{u}{v} = \dfrac{y}{x}, \\ u - v = \dfrac{y - x}{x^2 - y^2} = -\dfrac{1}{x + y}. \end{cases} \tag{4.1.14}$$

This system is already very easy to solve, for example by finding y from the first equation and inserting it into the second one. The solution has the form:

$$x = \frac{v}{v^2 - u^2}, \qquad y = \frac{u}{v^2 - u^2}. \tag{4.1.15}$$

Similarly as in (4.1.12), one can easily be convinced that in fact $x^2 - y^2 = 1$. For we have

$$\begin{aligned} x^2 - y^2 &= \left(\frac{v}{v^2 - u^2}\right)^2 - \left(\frac{u}{v^2 - u^2}\right)^2 = \frac{v^2 - u^2}{(v^2 - u^2)^2} \\ &= \frac{1}{v^2 - u^2} = \frac{1}{1} = 1. \end{aligned} \tag{4.1.16}$$

Thus, for each point (u, v) belonging to the hyperbola H we are able to find $(x, y) \in A$ such that $f(x, y) = (u, v)$. Therefore, one has $F(a) = H$.

4.2 Examining Limits and Continuity of Functions

Problem 1

The existence of limits:

$$\lim_{x\to 0} \lim_{y\to 0} f(x, y), \qquad \lim_{y\to 0} \lim_{x\to 0} f(x, y), \qquad \lim_{(x,y)\to(0,0)} f(x, y) \tag{4.2.1}$$

for the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by the formula:

$$f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 y^2 + (x - y)^2} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0 \end{cases} \tag{4.2.2}$$

will be examined.

Solution

The former two limits (4.2.1) to be found are called iterated limits. This term means that transitions $x \to 0$ and $y \to 0$ have to be performed in the indicated order, the other variable being temporarily treated as a constant parameter. The way of handling them is then identical to that known from one-variable functions. Let us consider

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}, \tag{4.2.3}$$

or, in other words:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right]. \tag{4.2.4}$$

In the first instance, x is treated as fixed and the limit is taken in the variable y. Using Heine's definition of the limit, well known from Part I and recalled in the theoretical introduction in this chapter, a sequence of points converging to $(x, 0)$ is chosen on the plane $\mathbb{R}^2$. It has the form $(x_n, y_n) = (x, y_n)$, with $\lim_{n\to\infty} y_n = 0$ and $y_n \neq 0$. The limit

$$\lim_{n\to\infty} f(x, y_n) = \lim_{n\to\infty}\frac{x^2y_n^2}{x^2y_n^2+(x-y_n)^2} \tag{4.2.5}$$

can be easily found. This is because for $x = 0$, the value of the function constantly equals zero. The same number must, therefore, be the limit (4.2.5). However, if $x \neq 0$, for suitably large n—that is, if y_n is sufficiently small—one has $|y_n| < |x|/2$. This inequality is a consequence of the fact that x is fixed and y_n tends to zero. If so, we have

$$0 < \frac{x^2y_n^2}{x^2y_n^2+(x-y_n)^2} < \frac{x^2y_n^2}{(x-y_n)^2} < \frac{x^2y_n^2}{(x/2)^2} = 4y_n^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{4.2.6}$$

This result indicates that regardless of the value of the variable x, one gets

$$\lim_{y\to 0} f(x, y) = 0, \tag{4.2.7}$$

which also entails

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = 0. \tag{4.2.8}$$

Note now that the function (4.2.2) is symmetrical under the replacement $x \leftrightarrow y$, and therefore, for the second limit of (4.2.1) the identical result must be obtained:

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = 0. \tag{4.2.9}$$

Thus, both iterated limits are equal to each other (and equal to 0). This does not mean, however, that there exists the third of the limits (4.2.1)—one could call it the "Limit" with capital "L." The ways of approaching the origin of the coordinate system on a plane are ultimately much more numerous (in fact they are infinitely many) than along the axes of the coordinate system only. To check the existence of this third limit, one must, therefore, choose a sequence of test points with coordinates (x_n, y_n), satisfying

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = 0 \quad \text{and} \quad (x_n, y_n) \neq (0, 0) \tag{4.2.10}$$

and examine whether for the limit

$$\lim_{n\to\infty} f(x_n, y_n) \tag{4.2.11}$$

one still gets the value 0. Let us take for example a sequence of points for which constantly $y_n = x_n$. We have then

$$f(x_n, x_n) = \frac{x_n^2 x_n^2}{x_n^2 x_n^2 + (x_n - x_n)^2} = 1 \tag{4.2.12}$$

and hence also

$$\lim_{n\to\infty} f(x_n, x_n) = 1 \neq 0. \tag{4.2.13}$$

This means that approaching the point $(0, 0)$ along the straight line $y = x$, a different value than the previous one is got. There is no need to consider other sequences. The result (4.2.13) already proves that the limit cannot exist. It should be stressed, however, that the applied approach (i.e., indicating two or more sequences of arguments converging to $(0, 0)$ for which function values go to different limits) is suitable only to show the *nonexistence* of a given limit. In order to prove its *existence*, one cannot choose any specific form of (x_n, y_n) except for requiring that $(x_n, y_n) \underset{n\to\infty}{\longrightarrow} (0, 0)$.

Problem 2

The existence of the limits:

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y), \qquad \lim_{y\to 0}\lim_{x\to 0} f(x, y), \qquad \lim_{(x,y)\to(0,0)} f(x, y) \tag{4.2.14}$$

of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by the formula:

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x^4 + y^4} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0 \end{cases} \tag{4.2.15}$$

will be examined.

Solution

As previously, we start by examining the iterated limits. In the first instance, we consider

$$\lim_{x\to 0} \lim_{y\to 0} \frac{xy^3}{x^4 + y^4}. \tag{4.2.16}$$

Let now x be fixed and the limit will be taken in the variable y. For this purpose, a sequence of points (x, y_n) converging to $(x, 0)$ is picked out on the plane. Then, obviously $\lim_{n\to\infty} y_n = 0$. In addition, it is assumed that $y_n \neq 0$. In accordance with Heine's definition of a limit, we now calculate

$$\lim_{n\to\infty} f(x, y_n) = \lim_{n\to\infty} \frac{xy_n^3}{x^4 + y_n^4}. \tag{4.2.17}$$

When $x = 0$, the value of the function equals zero. In turn, for $x \neq 0$ the following estimate can be used:

$$0 < \left| \frac{xy_n^3}{x^4 + y_n^4} \right| < \left| \frac{xy_n^3}{x^4} \right| = \left| \frac{y_n^3}{x^3} \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.2.18}$$

Thus, regardless of the value of the variable x, one has

$$\lim_{y\to 0} f(x, y) = 0, \tag{4.2.19}$$

which also entails

$$\lim_{x\to 0} \lim_{y\to 0} f(x, y) = 0. \tag{4.2.20}$$

The second iterated limit can be found in a similar manner. This time the limit in x is carried out, with y kept as a constant parameter. The sequence of points on the plane convergent to $(0, y)$ has now the form (x_n, y), where $\lim_{n\to\infty} x_n = 0$ and $x_n \neq 0$. Of course $f(x_n, 0) = 0$, so one can focus on $y \neq 0$ only. We then estimate

$$0 < \left|\frac{x_n y^3}{x_n^4 + y^4}\right| < \left|\frac{x_n y^3}{y^4}\right| = \left|\frac{x_n}{y}\right| \underset{n\to\infty}{\longrightarrow} 0, \tag{4.2.21}$$

which leads to

$$\lim_{x\to 0} f(x, y) = 0 \implies \lim_{y\to 0}\lim_{x\to 0} f(x, y) = 0. \tag{4.2.22}$$

This means that the two iterated limits are equal.

In order to demonstrate that the third of the limits (4.2.14) does not exist, such a sequence of points with coordinates (x_n, y_n) is chosen, for which $y_n = a x_n$ and $a \neq 0$. Now, if $\lim_{n\to\infty} x_n = 0$ and $x_n \neq 0$, then automatically

$$\lim_{n\to\infty} (x_n, y_n) = (0, 0) \quad \text{and} \quad (x_n, y_n) \neq (0, 0). \tag{4.2.23}$$

For this choice, it is easy to get

$$f(x_n, y_n) = \frac{a^3 x_n^4}{(1 + a^4) x_n^4} = \frac{a^3}{1 + a^4} \underset{n\to\infty}{\longrightarrow} \frac{a^3}{1 + a^4}. \tag{4.2.24}$$

The obtained result depends on a, i.e., on the direction of the straight line along which the sequence is converging to the origin. This means that the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ cannot exist.

Problem 3

The existence of the limit:

$$\lim_{(x,y,z)\to(0,0,0)} f(x, y, z) \tag{4.2.25}$$

will be examined for the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by the following formula:

$$f(x, y, z) = \begin{cases} \dfrac{\sqrt{|x|}\, y z^3}{x^2 + y^4 + z^6} & \text{for } x^2 + y^2 + z^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 + z^2 = 0. \end{cases} \tag{4.2.26}$$

Solution

First let us consider the iterated limits. y and z are kept fixed and x will tend to zero. If any of the parameters y, z has been already chosen equal to zero, then by virtue of formula (4.2.26)—regardless of x—the function f shall immediately have the same value and, consequently, its limit as $x \to 0$ too. In turn, if $y, z \neq 0$, one has

$$0 < f(x, y, z)| = \left| \frac{\sqrt{|x|}\, yz^3}{x^2 + y^4 + z^6} \right| < \sqrt{|x|} \left| \frac{yz^3}{y^4 + z^6} \right| \underset{x \to 0}{\longrightarrow} 0. \tag{4.2.27}$$

In the same way, it is easy to conclude that $\lim_{y \to 0} f(x, y, z) = 0$ and $\lim_{z \to 0} f(x, y, z) = 0$. As a result, all iterated limits (double and triple) vanish.

These results indicate that if the point $(0, 0, 0)$ is approached along any axis of the coordinate system, one will always get 0. Let us check now what will be obtained if one is going to the origin along the straight line defined by the equations $y = ax$ and $z = bx$, where $a, b \neq 0$, i.e., along any direction. We obtain

$$\begin{aligned} f(x, ax, bx) &= \frac{\sqrt{|x|}\,(ax)(bx)^3}{x^2 + (ax)^4 + (bx)^6} = \frac{ab^3 \sqrt{|x|}\, x^4}{(1 + a^4x^2 + b^6x^4)x^2} \\ &= \sqrt{|x|}\, x^2 \frac{ab^3}{1 + a^4x^2 + b^6x^4} \underset{x \to 0}{\longrightarrow} 0, \end{aligned} \tag{4.2.28}$$

i.e., again the same. Still it does not mean that the "Limit" (4.2.25) exists. In multidimensional space, a given point can be reached in infinitely many ways and not necessarily along straight lines. In order to determine the existence of the limit, one has to ascertain that each of them leads to the same result. It is easy to see that it is not the case for (4.2.25). Consider, for example: $x = z^3$, $y = |z|^{3/2}$ and let z run to 0. The curve obtained is not a straight line, but it still passes through the origin. One will then have

$$f(z^3, |z|^{3/2}, z) = \frac{|z|^{3/2}|z|^{3/2}z^3}{z^6 + z^6 + z^6} = \frac{1}{3} \cdot \left(\frac{|z|}{z} \right)^3 \underset{z \to 0}{\not\longrightarrow} 0. \tag{4.2.29}$$

The limit of the expression is not 0 (in fact it does not have any limit) in contrast to what was previously found for straight lines. This means that the function $f(x, y, z)$ has no limit at $(0, 0, 0)$.

Problem 4

The existence of limits:

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y), \qquad \lim_{y \to 0} \lim_{x \to 0} f(x, y), \qquad \lim_{(x,y) \to (0,0)} f(x, y) \tag{4.2.30}$$

will be examined for the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula:

$$f(x, y) = \begin{cases} y\,\dfrac{x^2 - y^2}{x^4 + y^2} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0. \end{cases} \tag{4.2.31}$$

Solution

Let us start with the iterated limits. In view of the presence of the variable y in front of the fraction in (4.2.30) it is very easy to check that, regardless of the value of x, one has

$$\lim_{y \to 0} f(x, y) = 0. \tag{4.2.32}$$

Consequently, the same result is obtained for the double limit:

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = 0. \tag{4.2.33}$$

While calculating the second iterated limit we first go down to zero with x, obtaining

$$\lim_{y \to 0} f(x, y) = -\frac{y^3}{y^2} = -y, \tag{4.2.34}$$

for $y \neq 0$, or immediately zero in the opposite case. Therefore,

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = 0. \tag{4.2.35}$$

Thus, both iterated limits do exist and are equal, but as we know, this does not prejudge yet the value nor even the existence of the last limit in (4.2.30). Let us see now what happens when the origin is approached not along the axes of the coordinate system, but along any straight line described by the equation $y = ax$ with $a \neq 0$. One readily gets the following result:

$$\lim_{x \to 0} f(x, ax) = \lim_{x \to 0} ax\,\frac{x^2 - (ax)^2}{x^4 + (ax)^2} = \lim_{x \to 0} \frac{x^3 a(1 - a^2)}{x^2(x^2 + a^2)}$$
$$= \lim_{x \to 0} x\,\frac{a(1 - a^2)}{x^2 + a^2} = 0. \tag{4.2.36}$$

It is consistent with those obtained in (4.2.33) and (4.2.35), but from the previous example, we know already that it still does not rule on the existence of the limit. The origin can be reached in an infinite number of other ways. For example, moving along a parabola $y = bx^2$ (where $b \neq 0$), one gets

$$\lim_{x\to 0} f(x, bx^2) = \lim_{x\to 0} \frac{bx^2(x^2 - b^2x^4)}{x^4 + b^2x^4} = \lim_{x\to 0} \frac{b(1 - b^2x^2)}{1 + b^2} = \frac{b}{1 + b^2} \neq 0. \tag{4.2.37}$$

This result is different from those obtained for straight lines, which shows that the last limit of (4.2.30) does not exist.

Problem 5

The continuity of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by the formula:

$$f(x, y) = \begin{cases} \dfrac{ax^4 + by^2}{x^2 + y^4} & \text{for } x^2 + y^2 \neq 0, \\ c & \text{for } x^2 + y^2 = 0, \end{cases} \tag{4.2.38}$$

will be examined, depending on the values of parameters $a, b, c \in \mathbb{R}$.

Solution

Anywhere other than the point $(0, 0)$, the continuity of the function (4.2.38) does not raise any doubts. It is a quotient of two polynomials (in two variables: x and y), i.e., two continuous functions, the denominator being different from zero. Thus we will focus only on examining the continuity at the origin, which is not obvious.

We start by investigating the limit of the function as $(x, y) \to (0, 0)$. First, let us consider the iterated limits. Letting x go to zero while keeping y fixed, one obtains

$$\lim_{x\to 0} f(x, y) = \lim_{x\to 0} \frac{ax^4 + by^2}{x^2 + y^4} = \begin{cases} \dfrac{b}{y^2} & \text{for } y \neq 0, \\ 0 & \text{for } y = 0. \end{cases} \tag{4.2.39}$$

The double limit $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ does, therefore, exist (and is then equal to zero) only if $b = 0$. Below it is assumed that this condition is met. Otherwise, the function $f(x, y)$ has no limit at $(0, 0)$ and cannot be continuous. To examine the second iterated limit, we treat x as a fixed parameter and let y go to zero:

$$\lim_{y\to 0} f(x, y) = \lim_{y\to 0} \frac{ax^4}{x^2 + y^4} = \begin{cases} ax^2 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{4.2.40}$$

It exists and equals zero regardless of the value of a: $\lim_{x\to 0} \lim_{y\to 0} f(x, y) = 0$.

To be certain of the existence of the double (non-iterated) limit for $(x, y) \to (0, 0)$, one picks out any sequence of points (x_n, y_n) satisfying

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = 0, \tag{4.2.41}$$

and $(x_n, y_n) \neq (0, 0)$ and then make the estimate:

$$0 \leq |f(x_n, y_n)| = \left| \frac{ax_n^4}{x_n^2 + y_n^4} \right| \leq |a| \frac{x_n^4}{x_n^2} = |a|x_n^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{4.2.42}$$

The squeeze theorem for sequences (see Sect. 5.1 in Part I) leads to the conclusion that $\lim_{n\to\infty} f(x_n, y_n) = 0$. Thereby one has:

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0, \tag{4.2.43}$$

for $b = 0$ and for any value of a. In order to ensure the continuity of the function it is still necessary to have $f(0, 0) = 0$. Thus we see that the function is continuous if $c = 0$.

4.3 Exercises for Independent Work

Exercise 1 Find the image of the set A under the function f

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y) = x^2 - 4y^2$, and A is a triangle with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$,
(b) $f : \mathbb{R}^3 \to \mathbb{R}$, where $f(x, y, z) = (x - y)^2 + z^2$, and
$A = \{(x, y, z) \in \mathbb{R} \mid x^2 + y^2 + x^2 = 4\}$.

Answers

(a) $f(A) = [-3, 4]$,
(b) $f(A) = [0, 8]$.

Exercise 2 Find iterated limits and a double limit at the point $(0, 0)$ and verify the continuity of functions:

(a) $\begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & \text{for } x^2+y^2 \neq 0, \\ 1 & \text{for } x^2+y^2 = 0, \end{cases}$

(b) $\begin{cases} \dfrac{(x-y)^3}{x^2+y^2} & \text{for } x^2+y^2 \neq 0, \\ c & \text{for } x^2+y^2 = 0. \end{cases}$

Answers

(a) $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 1$, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Discontinuous function.

(b) $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. The function is continuous for $c = 0$ and discontinuous (at the origin) in other cases.

Chapter 5
Investigating Derivatives of Multivariable Functions

In this chapter, we become acquainted with the differentiability and derivatives of functions in several dimensions. The dimensionality makes these notions much more complicated and difficult. Therefore, when solving the following problems, they will be dealt with in detail.

The **partial derivative** of a function $f(x_1, x_2, \ldots, x_N)$ with respect to x_i is defined as

$$\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_N) \tag{5.0.1}$$
$$= \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_N) - f(x_1, \ldots, x_i, \ldots, x_N)}{h}.$$

If the above limit exists, the function is said to be differentiable over the ith argument. The function differentiable with respect to each argument does not have to be differentiable in the "strong sense" as explained below.

The **Jacobian matrix** is a matrix composed of all partial derivatives. For a function $F : \mathbb{R}^N \to \mathbb{R}^M$ it is the $M \times N$ matrix:

$$\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_N \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_N \\ \cdots & \cdots & \cdots & \cdots \\ \partial f_M/\partial x_1 & \partial f_M/\partial x_2 & \cdots & \partial f_M/\partial x_N \end{bmatrix}, \tag{5.0.2}$$

where

$$F(x_1, x_2, \ldots, x_N) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_N) \\ f_2(x_1, x_2, \ldots, x_N) \\ \cdots \\ f_M(x_1, x_2, \ldots, x_N) \end{bmatrix}. \tag{5.0.3}$$

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The **directional derivative** of the function F in the direction of a certain vector $h = [h_1, h_2, \ldots, h_N]$ is defined as the limit

$$F'_h\Big|_x = \lim_{t\to 0} \frac{F(x+th) - F(x)}{t}, \tag{5.0.4}$$

where x stands for $(x_1, x_2, \ldots, x_N)$. If at a given point all such derivatives (i.e., in any direction) exist, we shall say that the **weak derivative** or **Gateaux derivative** of the function f exists too. It is worth noting that there are also other definitions of the Gateaux derivative, so the reader should be aware of this fact when studying various textbooks on mathematical analysis. In particular sometimes apart from the existence of F'_h, one also requires its linearity in h.

As for the **strong derivative**, i.e., the so-called **Fréchet derivative**, its definition is as follows. Let us write down the difference $F(x_0 + h) - F(x_0)$ in the form of the sum of a linear part and a certain remainder (note that $L(x_0)$ refers below to an array):

$$F(x_0 + h) - F(x_0) = L(x_0) \cdot h + r(h), \tag{5.0.5}$$

assuming about the latter that

$$\frac{\|r(h)\|}{\|h\|} \underset{h\to 0}{\longrightarrow} 0. \tag{5.0.6}$$

If these conditions are met, the strong derivative exists and

$$F'\Big|_{x_0} = L(x_0). \tag{5.0.7}$$

It can be proved that the function possessing all partial derivatives, which are continuous, is Fréchet-differentiable or F-differentiable. This notion will become more clear while solving problems of Sect. 5.2.

5.1 Calculating Partial and Directional Derivatives

Problem 1

All partial derivatives of the function $f : \mathbb{R}^3_+ \to \mathbb{R}$ given by the formula:

$$f(x, y, z) = x^y z^x, \tag{5.1.1}$$

will be found at the point of coordinates $(1, 2, 3)$.

Solution

Finding partial derivatives is not especially difficult if one is familiar with the differential calculus of one-variable functions. For example, when calculating the derivative over x, i.e., $\partial f/\partial x$, one should temporarily accept that other variables, i.e., y and z, are certain fixed parameters. To emphasize it let us for a moment denote them with symbols a and b. Then, according to the rule of calculating the derivative of a product, one finds

$$\frac{d}{dx} x^a b^x = a\, x^{a-1} b^x + x^a b^x \log b, \tag{5.1.2}$$

for positive values of x and b. This means that

$$\begin{aligned} &\frac{\partial f}{\partial x} = y\, x^{y-1} z^x + x^y z^x \log z \implies \\ &\left.\frac{\partial f}{\partial x}\right|_{(1,2,3)} = 2 \cdot 1^{2-1} \cdot 3^1 + 1^2 \cdot 3^1 \cdot \log 3 = 6 + 3 \log 3. \end{aligned} \tag{5.1.3}$$

This method becomes clear if we recall the definition of the partial derivative with respect to x of a function $f(x, y, z)$ at the point (x_0, y_0, z_0):

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} = \lim_{h\to 0} \frac{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)}{h}. \tag{5.1.4}$$

Similarly, but now without substituting auxiliary parameters a and b, one finds

$$\begin{aligned} &\frac{\partial f}{\partial y} = x^y \log x\, z^x \implies \left.\frac{\partial f}{\partial y}\right|_{(1,2,3)} = 1^2 \cdot \log 1 \cdot 3^1 = 0, \\ &\frac{\partial f}{\partial z} = x^y\, x\, z^{x-1} \implies \left.\frac{\partial f}{\partial z}\right|_{(1,2,3)} = 1^2 \cdot 1 \cdot 3^{1-1} = 1. \end{aligned} \tag{5.1.5}$$

Thus, the experience we have gained on how to calculate derivatives of one-variable function is sufficient to find all partial derivatives.

Problem 2

All partial derivatives and the directional derivative—in the direction of the vector $[1, 1, 2]$—of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by the formula:

$$f(x, y, z) = x^2 + 2xy + xz - z^2 \tag{5.1.6}$$

will be found at the point of coordinates (x_0, y_0, z_0).

Solution

The function dealt with here is a polynomial (in three variables), so all partial derivatives exist at any point of $\mathbb{R}^3$, and can be easily found as

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} &= 2x+2y+z\Big|_{(x_0,y_0,z_0)} = 2x_0+2y_0+z_0, \\ \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0,z_0)} &= 2x\Big|_{(x_0,y_0,z_0)} = 2x_0, \\ \left.\frac{\partial f}{\partial z}\right|_{(x_0,y_0,z_0)} &= x-2z\Big|_{(x_0,y_0,z_0)} = x_0-2z_0. \end{aligned} \tag{5.1.7}$$

Therefore, one can immediately go to the directional derivative for a certain vector $h = [h_x, h_y, h_z]$. In accordance with its definition (5.0.4) one has to find the following limit:

$$f'_h\big|_{(x_0,y_0,z_0)} = \lim_{t\to 0} \frac{f(x_0+th_x, y_0+th_y, z_0+th_z) - f(x_0,y_0,z_0)}{t}. \tag{5.1.8}$$

This quantity is helpful because it provides us with the information about the steepness of the graph at the point (x_0, y_0, z_0) if one moves in the direction established by the vector h.

Upon inserting the above definition into formula (5.1.6), one obtains

$$\begin{aligned} & f'_h\big|_{(x_0,y_0,z_0)} \\ &= \lim_{t\to 0}\left[\frac{(x_0+th_x)^2+2(x_0+th_x)(y_0+th_y)+(x_0+th_x)(z_0+th_z)}{t}\right. \\ &\qquad\left. + \frac{-(z_0+th_z)^2 - x_0^2 - 2x_0y_0 - 2x_0z_0 + z_0^2)}{t}\right] \\ &= \lim_{t\to 0}\,[2x_0h_x + 2x_0h_y + 2y_0h_x + x_0h_z + z_0h_x - 2z_0h_z \\ &\qquad + t(h_x^2 + 2h_xh_y + h_xh_z - h_z^2)] \\ &= 2x_0h_x + 2x_0h_y + 2y_0h_x + x_0h_z + z_0h_x - 2z_0h_z. \end{aligned} \tag{5.1.9}$$

This expression can also be given the form of the matrix multiplication:

$$f'_h\big|_{(x_0,y_0,z_0)} = [2x_0+2y_0+z_0, 2x_0, x_0-2z_0]\begin{bmatrix} h_x \\ hy \\ hz \end{bmatrix}, \tag{5.1.10}$$

in which (in the row) previously found values of partial derivatives are recognized. Thus, we have found the relation:

$$f'_h\big|_{(x_0,y_0,z_0)} = \underbrace{\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]}_{f'}\Big|_{(x_0,y_0,z_0)} \cdot h = f'\big|_{(x_0,y_0,z_0)} \cdot h. \tag{5.1.11}$$

As we will see in the exercises of the following section, it is not a coincidence.

Problem 3

The matrix of partial derivatives and the directional derivative for the vector $[-1, 2, 1]$ and the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula:

$$f(x, y, z) := \begin{bmatrix} f_x(x, y, z) \\ f_y(x, y, z) \\ f_z(x, y, z) \end{bmatrix} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \tag{5.1.12}$$

will be found at the point of coordinates (x_0, y_0, z_0).

Solution

Let us choose any vector $h = [h_x, h_y, h_z]$ and create the difference quotient, needed for the calculation of the derivative of the mapping f in accordance with formula (5.1.8):

$$\begin{aligned}
&\frac{1}{t}\big(f(x_0 + th_x, y_0 + th_y, z_0 + th_z) - f(x_0, y_0, z_0)\big) \\
&\quad = \frac{1}{t}\left(\begin{bmatrix} (y_0 + th_y)(z_0 + th_z) \\ (x_0 + th_x)(z_0 + th_z) \\ (x_0 + th_x)(y_0 + th_y) \end{bmatrix} - \begin{bmatrix} y_0 z_0 \\ x_0 z_0 \\ x_0 y_0 \end{bmatrix}\right) \\
&\quad = \frac{1}{t}\left(t \begin{bmatrix} y_0 h_z + z_0 h_y \\ x_0 h_z + z_0 h_x \\ x_0 h_y + y_0 h_x \end{bmatrix} + t^2 \begin{bmatrix} h_y h_z \\ h_x h_z \\ h_x h_y \end{bmatrix}\right).
\end{aligned} \tag{5.1.13}$$

Letting t go to zero, one gets the directional derivative:

$$f'_h\big|_{(x_0,y_0,z_0)} = \begin{bmatrix} y_0h_z + z_0h_y \\ x_0h_z + z_0h_x \\ x_0h_y + y_0h_x \end{bmatrix} = \begin{bmatrix} 0 & z_0 & y_0 \\ z_0 & 0 & x_0 \\ y_0 & x_0 & 0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = f'\big|_{(x_0,y_0,z_0)} \cdot h. \tag{5.1.14}$$

The matrix obtained contains all partial derivatives. Its subsequent columns are derivatives of the function (5.1.12) with respect to x, y, and z. For example:

$$\left.\frac{\partial f_x}{\partial y}\right|_{(x_0,y_0,z_0)} = z_0, \qquad \left.\frac{\partial f_y}{\partial z}\right|_{(x_0,y_0,z_0)} = x_0 \;\text{ etc.} \tag{5.1.15}$$

At the end, plugging into (5.1.14) the vector h given in the text of this exercise, i.e., $h = [-1, 2, 1]$, we find

$$f'_h\big|_{(x_0,y_0,z_0)} = \begin{bmatrix} y_0 + 2z_0 \\ x_0 - z_0 \\ 2x_0 - y_0 \end{bmatrix}. \tag{5.1.16}$$

5.2 Examining Differentiability of Functions

Problem 1

The differentiability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ of the formula:

$$f(x, y) = \begin{cases} \dfrac{x^2y^3}{x^4 + y^4} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0 \end{cases} \tag{5.2.1}$$

will be examined in the weak (i.e., Gateaux) and strong (i.e., Fréchet) sense.

Solution

In the previous problems we have learned how to find the directional derivative of a function f in the direction of a certain vector $h \neq 0$. In the theoretical introduction at the beginning of the present chapter the so-called Gateaux derivative was defined. Let us recall that this definition states that if at a given point all directional derivatives exist, the Gateaux derivative of the function f does too and that derivative is just the Jacobian matrix.

The notion of the so-called Fréchet derivative should be recalled as well. As told in the theoretical introduction, it appears while considering $f(x_0+h) - f(x_0)$. If it can be written in the form of the sum of a linear part and a remainder:

$$f(x_0+h) - f(x_0) = L(x_0) \cdot h + r(h), \tag{5.2.2}$$

where $L(x_0)$ is a matrix (sometimes trivial) and

$$\frac{\|r(h)\|}{\|h\|} \underset{h\to 0}{\longrightarrow} 0, \tag{5.2.3}$$

then the Fréchet derivative exists and

$$f'\big|_{x_0} = L(x_0). \tag{5.2.4}$$

Now let us examine the existence of both derivatives for the function given in this exercise. We are only concerned about the point $(0, 0)$, as in all other points of the plane $\mathbb{R}^2$, the function is a quotient of polynomials and the denominator is different from zero. It is then differentiable in both the weak and strong sense without any doubts. Let us start with checking if the function is continuous at the origin. Selecting the sequence of points of coordinates (x_n, y_n) convergent to $(0, 0)$ and such that $(x_n, y_n) \neq (0, 0)$, one obtains

$$0 \le |f(x_n, y_n)| = \left|\frac{x_n^2 y_n^3}{x_n^4 + y_n^4}\right| = \left|\frac{y_n}{2} \cdot \frac{2x_n^2 y_n^2}{x_n^4 + y_n^4}\right| \le \left|\frac{y_n}{2} \cdot 1\right| \underset{n\to\infty}{\longrightarrow} 0, \tag{5.2.5}$$

where the fact that

$$(x_n^2 - y_n^2)^2 \ge 0 \implies x_n^4 + y_n^4 \ge 2x_n^2 y_n^2 \implies \frac{2x_n^2 y_n^2}{x_n^4 + y_n^4} \le 1, \tag{5.2.6}$$

has been used. The above result indicates that the limit of the function at the origin exists and is equal to the value of the function. Thereby the function is continuous.

Now the derivative $f'_h\big|_{(0,0)}$ in the direction of the vector $h = [h_x, h_y]$ is calculated. By virtue of (5.1.8), one has

$$\begin{aligned} f'_h\Big|_{(0,0)} &= \lim_{t\to 0} \frac{f(0+th_x, 0+th_y) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{((th_x)^2(th_y)^3)/((th_x)^4 + (th_y)^4)}{t} = \lim_{t\to 0} \frac{t^5}{t^5} \cdot \frac{h_x^2 h_y^3}{h_x^4 + h_y^4} = \frac{h_x^2 h_y^3}{h_x^4 + h_y^4}. \end{aligned} \tag{5.2.7}$$

This expression undoubtedly exists for any nonzero vector h. In accordance with our definition, this proves the existence of the Gateaux derivative.

Continuing to the second part of the exercise, note that the result (5.2.7) is nonlinear in h. The question arises then as to whether the whole expression $h_x^2h_y^3/(h_x^4+h_y^4)$ constitutes simply the remainder $r(h)$ in the definition (5.2.2), or should one extract from it a certain linear part $L\cdot h$. Let us assume that this linear part exists and is equal to $L\cdot h = ah_x + bh_y$, a and b being certain constants. Then we would have

$$r(h) = \frac{h_x^2h_y^3}{h_x^4+h_y^4} - ah_x - bh_y = \frac{h_x^2h_y^3 - ah_x^5 - ah_xh_y^4 - bh_x^4h_y - bh_y^5}{h_x^4+h_y^4}. \tag{5.2.8}$$

Letting the vector h go to zero in such a way that always $h_y = 0$ and $h_x \to 0$ and requiring that $\|r(h)\|/\|h\| \underset{h\to 0}{\longrightarrow} 0$, one obtains the following condition:

$$\lim_{h\to 0} \frac{h_x^2h_y^3 - ah_x^5 - ah_xh_y^4 - bh_x^4h_y - bh_y^5}{(h_x^4+h_y^4)\sqrt{h_x^2+h_y^2}} = \lim_{h_x\to 0}\frac{-ah_x^5}{h_x^4|h_x|}$$
$$= -a\lim_{h_x\to 0}\frac{h_x}{|h_x|} = 0 \implies a = 0. \tag{5.2.9}$$

In turn, choosing $h_x = 0$ and $h_y \to 0$, we obtain in an analogous way:

$$\lim_{h\to 0} \frac{h_x^2h_y^3 - ah_x^5 - ah_xh_y^4 - bh_x^4y_y - bh_y^5}{(h_x^4+h_y^4)\sqrt{h_x^2+h_y^2}} = \lim_{h_y\to 0}\frac{-bh_y^5}{h_y^4|h_y|}$$
$$= -b\lim_{h_y\to 0}\frac{h_y}{|h_y|} = 0 \implies b = 0. \tag{5.2.10}$$

On the other hand, for $h_x = h_y \to 0$ one gets

$$\lim_{h\to 0} \frac{h_x^2h_y^3 - ah_x^5 - ah_xh_y^4 - bh_x^4y_y - bh_y^5}{(h_x^4+h_y^4)\sqrt{h_x^2+h_y^2}} = \lim_{h_x\to 0}\frac{(1-2a-2b)h_x^5}{2\sqrt{2}h_x^4|h_x|}$$
$$= \frac{(1-2a-2b)}{2\sqrt{2}}\lim_{h_x\to 0}\frac{h_x^5}{h_x^4|h_x|} = 0 \implies 1-2a-2b = 0, \tag{5.2.11}$$

which is at odds with the results (5.2.9) and (5.2.10). We conclude, therefore, that numbers a and b cannot be found and, consequently, the strong derivative does not exist.

It is worth asking at the end about the behavior of partial derivatives at the origin. For example, let us differentiate the function with respect to x. One obtains

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \frac{2xy^3(x^4+y^4)-4x^5y^3}{(x^4+y^4)^2} = \frac{2xy^3(y^4-x^4)}{(x^4+y^4)^2} \tag{5.2.12}$$

away from the point (0, 0) and

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{\triangle x\to 0} \frac{f(0+\triangle x, 0)-f(0,0)}{\triangle x} = 0 \tag{5.2.13}$$

at the origin. Expression (5.2.12), however, has no limit at this point because when moving along the axes of the coordinate system one gets 0 and, for instance, along the straight line $y = 2x$:

$$\lim_{x\to 0} \frac{2x(2x)^3((2x)^4-x^4)}{(x^4+(2x)^4)^2} = \frac{16\cdot 15}{17^2} \neq 0. \tag{5.2.14}$$

It is seen that the derivative $\partial f/\partial x$ is not continuous. The similar result can be found for the derivative with respect to y:

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(x,y)} &= \frac{3x^2y^2(x^4+y^4)-4x^2y^6}{(x^4+y^4)^2} = \frac{x^2y^2(3x^4-2y^4)}{(x^4+y^4)^2}, \\ \left.\frac{\partial f}{\partial y}\right|_{(0,0)} &= \lim_{\triangle y\to 0} \frac{f(0,0+\triangle y)-f(0,0)}{\triangle y} = 0. \end{aligned} \tag{5.2.15}$$

The upper expression has no limit for $x, y \to 0$. As before, limits obtained along the axes are equal to 0 and, for example, limit along the straight line $y = x$ has the value $1/4$. Thereby the partial derivative $\partial f/\partial y$ cannot be continuous. This result is not accidental, since as we know from the lecture of analysis, the continuity of all partial derivatives would imply the existence of the Fréchet derivative which—as it has already been proved—does not exist for the function under consideration.

Problem 2

The differentiability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula:

$$f(x, y) = \begin{cases} \dfrac{x^5+y^3}{x^4+y^2} & \text{for } x^2+y^2 \neq 0, \\ 0 & \text{for } x^2+y^2 = 0, \end{cases} \tag{5.2.16}$$

in the weak (i.e., Gateaux) and strong (i.e., Fréchet) sense, will be examined.

Solution

As in the previous example, we will focus on the point $(0, 0)$—the only one, in which the differentiability may be questionable. Let us begin by examining the continuity of the function. As usual, a sequence of points with coordinates (x_n, y_n) convergent to $(0, 0)$ such that $(x_n, y_n) \neq (0, 0)$ will be chosen. The following estimate can be made:

$$0 \le |f(x_n, y_n)| = \left|\frac{x_n^5 + y_n^3}{x_n^4 + y_n^2}\right| \le \left|\frac{x_n^5}{x_n^4 + y_n^2}\right| + \left|\frac{y_n^3}{x_n^4 + y_n^2}\right|. \tag{5.2.17}$$

The first fraction is either equal to zero (when $x_n = 0$) or is subject to the following inequality:

$$\left|\frac{x_n^5}{x_n^4 + y_n^2}\right| \le \left|\frac{x_n^5}{x_n^4 + 0}\right| = |x_n|. \tag{5.2.18}$$

The second fraction can also be equal to zero (for $y_n = 0$) or eventually satisfies

$$\left|\frac{y_n^3}{x_n^4 + y_n^2}\right| \le \left|\frac{y_n^3}{0 + y_n^2}\right| = |y_n|. \tag{5.2.19}$$

In any case, it is true that

$$0 \le |f(x_n, y_n)| \le |x_n| + |y_n| \underset{n\to\infty}{\longrightarrow} 0, \tag{5.2.20}$$

and, accordingly, the function is continuous.

In the next step, the directional derivative $f'_h|_{(0,0)}$ in the direction of $h = [h_x, h_y]$ is calculated, resulting with

$$\begin{aligned}
\lim_{t\to 0} \frac{f(0 + th_x, 0 + th_y) - f(0, 0)}{t} &= \lim_{t\to 0} \frac{((th_x)^5 + (th_y)^3)/((th_x)^4 + (th_y)^2)}{t} \\
&= \lim_{t\to 0} \frac{t^3}{t^3} \cdot \frac{t^2h_x^5 + h_y^3}{t^2h_x^4 + h_y^2} = \begin{cases} h_y & \text{for } h_y \neq 0, \\ h_x & \text{for } h_y = 0. \end{cases}
\end{aligned} \tag{5.2.21}$$

Thus, the directional derivative exists for any vector $h \neq 0$, and consequently the Gateaux derivative exists.

To verify the existence of the Fréchet derivative in a formal way, consider the difference $f(0 + h_x, 0 + h_y) - f(0, 0)$:

$$f(0 + h_x, 0 + h_y) - f(0, 0) = \frac{h_x^5 + h_y^3}{h_x^4 + h_y^2}. \tag{5.2.22}$$

This expression is nonlinear in h, so one has to try to extract the linear part $(ah_x + bh_y)$ out of it and find out whether the remainder is subject to the condition (5.2.3). In our case, we have

$$r(h) = \frac{h_x^5 + h_y^3}{h_x^4 + h_y^2} - ah_x - bh_y = \frac{(1-a)h_x^5 - ah_x h_y^2 - bh_x^4 h_y + (1-b)h_y^3}{h_x^4 + h_y^2}. \tag{5.2.23}$$

The requirement that $\|r(h)\|/\|h\| \underset{h\to\infty}{\longrightarrow} 0$, leads, however, to contradictory equations. When h_x is going to zero with the assumption that $h_y = 0$, one gets

$$\lim_{h\to 0} \frac{(1-a)h_x^5 - ah_x h_y^2 - bhx^4 h_y + (1-b)h_y^3}{(h_x^4 + h_y^2)\sqrt{h_x^2 + h_y^2}} = \lim_{h_x\to 0} \frac{(1-a)h_x^5}{h_x^4|h_x|}$$
$$= (1-a)\lim_{h_x\to 0} \frac{h_x}{|h_x|} = 0 \implies a = 1. \tag{5.2.24}$$

While performing this procedure in the reverse order, we find

$$\lim_{h\to 0} \frac{(1-a)h_x^5 - ah_x h_y^2 - bh_x^4 h_y + (1-b)h_y^3}{(h_x^4 + h_y^2)\sqrt{h_x^2 + h_y^2}} = \lim_{h_y\to 0} \frac{(1-b)h_y^3}{h_y^2|h_y|}$$
$$= (1-b)\lim_{h_y\to 0} \frac{h_y}{|h_y|} = 0 \implies b = 1. \tag{5.2.25}$$

Still another possibility constitutes $h_x = h_y$, which entails the equation:

$$\lim_{h\to 0} \frac{(1-a)h_x^5 - ah_x h_y^2 - bhx^4 h_y + (1-b)h_y^3}{(h_x^4 + h_y^2)\sqrt{h_x^2 + h_y^2}} = \lim_{h_x\to 0} \frac{h_x^3(h_x^2+1)(1-a-b)}{h_x^2|h_x|(h_x^2+1)\sqrt{2}}$$
$$= \frac{1-a-b}{\sqrt{2}} \lim_{h_x\to 0} \frac{h_x}{|h_x|} = 0 \implies a + b = 1. \tag{5.2.26}$$

It is not possible to comply with this condition for previously found values $a = b = 1$, and thereby the derivative in the strong sense does not exist.

As in the previous example, one can check that the partial derivatives are discontinuous at the origin. For one has

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \frac{5x^4(x^4+y^2) - 4x^3(x^5+y^3)}{(x^4+y^2)^2} = \frac{x^3(x^5 + 5xy^2 - 4y^3)}{(x^4+y^2)^2} \tag{5.2.27}$$

away from the origin and

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{\triangle x\to 0} \frac{f(0+\triangle x, 0) - f(0,0)}{\triangle x} = \lim_{\triangle x\to 0} \frac{\triangle x^5/\triangle x^4 - 0}{\triangle x} = 1 \tag{5.2.28}$$

at the point $(0,0)$. It is easy to observe that (5.2.27) has no limit at this point. Approaching it along the x-axis, we get 1, while along the y-axis—0 and, for instance, along the parabola $y = x^2$:

$$\lim_{x\to 0} \left.\frac{\partial f}{\partial x}\right|_{(x,x^2)} = \lim_{x\to 0} \frac{x^3(x^5 + 5x(x^2)^2 - 4(x^2)^3)}{(x^4 + (x^2)^2)^2} = \lim_{x\to 0} \frac{x^8(6-4x)}{4x^8} = \frac{3}{2}. \tag{5.2.29}$$

Thus, the derivative $\partial f/\partial x$ is discontinuous. In a straightforward way, one can check that the identical conclusion can be drawn as to the derivative over y.

Problem 3

The differentiability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by the formula:

$$f(x,y) = \begin{cases} x + y + \dfrac{x^3 y}{x^4 + y^2} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0, \end{cases} \tag{5.2.30}$$

in the weak (i.e., Gateaux) and strong (i.e., Fréchet) sense, will be examined.

Solution

The continuity of the function $f(x,y)$ is relatively easy to demonstrate. The only "dangerous" point is the coordinate-frame origin. However, using the obvious inequality:

$$(x^2 - |y|)^2 \geq 0 \implies 2x^2|y| \leq x^4 + y^2, \tag{5.2.31}$$

it is straightforward to justify that

$$\left|\frac{x^2 y}{x^4 + y^2}\right| \leq \frac{1}{2}. \tag{5.2.32}$$

This entails the following estimate:

$$0 \leq \left|x + y + \frac{x^3 y}{x^4 + y^2}\right| \leq |x| + |y| + \left|\frac{x^3 y}{x^4 + y^2}\right| \leq |x| + |y| + \frac{1}{2}|x| = \frac{3}{2}|x| + |y|. \tag{5.2.33}$$

This shows that if $(x,y) \to (0,0)$, then also $f(x,y) \to 0 = f(0,0)$, i.e., the function is continuous at $(0,0)$.

To examine the existence of the Gateaux derivative, let us find the directional derivative at the system origin along any vector $h = [h_x, h_y]$. In view of the definition (5.1.8), one has

$$
\begin{aligned}
f'_h\big|_{(0,0)} &= \lim_{t\to 0} \frac{f(0+th_x, 0+th_y) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{th_x + th_y + ((th_x)^3 th_y)/((th_x)^4 + (th_y)^2)}{t} \\
&= \lim_{t\to 0} \left[h_x + h_y + \frac{h_x^3 h_y}{t^2 h_x^4 + h_y^2}\, t \right] = h_x + h_y = [1\ 1] \cdot \begin{bmatrix} h_x \\ h_y \end{bmatrix}. \qquad (5.2.34)
\end{aligned}
$$

As one can see, the directional derivative exists for any vector h and is linear. In a moment, we will be convinced, however, that even this result does not guarantee the existence of the strong derivative. For this purpose let us write the increment of the function value in the form of the sum of the linear part and the remainder:

$$
\begin{aligned}
f(0+h_x, 0+h_y) - f(0,0) &= h_x + h_y + \frac{h_x^3 h_y}{h_x^4 + h_y^2} \\
&= \underbrace{[1\ 1]}_{L(0,0)} \cdot \begin{bmatrix} h_x \\ h_y \end{bmatrix} + \underbrace{\frac{h_x^3 h_y}{h_x^4 + h_y^2}}_{r(h)} \qquad (5.2.35)
\end{aligned}
$$

and examine the limit:

$$
\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = \lim_{h\to 0} \frac{h_x^3 h_y}{(h_x^4 + h_y^2)\sqrt{h_x^2 + h_y^2}}. \qquad (5.2.36)
$$

For the existence of the Fréchet derivative the above expression should vanish (cf. (5.2.3)). When one goes to the limit, assuming that h_y is fixed and equal to zero and $h_x \to 0$ or vice versa, the required result is actually obtained. However, attaining the limit $h \to 0$ in a different way, for example assuming that $h_y = h_x^2$ and $h_x \to 0$, yields the nonzero result:

$$
\begin{aligned}
\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} &= \lim_{h_x\to 0} \frac{h_x^3 \cdot (h_x^2)}{(h_x^4 + (h_x^2)^2)\sqrt{h_x^2 + (h_x^2)^2}} = \lim_{h_x\to 0} \frac{h_x^5}{2h_x^4 |h_x| \sqrt{1 + h_x^2}} \\
&= \frac{1}{2} \lim_{h_x\to 0} \frac{h_x}{|h_x|} \cdot \frac{1}{\sqrt{1 + h_x^2}} \neq 0. \qquad (5.2.37)
\end{aligned}
$$

One finally sees that $\lim_{h\to 0} \|r(h)\|/\|h\|$ does not exist and the function f defined by formula (5.2.30) is not Fréchet differentiable. When calculating partial derivatives, as in the previous problems, we would be convinced that they are discontinuous at the origin, although they do exist at all points of the $\mathbb{R}^2$ plane.

Problem 4

The differentiability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula:

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x^2 + y^2} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0, \end{cases} \tag{5.2.38}$$

in the weak (i.e., Gateaux) and strong (i.e., Fréchet) sense, will be examined.

Solution

Since $(|x| - |y|)^2 \geq 0$, i.e., $2|xy| \leq x^2 + y^2$, the following estimate is true:

$$0 \leq |f(x, y)| = \left|\frac{xy^3}{x^2 + y^2}\right| = y^2 \cdot \left|\frac{xy}{x^2 + y^2}\right| \leq \frac{1}{2}\, y^2 \underset{(x,y)\to(0.0)}{\longrightarrow} 0, \tag{5.2.39}$$

so the function $f(x, y)$ turns out to be continuous.

The directional derivative can be obtained in a standard way:

$$\begin{aligned} f'_h\big|_{(0,0)} &= \lim_{t\to 0} \frac{f(0 + th_x, 0 + th_y) - f(0, 0)}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \cdot \frac{th_x(th_y)^3}{(th_x)^2 + (th_y)^2} = \lim_{t\to 0} \frac{t^4}{t^3} \cdot \frac{h_x h_y^3}{h_x^2 + h_y^2} = 0 = \underbrace{[0\ 0]}_{L(0,0)} \cdot \begin{bmatrix} h_x \\ h_y \end{bmatrix}. \end{aligned} \tag{5.2.40}$$

Hence, the remainder $r(h)$ appearing in the formula:

$$f(0 + h_x, 0 + h_y) - f(0, 0) = L(0, 0) \cdot h + r(h) \tag{5.2.41}$$

is equal to

$$r(h) = \frac{h_x h_y^3}{h_x^2 + h_y^2}. \tag{5.2.42}$$

For the expression $\|r(h)\|/\|h\|$ one can use an estimate similar to that obtained in (5.2.39):

$$0 \leq \frac{\|r(h)\|}{\|h\|} = \left|\frac{h_x h_y^3}{h_x^2 + h_y^2}\right| \frac{1}{\sqrt{h_x^2 + h_y^2}} \leq \frac{h_y^2}{2\sqrt{h_x^2 + h_y^2}}. \tag{5.2.43}$$

Since

$$\lim_{h\to 0} \frac{h_y^2}{2\sqrt{h_x^2+h_y^2}} = 0, \tag{5.2.44}$$

then also $\lim_{h\to 0} \|r(h)\|/\|h\| = 0$ and we conclude that the strong derivative does exist.

Let us additionally examine the behavior of partial derivatives at $(0, 0)$. For $(x, y) \neq (0, 0)$ one has

$$\frac{\partial f}{\partial x} = \frac{y^3(x^2+y^2) - xy^3 \cdot 2x}{(x^2+y^2)^2} = \frac{y^3(y^2-x^2)}{(x^2+y^2)^2}. \tag{5.2.45}$$

For $y = 0$ one immediately gets $\partial f/\partial x = 0$, and if $y \neq 0$ the following estimate can be made:

$$\begin{aligned} \left|\frac{\partial f}{\partial x}\right| &= \left|\frac{y^3(y^2-x^2)}{(x^2+y^2)^2}\right| \leq \left|\frac{y^3(y^2+x^2)}{(x^2+y^2)^2}\right| \\ &= \left|\frac{y^3}{x^2+y^2}\right| \leq \left|\frac{y^3}{0+y^2}\right| = |y| \underset{(x,y)\to(0,0)}{\longrightarrow} 0. \end{aligned} \tag{5.2.46}$$

The same result is obtained for the value of the derivative at $(0, 0)$:

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h_x\to 0} \frac{f(0+h_x, 0) - f(0,0)}{h_x} = 0, \tag{5.2.47}$$

which means that the partial derivative with respect to x is continuous. The calculations of the derivative over y are similar and lead to the same conclusion. This is because

$$\frac{\partial f}{\partial y} = \frac{3xy^2(x^2+y^2) - xy^3 \cdot 2y}{(x^2+y^2)^2} = \frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}, \tag{5.2.48}$$

which implies that

$$\begin{aligned} \left|\frac{\partial f}{\partial y}\right| &= \left|\frac{xy^2(3x^2+y^2)}{(x^2+y^2)^2}\right| \leq \left|\frac{xy^2(3x^2+3y^2)}{(x^2+y^2)^2}\right| \\ &= 3|y| \left|\frac{xy}{x^2+y^2}\right| \leq \frac{3}{2}\,|y| \underset{(x,y)\to(0,0)}{\longrightarrow} 0. \end{aligned} \tag{5.2.49}$$

On the other hand, the derivative at this point is equal to

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h_y\to 0} \frac{f(0, 0+h_y) - f(0,0)}{h_y} = 0, \tag{5.2.50}$$

so the derivative $\partial f/\partial y$ is continuous.

5.3 Exercises for Independent Work

Exercise 1 Find all partial derivatives of the function f and the directional derivative along the vector v at the point A for:

(a) $f(x, y) = \sin(x^y + x)$, where $v = [1, -2]$ and $A(\pi, 1)$,

(b) $f(x, y) = e^{xy} \log(x + y - 1)$, where $v = [-1, 1]$ and $A(1, 2)$,

(c) $f(x, y) = \begin{bmatrix} xy \\ x + y \end{bmatrix}$, where $v = [1, 3]$ and $A(2, -1)$,

(d) $f(x, y, z) = \begin{bmatrix} xyz \\ xz + y \\ yz + x \end{bmatrix}$, where $v = [1, 2, 3]$ and $A(1, 1, 2)$.

Answers

(a) $f_x|_A = 2$, $f_y|_A = \pi \log \pi$, $f_v|_A = 2 - \pi \log \pi^2$,
(b) $f'_x|_A = e^2/2 + 2\log 2$, $f'_y|_A = e^2/2 + \log 2$, $f'_v|_A = -e^2 \log 2$,
(c) $f'_x|_A = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $f'_y|_A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $f'_v|_A = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$,
(d) $f'_x|_A = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $f'_y|_A = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $f'_z|_A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $f'_v|_A = \begin{bmatrix} 9 \\ 7 \\ 8 \end{bmatrix}$.

Exercise 2 Verify the continuity and the differentiability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ (in the Gateaux (G) and Fréchet (F) sense).

(a) $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0, \end{cases}$

(b) $f(x, y) = \begin{cases} \dfrac{x^4 y^2}{x^4 + y^4} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0. \end{cases}$

Answers

(a) Continuous function, G-differentiable, F-non-differentiable,
(b) Continuous function, G-differentiable, F-differentiable.

Chapter 6
Examining Higher Derivatives, Differential Expressions, and Taylor's Formula

Below we continue investigating the differential properties of functions of several variables. In the following sections, the derivatives of higher orders are explored, the chain rule is applied to practical calculations, and Taylor's formula is studied.

The **nth partial derivative** with respect to the variables $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ is denoted as

$$\frac{\partial}{\partial x_{i_n}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_1}} f(x_1, x_2, \ldots, x_N) = \frac{\partial^n}{\partial x_{i_n} \cdots \partial x_{i_2} \partial x_{i_1}} f(x_1, x_2, \ldots, x_N), \tag{6.0.1}$$

where $i_k \in \{1, 2, \ldots, N\}$ for $k = 1, 2 \ldots, n$. If all such derivatives exist and are continuous, the function is called **differentiable *n* times** or to belong to the **class *C*ⁿ** (in some textbooks this continuity is not required for a function differentiable n times). For such a function, the nth derivative (6.0.1) is symmetric with respect to the permutation of indices $\{i_1, i_2, \ldots, i_n\}$. In particular for a function $f(x, y)$, the matrix

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \tag{6.0.2}$$

is symmetric.

For the differentiation of a composite function $f(g_1(x), g_2(x), \ldots, g_k(x))$, where for abbreviation we denoted $x = (x_1, x_2, \ldots, x_N)$, the following **chain rule** applies:

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{k} \frac{\partial f}{\partial g_j} \cdot \frac{\partial g_j}{\partial x_i}, \tag{6.0.3}$$

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where the functions $g_1(x), g_2(x), \ldots, g_k(x)$ are assumed to be differentiable as well. We will come back to this expression in formula (6.2.4).

Given a function $f : \mathbb{R}^N \supset D \to \mathbb{R}$ of the class C^n. Then the following **Taylor's formula** holds:

$$f(x) = \sum_{l=0}^{n} \frac{1}{l!} \left[\sum_{i=1}^{k} h_i \frac{\partial}{\partial x_i} \right]^l f\Big|_{x=x_0} + R_n(x,h), \tag{6.0.4}$$

where $x_0, x \in D$ and $h = [x_1 - x_{01}, x_2 - x_{02}, \ldots, x_N - x_{0N}]$. The remainder R_n may be given various forms, but for our work it is sufficient to know that

$$\lim_{h\to 0} \frac{R_n(x,h)}{|h|^n} = 0. \tag{6.0.5}$$

6.1 Verifying the Existence of the Second Derivatives

Problem 1

The existence of the second derivative of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula:

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^2+y^2} & \text{for } x^2+y^2 \neq 0, \\ 0 & \text{for } x^2+y^2 = 0 \end{cases} \tag{6.1.1}$$

will be examined.

Solution

In the previous chapter, we learned how to examine first derivatives of functions and that is where one begins when solving this problem. One must verify whether at all points of the plane $\mathbb{R}^2$ the partial derivatives with respect to both x and y exist and are continuous. As we know, the fulfillment of these conditions will guarantee the differentiability of a function in the so-called "strong" sense.

At all points except for $(0, 0)$ both partial derivatives can be calculated from the formula for the differentiation of a quotient. One easily gets

$$f'_x\big|_{(x,y)} = \frac{\partial f}{\partial x}\bigg|_{(x,y)} = \frac{3x^2y(x^2+y^2) - 2x^4y}{(x^2+y^2)^2} = \frac{x^4y + 3x^2y^3}{(x^2+y^2)^2}, \tag{6.1.2}$$

$$f'_y\big|_{(x,y)} = \left.\frac{\partial f}{\partial y}\right|_{(x,y)} = \frac{x^3(x^2+y^2)-2x^3y^2}{(x^2+y^2)^2} = \frac{x^5-x^3y^2}{(x^2+y^2)^2}.$$

It is clear that these derivatives do exist, so one is left only with the origin of the coordinate system to study. At this point, the known and ready formulas for differentiation cannot be used, so one has to directly apply the definition:

$$f'_x\big|_{(0,0)} = \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{\triangle x\to 0}\frac{f(\triangle x,0)-f(0,0)}{\triangle x} = 0,$$
$$f'_y\big|_{(0,0)} = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{\triangle y\to 0}\frac{f(0,\triangle y)-f(0,0)}{\triangle y} = 0. \tag{6.1.3}$$

On the basis of these results, one can see that partial derivatives are defined at each point of the plane $\mathbb{R}^2$, which means that the first of the conditions is satisfied. We do not know yet whether they are continuous and this remains to be examined. However, out of the point $(0,0)$, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are rational functions so their continuity is evident (the denominators do not vanish). Thus it is sufficient to establish the limits of expressions (6.1.2) for $(x,y)\to(0,0)$. For this purpose, we will try to make the appropriate estimates. First we write

$$\left|\left.\frac{\partial f}{\partial x}\right|_{(x,y)}\right| = \left|\frac{x^4y+3x^2y^3}{(x^2+y^2)^2}\right| = \left|\frac{3x^2y(x^2+y^2)-2x^4y}{(x^2+y^2)^2}\right|$$
$$\leq \left|\frac{3x^2y(x^2+y^2)}{(x^2+y^2)^2}\right| + \left|\frac{2x^4y}{(x^2+y^2)^2}\right| = \left|\frac{3x^2y}{x^2+y^2}\right| + \left|\frac{2x^4y}{(x^2+y^2)^2}\right|$$
$$\leq \left|\frac{3x^2y}{x^2+0}\right| + \left|\frac{2x^4y}{(x^2+0)^2}\right| = 3|y|+2|y| = 5|y| \underset{(x,y)\to(0,0)}{\longrightarrow} 0, \tag{6.1.4}$$

which means that the derivative over x is continuous. The next estimation:

$$\left|\left.\frac{\partial f}{\partial y}\right|_{(x,y)}\right| = \left|\frac{x^3(x^2-y^2)}{(x^2+y^2)^2}\right| \leq \frac{|x^3|(x^2+y^2)}{(x^2+y^2)^2}$$
$$= \frac{|x^3|}{x^2+y^2} \leq \frac{|x^3|}{x^2+0} = |x| \underset{(x,y)\to(0,0)}{\longrightarrow} 0 \tag{6.1.5}$$

leads to the same conclusion as to the derivative with respect to y. Note that if one of the variables x, y equals zero in the first estimate (both cannot simultaneously) or $x=0$ in the second one, the above results still remain true, as then one immediately has $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$.

In this way, the differentiability of the function f in the Fréchet sense is confirmed. Now let us move on to the second derivative. Our study will be limited to the origin for the reasons that have already been repeatedly mentioned. If the second derivative at this point were to exist, this would entail the equality of second (partial) mixed derivatives:

$$f''_{yx} := \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} =: f''_{xy}. \tag{6.1.6}$$

However, one has

$$\begin{aligned} f''_{xy}\big|_{(0,0)} &= \lim_{\triangle y \to 0} \frac{f'_x(0, \triangle y) - f'_x(0,0)}{\triangle y} \lim_{\triangle y \to 0} \frac{0-0}{\triangle y} = 0, \\ f''_{yx}\big|_{(0,0)} &= \lim_{\triangle x \to 0} \frac{f'_y(\triangle x, 0) - f'_y(0,0)}{\triangle x} \lim_{\triangle x \to 0} \frac{\triangle x - 0}{\triangle x} = 1 \neq f''_{xy}\big|_{(0,0)}, \end{aligned} \tag{6.1.7}$$

where formulas (6.1.2) and (6.1.3) have been used. This result means that the second derivative cannot exist. So if we were studying the second partial derivatives, all would certainly not prove to be continuous because otherwise the condition for the existence of f'' would be met. The nonexistence of f'' can be easily established. Let us find the mixed derivative f''_{xy}, differentiating (6.1.2) over y:

$$f''_{xy}\big|_{(x,y)} = \frac{x^6 + 6x^4y^2 - 3x^2y^4}{(x^2+y^2)^3} \tag{6.1.8}$$

and examine its behavior as $(x, y) \to (0, 0)$. Approaching the origin along the x-axis, i.e., fixing first y to be zero, one obtains

$$f''_{xy}\big|_{(x,0)} = \frac{x^6 + 6x^4 \cdot 0 - 3x^2 \cdot 0}{(x^2+0)^3} = 1 \underset{x\to 0}{\longrightarrow} 1. \tag{6.1.9}$$

In turn when setting $x = 0$ and letting the variable y go to zero, one gets

$$f''_{xy}\big|_{(0,y)} = \frac{0 + 6 \cdot 0 \cdot y^2 - 3 \cdot 0 \cdot y^4}{(0+y^2)^3} = 0 \underset{x\to 0}{\longrightarrow} 0 \neq 1. \tag{6.1.10}$$

The case when one variable, x or y, was equal to zero has been omitted, since the estimates are then obvious. As one can see, the limit of f''_{xy} at the origin does not exist, and therefore—in accordance with our predictions—the second (mixed) derivative cannot be continuous.

Problem 2

The existence of the second derivatives of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by the formula:

$$f(x,y) = \begin{cases} \dfrac{x^2y^2 - y^4}{x^2+y^2} & \text{for } x^2+y^2 \neq 0, \\ 0 & \text{for } x^2+y^2 = 0 \end{cases} \tag{6.1.11}$$

will be examined.

Solution

First let us calculate the partial derivatives of the function f. Away from the origin, they are given by the formulas:

$$\begin{aligned} f'_x\big|_{(x,y)} &= \left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \frac{4xy^4}{(x^2+y^2)^2} \\ f'_y\big|_{(x,y)} &= \left.\frac{\partial f}{\partial y}\right|_{(x,y)} = \frac{2x^4y - 4x^2y^3 - 2y^5}{(x^2+y^2)^2}, \end{aligned} \tag{6.1.12}$$

while at the origin one has

$$\begin{aligned} f'_x\big|_{(0,0)} &= \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{\triangle x\to 0} \frac{f(\triangle x, 0) - f(0,0)}{\triangle x} = 0, \\ f'_y\big|_{(0,0)} &= \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{\triangle y\to 0} \frac{f(0, \triangle y) - f(0,0)}{\triangle y} = \lim_{\triangle y\to 0} \frac{-\triangle y^4/\triangle y^2}{\triangle y} = 0. \end{aligned} \tag{6.1.13}$$

Both these derivatives are continuous; one can easily be convinced of this by making the following estimates:

$$\begin{aligned} 0 \le \left| f'_x\big|_{(x,y)} \right| &= \left| \frac{4xy^4}{(x^2+y^2)^2} \right| \le \left| \frac{4xy^4}{(0+y^2)^2} \right| = 4|x| \underset{(x,y)\to(0,0)}{\longrightarrow} 0 \\ 0 \le \left| f'_y\big|_{(x,y)} \right| &= \left| \frac{2x^4y - 4x^2y^3 - 2y^5}{(x^2+y^2)^2} \right| \le \left| \frac{2x^4y}{(x^2+y^2)^2} \right| + \left| \frac{4x^2y^3}{(x^2+y^2)^2} \right| \\ &+ \left| \frac{2y^5}{(x^2+y^2)^2} \right| \le \left| \frac{2x^4y}{(x^2+0)^2} \right| + 2|y| \left| \frac{2x^2y^2}{(x^2+y^2)^2} \right| + \left| \frac{2y^5}{(0+y^2)^2} \right| \\ &\le 2|y| + 2|y| + 2|y| = 6|y| \underset{(x,y)\to(0,0)}{\longrightarrow} 0. \end{aligned} \tag{6.1.14}$$

We have used here the fact that

$$(x^2 - y^2)^2 \ge 0 \Longrightarrow 2x^2y^2 \le x^4 + y^4 \quad \Longrightarrow \quad 2x^2y^2 \le x^4 + y^4 + 2x^2y^2$$
$$\Longrightarrow 2x^2y^2 \le (x^2 + y^2)^2. \tag{6.1.15}$$

Hence, as we see, the limits of the partial derivatives are equal to their values. The continuity of both of them at the origin guarantees the existence of the strong derivative.

Now let us turn to higher derivatives. The second mixed derivatives prove to be identical:

$$f''_{xy}\big|_{(x,y)} = \frac{16x^3y^3}{(x^2+y^2)^3} = f''_{yx}\big|_{(x,y)}, \tag{6.1.16}$$

and for the others one obtains

$$f''_{xx}\big|_{(x,y)} = \frac{4y^4(y^2-3x^2)}{(x^2+y^2)^3},$$
$$f''_{yy}\big|_{(x,y)} = \frac{2x^6 - 18x^4y^2 - 6x^2y^4 - 2y^6}{(x^2+y^2)^3}. \tag{6.1.17}$$

In order to establish the continuity, one needs to examine their values at the point $(0, 0)$. Let us then calculate

$$f'_{xx}\big|_{(0,0)} = \lim_{\triangle x \to 0} \frac{f'_x(\triangle x, 0) - f'_x(0,0)}{\triangle x} = \lim_{\triangle x \to 0} \frac{0-0}{\triangle x} = 0,$$
$$f'_{yy}\big|_{(0,0)} = \lim_{\triangle y \to 0} \frac{f'_y(0, \triangle y) - f'_y(0,0)}{\triangle y} = \lim_{\triangle y \to 0} \frac{-2\triangle y^5/\triangle y^4 - 0}{\triangle y} = -2,$$
$$f'_{xy}\big|_{(0,0)} = \lim_{\triangle y \to 0} \frac{f'_x(0, \triangle y) - f'_x(0,0)}{\triangle y} = \lim_{\triangle y \to 0} \frac{0-0}{\triangle y} = 0,$$
$$f'_{yx}\big|_{(0,0)} = \lim_{\triangle x \to 0} \frac{f'_y(\triangle x, 0) - f'_y(0,0)}{\triangle x} = \lim_{\triangle x \to 0} \frac{0-0}{\triangle x} = 0, \tag{6.1.18}$$

and these results should be compared with the values of expressions (6.1.16) and (6.1.17). Already in the first limit, the discontinuity is encountered because approaching the origin along the straight line $y = x$, one finds

$$\lim_{x \to 0} f''_{xy}\big|_{(x,x)} = \lim_{x \to 0} \frac{16x^3x^3}{(x^2+x^2)^3} = \lim_{x \to 0} \frac{16x^6}{8x^6} = 2 \neq 0. \tag{6.1.19}$$

Similarly,

$$\lim_{x\to 0} f''_{xx}\big|_{(x,x)} = \lim_{x\to 0} \frac{4x^4(x^2-3x^2)}{(x^2+x^2)^3} = \lim_{x\to 0} \frac{-8x^6}{8x^6} = -1 \neq 0,$$
$$\lim_{x\to 0} f''_{yy}\big|_{(x,x)} = \lim_{x\to 0} \frac{2x^6-18x^4x^2-6x^2x^4-2x^6}{(x^2+x^2)^3} = \lim_{x\to 0} \frac{-24x^6}{8x^6} = -3 \neq -2. \tag{6.1.20}$$

As a consequence, it is seen that f'' does not exist at the point (0, 0).

6.2 Transforming Differential Expressions and Operators

Problem 1

The expression:

$$A := (x^2-y^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} \tag{6.2.1}$$

will be rewritten in polar coordinates, where $u(x, y)$ is a differentiable function of its arguments $x, y \in \mathbb{R}$.

Solution

As we know, the polar coordinates r and φ on the plane xy are introduced by the relations:

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{6.2.2}$$

In order to express $\partial u/\partial x$ and $\partial u/\partial y$ by the derivatives over variables r and φ, the so-called chain rule (6.0.3) is used. In our case it may be written as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x},$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y}. \tag{6.2.3}$$

These formulas constitute the well-known rule for the differentiation of the composite function. For, if one has a mapping $F(Y(X))$, where $F : \mathbb{R}^n \to \mathbb{R}^m$

and $Y : \mathbb{R}^k \to \mathbb{R}^n$, then the derivative of F over variables denoted collectively by X is given by

$$\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial X}. \tag{6.2.4}$$

The product above should be understood in the matrix sense: $\partial F/\partial X$ is an array of m rows and k columns, and those on the right-hand side respectively have dimensions $m \times n$ and $n \times k$. In the situation dealt with in the current problem, one has $m = 1$, $n = k = 2$, and

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad Y = \begin{bmatrix} r \\ \varphi \end{bmatrix}, \quad F = u \tag{6.2.5}$$

so the Eq. (6.2.4) reproduces formulas (6.2.3).

In order to calculate the derivatives $\partial r/\partial x$, $\partial\varphi/\partial x$ and similarly over y, it is a good idea to first reverse (6.2.2):

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\frac{y}{x}. \tag{6.2.6}$$

In the special case when $x = 0$, the alternative formula $\varphi = \operatorname{arccot}(x/y)$ can be used. Using now (6.2.6), one gets

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\varphi}{r} = \cos\varphi, \\
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\varphi}{r} = \sin\varphi, \\
\frac{\partial \varphi}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} = -\frac{r\sin\varphi}{r^2} = -\frac{\sin\varphi}{r}, \\
\frac{\partial \varphi}{\partial y} &= \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} = \frac{r\cos\varphi}{r^2} = \frac{\cos\varphi}{r},
\end{aligned} \tag{6.2.7}$$

which allows one to write

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\varphi - \frac{\partial u}{\partial \varphi}\cdot\frac{\sin\varphi}{r} \tag{6.2.8}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\varphi + \frac{\partial u}{\partial \varphi}\cdot\frac{\cos\varphi}{r}. \tag{6.2.9}$$

Plugging the obtained expressions into formula (6.2.1) and simplifying it, one can come to the final result:

$$\begin{aligned}
A = \ & (r^2\cos^2\varphi - r^2\sin^2\varphi)\left[\frac{\partial u}{\partial r}\cos\varphi - \frac{\partial u}{\partial\varphi}\cdot\frac{\sin\varphi}{r}\right] \\
& +2r^2\cos\varphi\sin\varphi\left[\frac{\partial u}{\partial r}\sin\varphi + \frac{\partial u}{\partial\varphi}\cdot\frac{\cos\varphi}{r}\right] \\
= \ & r^2(\cos^3\varphi - \sin^2\varphi\cos\varphi + 2\sin^2\varphi\cos\varphi)\frac{\partial u}{\partial r} \\
& +r(-\cos^2\varphi\sin\varphi + \sin^3\varphi + 2\cos^2\varphi\sin\varphi)\frac{\partial u}{\partial\varphi} \\
= \ & r^2\cos\varphi(\cos^2\varphi + \sin^2\varphi)\frac{\partial u}{\partial r} + r\sin\varphi(\cos^2\varphi + \sin^2\varphi)\frac{\partial u}{\partial\varphi} \\
= \ & r^2\cos\varphi\,\frac{\partial u}{\partial r} + r\sin\varphi\,\frac{\partial u}{\partial\varphi}. \qquad (6.2.10)
\end{aligned}$$

Problem 2

The following expression will be rewritten in the spherical coordinates:

$$B := \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2, \qquad (6.2.11)$$

where $u(x, y, z)$ is a differentiable function of its arguments $x, y, z \in \mathbb{R}$.

Solution

The spherical variables r, θ, φ defined in Fig. 6.1 are usually introduced via the relations:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \qquad (6.2.12)$$

x, y, z being Cartesian coordinates. Inverting (6.2.12) where possible, one finds

$$r = \sqrt{x^2 + y^2 + x^2}, \quad \theta = \arctan\frac{\sqrt{x^2 + y^2}}{z}, \quad \varphi = \arctan\frac{y}{x}. \qquad (6.2.13)$$

In a particular case when $z = 0$ or $x = 0$, one can use the formulas: $\theta = \operatorname{arccot}(z/\sqrt{x^2 + y^2})$ and $\varphi = \operatorname{arccot}(x/y)$ instead.

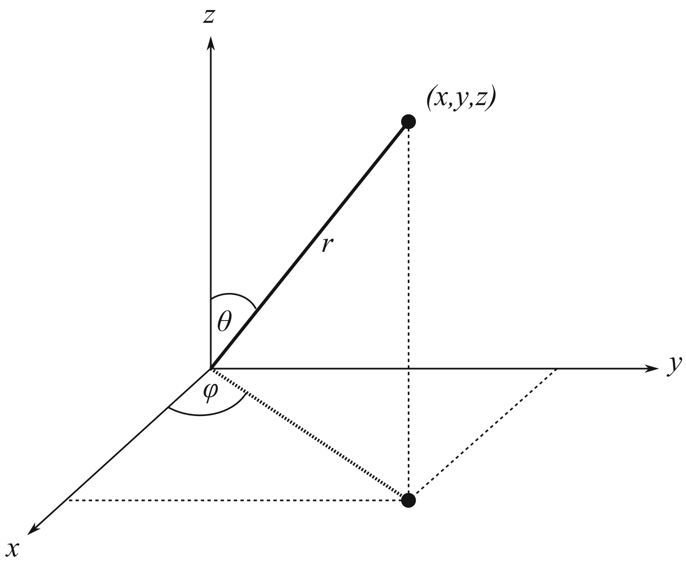

Fig. 6.1 Definitions of spherical coordinates r, θ, and φ

In order to appropriately transform the expression B, we can employ the formula for the differentiation of a composite function (6.2.4), in the present exercise substituting $m = 1$, $n = k = 3$:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad Y = \begin{bmatrix} r \\ \theta \\ \varphi \end{bmatrix}, \quad F = u. \tag{6.2.14}$$

It entails the following relations:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x},\\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y},\\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial z}.
\end{aligned} \tag{6.2.15}$$

Now let us calculate the derivatives of spherical variables r, θ, φ over the Cartesian ones: x, y, z. Since there are as many as nine derivatives, only the first 3 will be dealt with in detail, and for the others, for which the calculations are similar, only the final results are provided to be verified by the reader. Thus, one has

$$
\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\cos\varphi}{r} = \sin\theta\cos\varphi,\\
\frac{\partial\theta}{\partial x} &= \frac{x/(z\sqrt{x^2+y^2})}{1+(x^2+y^2)/z^2} = \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}\\
&= \frac{r^2\sin\theta\cos\theta\cos\varphi}{r^3\sin\theta} = \frac{\cos\theta\cos\varphi}{r},\\
\frac{\partial\varphi}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta\sin\varphi}{r^2\sin^2\theta} = -\frac{\sin\varphi}{r\sin\theta}
\end{aligned}
\tag{6.2.16}
$$

and

$$
\begin{aligned}
&\frac{\partial r}{\partial y} = \sin\theta\sin\varphi, \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\varphi}{r}, \quad \frac{\partial\varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}\\
&\frac{\partial r}{\partial z} = \cos\theta, \quad \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}, \quad \frac{\partial\varphi}{\partial z} = 0.
\end{aligned}
\tag{6.2.17}
$$

It remains only to substitute these results first into (6.2.15) and next into (6.2.11) and after the repeated application of the Pythagorean trigonometric identity and some procedural simplifications one comes to

$$
\begin{aligned}
B =& \left(\frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial\theta}\cdot\frac{\cos\theta\cos\varphi}{r} - \frac{\partial u}{\partial\varphi}\cdot\frac{\sin\varphi}{r\sin\theta}\right)^2\\
&+ \left(\frac{\partial u}{\partial r}\sin\theta\sin\varphi + \frac{\partial u}{\partial\theta}\cdot\frac{\cos\theta\sin\varphi}{r} - \frac{\partial u}{\partial\varphi}\cdot\frac{\cos\varphi}{r\sin\theta}\right)^2\\
&+ \left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\cdot\frac{\sin\theta}{r} - \frac{\partial u}{\partial\varphi}\cdot 0\right)^2\\
&= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial u}{\partial\varphi}\right)^2.
\end{aligned}
\tag{6.2.18}
$$

Problem 3

The new variables ξ, η, and ζ will be chosen in the expression:

$$
C := 3\frac{\partial^2 f}{\partial x^2} + 3\frac{\partial^2 f}{\partial y^2} + 3\frac{\partial^2 f}{\partial z^2} + 2\frac{\partial^2 f}{\partial x\partial y} + 2\frac{\partial^2 f}{\partial x\partial z} - 2\frac{\partial^2 f}{\partial y\partial z} \tag{6.2.19}
$$

so as to get rid of the mixed derivatives and convert it into the form:

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \zeta^2}. \tag{6.2.20}$$

Solution

In view of the fact that the coefficients in the expression for C are fixed numbers, independent of $x,\ y,\ z$, and all derivatives, both in (6.2.19) and in (6.2.20), are of the second order, these new variables $\xi,\ \eta$ and ζ should be related to the old ones by linear relations:

$$\xi = \alpha_1 x + \beta_1 y + \gamma_1 z, \quad \eta = \alpha_2 x + \beta_2 y + \gamma_2 z, \quad \zeta = \alpha_3 x + \beta_3 y + \gamma_3 z. \tag{6.2.21}$$

Our goal is simply to find all constants α_i, β_i, and γ_i.

When changing variables as defined by (6.2.21), the chain rule takes here the particularly simple form:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \alpha_1 \frac{\partial f}{\partial \xi} + \alpha_2 \frac{\partial f}{\partial \eta} + \alpha_3 \frac{\partial f}{\partial \zeta}, \\
\frac{\partial f}{\partial y} &= \beta_1 \frac{\partial f}{\partial \xi} + \beta_2 \frac{\partial f}{\partial \eta} + \beta_3 \frac{\partial f}{\partial \zeta}, \\
\frac{\partial f}{\partial z} &= \gamma_1 \frac{\partial f}{\partial \xi} + \gamma_2 \frac{\partial f}{\partial \eta} + \gamma_3 \frac{\partial f}{\partial \zeta}.
\end{aligned} \tag{6.2.22}$$

In order to calculate the second derivatives, required in (6.2.19), this rule has to be applied once again, but this time with reference to the expressions $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$. In this way, one gets

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \alpha_1^2 \frac{\partial^2 f}{\partial \xi^2} + \alpha_2^2 \frac{\partial^2 f}{\partial \eta^2} + \alpha_3^2 \frac{\partial^2 f}{\partial \zeta^2} + 2\alpha_1\alpha_2 \frac{\partial^2 f}{\partial \xi \partial \eta} \\
&\quad + 2\alpha_1\alpha_3 \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\alpha_2\alpha_3 \frac{\partial^2 f}{\partial \eta \partial \zeta}, \\
\frac{\partial^2 f}{\partial y^2} &= \beta_1^2 \frac{\partial^2 f}{\partial \xi^2} + \beta_2^2 \frac{\partial^2 f}{\partial \eta^2} + \beta_3^2 \frac{\partial^2 f}{\partial \zeta^2} + 2\beta_1\beta_2 \frac{\partial^2 f}{\partial \xi \partial \eta} \\
&\quad + 2\beta_1\beta_3 \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\beta_2\beta_3 \frac{\partial^2 f}{\partial \eta \partial \zeta},
\end{aligned}$$

$$\frac{\partial^2 f}{\partial z^2} = \gamma_1^2 \frac{\partial^2 f}{\partial \xi^2} + \gamma_2^2 \frac{\partial^2 f}{\partial \eta^2} + \gamma_3^2 \frac{\partial^2 f}{\partial \zeta^2} + 2\gamma_1\gamma_2 \frac{\partial^2 f}{\partial \xi \partial \eta}$$
$$+2\gamma_1\gamma_3 \frac{\partial^2 f}{\partial \xi \partial \zeta} + 2\gamma_2\gamma_3 \frac{\partial^2 f}{\partial \eta \partial \zeta}, \tag{6.2.23}$$

and for the mixed derivatives:

$$\frac{\partial^2 f}{\partial x \partial y} = \alpha_1\beta_1 \frac{\partial^2 f}{\partial \xi^2} + \alpha_2\beta_2 \frac{\partial^2 f}{\partial \eta^2} + \alpha_3\beta_3 \frac{\partial^2 f}{\partial \zeta^2} + (\alpha_1\beta_2 + \alpha_2\beta_1) \frac{\partial^2 f}{\partial \xi \partial \eta}$$
$$+(\alpha_1\beta_3 + \alpha_3\beta_1) \frac{\partial^2 f}{\partial \xi \partial \zeta} + (\alpha_2\beta_3 + \alpha_3\beta_2) \frac{\partial^2 f}{\partial \eta \partial \zeta},$$
$$\frac{\partial^2 f}{\partial x \partial z} = \alpha_1\gamma_1 \frac{\partial^2 f}{\partial \xi^2} + \alpha_2\gamma_2 \frac{\partial^2 f}{\partial \eta^2} + \alpha_3\gamma_3 \frac{\partial^2 f}{\partial \zeta^2} + (\alpha_1\gamma_2 + \alpha_2\gamma_1) \frac{\partial^2 f}{\partial \xi \partial \eta}$$
$$+(\alpha_1\gamma_3 + \alpha_3\gamma_1) \frac{\partial^2 f}{\partial \xi \partial \zeta} + (\alpha_2\gamma_3 + \alpha_3\gamma_2) \frac{\partial^2 f}{\partial \eta \partial \zeta},$$
$$\frac{\partial^2 f}{\partial y \partial z} = \beta_1\gamma_1 \frac{\partial^2 f}{\partial \xi^2} + \beta_2\gamma_2 \frac{\partial^2 f}{\partial \eta^2} + \beta_3\gamma_3 \frac{\partial^2 f}{\partial \zeta^2} + (\beta_1\gamma_2 + \beta_2\gamma_1) \frac{\partial^2 f}{\partial \xi \partial \eta}$$
$$+(\beta_1\gamma_3 + \beta_3\gamma_1) \frac{\partial^2 f}{\partial \xi \partial \zeta} + (\beta_2\gamma_3 + \beta_3\gamma_2) \frac{\partial^2 f}{\partial \eta \partial \zeta}. \tag{6.2.24}$$

These expressions will be now inserted into (6.2.19), and the lacking constants can be established from the requirement that the coefficients accompanying the derivatives $\partial^2 f/\partial \xi^2$, $\partial^2 f/\partial \eta^2$, and $\partial^2 f/\partial \zeta^2$ equal 1, and those in front of the mixed derivatives—0. The whole expression for C obtained in that way is too long to be written down here, but it is straightforward to provide single equations. There are six in total. The first three have the following form:

$$3\alpha_1^2 + 3\beta_1^2 + 3\gamma_1^2 + 2\alpha_1\beta_1 + 2\alpha_1\gamma_1 - 2\beta_1\gamma_1 = 1,$$
$$3\alpha_2^2 + 3\beta_2^2 + 3\gamma_2^2 + 2\alpha_2\beta_2 + 2\alpha_2\gamma_2 - 2\beta_2\gamma_2 = 1,$$
$$3\alpha_3^2 + 3\beta_3^2 + 3\gamma_3^2 + 2\alpha_3\beta_3 + 2\alpha_3\gamma_3 - 2\beta_3\gamma_3 = 1, \tag{6.2.25}$$

and the remaining ones:

$$6\alpha_1\alpha_2 + 6\beta_1\beta_2 + 6\gamma_1\gamma_2 + 2\alpha_1\beta_2 + 2\alpha_2\beta_1 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1$$
$$- 2\beta_1\gamma_2 - 2\beta_2\gamma_1 = 0,$$
$$6\alpha_1\alpha_3 + 6\beta_1\beta_3 + 6\gamma_1\gamma_3 + 2\alpha_1\beta_3 + 2\alpha_3\beta_1 + 2\alpha_1\gamma_3 + 2\alpha_3\gamma_1$$
$$- 2\beta_1\gamma_3 - 2\beta_3\gamma_1 = 0,$$
$$6\alpha_2\alpha_3 + 6\beta_2\beta_3 + 6\gamma_2\gamma_3 + 2\alpha_2\beta_3 + 2\alpha_3\beta_2 + 2\alpha_2\gamma_3 + 2\alpha_3\gamma_2$$
$$- 2\beta_1\gamma_3 - 2\beta_3\gamma_2 = 0. \tag{6.2.26}$$

We have finally six equations for nine parameters, so we are provided with some freedom. Therefore, three unknowns can arbitrarily be fixed, and as long as the obtained equations are not contradictory, the obtained parameters are as equally good as any others. One should remember that what we want to find is *any* solution of the system and not *the most general* one. Preferably, of course, would be to fix parameters so as to obtain relatively simple equations to solve. For example, let us assume that $\gamma_1 = 0$. The first of equations (6.2.25) can then be given the form:

$$3\alpha_1^2 + 3\beta_1^2 + 2\alpha_1\beta_1 = 1, \quad \text{i.e.,} \quad (\alpha_1 + \beta_1)^2 + 2(\alpha_1^2 + \beta_1^2) = 1. \tag{6.2.27}$$

Additionally letting $\beta_1 = -\alpha_1$, one can get rid of the first term, which yields $\alpha_1 = 1/2$ and $\beta_1 = -1/2$ (the signs may also be chosen vice versa). In this way the expression for the variable ξ is obtained:

$$\xi = \frac{1}{2}(x - y). \tag{6.2.28}$$

Plugging the obtained values of α_1, β_1, and γ_1 into the first of the equations (6.2.26), one obtains

$$\alpha_2 - \beta_2 + \gamma_2 = 0. \tag{6.2.29}$$

Now let us set $\alpha_2 = 0$, which leads to $\beta_2 = \gamma_2$ and the second equation from (6.2.25) takes the form:

$$4\beta_2^2 = 1 \implies \beta_2 = \pm\frac{1}{2}. \tag{6.2.30}$$

Choosing (arbitrarily) $\beta_2 = 1/2$, and therefore, also $\gamma_2 = 1/2$, we see that the variable η may be fixed by the formula:

$$\eta = \frac{1}{2}(y + z). \tag{6.2.31}$$

The second and third of the equations (6.2.26), after having used the values found above, now take the form:

$$\alpha_3 - \beta_3 + \gamma_3 = 0 \quad \text{and} \quad \alpha_3 + \beta_3 + \gamma_3 = 0. \tag{6.2.32}$$

By adding them to each other and subtracting from each other, it can easily be seen that they can be simultaneously fulfilled only if $\beta_3 = 0$ and $\alpha_3 = -\gamma_3$. Then the last of the equations (6.2.25)—and simultaneously the last equation to be considered—takes the form:

$$4\gamma_3^2 = 1 \implies \gamma_3 = \pm\frac{1}{2}. \tag{6.2.33}$$

Again arbitrarily γ_3 can be fixed as equal to $1/2$, which entails at the same time the missing expression for the variable ζ:

$$\zeta = \frac{1}{2}(-x+z). \tag{6.2.34}$$

It is worth noting at the end that the freedom in the selection of signs is due to the fact that in the expression C there appear only derivatives of the even (second) order.

Problem 4

The expression:

$$V(x) := x^2 f''(x) + x f'(x) + f(x), \tag{6.2.35}$$

where $x > 0$ and f is a twice differentiable function, will be transformed by introducing a new independent variable $t = \log x$.

Solution

In the example below, one can again be convinced that the appropriate choice of the independent variable happens to significantly simplify the expressions. This is of vital importance when solving complicated differential equations, and the procedure detailed here will be useful in the subsequent problem.

Passing from the variable x to t, we will use below the notation : $f(x(t)) =: \tilde{f}(t)$. The derivative of the function is first transformed in the following way:

$$f'(x) = \frac{d}{dx} f(x) = \frac{dt}{dx} \cdot \frac{d}{dt} f(x(t)) = \frac{1}{x} \cdot \frac{d}{dt} f(x(t)) = e^{-t} \frac{d}{dt} \tilde{f}(t). \tag{6.2.36}$$

One could write that the differential operator is converted as follows:

$$\frac{d}{dx} \longmapsto e^{-t} \frac{d}{dt}. \tag{6.2.37}$$

In view of the presence of the second derivative in expression (6.2.35) one also has to calculate

$$\left(\frac{d}{dx}\right)^2 f(x) = \left(e^{-t} \frac{d}{dt}\right)^2 f(x(t)) = e^{-t} \frac{d}{dt} \left(e^{-t} \frac{d}{dt} \tilde{f}(t)\right)$$
$$= -e^{-2t} \frac{d}{dt} \tilde{f}(t) + e^{-2t} \frac{d^2}{dt^2} \tilde{f}(t). \tag{6.2.38}$$

Using differential operators, the outcome would be written as

$$\left(\frac{d}{dx}\right)^2 \longmapsto -e^{-2t}\,\frac{d}{dt} + e^{-2t}\,\frac{d^2}{dt^2}. \tag{6.2.39}$$

Plugging these partial results into the expression for $V(x)$, which is now renamed to $\tilde{V}(t)$, one obtains

$$\begin{aligned}\tilde{V}(t) &= e^{2t}\left[-e^{-2t}\,\frac{d}{dt}\,\tilde{f}(t) + e^{-2t}\,\frac{d^2}{dt^2}\,\tilde{f}(t)\right] + e^{t}\left[e^{-t}\,\frac{d}{dt}\,\tilde{f}(t)\right] + \tilde{f}(t)\\ &= \frac{d^2}{dt^2}\,\tilde{f}(t) + \tilde{f}(t).\end{aligned} \tag{6.2.40}$$

The benefit resulting from these transformations is clearly seen. For instance, if our goal was to solve the differential equation in the form

$$x^2 f''(x) + xf'(x) + f(x) = 0, \tag{6.2.41}$$

then, after this significant simplification, it would boil down to the well-known and easily solvable harmonic oscillator equation:

$$\frac{d^2}{dt^2}\,\tilde{f}(t) + \tilde{f}(t) = 0, \tag{6.2.42}$$

the solution of which is

$$\tilde{f}(t) = A\sin t + B\cos t, \tag{6.2.43}$$

with any constants A and B (see Problem 2 in Sect. 10.1).

Problem 5

The three-dimensional Laplace operator in cylindrical variables will be derived.

Solution

In Cartesian coordinates, the Laplace operator has the following form:

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{6.2.44}$$

Our goal is to rewrite it with the cylindrical variables $r,\ \varphi,\ z$, related to the Cartesian ones by the equations:

$$x = r\cos\varphi, \qquad y = \sin\varphi, \tag{6.2.45}$$

the coordinate z remaining unchanged (one could write $z' = z$ and then omit the prime).

In order to express the derivatives over x and y via those with respect to r and φ, one has to apply the chain rule to both of them which takes here the following form:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial x}\cdot\frac{\partial}{\partial\varphi}, \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial}{\partial\varphi}. \end{aligned} \tag{6.2.46}$$

It should be noted that the factors in these expressions—similarly as in the previous example—were written in reverse order from what was done in formulas (6.2.3) or (6.2.15). The notation:

$$\frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial x}$$

could lead to an erroneous belief that the derivatives standing on the left act also on the expressions written on the right. Actually, subject to differentiation is only certain hypothetical function f, on which the above expression is assumed to operate (and which, in this example, is not explicitly put), and for this reason it seems better to move the operators $\partial/\partial r$ and $\partial/\partial\varphi$ to the right.

The change of variables $x, y \mapsto r, \varphi$ on the xy plane corresponds to introducing polar coordinates, and thus one can use the results obtained in Problem 1 (see formulas (6.2.8) and (6.2.9)). We obtain, therefore,

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\cdot\frac{\partial}{\partial\varphi}, \\ \frac{\partial}{\partial y} &= \sin\varphi\,\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\cdot\frac{\partial}{\partial\varphi}. \end{aligned} \tag{6.2.47}$$

Now one has to calculate

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\cdot\frac{\partial}{\partial\varphi}\right)^2 \\ &= \cos^2\varphi\,\frac{\partial^2}{\partial r^2} + \frac{\cos\varphi\sin\varphi}{r^2}\cdot\frac{\partial}{\partial\varphi} - \frac{\cos\varphi\sin\varphi}{r}\cdot\frac{\partial^2}{\partial r\partial\varphi} \end{aligned}$$

$$
\begin{aligned}
&-\frac{\sin\varphi\cos\varphi}{r}\cdot\frac{\partial^2}{\partial\varphi\partial r}+\frac{\sin^2\varphi}{r}\cdot\frac{\partial}{\partial r}+\frac{\sin^2\varphi}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}+\frac{\cos\varphi\sin\varphi}{r^2}\cdot\frac{\partial}{\partial\varphi}\\
=&\cos^2\varphi\,\frac{\partial^2}{\partial r^2}+\frac{2\cos\varphi\sin\varphi}{r^2}\cdot\frac{\partial}{\partial\varphi}-\frac{2\cos\varphi\sin\varphi}{r}\cdot\frac{\partial^2}{\partial r\partial\varphi}\\
&+\frac{\sin^2\varphi}{r}\cdot\frac{\partial}{\partial r}+\frac{\sin^2\varphi}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}
\end{aligned}
\tag{6.2.48}
$$

and similarly

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2}=&\left(\sin\varphi\,\frac{\partial}{\partial r}+\frac{\cos\varphi}{r}\cdot\frac{\partial}{\partial\varphi}\right)^2\\
=&\sin^2\varphi\,\frac{\partial^2}{\partial r^2}-\frac{\sin\varphi\cos\varphi}{r^2}\cdot\frac{\partial}{\partial\varphi}+\frac{\sin\varphi\cos\varphi}{r}\cdot\frac{\partial^2}{\partial r\partial\varphi}\\
&+\frac{\cos^2\varphi}{r}\cdot\frac{\partial}{\partial r}+\frac{\cos\varphi\sin\varphi}{r}\cdot\frac{\partial^2}{\partial\varphi\partial r}-\frac{\sin\varphi\cos\varphi}{r^2}\cdot\frac{\partial}{\partial\varphi}+\frac{\cos^2\varphi}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}\\
=&\sin^2\varphi\,\frac{\partial^2}{\partial r^2}-\frac{2\sin\varphi\cos\varphi}{r^2}\cdot\frac{\partial}{\partial\varphi}+\frac{2\sin\varphi\cos\varphi}{r}\cdot\frac{\partial^2}{\partial r\partial\varphi}\\
&+\frac{\cos^2\varphi}{r}\cdot\frac{\partial}{\partial r}+\frac{\cos^2\varphi}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}.
\end{aligned}
\tag{6.2.49}
$$

By adding expressions (6.2.48), (6.2.49), and $\partial^2/\partial z^2$, one obtains after some simplifications the needed form of the Laplace operator in cylindrical coordinates:

$$
\triangle=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}.
\tag{6.2.50}
$$

6.3 Expanding Functions

Problem 1

The Taylor formula for the function $f(x, y) = \sin x \cos y$ in the neighborhood of the point $(\pi/4, \pi/4)$ will be derived up to fourth order (inclusive).

Solution

The Taylor expansion for a function f of several variables can be found using the formula of the type (6.0.4) or (6.3.6) containing the higher order derivatives, known from the lecture of analysis. Its application, however, is quite cumbersome in practice, especially if derivatives of high order are needed. Wherever possible, it is much simpler to solve the problem using the well-known one-variable Taylor formulas—in our case those for sine and cosine functions. By expanding these functions around zero up to the fourth order inclusive (i.e., leaving the terms containing at most x^4), one has

$$\begin{aligned}\sin x &= \frac{x}{1!} - \frac{x^3}{3!} + O(x^5),\\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^5).\end{aligned} \tag{6.3.1}$$

The symbol $O(x^5)$ denotes a certain small quantity of the order of at least x^5 and thus satisfying the condition:

$$\lim_{x\to 0} \frac{O(x^5)}{x^4} = 0. \tag{6.3.2}$$

The reader might object at this point, noting that the literal use of Taylor's formula in the first of the equations (6.3.1) requires there to be $O(x^4)$ rather than $O(x^5)$. However, we know that sine is an odd function, and therefore, its expansion around zero contains only odd powers of x. The subsequent omitted term must then be of order x^5 and not x^4. The same applies to the cosine function which is even and its expansion has only even powers of x. Instead of $O(x^5)$, one could equally write $O(x^6)$. For some reasons, however, we prefer the symmetrical form of formulas (6.3.1).

These expressions can be used to find the needed expansions around the point $\pi/4$ if they are rewritten with the replacement of the variable x with $x - \pi/4 + \pi/4$ and likewise y with $y - \pi/4 + \pi/4$ and with the exploitation of the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$. The expression for the sine function reads

$$\begin{aligned}\sin x &= \sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right) = \sin\left(x - \frac{\pi}{4}\right)\cos\frac{\pi}{4} + \cos\left(x - \frac{\pi}{4}\right)\sin\frac{\pi}{4}\\ &= \cos\frac{\pi}{4}\left[\frac{(x-\pi/4)}{1!} - \frac{(x-\pi/4)^3}{3!} + O\left(\left(x - \frac{\pi}{4}\right)^5\right)\right]\\ &\quad + \sin\frac{\pi}{4}\left[1 - \frac{(x-\pi/4)^2}{2!} + \frac{(x-\pi/4)^4}{4!} + O\left(\left(x - \frac{\pi}{4}\right)^5\right)\right]\\ &= \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right.\\ &\quad \left. + \frac{1}{24}\left(x - \frac{\pi}{4}\right)^4 + O\left(\left(x - \frac{\pi}{4}\right)^5\right)\right],\end{aligned} \tag{6.3.3}$$

and that for the cosine

$$\begin{aligned}
\cos y &= \cos\left(y-\frac{\pi}{4}+\frac{\pi}{4}\right) = \cos\left(y-\frac{\pi}{4}\right)\cos\frac{\pi}{4} - \sin\left(y-\frac{\pi}{4}\right)\sin\frac{\pi}{4} \\
&= \cos\frac{\pi}{4}\left[1-\frac{(y-\pi/4)^2}{2!}+\frac{(y-\pi/4)^4}{4!}+O\left(\left(y-\frac{\pi}{4}\right)^5\right)\right] \\
&\quad -\sin\frac{\pi}{4}\left[\frac{(y-\pi/4)}{1!}-\frac{(y-\pi/4)^3}{3!}+O\left(\left(y-\frac{\pi}{4}\right)^5\right)\right] \\
&= \frac{\sqrt{2}}{2}\left[1-\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(y-\frac{\pi}{4}\right)^2+\frac{1}{6}\left(y-\frac{\pi}{4}\right)^3\right. \\
&\quad \left.+\frac{1}{24}\left(y-\frac{\pi}{4}\right)^4+O\left(\left(y-\frac{\pi}{4}\right)^5\right)\right]. \qquad (6.3.4)
\end{aligned}$$

It should be noted that these expressions already comprise *all* powers of $(x - \pi/4)$ or $(y - \pi/4)$ and not only *odd* or *even*, what was mentioned earlier. There is nothing wrong with that. The graphs of the sine and cosine functions have a symmetry property neither with respect to the point $(\pi/4, \sqrt{2}/2)$ nor to the straight line $x = \pi/4$ (as opposed to the point $(0, 0)$ and the line $x = 0$).

To expand the function $f(x, y) = \sin x \cos y$ up to the fourth order, expressions (6.3.3) and (6.3.4) are simply multiplied, leaving only terms of at most fourth degree in the products of factors $(x - \pi/4)$ and $(y - \pi/4)$.

It is convenient to introduce in the space $\mathbb{R}^2$ the vector $h = [x - \pi/4, y - \pi/4]$. Then the omitted terms can be collectively denoted with the symbol $O(h^5)$, and as a consequence, the desired formula will take the form:

$$\begin{aligned}
f(x,y) = \frac{1}{2}\Bigg[&1+\left(x-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{2}\left(y-\frac{\pi}{4}\right)^2 \\
&-\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)-\frac{1}{6}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{6}\left(y-\frac{\pi}{4}\right)^3 \\
&+\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)^2 \\
&+\frac{1}{24}\left(x-\frac{\pi}{4}\right)^4+\frac{1}{24}\left(y-\frac{\pi}{4}\right)^4+\frac{1}{6}\left(x-\frac{\pi}{4}\right)^3\left(y-\frac{\pi}{4}\right) \\
&+\frac{1}{6}\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)^3+\frac{1}{4}\left(x-\frac{\pi}{4}\right)^2\left(y-\frac{\pi}{4}\right)^2\Bigg]+O(h^5).
\end{aligned} \qquad (6.3.5)$$

As mentioned at the beginning, this expression could also be obtained directly from the two-variable Taylor formula by differentiation. In our case, it would be the easiest to make use of the following general form (6.0.4):

$$f(x_1, x_2, \ldots, x_k) = \sum_{l=0}^{n} \frac{1}{l!} \left[\sum_{i=1}^{k} \underbrace{(x_i - x_{i0})}_{h_i} \frac{\partial}{\partial \xi_i} \right]^l f(\underbrace{\xi_1, \xi_2, \ldots, \xi_k}_{\xi}) \Big|_{\xi = x_0}$$
$$+ O(h^{n+1}), \tag{6.3.6}$$

in which one must plug in $k = 2$, $n = 4$, $x_{10} = x_{20} = \pi/4$, $\xi_1 = \xi$, $\xi_2 = \eta$, $x_1 = x$, and $x_2 = y$, getting

$$\begin{aligned} f(x, y) = f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) &+ \frac{1}{1!} \left[\left(x - \frac{\pi}{4}\right) \frac{\partial}{\partial \xi} + \left(y - \frac{\pi}{4}\right) \frac{\partial}{\partial \eta} \right] f(\xi, \eta) \Big|_{\xi, \eta = \pi/4} \\ &+ \frac{1}{2!} \left[\left(x - \frac{\pi}{4}\right) \frac{\partial}{\partial \xi} + \left(y - \frac{\pi}{4}\right) \frac{\partial}{\partial \eta} \right]^2 f(\xi, \eta) \Big|_{\xi, \eta = \pi/4} \\ &+ \frac{1}{3!} \left[\left(x - \frac{\pi}{4}\right) \frac{\partial}{\partial \xi} + \left(y - \frac{\pi}{4}\right) \frac{\partial}{\partial \eta} \right]^3 f(\xi, \eta) \Big|_{\xi, \eta = \pi/4} \\ &+ \frac{1}{4!} \left[\left(x - \frac{\pi}{4}\right) \frac{\partial}{\partial \xi} + \left(y - \frac{\pi}{4}\right) \frac{\partial}{\partial \eta} \right]^4 f(\xi, \eta) \Big|_{\xi, \eta = \pi/4} + O(h^5). \end{aligned} \tag{6.3.7}$$

However, a technical difficulty arises here: the necessity of finding a number of higher-order derivatives.

In the approach presented earlier, based on the known expansions of elementary functions, this kind of a trouble is avoided. One needs to be aware, however, that the use of this method is limited to a certain class of functions.

Problem 2

The Taylor formula for the function:

$$f(x, y, z) = x^3 + y^3 + z^3 + xy^2 + yz^2 + zx^2 + xyz, \tag{6.3.8}$$

in the neighborhood of the point (1, 1, 1) will be written out with all nonvanishing terms included.

Solution

When the considered function is a polynomial (of one or more variables), the complete Taylor formula is most easily obtained not by performing any differentiations but simply by regrouping the terms. The word "regrouping" also ensures that the result will be exact and no small "higher order" appears. This is a characteristic of polynomials (and only of them).

Since in this exercise we are interested in the neighborhood of the point $(1, 1, 1)$, in Taylor's formula only expressions $(x-1)$, $(y-1)$, and $(z-1)$ occur, together with their products and natural powers, just as in the previous problem where one had $(x-\pi/4)$ and $(y-\pi/4)$. This effect can be achieved if anywhere in formula (6.3.8), the arguments are shifted as follows:

$$x \mapsto \underline{(x-1)}+1, \quad y \mapsto \underline{(y-1)}+1, \quad z \mapsto \underline{(z-1)}+1, \tag{6.3.9}$$

and all terms are sorted out without separating the underlined addends. For example, the expression x^3 will be written down in the form:

$$X^3 = [(x-1)+1]^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1,$$

and for the expression xy^2, one gets

$$\begin{aligned} xy^2 &= [(x-1)+1]\cdot[(y-1)+1]^2 = [(x-1)+1]\cdot[(y-1)^2+2(y-1)+1] \\ &= (x-1)(y-1)^2 + 2(x-1)(y-1) + (y-1)^2 + (x-1) + 2(y-1) + 1. \end{aligned}$$

Similarly proceeding with each term in (6.3.8), we obtain

$$\begin{aligned} f(x,y,z) &= x^3+y^3+z^3+xy^2+yz^2+zx^2+xyz = [(x-1)+1]^3 \\ &+[(y-1)+1]^3+[(z-1)+1]^3+[(x-1)+1][(y-1)+1]^2 \\ &+[(y-1)+1][(z-1)+1]^2+[(z-1)+1][(x-1)+1]^2 \\ &+[(x-1)+1][(y-1)+1][(z-1)+1] = (x-1)^3+3(x-1)^2 \\ &+3(x-1)+1+(y-1)^3+3(y-1)^2+3(y-1)+1+(z-1)^3 \\ &+3(z-1)^2+3(z-1)+1+(x-1)(y-1)^2+(y-1)^2 \\ &+2(x-1)(y-1)+2(y-1)+(x-1)+1+(y-1)(z-1)^2 \\ &+(z-1)^2+2(y-1)(z-1)+2(z-1)+(y-1)+1 \\ &+(z-1)(x-1)^2+(x-1)^2+2(z-1)(x-1)+2(x-1) \\ &+(z-1)+1+(x-1)(y-1)(z-1)+(x-1)(y-1) \end{aligned}$$

$$+ (y-1)(z-1) + (x-1)(z-1) + (x-1) + (y-1)$$
$$+ (z-1) + 1 = (x-1)^3 + (y-1)^3 + (z-1)^3 + (x-1)(y-1)^2$$
$$+ (y-1)(z-1)^2 + (z-1)(x-1)^2 + (x-1)(y-1)(z-1)$$
$$+ 4(x-1)^2 + 4(y-1)^2 + 4(z-1)^2 + 3(x-1)(y-1)$$
$$+ 3(y-1)(z-1) + 3(x-1)(z-1) + 7(x-1) + 7(y-1)$$
$$+ 7(z-1) + 7. \tag{6.3.10}$$

It is interesting to note the symmetry of the coefficients of the above expression (starting with the highest powers, they sequentially equal: 1, 1, 1; 1, 1, 1; 1; 4, 4, 4; 3, 3, 3; 7, 7, 7; 7) which is a consequence of the symmetry of the initial expression (6.3.8) while replacing: $x \mapsto y$, $y \mapsto z$, and $z \mapsto x$. These symmetries should be traced while performing the transformations, since their unexpected absence would suggest some computational error.

Naturally, the same result would be obtained with the use of formula (6.3.6) for $n = 3$, $k = 3$, $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_{10} = x_{20} = x_{30} = 1$ and $\xi_1 = \xi$, $\xi_2 = \eta$, $\xi_3 = \zeta$:

$$f(x, y, z) = f(1, 1, 1)$$
$$+ \frac{1}{1!}\left[(x-1)\frac{\partial}{\partial\xi} + (y-1)\frac{\partial}{\partial\eta} + (z-1)\frac{\partial}{\partial\zeta}\right] f(\xi, \eta, \zeta)\bigg|_{\xi,\eta,\zeta=1}$$
$$+ \frac{1}{2!}\left[(x-1)\frac{\partial}{\partial\xi} + (y-1)\frac{\partial}{\partial\eta} + (z-1)\frac{\partial}{\partial\zeta}\right]^2 f(\xi, \eta, \zeta)\bigg|_{\xi,\eta,\zeta=1}$$
$$+ \frac{1}{3!}\left[(x-1)\frac{\partial}{\partial\xi} + (y-1)\frac{\partial}{\partial\eta} + (z-1)\frac{\partial}{\partial\zeta}\right]^3 f(\xi, \eta, \zeta)\bigg|_{\xi,\eta,\zeta=1}. \tag{6.3.11}$$

All higher derivatives identically vanish—as the function f is a polynomial of the *third* degree—so the above expression is exact and the remainder of formula (6.3.6) (i.e., $O(h^{n+1})$) is strictly equal to zero.

Problem 3

The Taylor formula for the function $f(x, y, z) = \log(1 + x + y + z)$ in the neighborhood of the point $(0, 0, 0)$ will be written out up to the third order (inclusive).

Solution

This problem will be solved in two ways. In the first approach, the well-known Taylor formula for the logarithmic function (of one variable) is used, which significantly simplifies our calculations:

$$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + O(u^4). \tag{6.3.12}$$

Since we are looking for the expansion around the point (0, 0, 0), it is reasonable to put $u = x + y + z$ and to sort all terms (denoting $h = [x, y, z]$). Then one obtains

$$\begin{aligned}\log(1+x+y+z) &= (x+y+z) - \frac{(x+y+z)^2}{2} + \frac{(x+y+z)^3}{3} + O(h^4)\\ &= x+y+z - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}y^2 - xy - xz - yz + \frac{1}{3}x^3\\ &\quad + \frac{1}{3}y^3 + \frac{1}{3}z^3 + x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2\\ &\quad + 2xyz + O(h^4).\end{aligned} \tag{6.3.13}$$

In the second approach, the general formula (6.3.6) will be used, for $n = 3$, $k = 3$, $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_{10} = x_{20} = x_{30} = 0$ and $\xi_1 = \xi$, $\xi_2 = \eta$, $\xi_3 = \zeta$. In order to abbreviate the expressions, we will use the denotation $f'_\xi := \partial f/\partial\xi$, $f''_{\xi,\eta} := \partial^2 f/\partial\eta\partial\xi$, etc. exploited already in the previous exercises of this chapter, with the modification that now all derivatives will be calculated at $\xi = \eta = \zeta = 0$, which will not be especially indicated. Formula (6.3.6) will then take the form:

$$\begin{aligned}\log(1+x+y+z) &= \log(1+0+0+0)\\ &+ \frac{1}{1!}\left[x\frac{\partial}{\partial\xi} + y\frac{\partial}{\partial\eta} + z\frac{\partial}{\partial\zeta}\right]\log(1+\xi+\eta+\zeta)\Big|_{\xi,\eta,\zeta=0}\\ &+ \frac{1}{2!}\left[x\frac{\partial}{\partial\xi} + y\frac{\partial}{\partial\eta} + z\frac{\partial}{\partial\zeta}\right]^2\log(1+\xi+\eta+\zeta)\Big|_{\xi,\eta,\zeta=0}\\ &+ \frac{1}{3!}\left[x\frac{\partial}{\partial\xi} + y\frac{\partial}{\partial\eta} + z\frac{\partial}{\partial\zeta}\right]^3\log(1+\xi+\eta+\zeta)\Big|_{\xi,\eta,\zeta=0} + O(h^4)\\ &= xf'_\xi + yf'_\eta + zf'_\zeta + \frac{1}{2}x^2f''_{\xi\xi} + \frac{1}{2}y^2f''_{\eta\eta} + \frac{1}{2}y^2f''_{\eta\eta}\\ &\quad + \frac{1}{2}z^2f''_{\zeta\zeta} + xyf''_{\xi\eta} + xzf''_{\xi\zeta} + yzf''_{\eta\zeta} + \frac{1}{6}x^3f'''_{\xi\xi\xi} + \frac{1}{6}y^3f'''_{\eta\eta\eta}\\ &\quad + \frac{1}{6}z^3f'''_{\zeta\zeta\zeta} + \frac{1}{2}x^2yf'''_{\xi\xi\eta} + \frac{1}{2}xy^2f'''_{\xi\eta\eta} + \frac{1}{2}x^2zf'''_{\xi\xi\zeta} + \frac{1}{2}xz^2f'''_{\xi\zeta\zeta}\\ &\quad + \frac{1}{2}y^2zf'''_{\eta\eta\zeta} + \frac{1}{2}yz^2f'''_{\eta\zeta\zeta} + xyzf'''_{\xi\eta\zeta} + O(h^4).\end{aligned} \tag{6.3.14}$$

Now one needs to find all the derivatives. Because of the symmetry in exchanging variables ξ, η, ζ and of the linearity of the argument in the logarithm this task is quite simple and one gets

$$\begin{aligned}
&f'_\xi = f'_\eta = f'_\zeta = \left.\frac{1}{1+\xi+\eta+\zeta}\right|_{\xi,\eta,\zeta=0} = 1,\\
&f''_{\xi\xi} = f''_{\eta\eta} = f''_{\zeta\zeta} = f''_{\xi\eta} = f''_{\xi\zeta} = f''_{\eta\zeta} = -\left.\frac{1}{(1+\xi+\eta+\zeta)^2}\right|_{\xi,\eta,\zeta=0} = -1,\\
&f'''_{\xi\xi\xi} = f'''_{\eta\eta\eta} = \ldots = f'''_{\xi\eta\zeta} = \left.\frac{2}{(1+\xi+\eta+\zeta)^3}\right|_{\xi,\eta,\zeta=0} = 2.
\end{aligned} \tag{6.3.15}$$

After plugging them into formula (6.3.14) in this slightly longer way one comes to the result obtained earlier in (6.3.13).

6.4 Exercises for Independent Work

Exercise 1 Examine the existence of the second derivatives of the following functions at the point (0, 0):

(a) $f(x, y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0, \end{cases}$

(b) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{for } x^2 + y^2 \neq 0, \\ 0 & \text{for } x^2 + y^2 = 0. \end{cases}$

Answers

(a) f'' does not exist,
(b) f'' does not exist.

Exercise 2 Transform (and simplify, if possible) the differential expression L by introducing independent variables x and y and the dependent variable f, where:

(a) $L = 2u\dfrac{\partial g}{\partial u} - 2v\dfrac{\partial g}{\partial v} + \dfrac{1}{2}(u^2 - v^2)\left(\dfrac{\partial^2 g}{\partial u^2} + \dfrac{\partial^2 g}{\partial v^2}\right),$
$x = u + v,\ y = u - v,\ f = xyg,$

(b) $L = \dfrac{\partial^2 g}{\partial u^2} + \dfrac{\partial^2 g}{\partial v^2},\ u = x\cos y,\ v = x\sin y,\ f = xg.$

Answers

(a) $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$,
(b) $(x^2\partial^2 f/\partial x^2 - x\partial f/\partial x + f + \partial^2 f/\partial y^2)/x^3$.

Exercise 3 Write Taylor's formula for the function f, where

(a) $f(x, y) = e^{\sin x \cos y}$, in the neighborhood of $(0, 0)$, up to the third order (inclusive),
(b) $f(x, y, z) = \log[(z+1)\cos(x+y)+(z+x)y]$, in the neighborhood of $(0, 0, 0)$, up to the second order (inclusive).

Answers

(a) $f(x, y) = 1 + x + x^2/2 - xy^2/2 + O(h^4)$,
(b) $f(x, y, z) = z - x^2/2 - y^2/2 - z^2/2 + yz + O(h^3)$.

Chapter 7
Examining Extremes and Other Important Points

The main ideas of the present chapter are extremes of multivariable functions. We will learn how to find and investigate the critical points and how to resolve whether they correspond to maxima, minima, or saddle points. We will also look for global maxima and minima of functions defined on compact sets.

The so-called **critical** or **stationary** points of a certain function $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^N$ is an open set, are those for which the following set of equations is satisfied:

$$\frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_N) = 0, \quad \text{for } i = 1, 2, \ldots, N. \tag{7.0.1}$$

Such points can correspond to local minima, maxima, or to the so-called **saddle points** which are counterparts to the inflection points in one dimension (see Chap. 12 of Part I). To rule on the specific character of a stationary point, the matrix of the second partial derivatives should be found at a given point (assuming that the function is twice differentiable):

$$M = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \ldots & \frac{\partial^2 f}{\partial x_2 \partial x_N} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \frac{\partial^2 f}{\partial x_N \partial x_2} & \ldots & \frac{\partial^2 f}{\partial x_N^2} \end{bmatrix}. \tag{7.0.2}$$

After having substituted the coordinates of a given stationary point P, it becomes a symmetric matrix of real numbers. Now, if the matrix M is

- positive-definite, then the function f has a minimum at P,
- negative-definite, then the function f has a maximum at P,

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- indefinite, then the function f has s saddle point at P.

By **positive-definite** matrix, we mean such that for any vector $h \in \mathbb{R}^N$ and $h \neq 0$ the inequality $h^{\mathrm{T}} \cdot M \cdot h > 0$ holds. In turn the **negative-definite** matrix satisfies the inequality $h^{\mathrm{T}} \cdot M \cdot h < 0$ for any nonzero h. For the **indefinite** matrix, these signs are violated for some vectors h.

Another case may occur in which only inequality $h^{\mathrm{T}} \cdot M \cdot h \geq 0$ or $h^{\mathrm{T}} \cdot M \cdot h \leq 0$ is satisfied, but the remaining signs are correct. Then one has a **semi-definite** matrix. In this case, the criterion does not rule on extremes.

The definiteness of the matrix M may also be verified by the calculation of the following determinants called **minors**:

$$d_i = \det M_i, \quad i = 1, 2, \ldots, N, \tag{7.0.3}$$

where matrices M_i are subsequent $i \times i$ submatrices of the matrix M according to the rule:

$$(M_i)_{jk} = M_{jk}, \quad \text{for } \ j, k = 1, 2, \ldots, i. \tag{7.0.4}$$

Then,

- if $d_i > 0$ for $i = 1, 2, \ldots, N$, the matrix M is positive-definite,
- if $(-1)^i d_i > 0$ for $i = 1, 2, \ldots, N$, the matrix M is negative-definite.

The entire procedure is explained once again in more detail in Sect. 7.2.

7.1 Looking for Global Maxima and Minima of Functions on Compact Sets

Problem 1

The maximum and minimum values taken by the function:

$$f(x, y) = \frac{x^3}{2} - x^3 y^2 - x^2 y^2 - x \tag{7.1.1}$$

on the set $[-1, 1] \times [-1, 1]$ will be found.

Solution

When we are looking for extreme values of a function on a compact set—such as the square $[-1, 1] \times [-1, 1]$—they can appear inside as local extremes or on the edge of the set. These cases will be addressed one by one.

(a) *Interior* As it is known, the local extremes of a differentiable function appear at points for which all partial derivatives vanish (as we already know they are called critical or stationary points). For a function of two variables this means that tangents issued at extremes in any direction are horizontal. This condition must be fulfilled (it is a necessary condition), but does not prejudge the existence of an extreme. One can still have the so-called saddle point. An example of such a point, although not the only one, is a mountain pass. This situation is analogous to the one-variable case for which the equation $f'(x) = 0$ is met in extremes but at points of inflection too. Looking for extreme values on a compact set, one need not, however, resolve which type of point one is dealing with. It is just added to the list of "suspicious points" and the corresponding value of the function is calculated. If it is found not to be extremal, it is simply rejected without clarifying whether it was a *local* extreme (but certainly not *global*) or a saddle point. Such a procedure is much simpler than the detailed examination of the "suspicious" point.

We, therefore, start by solving the equations:

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0, \tag{7.1.2}$$

i.e.,

$$\begin{cases} \dfrac{3}{2}x^2 - 3x^2y^2 - 2xy^2 - 1 = 0, \\ -2x^3y - 2x^2y = 0 \end{cases} \tag{7.1.3}$$

on the set $]-1, 1[\times]-1, 1[$. If the second equation is rewritten in the form:

$$-2x^2(x+1)y = 0, \tag{7.1.4}$$

it is obvious that it has the following solutions: $x = 0 \;\vee\; x = -1 \;\vee\; y = 0$. Let us examine them in detail.

- $x = 0$. After plugging this value into the first of equations (7.1.3) one comes to the contradiction: $-1 = 0$. Thereby this solution must be rejected.
- $x = -1$. This value is situated at the edge of the square, so it will be analyzed in item (b).
- $y = 0$. The first equation gives

$$\frac{3}{2}x^2 = 1 \implies x = \sqrt{\frac{2}{3}} \;\vee\; x = -\sqrt{\frac{2}{3}}. \tag{7.1.5}$$

Points of coordinates $(\sqrt{2/3}, 0)$ and $(-\sqrt{2/3}, 0)$ belong to the interior of the square, so these are our "suspicious" points. The values of the function f at these points are as follows:

$$f\left(\sqrt{\frac{2}{3}}, 0\right) = -\frac{2}{3}\sqrt{\frac{2}{3}}, \quad f\left(-\sqrt{\frac{2}{3}}, 0\right) = \frac{2}{3}\sqrt{\frac{2}{3}}. \tag{7.1.6}$$

These numbers are candidates (but only *candidates* for a while) for the extreme values of the function. No other important points inside appear, so one can now continue and investigate the edge.

(b) *Edge* The square are four sides corresponding to the values $x = \pm 1$ and $y = \pm 1$.

- $x = -1$. In this case, one has

$$f(-1, y) = -\frac{1}{2} + y^2 - y^2 + 1 = \frac{1}{2}. \tag{7.1.7}$$

  The function is then constant on this edge. The largest and at the same time the smallest values are then equal to each other and amount to $1/2$.
- $x = 1$. This time we obtain the formula

$$f(1, y) = \frac{1}{2} - y^2 - y^2 - 1 = -2y^2 - \frac{1}{2}, \tag{7.1.8}$$

  which describes the parabola taking the greatest value equal to $-1/2$ at $y = 0$ and the smallest one equal to $-5/2$ for $y = \pm 1$ (since one is dealing with the interval $[-1, 1]$ only).
- $y = -1$. For this value of y, the polynomial of the third degree must be investigated:

$$f(x, -1) = -\frac{1}{2}x^3 - x^2 - x. \tag{7.1.9}$$

  This function is monotonic in the variable x because its derivative

$$\frac{d}{dx} f(x, -1) = -\frac{3}{2}x^2 - 2x - 1 \tag{7.1.10}$$

  is constantly negative, which can be easily established by calculating the discriminant

$$\triangle = 4 - 6 = -2 < 0.$$

The function (7.1.9) assumes its extremal values at the ends of the interval $[-1, 1]$: $f(-1, -1) = 1/2$, $f(1, -1) = -5/2$. Both of these points have already been covered (these are the tips of the square, i.e., points belonging simultaneously to two different sides).

- $y = 1$. In this case the identical polynomial of the third degree is obtained. Again it takes the extremal values at the ends of the interval $[-1, 1]$: $f(-1, 1) = 1/2$ and $f(1, 1) = -5/2$.

In conclusion, we can see that the minimal value of the function on the square $[-1, 1] \times [-1, 1]$ equals $-5/2$, and is achieved at the two points: $(1, -1)$ and $(1, 1)$. As for the maximal value, it equals $2/3 \cdot \sqrt{2/3}$ at the point $(-\sqrt{2/3}, 0)$. It is easy to notice by squaring both sides that the following inequality is satisfied:

$$\frac{2}{3}\sqrt{\frac{2}{3}} > \frac{1}{2}. \tag{7.1.11}$$

Hence, the points one has found, for which the value of the function is equal to $1/2$, cannot be extremes.

The graph of the function $f(x, y)$, limited to the set $[-1, 1] \times [-1, 1]$, is shown in Fig. 7.1.

Problem 2

The global maxima and minima taken by the function:

$$f(x, y) = (x^2 + y^2)\, e^{-2x^2-y^2} \tag{7.1.12}$$

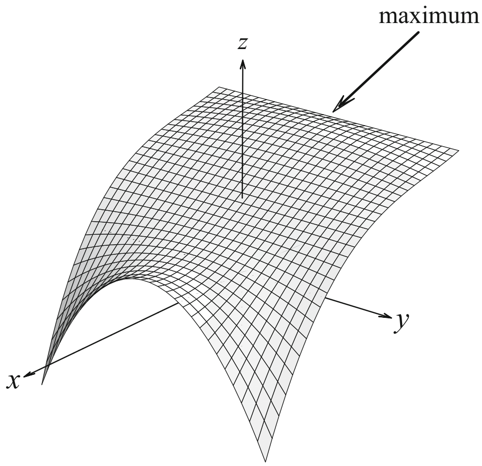

Fig. 7.1 The graph of the function (7.1.1) for $(x, y) \in [-1, 1] \times [-1, 1]$

on the set:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

will be found.

Solution

Global extremes on the closed and bounded set (i.e., the compact one) are looked for, so we can proceed as in the previous problem. First one should find the critical points of the function f lying inside the set A and then we will concentrate on the border, which is the circle with the center located at the origin and radius equal to 1.

(a) *Interior* The critical points have coordinates satisfying the set of equations:

$$\begin{cases} \dfrac{\partial f}{\partial x} = (2x - 4x(x^2 + y^2))e^{-2x^2 - y^2} = -4x\left(x^2 + y^2 - \frac{1}{2}\right)e^{-2x^2 - y^2} = 0, \\ \dfrac{\partial f}{\partial y} = (2y - 2y(x^2 + y^2))e^{-2x^2 - y^2} = -2y\left(x^2 + y^2 - 1\right)e^{-2x^2 - y^2} = 0. \end{cases} \tag{7.1.13}$$

The factor $e^{-2x^2 - y^2}$ is always positive, so it can be omitted. One is then left with two simple equations:

$$x\left(x^2 + y^2 - \frac{1}{2}\right) = 0, \quad y\left(x^2 + y^2 - 1\right) = 0, \tag{7.1.14}$$

and their solutions are given below.

- $x = 0$ and $y = 0$. This point belongs to the set A, so it should be taken into account. Leaving aside the question whether it is a saddle or extreme point, one can calculate the value of the function there: $f(0, 0) = 0$. Because the function f is nonnegative it constitutes the global minimum without a doubt.
- $x = 0$ and $x^2 + y^2 = 1$. There are two points satisfying these equations: $(0, 1)$ and $(0, -1)$. Both are situated at the edge, so they will be of interest in item (b).
- $y = 0$ and $x^2 + y^2 = 1/2$. Again there are two points satisfying the equation: $(1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 0)$, but this time they are located in the interior of the set A. We calculate then

$$f\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2e} \quad \text{and} \quad f\left(-\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2e}. \tag{7.1.15}$$

  Both these values are equal to each other, since the function f is even in the variable x.

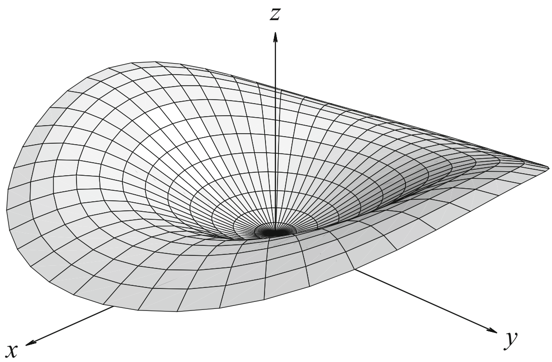

Fig. 7.2 Graph of the function (7.1.12) for $(x, y) \in A$

- $x^2 + y^2 = 1$ and $x^2 + y^2 = 1/2$. These equations are inconsistent.

(b) *Edge* At the edge of the domain A (let us denote it with ∂A) the equation $x^2 + y^2 = 1$ is fulfilled. The function f, cut down to ∂A, can be written as a function of one variable:

$$f(x, y)\big|_{\partial A} = 1 \cdot e^{-1-x^2} = \frac{1}{e}\, e^{-x^2}, \tag{7.1.16}$$

where $x \in [-1, 1]$. The largest value equal to $1/e$ is achieved at $x = 0$. There are two such points on the edge: $(0, 1)$ and $(0, -1)$. In turn, the smallest value $(1/e^2)$ is obtained for $(1, 0)$ and $(-1, 0)$.

The comparison of the numbers found in items (a) and (b) indicates that the largest value of the function f on the set A equals $1/e$, and the smallest one—0. The graph of this function, cut down to set A, is shown in Fig. 7.2.

Problem 3

The global maxima and minima taken by the function:

$$f(x, y, z) = (x^2 + y^2 + z^2)e^{-2x^2-y^2-z^2} \tag{7.1.17}$$

on the set

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

will be found.

Solution

This problem constitutes an extension of the previous case to three dimensions. The set B—a closed ball—is compact, so the way one proceeds is the same: our first concern will be critical points of the function f lying inside B, and then we will concentrate on the boundary.

(a) *Interior* The coordinates of the "suspicious" points satisfy the equations:

$$\begin{cases} \dfrac{\partial f}{\partial x} = -4x\left(x^2+y^2+z^2-\frac{1}{2}\right)e^{-2x^2-y^2-z^2} = 0, \\[2ex] \dfrac{\partial f}{\partial y} = -2y\left(x^2+y^2+z^2-1\right)e^{-2x^2-y^2} = 0, \\[2ex] \dfrac{\partial f}{\partial z} = -2z\left(x^2+y^2+z^2-1\right)e^{-2x^2-y^2} = 0. \end{cases} \tag{7.1.18}$$

The exponential factor $e^{-2x^2-y^2-z^2}$ can be omitted because it never vanishes, and one gets three polynomial equations:

$$\begin{cases} x\left(x^2+y^2+z^2-\dfrac{1}{2}\right) = 0, \\ y\left(x^2+y^2+z^2-1\right) = 0, \\ z\left(x^2+y^2+z^2-1\right) = 0, \end{cases}$$

the solutions of which are found below. The following possibilities should be explored:

- $x = 0$, $y = 0$, and $z = 0$. This point belongs to the set B. It is not known whether it is a saddle point or a local extreme, but for the time being it is enough to calculate the value of the function: $f(0,0,0) = 0$.
- $x = 0$, $y = 0$, and $x^2+y^2+z^2 = 1$. There are two points satisfying these equations: $(0,0,1)$ and $(0,0,-1)$. However, they are situated on the edge (∂B), so they are left for consideration in item (b).
- $x = 0$, $x^2+y^2+z^2 = 1$, and $z = 0$. Again two points satisfy these conditions. They are: $(0,1,0)$ and $(0,-1,0)$ so they belong to ∂B too.
- $x = 0$, $x^2+y^2+z^2 = 1$, and $x^2+y^2+z^2 = 1$. All points satisfying these equations—i.e., the entire circle lying in a plane yz—belong to the border.
- $x^2+y^2+z^2 = 1/2$, $y = 0$, and $z = 0$. This time two points lying in the interior of B are found: $(1/\sqrt{2},0,0)$ and $(-1/\sqrt{2},0,0)$. Thus

$$f\left(\frac{1}{\sqrt{2}},0,0\right) = \frac{1}{2e} \quad \text{and} \quad f\left(-\frac{1}{\sqrt{2}},0,0\right) = \frac{1}{2e}. \tag{7.1.19}$$

- All other cases lead to contradictory equations.

(b) *Edge* At the edge of the area B, the equation defining the sphere ($x^2+y^2+z^2=1$) is satisfied. This ensures that the function f, reduced to ∂B, can be written as a function of one variable:

$$f(x, y, z)\big|_{\partial B} = 1 \cdot e^{-1-x^2} = \frac{1}{e}\, e^{-x^2}, \tag{7.1.20}$$

where $x \in [-1, 1]$. The highest value, which is equal to $1/e$, is taken by the expression at $x = 0$. On ∂B there is an infinite number of such points—all satisfying the equation of the circle: $y^2 + z^2 = 1$. In turn the smallest value (equal to $1/e^2$) is obtained for $(1, 0, 0)$ and $(-1, 0, 0)$.

The results derived in items (a) and (b) show that the largest value of the function on the entire set B equals $1/e$, and the smallest one—0.

At this point, it should be stressed that, in a more general case, the cut of the domain to a *two-dimensional* border will lead to the function of *two* variables. In such a situation, it would be necessary to perform a separate and full examination of this reduced function in a manner similar to that of the previous problems. One can treat it as an independent exercise. Still another case can be met when the equation for the edge is so complex that it is unfeasible (or unviable) to explicitly find the formula for such a reduced function. In this case, one should look for the so-called conditional extremes. Fortunately, there are mathematical procedures to cope with this trouble and they will be examined in Part III of this book series.

7.2 Examining Local Extremes and Saddle Points of Functions

Problem 1

The critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by the formula:

$$f(x, y) = 3x^3 - y^2x + 8y \tag{7.2.1}$$

will be found and examined.

Solution

The critical or stationary points can be found as usual as the solutions of the equations:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 9x^2 - y^2 = 0, \\ \dfrac{\partial f}{\partial y} = -2xy + 8 = 0. \end{cases} \tag{7.2.2}$$

The former gives $y = 3x$ or $y = -3x$, but this second case must be rejected, since after plugging it into the latter equation of (7.2.2) one comes to a contradiction: $3x^2 + 4 = 0$. The first case, $y = 3x$, leads to

$$3x^2 - 4 = 0 \implies x = \pm\frac{2\sqrt{3}}{3}. \tag{7.2.3}$$

The function has, therefore, two critical points:

$$\left(\frac{2\sqrt{3}}{3}, 2\sqrt{3}\right) \quad \text{and} \quad \left(-\frac{2\sqrt{3}}{3}, -2\sqrt{3}\right), \tag{7.2.4}$$

to be examined. In particular, one would like to establish whether one is dealing with extremes or saddle points. As we remember in the case of one-variable function the existence of solutions of the equation $f'(x) = 0$ did not imply the existence of extremes: they might be points of inflection equally well. A tool allowing to resolve this issue was the sign of the second derivative calculated at the "suspicious" points: negative in a maximum, positive in a minimum. The vanishing of $f''(x)$ together with the condition $f'''(x) \neq 0$ led to the conclusion that it was a point of inflection.

As it turns out and as it was signaled in the theoretical introduction at the beginning of the chapter, in the case of many-variables function, the second derivative remains a useful tool, although the issue is much more complicated. First of all, one knows that f'' is now not the number, but the entire matrix whose elements are second partial derivatives. For example, for a function $f(x_1, x_2, \ldots, x_N)$ one would have

$$f'' = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_N} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_N} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 f}{\partial x_N \partial x_1} & \dfrac{\partial^2 f}{\partial x_N \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_N^2} \end{bmatrix}. \tag{7.2.5}$$

What could now mean the inequality $f'' > 0$ ($f'' < 0$)? Or stating it a little differently: what should be now positive (negative)? To answer this question, one must try to understand—first for the one-variable function—why, in general, the sign of the second derivative is to provide the information on extremes. The required explanation constitutes the Taylor formula known to us from Part I (see Sect. 11.2):

$$f(x_0+h)-f(x_0)=\frac{h}{1!}\,f'(x_0)+\frac{h^2}{2!}\,f''(x_0)+\ldots+\frac{h^n}{n!}\,f^{(n)}(x_0)+R_n(x_0,h). \tag{7.2.6}$$

Imagine that we are standing at x_0 and we would like to move a little bit to the left or to the right by a very small value h. If at x_0 the function has maximum, i.e., we are "at the top of the hill," then certainly we move down. The function f should lose its value, and therefore, the sum on the right-hand side must be negative. For sufficiently small values of h, it is the first term that decides about this sign (the subsequent ones contain higher powers of h). However, if x_0 is a stationary point under study, $f'(x_0)$ vanishes by definition. Therefore, the nonzero term (and hence the decisive one) is that containing the second derivative, and its sign is the same as that of $f''(x_0)$, as the coefficient $h^2/2!$ is positive. For a maximum at the point x_0, one thus obtains the condition $f''(x_0) < 0$.

If, on the other hand, we were standing at a minimum, i.e., "at the very bottom of the valley," then moving by h in any direction, we climb up. The function f should then gain in value. This entails the condition $f''(x_0) > 0$. In a theoretically possible situation that $f''(x_0) = 0$ the sign will be determined by the following terms of (7.2.6), but it will not be analyzed here.

For a function of many (N) variables, one has a similar situation. At a "suspicious" point $f' = 0$ and the first term of Taylor's formula again vanishes, so that the sign of $f(x_0+h) - f(x_0)$ comes from the following term (it is assumed here that $f''(x_0) \neq 0$). This expression is a quadratic form well known from algebra (the irrelevant factor $1/2$ is omitted):

$$h^{\mathrm{T}}\cdot f''(x_0)\cdot h,\quad \text{where}\quad h=\begin{bmatrix} h_1\\ h_2\\ \vdots\\ h_N\end{bmatrix}. \tag{7.2.7}$$

Thereby the answer to the earlier question has been found: it is the quadratic form $h^{\mathrm{T}}\cdot f''(x_0)\cdot h$ that must be positive (negative) if at the point x_0 the function has a minimum (maximum).

This conclusion will now be applied to our test function. To start one has to find the matrix of second derivatives at the first stationary point:

$$f''\left(\frac{2\sqrt{3}}{3},2\sqrt{3}\right)=\begin{bmatrix} 18x & -2y\\ -2y & -2x\end{bmatrix}\Bigg|_{\left(2\sqrt{3}/3,2\sqrt{3}\right)}=\begin{bmatrix} 12\sqrt{3} & -4\sqrt{3}\\ -4\sqrt{3} & -4\sqrt{3}/3\end{bmatrix}, \tag{7.2.8}$$

and then one creates the quadratic form:

$$h^{\mathrm{T}}\cdot f''\cdot h=\begin{bmatrix} h_x, h_y\end{bmatrix}\begin{bmatrix} 12\sqrt{3} & -4\sqrt{3}\\ -4\sqrt{3} & -4\sqrt{3}/3\end{bmatrix}\begin{bmatrix} h_x\\ h_y\end{bmatrix}$$

$$= 4\sqrt{3}\left[h_x, h_y\right]\begin{bmatrix} 3h_x - h_y \\ -h_x + h_y/3 \end{bmatrix}$$
$$= 4\sqrt{3}\left(-3h_x^2 - h_x h_y - h_x h_y + \frac{1}{3}h_y^2\right)$$
$$= 4\sqrt{3}\left[-3\left(h_x + \frac{1}{3}h_y\right)^2 + \frac{2}{3}h_y^2\right]. \tag{7.2.9}$$

In order to make the analysis of the sign easier, $h^{\mathrm{T}} \cdot f'' \cdot h$ has been written in the form of square terms (this is called the "diagonal" form, the reader should be familiar with, from the lecture of algebra). The coefficients on these square terms have different signs, which means that the considered expression can be made either positive or negative, depending on the direction of the vector h. Well, one can see that if we are moving along the vector $[-1, 3]$ for which $h_x + h_y/3 = 0$, (7.2.9) is positive, which means that the value of the function is increasing. Therefore, we are climbing up. But if we moved along the vector $[1, 0]$, then in (7.2.9) only the negative term would be preserved, which would mean that we are coming down. This is a typical situation for the so-called saddle point, which, with the use of the mountain terminology, could be called "a pass." The situation one is dealing with is shown in Fig. 7.3.

The question arises whether one can always reduce the quadratic form to fully diagonal shape and establish what type of a point one has. The answer to the first part of this question is affirmative. The appropriate algebraic theorem says that any quadratic form can be transformed into a diagonal one (for example, with the use of the so-called Lagrange method). However, the problem lies in the fact that these diagonal terms may be fewer than the dimension of the space (which means that some of them equal zero). Then, to determine whether one is dealing with extremes

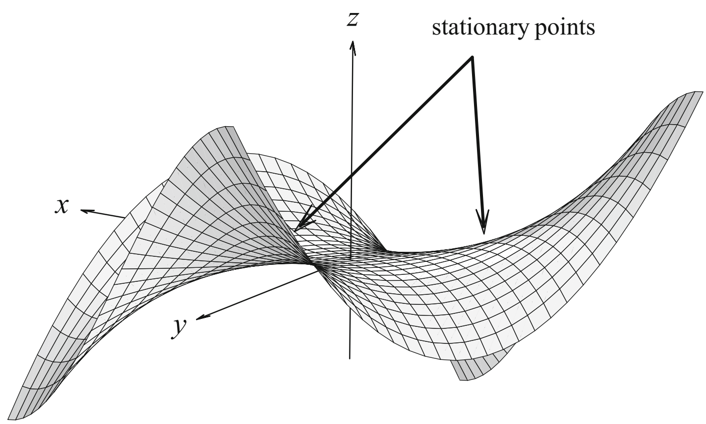

Fig. 7.3 The stationary points of the function (7.2.1)

or saddle points, it can turn out to be desirable (but not necessarily) to refer to subsequent terms of the Taylor formula.

One should add at this place that, in addition to diagonalization, we have at our disposal another method (mentioned in the theoretical introduction) to determine the sign of a quadratic form $h^{\mathrm{T}} \cdot M \cdot h$, M being a symmetric matrix:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1N} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2N} \\ m_{31} & m_{32} & m_{33} & \cdots & m_{3N} \\ \dots & \dots & \dots & \dots & \dots \\ m_{N1} & m_{N2} & m_{N3} & \cdots & m_{NN} \end{bmatrix}. \tag{7.2.10}$$

To this end we calculate the determinants of square submatrices obtained from M as follows:

$$M_1 = \begin{bmatrix} m_{11} \end{bmatrix}, \quad M_2 = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$
$$M_3 = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}, \quad \dots, \quad M_N = M. \tag{7.2.11}$$

Such determinants are called *minors*. If one denotes $d_i = \det M_i$, for $i = 1, \dots, N$, the appropriate theorem of the algebra says that

- if all these minors are positive, the form is positively defined (and therefore, the function f has a minimum at a given point),
- if, on the other hand, for $i = 1, 2, \dots, N$ the conditions $(-1)^i d_i > 0$ are met (which means that odd minors are negative, and even ones positive), the quadratic form is negatively defined (therefore, f has a maximum).

If these signs were violated, one would have a saddle point. The case of some determinants equal to zero (and the remaining ones having correct signs) requires a separate study, which will not be addressed here.

For the matrix obtained in this exercise

$$M = \begin{bmatrix} 12\sqrt{3} & -4\sqrt{3} \\ -4\sqrt{3} & -4\sqrt{3}/3 \end{bmatrix} \tag{7.2.12}$$

it is sufficient to check only two minors:

$$d_1 = 12\sqrt{3} > 0,$$
$$d_2 = 12\sqrt{3} \cdot \left(-\frac{4\sqrt{3}}{3} \right) - \left(-4\sqrt{3} \right)^2 = -96 < 0. \tag{7.2.13}$$

The signs are "incorrect," which again entails the conclusion drawn earlier: at the investigated point, the function has a saddle point. The value of the function can be easily calculated to be $32\sqrt{3}/3$.

Now let us repeat this analysis for the second critical point. We find

$$f''\left(\frac{-2\sqrt{3}}{3}, -2\sqrt{3}\right) = \begin{bmatrix} 18x & -2y \\ -2y & -2x \end{bmatrix}\Bigg|_{\left(-2\sqrt{3}/3,-2\sqrt{3}\right)} = \begin{bmatrix} -12\sqrt{3} & 4\sqrt{3} \\ 4\sqrt{3} & 4\sqrt{3}/3 \end{bmatrix} \tag{7.2.14}$$

and the quadratic form is

$$\begin{aligned} h^{\mathrm{T}} \cdot f'' \cdot h &= \left[h_x, h_y\right]\begin{bmatrix} -12\sqrt{3} & 4\sqrt{3} \\ 4\sqrt{3} & 4\sqrt{3}/3 \end{bmatrix}\begin{bmatrix} h_x \\ h_y \end{bmatrix} \\ &= -4\sqrt{3}\left[h_x, h_y\right]\begin{bmatrix} 3h_x - h_y \\ -h_3 + h_y/3 \end{bmatrix} \\ &= -4\sqrt{3}\left(-3h_x^2 - h_x h_y - h_x h_y + \frac{1}{3}h_y^2\right) \\ &= 4\sqrt{3}\left[3\left(h_x + \frac{1}{3}h_y\right)^2 - \frac{2}{3}h_y^2\right]. \end{aligned} \tag{7.2.15}$$

The signs one has again indicate a saddle point (where $f(-2\sqrt{3}/3, -2\sqrt{3}) = -32\sqrt{3}/3)$. The same result would be found by examining minors of the matrix:

$$\begin{bmatrix} -12\sqrt{3} & 4\sqrt{3} \\ 4\sqrt{3} & 4\sqrt{3}/3 \end{bmatrix}. \tag{7.2.16}$$

They are equal to

$$\begin{aligned} d_1 &= -12\sqrt{3} < 0, \\ d_2 &= -12\sqrt{3} \cdot \frac{4\sqrt{3}}{3} - \left(4\sqrt{3}\right)^2 = -96 < 0. \end{aligned} \tag{7.2.17}$$

The existence of this second saddle point could be anticipated, thanks to the observation that $f(-x, -y) = f(x, y)$. The function f then has no local extremes but two saddle points.

Problem 2

The critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by the formula:

$$f(x, y) = (x^2 + 4y^2)e^{-2xy}, \tag{7.2.18}$$

will be found and examined.

Solution

We begin with identifying the stationary points of the function f by solving the equations:

$$\begin{cases} \dfrac{\partial f}{\partial x} = [2x - 2y(x^2 + 4y^2)]e^{-2xy} = 0, \\ \dfrac{\partial f}{\partial y} = [8y - 2x(x^2 + 4y^2)]e^{-2xy} = 0. \end{cases} \tag{7.2.19}$$

The exponential factor is always positive, so the above set is equivalent to

$$\begin{cases} x - y(x^2 + 4y^2) = 0, \\ 4y - x(x^2 + 4y^2)] = 0. \end{cases} \tag{7.2.20}$$

One obvious solution constitutes the point (0, 0) and accordingly it must be a desired stationary point. The coordinates of the other points—in so far as they exist—must be (both) different from zero (from the system (7.2.20), it is visible that $x = 0$ implies also $y = 0$ and vice versa) and it is assumed below. By eliminating the factor $(x^2 + 4y^2)$, one gets

$$x^2 = 4y^2, \tag{7.2.21}$$

which leads to $x = 2y$ or $x = -2y$. In the first case, any of the equations (7.2.20) gives

$$2y - y((2y)^2 + 4y^2) = 0 \implies 2y(1 - 4y^2) = 0, \tag{7.2.22}$$

and therefore, $y = \pm 1/2$ and $x = \pm 1$. In this way, two other critical points have been found: $(1, 1/2)$ and $(-1, -1/2)$.

The case $x = -2y$ is not interesting because it leads to the contradictory equation (assuming $y \neq 0$):

$$-2y - y((2y)^2 + 4y^2) = 0 \implies -2y(1 + 4y^2) = 0. \tag{7.2.23}$$

It does not lead to any new stationary point. In conclusion, one sees that the function under consideration has ultimately three "suspicious" points. Now we are going to write down the expression for the matrix of second derivatives:

$$f'' = e^{-2xy} \begin{bmatrix} 2 - 8xy + 4x^2y^2 + 16y^4 & -6x^2 - 24y^2 + 4x^3y + 16xy^3 \\ -6x^2 - 24y^2 + 4x^3y + 16xy^3 & 8 - 32xy + 4x^4 + 16x^2y^2 \end{bmatrix}, \tag{7.2.24}$$

and examine all these points in turn.

- $(x, y) = (0, 0)$. At this point, the matrix of second derivative becomes

$$f''(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \tag{7.2.25}$$

and apparently is positively defined. The appropriate minors can be found mentally: $d_1 = 2$, $d_2 = 16$, and are both positive. This means that the function has a minimum at the point $(0, 0)$ (and $f(0, 0) = 0$). The same conclusion might be immediately drawn by writing the appropriate quadratic form:

$$h^{\mathrm{T}} \cdot f''(0, 0) \cdot h = \begin{bmatrix} h_x, h_y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix} = 2h_x^2 + 8h_y^2 > 0. \tag{7.2.26}$$

- $(x, y) = (1, 1/2)$. The matrix of second derivatives has now the form:

$$f''\left(1, \frac{1}{2}\right) = \begin{bmatrix} 0 & -8/e \\ -8/e & 0 \end{bmatrix} \tag{7.2.27}$$

and the values of minors are: $d_1 = 0$, $d_2 = -64/e^2 < 0$. The zero value of the former and "bad" sign of the latter indicate nonexistence of the extreme. If we prefer to use the quadratic form:

$$h^{\mathrm{T}} \cdot f''\left(1, \frac{1}{2}\right) \cdot h = \begin{bmatrix} h_x, h_y \end{bmatrix} \begin{bmatrix} 0 & -8/e \\ -8/e & 0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix} = -\frac{16}{e} h_x h_y, \tag{7.2.28}$$

it is clear that it can have both positive (e.g., when $h_x = -h_y$) and negative values (e.g., when $h_x = h_y$). It must be then a saddle point of the function f. The value of the function at that point is $f(1, 1/2) = 2/e$.
- $(x, y) = (-1, -1/2)$. It turns out that the matrix of second derivatives has again the same form:

$$f''\left(-1, -\frac{1}{2}\right) = \begin{bmatrix} 0 & -8/e \\ -8/e & 0 \end{bmatrix}, \tag{7.2.29}$$

so our conclusions will be the same: $(-1, -1/2)$ is the saddle point and $f(-1, -1/2) = 2/e$.

The function f is sketched in Fig. 7.4.

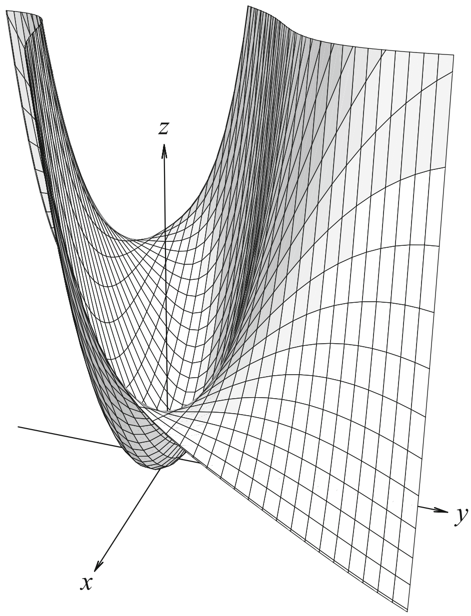

Fig. 7.4 The graph of the function (7.2.18)

Problem 3

The stationary points of the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by the formula:

$$f(x, y, z) = 16xy + z(2x + y + 2z)^2 \tag{7.2.30}$$

will be found and examined.

Solution

The developed algorithm will be followed. The stationary points are found by solving the set of equations:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 16y + 4z(2x + y + 2z) = 0, \\ \dfrac{\partial f}{\partial y} = 16x + 2z(2x + y + 2z) = 0, \\ \dfrac{\partial f}{\partial z} = (2x + y + 2z)^2 + 4z(2x + y + 2z) = 0. \end{cases} \tag{7.2.31}$$

From the first two, one can infer that $y = 2x$ and after plugging that into the third one gets

$$(4x + 2z)^2 + 4z(4x + 2z) = 0 \Longrightarrow (4x + 2z)(4x + 6z) = 0$$
$$\Longrightarrow z = -2x \ \vee z = -\frac{2}{3}x. \qquad (7.2.32)$$

Thus the following cases arise:

- $y = 2x \ \wedge \ z = -2x$. After having inserted these relations into the second of equations (7.2.31), one obtains $x = 0$. The first critical point is then: $(0, 0, 0)$.
- $y = 2x \ \wedge \ z = -2x/3$. Now the second of equations (7.2.31)—after some simplifications—yields the condition $x(x-9/2) = 0$. The case $x = 0$ has already been considered above, so it remains $x = 9/2$, which implies $y = 9$ and $z = -3$. Thus, the second critical point has the coordinates: $(9/2, 9, -3)$.

Both these points are examined below. The general expression for the second derivative has the form:

$$f'' = \begin{bmatrix} 8z & 16 + 4z & 8x + 4y + 16z \\ 16 + 4z & 2z & 4x + 2y + 8z \\ 8x + 4y + 16z & 4x + 2y + 8z & 16x + 8y + 24z \end{bmatrix}. \qquad (7.2.33)$$

Plugging in the coordinates of the first stationary point we get

$$f''(0, 0, 0) = \begin{bmatrix} 0 & 16 & 0 \\ 16 & 0 & 0 \\ 0 & 0 & 0, \end{bmatrix} \qquad (7.2.34)$$

and it is easy to observe that $d_1 = 0$, $d_2 = -16^2 = -256$, and $d_3 = 0$. This result indicates that one is dealing with a saddle point, which can be easily confirmed by considering the quadratic form:

$$h^{\mathrm{T}} \cdot f''(0, 0, 0) \cdot h = \begin{bmatrix} h_x, h_y, h_z \end{bmatrix} \begin{bmatrix} 0 & 16 & 0 \\ 16 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = 32 h_x h_y. \qquad (7.2.35)$$

A similar expression appeared already in the previous exercise and we know that it does not have a definite sign (for $h_x = h_y \neq 0$ the form is positive, and for $h_x = -h_y \neq 0$ it is negative). As it has been suspected, the function f has a saddle point here and its value is $f(0, 0, 0) = 0$.

It is worth noting that the result (7.2.35) could also be given the diagonal form:

$$h^{\mathrm{T}} \cdot f''(0, 0, 0) \cdot h = 8(h_x + h_y)^2 - 8(h_x - h_y)^2. \qquad (7.2.36)$$

The absence of the third term in this formula means that the quadratic form is degenerated, and the "saddle" itself is two-dimensional. The infinitesimal shift by the vector (0, 0, 1) does not change the value of (7.2.36). This fact was reflected by vanishing of the minor d_3.

For the point $(9/2, 9, -3)$, one obtains

$$f''\left(\frac{9}{2}, 9, -3\right) = \begin{bmatrix} -24 & 4 & 24 \\ 4 & -6 & 12 \\ 24 & 12 & 72 \end{bmatrix}, \tag{7.2.37}$$

and the needed minors have the following values: $d_1 = -24 < 0$, $d_2 = 128 > 0$, and $d_3 = 18432 > 0$. By comparing them with general conditions for an extreme set forth in Problem 1, we see that the signs are "wrong." Thus the function here has a saddle point. This result could be obtained equally well from the quadratic form:

$$\begin{aligned} h^{\mathrm{T}} \cdot f''\left(\frac{9}{2}, 9, -3\right) \cdot h &= \left[h_x, h_y, h_z\right] \begin{bmatrix} -24 & 4 & 24 \\ 4 & -6 & 12 \\ 24 & 12 & 72 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \\ &= -24h_x^2 - 6h_y^2 + 72h_z^2 + 8h_x h_y + 48h_x h_z + 24h_y h_z, \end{aligned} \tag{7.2.38}$$

if it is rewritten as

$$\begin{aligned} h^{\mathrm{T}} \cdot f''\left(\frac{9}{2}, 9, -3\right) \cdot h &= -24\left(h_x^2 - \frac{1}{3}h_x h_y - 2h_x h_z\right) - 6h_y^2 + 72h_z^2 + 24h_y h_z \\ &= -24\left(h_x - \frac{1}{6}h_y - h_z\right)^2 - \frac{16}{3}h_y^2 + 96h_z^2 + 32h_y h_z \\ &= -24\left(h_x - \frac{1}{6}h_y - h_z\right)^2 + 96\left(h_z^2 + \frac{1}{3}h_y h_z\right) - \frac{16}{3}h_y^2 \\ &= -24\left(h_x - \frac{1}{6}h_y - h_z\right)^2 + 96\left(h_z + \frac{1}{6}h_y\right)^2 - 8h_y^2. \end{aligned} \tag{7.2.39}$$

If one now chooses a vector h such that $h_y = 0$ and additionally the expression in the first brackets vanishes (i.e., $h_z = h_x$), one obtains

$$h^{\mathrm{T}} \cdot f''\left(\frac{9}{2}, 9, -3\right) \cdot h = 96h_z^2 > 0. \tag{7.2.40}$$

On the other hand, if $h_x = h_z = 0$, and $h_y \neq 0$, then

$$h^{\mathrm{T}} \cdot f''\left(\frac{9}{2}, 9, -3\right) \cdot h = -6h_y^2 < 0. \tag{7.2.41}$$

It is obvious that the sign of the quadratic form is indefinite, which means that one has a saddle point. At the end, we calculate the value of the function: $f(9/2, 9, -3) = 216$.

7.3 Exercises for Independent Work

Exercise 1 Find global maxima and minima of the function f on the set X, for

(a) $f(x, y) = x^3 - x^2y^2 + 2x^2y - x$, $X = [-2, 2] \times [-2, 2]$,
(b) $f(x, y, z) = (x + y + 2z)^2 - 16xyz$, $X = [-1, 1] \times [-1, 1] \times [-1, 1]$.

Answers

(a) $f_{\min} = -38$, $f_{\max} = 10$,
(b) $f_{\min} = -16$, $f_{\max} = 32$.

Exercise 2 Find and examine critical points of the functions:

(a) $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y) = xy^2 + x^2y - x - y$,
(b) $f : \mathbb{R}^3 \to \mathbb{R}$, where $f(x, y, z) = \frac{1}{4}x - y + z + \frac{1}{2}xy + x^2z - xz^2$,
(c) $f : \mathbb{R}^3 \to \mathbb{R}$, where $f(x, y, z) = x^3 + x^2 - y^2 + 3zy + z^2 - z^3$.

Answers

(a) No local extremes; saddle points: $(1/\sqrt{3}, 1/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3})$,
(b) No local extremes; saddle point at $(2, -59/8, 5/4)$,
(c) Maximum at the point $(-2/3, 13/4, 13/6)$; saddle points: $(0, 0, 0)$, $(-2/3, 0, 0)$, and $(0, 13/4, 13/6)$.

Chapter 8
Examining Implicit and Inverse Functions

The present chapter is devoted to the investigation of the so-called implicit functions and inverse functions. We will learn how to check if a given equation defines the implicit or inverse function and we will become acquainted with the problem of finding extremes of such functions.

The **implicit function theorem** has the following form:

Let a given function F be defined on an open set $U \subset \mathbb{R}^{n+m}$ with the values in $\mathbb{R}^m$. Let us denote the first n variables $x_1, x_2, \ldots, x_n$ with the aggregate symbol X, and further m variables $y_1, y_2, \ldots, y_m$ with the symbol Y. Then the equation

$$F(X, Y) = 0 \tag{8.0.1}$$

can eventually be solved for m unknowns called y_i. Now, if the function F is of the class C^1 and at a given point (X_0, Y_0), for which

$$F(X_0, Y_0) = 0, \tag{8.0.2}$$

the following condition is met:

$$\det \left.\frac{\partial F}{\partial Y}\right|_{(X_0, Y_0)} \neq 0. \tag{8.0.3}$$

Then, on a certain neighborhood of this point, the Eq. (8.0.1) defines Y as a function of X and the following formula holds:

$$\frac{\partial Y}{\partial X} = -\left(\frac{\partial F}{\partial Y}\right)^{-1} \frac{\partial F}{\partial X}. \tag{8.0.4}$$

The **inverse function** theorem may be formulated as follows:

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T. Radożycki, *Solving Problems in Mathematical Analysis, Part II*,
Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-36848-7_8

Let the function $F : \mathcal{U} \to \mathbb{R}^n$, where $\mathcal{U} \subset \mathbb{R}^n$ is an open set, be of the class C^1. Assume that for a certain $X_0 \in \mathcal{U}$ the condition $\det F'(X_0) \neq 0$ is fulfilled. Then, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of the point X_0 such that the function $F : \mathcal{V} \to F(\mathcal{V})$ is invertible. The inverse function is also of class C^1 and its derivative at $F(X_0)$ is given by the formula:

$$(F^{-1})'(F(X_0)) = F'(X_0)^{-1}. \tag{8.0.5}$$

A reversible function of the class C^1 such that its inverse is of the class C^1 too is called a **diffeomorphism**.

8.1 Investigating the Existence and Extremes of Implicit Functions

Problem 1

It will be assessed whether the equation $F(x, y) = 0$, where the function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is given by the formula:

$$F(x, y) = \log(x^2 + y^2) - 2\sqrt{3} \arctan \frac{y}{x}, \tag{8.1.1}$$

defines implicit functions $y(x)$ and $x(y)$. In the domains where they exist, their extremes will be found.

Solution

At the beginning, the point of the implicit function theorem formulated above should be explained. One deals with a function $F(X, Y)$ which is defined on an open set $U \subset \mathbb{R}^{n+m}$ and with the values in $\mathbb{R}^m$, where the variables $x_1, x_2, \ldots, x_n$ have been collectively denoted as X, and $y_1, y_2, \ldots, y_m$ as Y. The main question is to establish whether the equation

$$F(X, Y) = 0 \tag{8.1.2}$$

may be treated as defining the function $Y(X)$. This equation constitutes, in fact, m distinct equations which, at least formally, can be solved for m unknowns. These unknowns that for some reason we wish to determine have been denoted with symbols y_i and will be called variables "of type y." For example, the equation $F(x, y, z) = 0$, where F is a scalar function, allows us to find out only one of the variables x, y, z as a function of the remaining two. If one is interested in the dependence $y(x, z)$, one chooses $X = (x, z)$ and $Y = y$. But if one wants to obtain the function $x(y, z)$ then $X = (y, z)$ and $Y = x$.

It is possible that when solving (8.1.2), all variables $y_1, y_2, \ldots, y_m$ can be found explicitly. Then one does not actually deal with an *implicit* function and this case will not be of interest to us. We will focus on the situation when the equation cannot be explicitly solved for essential variables or when these solutions are too intricate and their use is not viable. However, even in this case, one needs to know if (8.1.2) may at all be considered as an equation which defines the function $Y(X)$. And that is just what the implicit function theorem, stated in the theoretical introduction, helps to resolve. It also allows us to find the derivative of the function $Y(X)$ without knowing the formula for this function explicitly. Notice that the Jacobian matrix $\partial F/\partial Y$ is a square matrix ($m \times m$), and therefore, it makes sense both to compute its determinant and to find the inverse matrix. The multiplication on the right-hand side of the Eq. (8.0.4) is, of course, a matrix multiplication.

The justification of the validity of formula (8.0.4) is quite simple. Well, if we assume that the Eq. (8.1.2) defines a function $Y(X)$, then by inserting it in place of the variable Y, we inevitably get an *identity*:

$$F(X, Y(X)) = 0, \tag{8.1.3}$$

i.e., the constant equal to zero regardless of the values of the arguments $x_1, x_2, \ldots, x_n$. The derivative with respect to these variables must, therefore, vanish too. On the other hand, it can also be calculated with the use of the chain rule for a composite function (see Sect. 6.2). Thus one gets the equation:

$$\frac{\partial F}{\partial X} + \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial X} = 0 \tag{8.1.4}$$

which is equivalent to (8.0.4).

Below this knowledge will be applied to the function given in the current problem. It has a scalar character (so $m = 1$) and depends on two variables, i.e., $n = 2 - 1 = 1$. Let us first consider the question of existence (or nonexistence) of the function $y(x)$. We choose any point $(x, y) \in \mathbb{R}_+ \times \mathbb{R}$ for which the equation $F(x, y) = 0$ is satisfied and calculate the partial derivative $\partial F/\partial y$:

$$\frac{\partial F}{\partial y} = \frac{2y}{x^2 + y^2} - 2\sqrt{3}\,\frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{2(y - x\sqrt{3})}{x^2 + y^2}. \tag{8.1.5}$$

The other derivative might come in handy later:

$$\frac{\partial F}{\partial x} = \frac{2x}{x^2 + y^2} - 2\sqrt{3}\,\frac{1}{1 + (y/x)^2} \cdot \frac{-y}{x^2} = \frac{2(x + y\sqrt{3})}{x^2 + y^2}. \tag{8.1.6}$$

As one can see, beyond the straight line $y = \sqrt{3}\,x$ (or more precisely the half-line since $x > 0$), the Eq. (8.1.1) defines the function $y(x)$. Now, we are going to find such points on the surface $F(x, y) = 0$ for which the function is not defined in their neighborhoods. They constitute the solutions of the set of equations:

$$\begin{cases} \log(x^2 + y^2) - 2\sqrt{3}\arctan\dfrac{y}{x} = 0, \\ y = \sqrt{3}\,x\ . \end{cases} \tag{8.1.7}$$

Eliminating y from the first one, one comes to

$$\begin{aligned} \log(x^2 + (\sqrt{3}\,x)^2) \ &- \ 2\sqrt{3}\arctan\frac{\sqrt{3}\,x}{x} = 0 \implies \log(4x^2) = 2\sqrt{3}\arctan\sqrt{3} \\ &\implies \log(4x^2) = 2\pi\,\frac{\sqrt{3}}{3} \implies x = \frac{1}{2}\,e^{\pi\sqrt{3}/3}. \end{aligned} \tag{8.1.8}$$

Hence, apart from the point of coordinates

$$(1/2\exp(\pi\sqrt{3}/3),\ \sqrt{3}/2\exp(\pi\sqrt{3}/3)),$$

the function $y(x)$ is defined and its extremes can be found with the use of (8.0.4). For one has

$$y'(x) = -\left(\frac{\partial F}{\partial y}\right)^{-1}\cdot\frac{\partial F}{\partial x} = -\frac{x^2+y^2}{2(y - x\sqrt{3})}\cdot\frac{2(x + y\sqrt{3})}{x^2+y^2} = -\frac{x + y\sqrt{3}}{y - x\sqrt{3}}. \tag{8.1.9}$$

The condition for the existence of the extremes, $y'(x) = 0$, is fulfilled for the points constituting the solutions of the set:

$$\begin{cases} \log(x^2 + y^2) - 2\sqrt{3}\arctan\dfrac{y}{x} = 0, \\ x = -\sqrt{3}\,y\ . \end{cases} \tag{8.1.10}$$

Following exactly the same way as above, one finds

$$\begin{aligned} &\log(x^2 + (-x/\sqrt{3})^2) - 2\sqrt{3}\arctan\frac{-x/\sqrt{3}}{x} = 0 \\ &\implies \log\left(\frac{4}{3}\,x^2\right) = -2\sqrt{3}\arctan\frac{1}{\sqrt{3}} \implies x = \frac{\sqrt{3}}{2}\,e^{-\pi\sqrt{3}/6}. \end{aligned} \tag{8.1.11}$$

In order to find out whether at this point one is indeed dealing with the extreme, one has to calculate the value of the second derivative. To this end, one can differentiate (8.1.9), treating y as a function of x, and wherever the derivative $y'(x)$ appears, one reinserts the expression (8.1.9):

$$y''(x) = -\left[\frac{x + y\sqrt{3}}{y - x\sqrt{3}}\right]' = -\frac{(1 + y'\sqrt{3})(y - x\sqrt{3}) - (x + y\sqrt{3})(y' - \sqrt{3})}{(y - x\sqrt{3})^2}. \tag{8.1.12}$$

However, for our purposes, it is enough to find only the value of the second derivative at the "suspicious" point, where $y' = 0$, by definition. Therefore, after plugging in the values of x and y found above, we obtain

$$y'' = -\frac{4y}{(y - x\sqrt{3})^2} = \frac{1}{2}\, e^{\pi\sqrt{3}/6} > 0. \tag{8.1.13}$$

This result indicates that at the point $x = \sqrt{3}/2\, e^{-\pi\sqrt{3}/6}$ the function $y(x)$ has a minimum.

Now we are going to tackle the implicit function $x(y)$. This time it does not exist in the neighborhood of a point for which $\partial F/\partial x = 0$. This equation has already been solved when looking for the extremes of the function $y(x)$ and we know that it is satisfied for $x = \sqrt{3}/2\, e^{-\pi\sqrt{3}/6}$ and $y = -1/2\, e^{-\pi\sqrt{3}/6}$. On certain environments of other points of the curve $F(x, y) = 0$ the implicit function $x(y)$ exists and its derivative is given by the formula:

$$x'(y) = -\left(\frac{\partial F}{\partial x}\right)^{-1} \cdot \frac{\partial F}{\partial y} = -\frac{x^2 + y^2}{2(x + y\sqrt{3})} \cdot \frac{2(y - x\sqrt{3})}{x^2 + y^2} = -\frac{y - x\sqrt{3}}{x + y\sqrt{3}}. \tag{8.1.14}$$

It has a zero value (which is known from our previous calculations) for $y = \sqrt{3}/2 \exp(\pi\sqrt{3}/3)$. Calculating the value of the second derivative at this point in the same way as we did in formulas (8.1.12) and (8.1.13), one finds

$$x'' = -\frac{4x}{(x + y\sqrt{3})^2} = -\frac{1}{2}\, e^{-\pi\sqrt{3}/3} < 0. \tag{8.1.15}$$

This means that at this point one is dealing with a maximum.

The curve $F(x, y) = 0$ in the domain of interest is shown in Fig. 8.1. It constitutes a portion of an exponential spiral; the other fragments (inessential for our problem) could not fit in the graph because the helix is growing wider very quickly (exponentially). The point A is the place where the derivative $\partial F/\partial y$ is vanishing, and therefore, in the neighboring area, one cannot define the function $y(x)$. It simultaneously constitutes the extreme for $x(y)$. In turn, in the area surrounding the point B, the function $x(y)$ cannot be defined and $y(x)$ has an extreme.

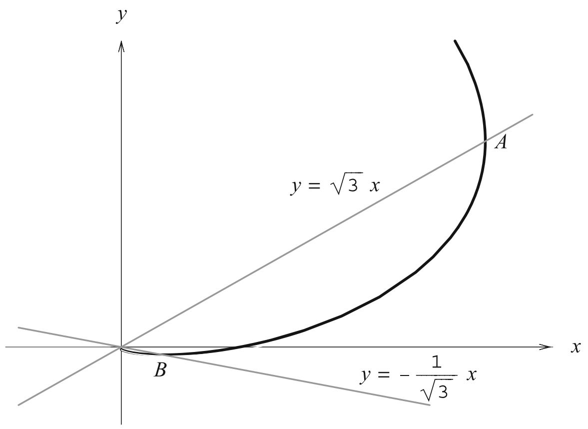


Fig. 8.1 A portion of the graph of the curve defined by the equation $F(x, y) = 0$. The scales adopted on both axes are different, so the coefficients are not tangents of the slope angles in the diagram

Problem 2

It will be examined whether the equation $F(x, y) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is given by the formula:

$$F(x, y) = x^2 - xy^2 + 2y, \tag{8.1.16}$$

defines implicit functions $y(x)$ and $x(y)$. In the regions where they exist, their extremes will be found.

Solution

In contrast to the previous exercise, one could now seek to solve the equation $F(x, y) = 0$ for x or y and to examine the explicit functions instead of implicit ones. The quantity F is quadratic both in variables x and y. However, we are learning how to deal with implicit functions, so the previous method will be followed. There are two variables and one equation, which means that exactly one variable is of type x (i.e., an independent variable) and exactly one of type y (i.e., a dependent variable) and consequently $n = m = 1$.

We start with calculating both partial derivatives, the values of which, as we know, are essential both for the existence of the implicit functions and to investigate their extremes:

$$\frac{\partial F}{\partial x} = 2x - y^2, \quad \frac{\partial F}{\partial y} = -2xy + 2. \tag{8.1.17}$$

On the neighborhood of a point belonging to the curve for which $\partial F/\partial y = 0$, the function $y(x)$ cannot be defined. By solving the set of equations:

$$\begin{cases} -2xy + 2 = 0, \\ x^2 - xy^2 + 2y = 0, \end{cases} \tag{8.1.18}$$

one checks whether such points exist. The former equation means that $xy = 1$, i.e., $y = 1/x$. By inserting it into the equation one gets

$$x^2 - x\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) = 0 \implies x^2 + \left(\frac{1}{x}\right) = 0 \implies x = -1 \tag{8.1.19}$$

and consequently also $y = -1$. On certain neighborhoods of other points of the curve, the function $y(x)$ exists. In order to find its extremes, the equation $y'(x) = 0$ must be solved. The needed formula for a derivative is known already from the theoretical introduction (cf. (8.0.4)). Applied to the current function, it takes the form:

$$y'(x) = -\left(\frac{\partial F}{\partial y}\right)^{-1} \frac{\partial F}{\partial x} = -\frac{2x - y^2}{-2xy + 2}. \tag{8.1.20}$$

The necessary condition for the existence of the extremes boils down to the equation $2x - y^2 = 0$ or $x = y^2/2$. After plugging x in this form into (8.1.16) one obtains

$$\left(\frac{y^2}{2}\right)^2 - \left(\frac{y^2}{2}\right)y^2 + 2y = 0 \implies -\frac{y^4}{4} + 2y = 0 \implies y(y^3 - 8) = 0$$
$$\implies y = 0 \ \vee \ y = 2. \tag{8.1.21}$$

Thus, there are two points on the curve in which one can expect the extremes of the function $y(x)$: (0, 0) and (2, 2). In order to find out whether they really are extremes, one should calculate the second derivative of the function, differentiating (8.1.20):

$$y''(x) = \frac{1}{2} \cdot \frac{(2 - 2yy')(xy - 1) - (2x - y^2)(y + xy')}{(xy - 1)^2}. \tag{8.1.22}$$

In both points in question, the zero value can replace the y'. The coordinates x and y are known too, so one can easily obtain

$$y''\big|_{(0,0)} = -1 < 0, \quad y''\big|_{(2,2)} = \frac{1}{3} > 0. \tag{8.1.23}$$

This means that in the first point the maximum and in the second one the minimum of the implicit function are encountered. In the graph of the equation $F(x, y) = 0$, a piece of which is sketched in Fig. 8.2, they are marked with B and C. At the point A, the function $y(x)$ does not exist.

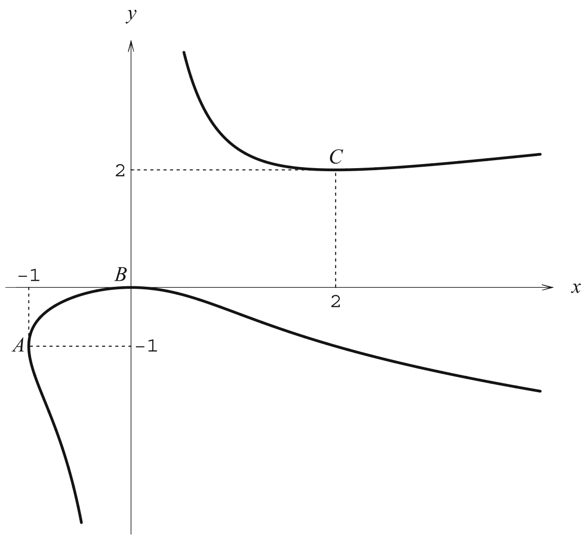


Fig. 8.2 A portion of the graph of the curve defined by the equation $F(x, y) = 0$

To examine the existence or nonexistence of the function $x(y)$, one must look for the solutions of the equations:

$$\begin{cases} \dfrac{\partial F}{\partial x} = 2x - y^2 = 0, \\ x^2 - xy^2 + 2y = 0. \end{cases} \tag{8.1.24}$$

This system has already been addressed above when looking for extremes of the function $y(x)$ (cf. (8.1.21)). We know, therefore, that the implicit function $x(y)$ does not exist in the neighborhoods of $y = 0$ and $y = 2$. On the other hand, an extreme can appear only at A because just there the derivative $\partial F/\partial y$ vanished and thereby also $x'(y) = 0$. The full expression for x' is given by

$$x'(y) = -\left(\frac{\partial F}{\partial x}\right)^{-1} \frac{\partial F}{\partial y} = -\frac{-2xy + 2}{2x - y^2}. \tag{8.1.25}$$

After differentiating it again, one gets:

$$x''(y) = 2\,\frac{(x + x'y)(2x - y^2) - (xy - 1)(2x' - 2y)}{(2x - y^2)^2}. \tag{8.1.26}$$

At the point A, one has $x = y = -1$ and $x' = 0$, so

$$x''\big|_{(-1,-1)} = \frac{2}{3} > 0 \tag{8.1.27}$$

and the minimum is encountered.

Problem 3

It will be examined whether the equation $F(x, y, z) = 0$, where $F : \mathbb{R}^3 \to \mathbb{R}$ is given by the formula:

$$F(x, y, z) = 2x^2 + 2y^2 + z^2 + 8xz - z + 8, \tag{8.1.28}$$

defines an implicit function $z(x, y)$. In the domains where it is determined, its extremes will be found.

Solution

Let us write down the set of partial derivatives of the function F:

$$\frac{\partial F}{\partial x} = 4x + 8z, \quad \frac{\partial F}{\partial y} = 4y, \quad \frac{\partial F}{\partial z} = 2z + 8x - 1. \tag{8.1.29}$$

There are two variables of type "x" ($X = (x, y)$) and one of type "y" ($Y = z$), and therefore,

$$\frac{\partial F}{\partial X} = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right] = [4x + 8z, 4y], \quad \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial z} = 2z + 8x - 1. \tag{8.1.30}$$

For the existence of the implicit function $z(x, y)$, it is essential that the last of these derivatives be nonzero. In the neighborhoods of points constituting the solution of the system:

$$\begin{cases} 2z + 8x - 1 = 0, \\ 2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0, \end{cases} \tag{8.1.31}$$

this function does not exist. From the first equation, one gets $z = -4x + 1/2$ and plugging this into the second one, one finds

$$2x^2 + 2y^2 + \left(-4x + \frac{1}{2}\right)^2 + 8x\left(-4x + \frac{1}{2}\right) - \left(-4x + \frac{1}{2}\right) + 8 = 0$$
$$\Longrightarrow \quad -14x^2 + 4x + 2y^2 + \frac{31}{4} = 0. \tag{8.1.32}$$

This implies that x and y are linked to each other by the equation of a hyperbola. Thus, there exists an infinite number of points of coordinates $(x, y, -4x + 1/2)$ in the neighborhoods on which the function $z(x, y)$ cannot be defined. Away from these points, this function does exist and its derivative is given by the formula:

$$\frac{\partial Y}{\partial X} = -\left(\frac{\partial F}{\partial Y}\right)^{-1}\frac{\partial F}{\partial X}, \tag{8.1.33}$$

i.e.,

$$\left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right] = -\left(\frac{\partial F}{\partial z}\right)^{-1}\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right] = -\left[\frac{4x+8z}{2z+8x-1}, \frac{4y}{2z+8x-1}\right]. \tag{8.1.34}$$

As we know from Sect. 7.1, at extremes both partial derivatives must vanish and since such points are searched on the considered surface, the following set of equations must be met:

$$\begin{cases} 4x + 8z = 0, \\ 4y = 0, \\ 2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0. \end{cases} \tag{8.1.35}$$

The first of them gives $x = -2z$ and by inserting it into the last one together with $y = 0$, one finds

$$-7z^2 - z + 8 = 0 \implies z = 1 \vee z = -\frac{8}{7}. \tag{8.1.36}$$

Thus, there are two "suspicious" points: $(-2, 0, 1)$ and $(16/7, 0, -8/7)$, but one still has to make sure that they do not lie on the curve (8.1.31), where the function $z(x, y)$ is not defined. However, it is easy to check by substituting the coordinates that the equation of the plane ($z = -4x+1/2$) is not satisfied, so this possibility is excluded.

In order to determine if the stationary points that have been found are really extremes, we are going to examine the matrix of second derivatives:

$$z'' = \begin{bmatrix} \dfrac{\partial^2 z}{\partial x^2} & \dfrac{\partial^2 z}{\partial x \partial y} \\ \dfrac{\partial^2 z}{\partial y \partial x} & \dfrac{\partial^2 z}{\partial y^2} \end{bmatrix}. \tag{8.1.37}$$

Its elements can be found by differentiating (8.1.34) over x and y:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= -\frac{(4 + 8\,\partial z/\partial x)(2z+8x-1) - (4x+8z)(2\,\partial z/\partial x + 8)}{(2z+8x-1)^2}, \\ \frac{\partial^2 z}{\partial x \partial y} &= -\frac{8\,\partial z/\partial y(2z+8x-1) - (4x+8z)\,2\,\partial z/\partial y}{(2z+8x-1)^2} = \frac{\partial^2 z}{\partial y \partial x}, \\ \frac{\partial^2 z}{\partial y^2} &= -\frac{4(2z+8x-1) - 4y \cdot 2\,\partial z/\partial y}{(2z+8x-1)^2}. \end{aligned} \tag{8.1.38}$$

After plugging in the coordinates of the first point: $(-2, 0, 1)$—remember that both partial derivatives of the function $z(x, y)$ vanish therein—one gets

$$z'' = \begin{bmatrix} 4/15 & 0 \\ 0 & 4/15 \end{bmatrix}. \tag{8.1.39}$$

This matrix is clearly positively definite. When calculating minors in accordance with the rule explained in Sect. 7.2, one would find $d_1 = 4/15 > 0$ and $d_2 = (4/15)^2 > 0$, which means that the function has a minimum at the point in question.

By inserting into (8.1.37) and (8.1.38) the coordinates of the second "suspicious" point: $(16/7, 0, -8/7)$, one similarly comes to

$$z'' = \begin{bmatrix} -4/15 & 0 \\ 0 & -4/15 \end{bmatrix} \tag{8.1.40}$$

which entails $d_1 = -4/15 < 0$ and $d_2 = (-4/15)^2 > 0$. This means that one is dealing with a maximum. The appropriate part of the graph of the Eq. (8.1.28) is shown in Fig. 8.3.

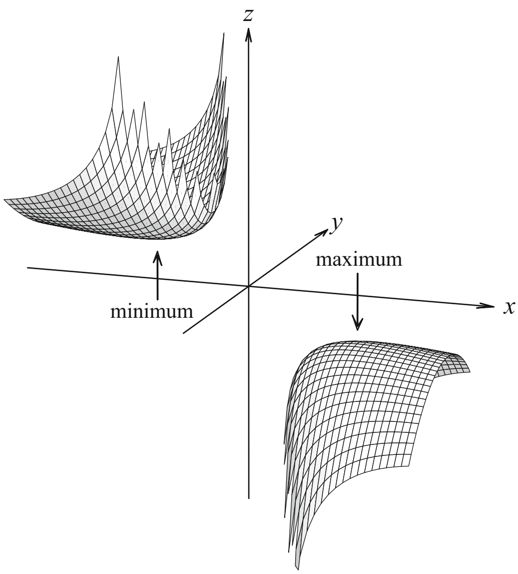


Fig. 8.3 The graph of the Eq. (8.1.28)

Problem 4

It will be examined whether the equation $F(x, y, z, w) = 0$ defines an implicit function $w(x, y, z)$. In the domains where it does, partial derivatives $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$ will be found, as well as stationary points. The function $F : \mathbb{R}^4 \to \mathbb{R}$ is given by the formula

$$F(x, y, z, w) = w^3 - 3xyzw - 1. \tag{8.1.41}$$

Solution

This time one has $X = (x, y, z)$ and $Y = w$. First all partial derivatives of the function F should be calculated:

$$\begin{aligned} \frac{\partial F}{\partial x} &= -3yzw, \quad \frac{\partial F}{\partial y} = -3xzw, \\ \frac{\partial F}{\partial z} &= -3xyw, \quad \frac{\partial F}{\partial w} = 3w^2 - 3xyz, \end{aligned} \tag{8.1.42}$$

that is

$$\frac{\partial F}{\partial X} = [-3yzw, -3xzw, -3xyw] \quad \text{and} \quad \frac{\partial F}{\partial Y} = 3w^2 - 3xyz. \tag{8.1.43}$$

As we know from previous examples, the implicit function $w(x, y, z)$ exists in neighborhoods of points for which the last of the derivatives of (8.1.43) does not vanish. Suppose for the sake of the further part of the exercise that $w^2 \neq xyz$, and for all essential points found, this condition will be verified. Thus, in accordance with formula (8.0.4), one has

$$\left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right] = -\frac{1}{3w^2 - 3xyz}\,[-3yzw, -3xzw, -3xyw]\,. \tag{8.1.44}$$

The coordinates of stationary points have to satisfy the equations:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial z} = 0 \implies \begin{cases} yzw = 0, \\ xzw = 0, \\ xyw = 0, \end{cases} \tag{8.1.45}$$

together with $w^3 - 3xyzw - 1 = 0$. This last equation (after (8.1.45)) means that $w^3 = 1$, i.e., simply $w = 1$. Then, the first three equations reduce to

$$yz = 0, \quad xz = 0, \quad xy = 0 \tag{8.1.46}$$

and are met when at least two of variables x, y, z are simultaneously equal to zero. Any points lying on the axes of the coordinate system are, therefore, stationary points of the function $w(x, y, z)$, and no others exist. At all of them, the value of the function w is the same and equal to 1. It is clear that for all of these points

$$\frac{\partial F}{\partial Y} = 3w^2 - 3xyz = 3 \neq 0, \tag{8.1.47}$$

and the assumptions of the implicit function theorem are fulfilled.

Problem 5

Given the function $F : \mathbb{R}^4 \to \mathbb{R}^2$, where

$$F(x, y, z, w) = \begin{bmatrix} x^2 + y^2 + z^2 + w^2 - 4 \\ x^2 + 2y^2 + 3z^2 + 4w^2 - 10 \end{bmatrix}. \tag{8.1.48}$$

It will be examined whether the equation $F(x, y, z, w) = 0$ defines the implicit function $f(x, y)$, with

$$f(x, y) := \begin{bmatrix} z(x, y) \\ w(x, y) \end{bmatrix}. \tag{8.1.49}$$

In the domains where it exists, the matrix of derivatives (the Jacobian matrix) will be found.

Solution

This exercise will be solved in a manner very similar to the previous ones, with the difference that now one has to work with true matrices. This is because there are *two* variables of type "x" (i.e., $X = (x, y)$) and *two* of type "y" (i.e., $Y = (z, w)$). This means that $\partial F/\partial X$, $\partial F/\partial Y$, and $\partial Y/\partial X$ (if this implicit function exists) are also matrices with dimensions 2×2. Referring to the notation of Exercise 1, one can write that $m = n = 2$.

It occurs that

$$\frac{\partial F}{\partial Y} = \begin{bmatrix} 2z & 2w \\ 6z & 8w \end{bmatrix}. \tag{8.1.50}$$

One needs to investigate where this matrix is nonsingular—then the function $Y(X)$ will exist. To this end, let us calculate the determinant:

$$\det \frac{\partial F}{\partial Y} = 2z \cdot 8w - 2w \cdot 6z = 4zw. \tag{8.1.51}$$

It is visible that in some neighborhoods of such points that the condition $F(x, y, z, w) = 0$ is met, and also $z \neq 0$ and $w \neq 0$, the implicit function $Y(X)$ may be defined. There is no doubt that such areas exist. Considering the set of equations:

$$\begin{cases} x^2 + y^2 + z^2 + w^2 - 4 = 0, \\ x^2 + 2y^2 + 3z^2 + 4w^2 - 10 = 0, \end{cases} \tag{8.1.52}$$

and eliminating x, one gets

$$y^2 + 2z^2 + 3w^2 = 6 \iff y^2 = 6 - 2z^2 - 3w^2, \tag{8.1.53}$$

and after eliminating y:

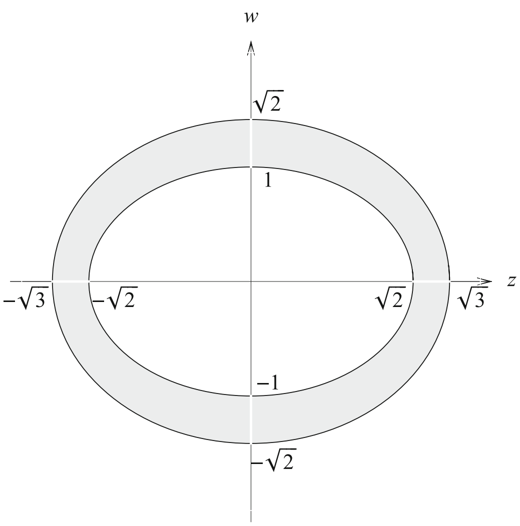
$$-x^2 + z^2 + 2w^2 = 2 \iff x^2 = z^2 + 2w^2 - 2. \tag{8.1.54}$$

If the right-hand side is positive, both equations possess solutions for y and x. There remains a question if the two conditions (with nonzero z and w) can simultaneously be satisfied:

$$\begin{cases} 2z^2 + 3w^2 < 6, \\ z^2 + 2w^2 > 2. \end{cases} \tag{8.1.55}$$

These inequalities on the plane zw specify the following: the first one—the interior of an ellipse, and the second one—the exterior of a different ellipse, the latter being contained in the former. This situation is shown in Fig. 8.4. The area marked in

Fig. 8.4 The areas for which there exist solutions of the system (8.1.52) and simultaneously the matrix (8.1.50) is nonsingular



gray indicates all such pairs (z, w) where the system (8.1.52) has a solution. In accordance with our previous findings, all points for which one of the coordinates vanishes are additionally excluded from these areas.

To find the derivative $\partial Y/\partial X$ one needs—in compliance with formula (8.0.4)—the matrix inverse to (8.1.50). It is given by

$$\left(\frac{\partial F}{\partial Y}\right)^{-1} = \begin{bmatrix} 2z & 2w \\ 6z & 8w \end{bmatrix}^{-1} = \frac{1}{2zw}\begin{bmatrix} 4w & -w \\ -3z & z \end{bmatrix} \tag{8.1.56}$$

and after plugging it into (8.0.4), one finally gets

$$\frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = -\frac{1}{2zw}\begin{bmatrix} 4w & -w \\ -3z & z \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 2x & 2y \\ 2x & 4y \end{bmatrix}}_{\partial F/\partial X}$$

$$= \frac{1}{zw}\begin{bmatrix} -3wx & -2wy \\ 2zx & zy \end{bmatrix}. \tag{8.1.57}$$

8.2 Finding Derivatives of Inverse Functions

Problem 1

It will be checked if the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula:

$$F(u, v) = \begin{bmatrix} u \cosh v \\ u \sinh v \end{bmatrix} \tag{8.2.1}$$

is locally invertible, and if so, the inverse function will be found.

Solution

In this problem, the inverse function theorem formulated in the theoretical summary will be used. Accordingly the fulfillment of the condition $\det F'(X_0) \neq 0$ (where X_0 denotes here a point on the plane (u, v)) should be verified first. Then, formula (8.0.5) for the derivative of the inverse function at $F(X_0)$ may be used.

It is obvious that one should start with analyzing the matrix of derivatives (the Jacobian matrix) for (8.2.1). Let us denote

$$F(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \end{bmatrix} = \begin{bmatrix} u \cosh v \\ u \sinh v \end{bmatrix}, \tag{8.2.2}$$

so

$$F'(u,v) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \end{bmatrix}. \tag{8.2.3}$$

Calculating the determinant of this matrix, one finds

$$\det F'(u,v) = u \cosh^2 v - u \sinh^2 v = u, \tag{8.2.4}$$

having used the hyperbolic version of the Pythagorean trigonometric identity. The obtained result shows that all points of the plane uv not on the v-axis ($u \neq 0$) have neighborhoods in which the function is locally reversible. It is clear why around the point $F(0,v) = (0,0)$ the function cannot be inverted: one gets the same value for any argument $(0,v)$ and therefore, the function is not single-valued, which is necessary for reversibility.

However, if the domain of the function F is cut down either to $D_+ := \mathbb{R}_+ \times \mathbb{R}$ or to $D_- := \mathbb{R}_- \times \mathbb{R}$, the function becomes locally invertible in each subdomain. One should recall here that a reversible function f of the class C^1, such that its inverse f^{-1} is of the class C^1 too, is called a "diffeomorphism." This means that the function F cut down to one of the above sub-domains is a *local diffeomorphism*. Its derivative, in accordance with (8.0.5), is a matrix:

$$(F^{-1})'(F(u,v)) = \begin{bmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \end{bmatrix}^{-1} = \frac{1}{u} \begin{bmatrix} u \cosh v & -u \sinh v \\ -\sinh v & \cosh v \end{bmatrix}. \tag{8.2.5}$$

It must still be established whether F is also a *global diffeomorphism*. To this end, it should be examined whether this function is single-valued on the entire set D_+ (and not only on some, typically small, neighborhood of a given point) and D_-. Are there then two different points (u_1, v_1) and (u_2, v_2), both belonging to D_+, for which the function F takes the same value:

$$F(u_1, v_1) = F(u_2, v_2)? \tag{8.2.6}$$

This equation constitutes, in fact, the set of equations:

$$\begin{cases} u_1 \cosh v_1 = u_2 \cosh v_2, \\ u_1 \sinh v_1 = u_2 \sinh v_2. \end{cases} \tag{8.2.7}$$

For the points of the set D_+ (and identically D_-), the left-hand and right-hand sides of the first equation are different from zero (remember that hyperbolic cosine is always positive), and therefore, the equations can be divided by each other (the second one by the first), which leads to

$$\tanh v_1 = \tanh v_2 \implies v_1 = v_2. \tag{8.2.8}$$

The conclusion comes from the fact that the hyperbolic tangent is single-valued. This can be easily found by using the definition $\tanh x := (e^x - e^{-x})/(e^x + e^{-x})$ and by solving the equation:

$$\frac{e^{v_1} - e^{-v_1}}{e^{v_1} + e^{-v_1}} = \frac{e^{v_2} - e^{-v_2}}{e^{v_2} + e^{-v_2}}, \tag{8.2.9}$$

which can be transformed into

$$(e^{v_1} - e^{-v_1})(e^{v_2} + e^{-v_2}) = (e^{v_1} + e^{-v_1})(e^{v_2} - e^{-v_2}) \iff e^{2(v_1 - v_2)} = 1. \tag{8.2.10}$$

Thereby, one gets $v_1 = v_2$. Plugging that into the first of the equations (8.2.7), one obtains $u_1 = u_2$. This means that on the set D_+ the function is single-valued. The similar line of thought leads to the identical conclusion for D_-. For each of these sub-domains, the function F is then the *local* and *global* diffeomorphism.

Problem 2

It will be examined whether the function $F : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ given by the formula:

$$F(t,\phi) = \begin{bmatrix} e^t \cos\phi \\ e^t \sin\phi \end{bmatrix} \tag{8.2.11}$$

is a diffeomorphism.

Solution

The matrix of partial derivatives for the function (8.2.11) has the form

$$F'(t,\phi) = \begin{bmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} e^t \cos\phi & -e^t \sin\phi \\ e^t \sin\phi & e^t \cos\phi \end{bmatrix}, \tag{8.2.12}$$

where it has been denoted

$$x(t,\phi) = e^t \cos\phi \quad \text{and} \quad y(t,\phi) = e^t \sin\phi. \tag{8.2.13}$$

Its determinant never vanishes:

$$\det F'(t,\phi) = e^{2t}\cos^2\phi + e^{2t}\sin^2\phi = e^{2t} > 0. \tag{8.2.14}$$

The function is, therefore, a local diffeomorphism. Accordingly, for the derivative of the inverse function, one gets

$$\begin{aligned}(F^{-1})'(F(t,\phi)) &= \begin{bmatrix} e^t\cos\phi & -e^t\sin\phi \\ e^t\sin\phi & e^t\cos\phi \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{-t}\cos\phi & e^{-t}\sin\phi \\ -e^{-t}\sin\phi & e^{-t}\cos\phi \end{bmatrix}.\end{aligned} \tag{8.2.15}$$

Now let us verify whether F is also a global diffeomorphism by solving the equation:

$$F(t_1,\phi_1) = F(t_2,\phi_2). \tag{8.2.16}$$

It is equivalent to the following set:

$$\begin{cases} e^{t_1}\cos\phi_1 = e^{t_2}\cos\phi_2, \\ e^{t_1}\sin\phi_1 = e^{t_2}\sin\phi_2. \end{cases} \tag{8.2.17}$$

By adding their squares to each other and using the Pythagorean trigonometric identity, one finds

$$e^{2t_1} = e^{2t_2} \implies e^{2(t_1-t_2)} = 1 \implies t_1 = t_2. \tag{8.2.18}$$

This result may now be plugged into (8.2.17) leading to the system

$$\begin{cases} \cos\phi_1 = \cos\phi_2, \\ \sin\phi_1 = \sin\phi_2, \end{cases} \tag{8.2.19}$$

which has an infinite number of solutions in the form of $\phi_2 = \phi_1 + 2n\pi$, where n is an integer number. Therefore, if one wants the function F to be a global diffeomorphism, one needs to cut down the domain to the set $\mathbb{R}\times[0, 2\pi[$ or more generally to $\mathbb{R}\times[\alpha, 2\pi+\alpha[$, where α is any real number. In this new domain, the system (8.2.19) has a unique solution: $\phi_1 = \phi_2$.

8.3 Exercises for Independent Work

Exercise 1 Examine on what sets the equation $F(x, y) = 0$ defines implicit functions $y(x)$ and $x(s)$ and find their extremes if

(a) $F : \mathbb{R}^2 \to \mathbb{R}$, where $F(x, y) = x^2 - y^2 + 2xy + 1$,
(b) $F : \mathbb{R}^2 \to \mathbb{R}$, where $F(x, y) = x^3 - y^2 - 3x + y + 4$.

Answers

(a) $y(x)$ exists in neighborhoods of any point of the curve $F(x, y) = 0$. Extremes: a minimum for $x = -1/\sqrt{2}$, a maximum for $x = 1/\sqrt{2}$. $x(y)$ exists in neighborhoods of any point of the curve $F(x, y) = 0$ except $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$. No extremes.
(b) $y(x)$ exists in neighborhoods of any point on the curve $F(x, y) = 0$ except $(-\sqrt[3]{4} - 1/\sqrt[3]{4}, 1/2)$. Extremes: minima at $(-1, -2)$ and $(1, 2)$, maxima at $(-1, 3)$ and $(1, -1)$. $x(y)$ exists in neighborhoods of any point of the curve $F(x, y) = 0$ except $(-1, -2)$, $(1, -1)$, $(-1, 3)$, and $(1, 2)$. Minimum at $(-\sqrt[3]{4} - 1/\sqrt[3]{4}, 1/2)$.

Exercise 2 Examine where the equation $F(x, y, z) = 0$ defines an implicit function $z(x, y)$ and find its extremes for

(a) $F : \mathbb{R}^3 \to \mathbb{R}$, where $F(x, y, z) = 2x^2 - y^2 + z^2 + 2x - z$,
(b) $F : \mathbb{R}^3 \to \mathbb{R}$, where $F(x, y, z) = x^2y - 2yz + 4xz - 2$.

Answers

(a) $z(x, y)$ exists in neighborhoods of any point of the surface $F(x, y, z) = 0$ except those lying on the hyperbola $(x + 1/2)^2 - y^2/2 = 3/8$, $z = 1/2$. No extremes.
(b) $z(x, y)$ exists in neighborhoods of any point of the surface $F(x, y, z) = 0$ except those lying on the straight line $x = 1$, $y = 2$. No extremes.

Chapter 9
Solving Differential Equations of the First Order

The following three chapters are devoted to one of the most important topics in calculus: solving ordinary differential equations. In the present chapter, we deal with the equations of the first order and learn how to identify their special types and apply the solving procedure to each of them.

An **ordinary differential equation of the first order** for the unknown function $y(x)$ is the equation of the form

$$F(x, y(x), y'(x)) = 0, \tag{9.0.1}$$

where $F : \mathbb{R}^3 \supset D \to \mathbb{R}$. This form is called **implicit**. Often one has to work with equations given in the **explicit form**

$$\frac{dy}{dx} = f(x, y(x)), \tag{9.0.2}$$

where f is defined on a certain domain in $\mathbb{R}^2$. The solution of (9.0.2) constitutes a family of certain curves on the plane. Such curves are called **integral curves**. Each of them at any point (ξ, η) in the considered domain that belongs to the curve in question is tangent to the straight line of the slope equal to $f(\xi, \eta)$ also passing through that point.

The differential equation (9.0.2) can be supplemented by the **initial condition** in the form $y(x_0) = y_0$, which specifies the value of the function $y(x)$ for a given argument x_0 (it is the so-called **Cauchy problem** or **initial value problem**). Then the following **theorem on existence and uniqueness of solutions** applies:

If the function f is continuous in a certain domain $D \subset \mathbb{R}^2$ and its partial derivative with respect to y is continuous too, then for each point $(\xi, \eta) \in D$ there exists a unique integral curve of (9.0.2) passing through it.

The assumption as to the continuity of $\partial f/\partial y$ is sometimes softened and replaced with the so-called **Lipschitz condition** that the difference quotient with respect to y is bounded.

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In the following sections, some special types of the first-order differential equations are discussed. They are

- the **separable equation** of the general form (9.1.2),
- the **homogeneous equation** of the general form (9.2.1),
- the **linear nonhomogeneous equation** of the general form (9.3.2),
- the **Riccati equation** of the general form (9.3.14),
- the **Lagrange equation** of the general form (9.3.33),
- the **Bernoulli equation** of the general form (9.3.63), and
- the **exact equation** of the general form (9.4.5), provided the condition (9.4.7) is satisfied.

As the reader can see, the Lagrange equation is formulated in the implicit form (9.0.1).

9.1 Finding Solutions of Separable Equations

Problem 1

The general solution of the differential equation:

$$y' \sin y = xe^{x^2+y} \tag{9.1.1}$$

will be found on the plane $\mathbb{R}^2$.

Solution

As one well knows from the lecture of analysis, no universal method to solve all differential equations can be formulated. However, there exist certain types, one should be aware of, for which the solving procedure is known. In this section, we will discuss some of them, restricting ourselves to the first-order equations. Those containing higher derivatives will be addressed in Chap. 10.

Equation (9.1.1) belongs to the class of the so-called separable equations, i.e., those that can be written in the form:

$$y' f(y) = g(x), \tag{9.1.2}$$

f and g being certain functions of one variable. Solving the equations of that kind boils down to two steps:

1. to bring the equation to the form (9.1.2) and

2. to integrate both sides over x, performing on the left-hand side the formal substitution of variables.

Equation (9.1.1) has actually the form (9.1.2), which can be clearly seen if the exponential factor is rewritten as $e^{x^2+y} = e^{x^2} \cdot e^y$. If so, one obtains

$$y'e^{-y}\sin y = xe^{x^2} \quad\Longrightarrow\quad \int y'e^{-y}\sin y\,dx = \int xe^{x^2}dx, \tag{9.1.3}$$

and after passing from the integration over x to that over y on the left-hand side, one has

$$\int e^{-y}\sin y\,dy = \int xe^{x^2}dx. \tag{9.1.4}$$

As one can see, the variables are actually separated (on the left-hand side we have only y, and on the right-hand side only x), which justifies the name of the equation ("separable"). What remains is now to calculate both indefinite integrals which will be denoted with symbols I_1 and I_2. The former can be found when integrating by parts twice:

$$\begin{aligned}
I_1 &= \int e^{-y}\sin y\,dy = \int \left[-e^{-y}\right]'\sin y\,dy = -e^{-y}\sin y + \int e^{-y}\cos y\,dy\\
&= -e^{-y}\sin y + \int \left[-e^{-y}\right]'\cos y\,dy\\
&= -e^{-y}\sin y - e^{-y}\cos y - \int e^{-y}\sin y\,dy\\
&= -e^{-y}(\sin y + \cos y) - I_1.
\end{aligned} \tag{9.1.5}$$

An equation for I_1 has been obtained, which implies that

$$I_1 = \int e^{-y}\sin y dy = -\frac{1}{2}\,e^{-y}(\sin y + \cos y). \tag{9.1.6}$$

The integration constant may be taken into account only in one of the integrals occurring in (9.1.4) and it will be done below. The integral on the right-hand side can be calculated with the use of the substitution $t = x^2$:

$$I_2 = \int xe^{x^2}dx = \frac{1}{2}\int e^t dt = \frac{1}{2}\,e^t + C = \frac{1}{2}\,e^{x^2} + C. \tag{9.1.7}$$

Thus, the solution of the Eq. (9.1.1) constitutes the implicit function $y(x)$ defined by the relation:

$$e^{-y}(\sin y + \cos y) + e^{x^2} = -2C = B. \tag{9.1.8}$$

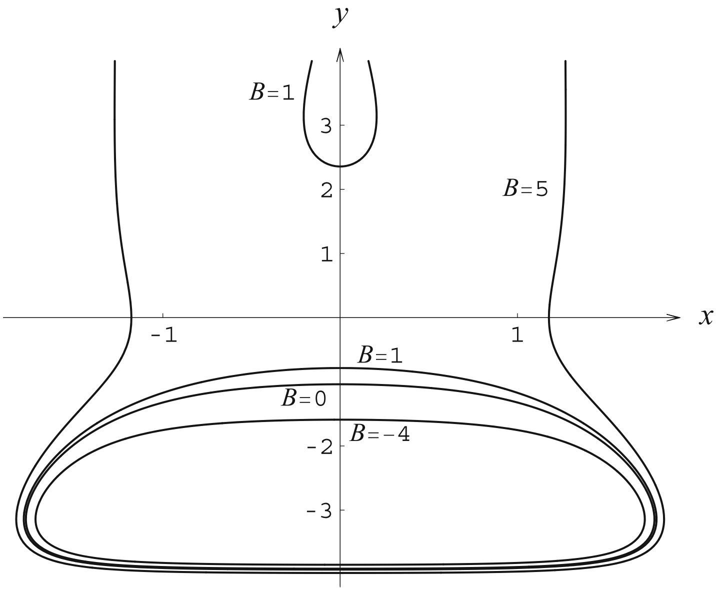

Fig. 9.1 Curves (9.1.8) for some values of the constant B

The inverse function $(x(y))$—being a solution of (9.1.1)—can be explicitly determined. The minus sign at C in (9.1.8) is simply a consequence of its earlier definition and can be omitted by changing the name of this constant; "2" is also inconsequential and can be "absorbed" into the new constant denoted as B. At this stage, this constant is totally arbitrary and a specific value would be given to it only after incorporating an initial condition which is absent in this problem. Without fixing its value, the solution to the problem constitutes the entire family of curves defined by formula (9.1.8) and shown in Fig. 9.1.

Problem 2

The general solution of the differential equation:

$$y' = \frac{2xy}{x^2 + y^2} \tag{9.1.9}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}$.

Solution

The first step that should be carried out when dealing with a differential equation is to check whether it belongs to one of the known categories. In this and in the following chapters, we will become accustomed to different types of equations, but for now, we have learned how to solve a separable one. Thus the question arises whether (9.1.9) is an equation of a separable kind. It is easy to determine that it is not. In the numerator, admittedly one has the product of expressions dependent individually on x and y and it would be easy to separate the variables by dividing both sides by y, but their further separation is not possible due to the denominator in which the sum of terms and not the product is seen. As it turns out, however, a simple step can help to transform this equation into the useful form. Rather than looking for $y(x)$, let us derive an equation for the function:

$$z(x) := \frac{y(x)}{x}. \tag{9.1.10}$$

Because of this substitution, we restrict our interest to the domains for which $x \neq 0$ (remember that $x > 0$ in line with the text of the exercise). If we succeed in finding explicitly the function $z(x)$, thereby, the function $y(x)$ will be obtained. To see what advantage is gained for using the substitution (9.1.10), one needs to eliminate $y(x)$ from the Eq. (9.1.9) by inserting $y(x) = x \cdot z(x)$. In that way, one obtains:

$$[x\, z(x)]' = \frac{2x(x\, z(x))}{x^2 + (x\, z(x))^2} \implies xz' + z = \frac{2z}{1+z^2}. \tag{9.1.11}$$

This equation is no longer a linear separable one, which becomes apparent when written down in the form:

$$xz' = \frac{2z}{1+z^2} - z = \frac{z(1-z^2)}{1+z^2}, \tag{9.1.12}$$

i.e.,

$$\frac{1+z^2}{z(1-z^2)}\, z' = \frac{1}{x}. \tag{9.1.13}$$

In conclusion, (9.1.9) is not a true separable equation, but it is *reducible* to this form! As we know from the previous problem, after having separated the variables, it remains only to integrate both sides over x, getting the equation:

$$\int \frac{1+z^2}{z(1-z^2)}\, dz = \int \frac{1}{x}\, dx = \log\left|\frac{x}{C}\right|. \tag{9.1.14}$$

For later convenience, a constant of integration has been marked here not with C, but with $\log|C|$. This is eligible, since the image of the logarithmic function is the whole set of real numbers, so that an arbitrary number can be written as a logarithm of a certain other (positive) number.

The integral on the left-hand side of (9.1.14) can be found by the well-known method of decomposing a rational function into simple fractions (see Sect. 14.3 in Part I). The integrand can be written in the form:

$$\frac{1+z^2}{z(1-z)(z+1)} = -\frac{1+z^2}{z(z-1)(z+1)} = \frac{\alpha}{z} + \frac{\beta}{z-1} + \frac{\gamma}{z+1} \tag{9.1.15}$$

and then, upon bringing fractions to a common denominator, the following equations for constants α, β, and γ arise:

$$\begin{aligned} -\frac{1+z^2}{z(z-1)(1+z)} &= \frac{z^2(\alpha+\beta+\gamma)+z(\beta-\gamma)-\alpha}{z(z-1)(z+1)} \\ &\Longrightarrow \begin{cases} \alpha+\beta+\gamma=-1, \\ \beta-\gamma=0, \\ -\alpha=-1. \end{cases} \end{aligned} \tag{9.1.16}$$

By solving them, one finds $\alpha = 1$, $\beta = \gamma = -1$ and the integral is given by the formula (an integration constant has already been included in (9.1.14)):

$$\begin{aligned} \int \frac{1+z^2}{z(1-z^2)}\,dz &= \int \left(\frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1}\right) dz \\ &= \log|z| - \log|z-1| - \log|z+1| = \log\left|\frac{z}{z^2-1}\right|. \end{aligned} \tag{9.1.17}$$

By plugging the obtained result into (9.1.14), one gets the (implicit) solution for $z(x)$:

$$\frac{z}{z^2-1} = \frac{x}{C}. \tag{9.1.18}$$

The symbols of the absolute values on both sides can be omitted due to the fact that the constant C is allowed, if necessary, to take both positive and negative values.

Remember, however, that we are looking not for the function $z(x)$, but rather $y(x)$. Substituting $z = y/x$ into the above equation, one finds

$$\frac{y/x}{(y/x)^2-1} = \frac{x}{C} \implies Cy = y^2 - x^2. \tag{9.1.19}$$

Except for the case $C = 0$, this is an equation for a hyperbola since it may be given a well-known form:

$$\left(y - \frac{C}{2}\right)^2 - x^2 = \frac{C^2}{4}. \tag{9.1.20}$$

If $C = 0$, our equations starting from (9.1.14) cease to be correct. However, one can check by a direct insertion into (9.1.9) that, in addition to hyperbolas, an allowed solution is also the set of two lines $y = \pm x$, given by the Eq. (9.1.19) for $C = 0$ (or more precisely a set of half-lines because $x > 0$).

At the end, let us add that the considered equation is an example of a so-called homogeneous differential equation, and the applied substitution (9.1.10) is universally applicable for this type of equation. The issue will be covered in more detail in Sect. 9.2.

Problem 3

The differential equation:

$$y' = \tan(x + y) \tag{9.1.21}$$

will be solved in the domain $]0, \pi/4[\times]0, \pi/4[$ with the "initial" condition $y(\pi/8) = \pi/8$.

Solution

Equation (9.1.21) is not a separable equation since the right-hand side is not a product of factors depending individually on x and on y. However, after having solved the previous example, we know that sometimes an equation can be brought into the desired form by performing a substitution. The right-hand side of (9.1.21) suggests its form. For, introducing a new function $z(x)$ as

$$z(x) = x + y(x), \tag{9.1.22}$$

the most troublesome part is simplified. On the left-hand side, admittedly, rather than derivative y' the expression $z' - 1$ appears, but the right-hand side gets reduced to $\tan z$. The equation obtained in this way,

$$z' = \tan z + 1, \tag{9.1.23}$$

is already separable. It can be now rewritten as

$$\frac{1}{\tan z + 1} z' = 1, \tag{9.1.24}$$

and then integrated over x. On the right-hand side, one gets $x + C$, where C is a constant, and the left-hand side, after replacing the integration variable, takes the form:

$$I := \int \frac{1}{\tan z + 1} dz. \tag{9.1.25}$$

The integrand expression can be easily transformed into a rational function if one puts $\tan z = t$, i.e., $z = \arctan t$. Merely, the following replacements should be made under the integral:

$$\frac{1}{\tan z + 1} \longmapsto \frac{1}{t+1} \quad \text{and} \quad dz \longmapsto \frac{1}{t^2+1} dt, \tag{9.1.26}$$

and one comes to the integral:

$$I = \int \frac{1}{(t+1)(t^2+1)} dt = \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{t-1}{t^2+1} \right) dt. \tag{9.1.27}$$

One can readily find it as one knows the integrals of individual fractions (an integration constant has been included previously):

$$\begin{aligned} \int \frac{1}{t+1} dt &= \log|t+1|, \\ \int \frac{t}{t^2+1} dt &= \frac{1}{2} \log(t^2+1), \\ \int \frac{1}{t^2+1} dt &= \arctan t. \end{aligned} \tag{9.1.28}$$

The result is

$$I = \frac{1}{4} \log \frac{(t+1)^2}{t^2+1} + \frac{1}{2} \arctan t = \frac{1}{4} \log \frac{(\tan z + 1)^2}{\tan^2 z + 1} + \frac{1}{2} z. \tag{9.1.29}$$

Transforming the expression under the logarithm as follows:

$$\frac{(\tan z + 1)^2}{\tan^2 z + 1} = \frac{(\sin z / \cos z + 1)^2}{\sin^2 z / \cos^2 z + 1} = \frac{(\sin z + \cos z)^2}{\sin^2 z + \cos^2 z} = 1 + \sin(2z),$$

one comes to the implicit solution of the Eq. (9.1.23):

$$\frac{1}{4}\log(1+\sin(2z)) + \frac{1}{2}z = x + C. \tag{9.1.30}$$

Given $z = x + y$, one has

$$\frac{1}{4}\log(1+\sin(2x+2y)) + \frac{1}{2}(x+y) = x + C, \quad \text{or} \tag{9.1.31}$$
$$\log(1+\sin(2x+2y)) + 2y - 2x = 4C.$$

A constant C can be determined by the requirement that $y(\pi/8) = \pi/8$. One then obtains

$$4C = \log\left[1+\sin\left(2\frac{\pi}{8}+2\frac{\pi}{8}\right)\right] + 2\frac{\pi}{8} - 2\frac{\pi}{8} = \log 2$$
$$\Longrightarrow \quad C = \frac{1}{4}\log 2. \tag{9.1.32}$$

In Fig. 9.2, the curve (9.1.31) satisfying the initial condition is drawn.

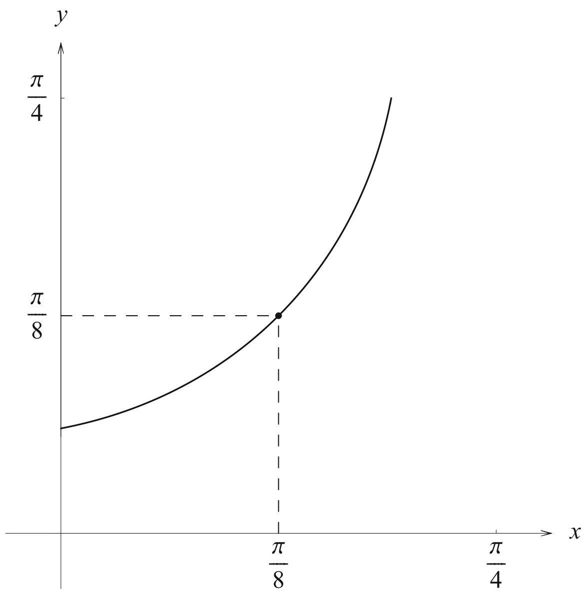

Fig. 9.2 The curve representing the solution of the Eq. (9.1.21) with the "initial" point marked

9.2 Solving Homogeneous Equations

Problem 1

The general solution of the differential equation:

$$y' = \frac{x}{y} + \frac{y}{x} \tag{9.2.1}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}_+$.

Solution

In Problem 2 of the previous section, we became familiar with an equation which, after the substitution $z(x) = y(x)/x$, could be transformed into the separable form. Its main feature was that the right-hand side did not depend separately on y and on x but on the quotient y/x (or in any case, it could be given this form). This type of equation, i.e.,

$$y' = f\left(\frac{y}{x}\right), \tag{9.2.2}$$

where f is a continuous function, is called a *homogeneous differential equation*. It appears that the substitution used is universal: each equation of this type can be solved with the considered method (in the worst case, in the integral form). For, using the relation $y' = xz' + z$, one can write

$$xz' + z = f(z) \implies z' = \frac{f(z) - z}{x}. \tag{9.2.3}$$

Thereby, a separable equation has been obtained regardless of the function f. In the case of (9.2.1), the function $f(z)$ is given by the formula:

$$f(z) = \frac{1}{z} + z, \tag{9.2.4}$$

so our equation takes the form:

$$z' = \frac{1}{xz}, \quad \text{i.e.,} \quad zz' = \frac{1}{x}. \tag{9.2.5}$$

Integrating both sides over the variable x in such a way, as it was done in the problems of Sect. 9.1, leads to an implicit solution:

$$\frac{1}{2}z^2 - \log|x| = C, \tag{9.2.6}$$

which when multiplied by 2 gives

$$\frac{y^2}{x^2} - \log(x^2) = 2C. \tag{9.2.7}$$

The symbol C, as usual, denotes a constant whose value can eventually be set by exploiting the initial condition (if specified). On the assumption that $x, y > 0$, the Eq. (9.2.7) can be unraveled for the variable y:

$$y(x) = x\sqrt{2(C + \log x)}. \tag{9.2.8}$$

Problem 2

The general solution of the differential equation:

$$y' = e^{y/x} + \frac{y}{x} \tag{9.2.9}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}$.

Solution

A glance at the Eq. (9.2.9) immediately leads to the conclusion that it already has the form (9.2.2), where the function $f(z)$ is given by the formula:

$$f(z) = e^z + z \tag{9.2.10}$$

after having applied the substitution $z = y/x$. Thus one has to work with a homogeneous equation. Using (9.2.3), we come to the very simple equation:

$$z' = \frac{e^z}{x} \implies e^{-z}z' = \frac{1}{x}, \tag{9.2.11}$$

and after integration of both sides with respect to x, one gets

$$e^{-z}\log|x| = C. \tag{9.2.12}$$

Rewriting (9.2.12) in the variable y and keeping in mind that according to the text of the problem $x > 0$, one obtains

$$e^{-y/x} + \log x = C \implies y(x) = -x\,\log(C - \log x). \tag{9.2.13}$$

Problem 3

The solution of the differential equation

$$y' = \frac{3x - y + 2}{x + y} \tag{9.2.14}$$

satisfying the condition $y(0) = \sqrt{2}$ will be found in the domain $]-1/2, \infty[\times]1/2, \infty[$.

Solution

Equation (9.2.14) in the given form *is not* a homogeneous equation. However, rewriting its right-hand side as

$$\frac{3x - y + 2}{x + y} = \frac{3 - y/x + 2/x}{1 + y/x}, \tag{9.2.15}$$

one sees that it is not far from it. If one managed to get rid of the "2" from the numerator of (9.2.15), the right-hand side would become a function of the quotient y/x only, and thus the equation would turn into the homogeneous form. So the question arises whether some smart provisional substitution can help to bring the equation to the desired form. However, this substitution must be simple in order not to overcomplicate the left-hand side of (9.2.14).

The "2" can be easily absorbed into y by introducing a new function $u(x)$ with $u(x) = y(x) - 2$. What is more, the left-hand side does not change as $y' = u'$. The price to be paid for it would be, however, "spoiling" the denominator, which would now take the form: $x + u + 2$. Therefore, we will try to simultaneously shift the response variable y and the independent variable x by some numbers α and β and determine their values so that no constant survives, neither in the numerator nor in the denominator of (9.2.14). Thus, we put

$$y = u + \alpha, \quad x = z + \beta, \tag{9.2.16}$$

where u will now be treated as a function of the new variable z and will require that

$$3(z + \beta) - (u + \alpha) + 2 = 3z - u \quad \text{and} \quad z + \beta + u + \alpha = z + u. \tag{9.2.17}$$

Thus, the constants α and β have to satisfy the equations:

$$\begin{cases} -\alpha + 3\beta + 2 = 0, \\ \alpha + \beta = 0, \end{cases} \tag{9.2.18}$$

from which it is found that $\alpha = -\beta = 1/2$. With this choice, the right-hand side of the Eq. (9.2.14) will take the form:

$$\frac{3z-u}{z+u} = \frac{3-u/z}{1+u/z},$$

but one must still check what happens to the left. Fortunately, we obtain

$$y' = \frac{dy}{dx} = \frac{d(u+\alpha)}{dx} = \underbrace{\frac{d(u+\alpha)}{dz}}_{u'} \cdot \underbrace{\frac{dz}{dx}}_{1} = u', \tag{9.2.19}$$

and the homogeneous equation for the function u is obtained:

$$u' = \frac{3-u/z}{1+u/z}. \tag{9.2.20}$$

According to the method elaborated in the previous problems, a new function $v(z) = u(z)/z$ is introduced, and, reading the function f from the right-hand side of (9.2.20):

$$f(v) = \frac{3-v}{1+v}, \tag{9.2.21}$$

one can utilize (9.2.3) with the obvious adaptation of the notation (i.e., $x \mapsto z$, $u(x) \mapsto v(z)$):

$$v' = \frac{f(v)-v}{z} = \frac{(3-v)/(1+v)-v}{z} = -\frac{v^2+2v-3}{(v+1)z}. \tag{9.2.22}$$

Now the variables can be separated:

$$\frac{v+1}{v^2+2v-3}\, v' = -\frac{1}{z}, \tag{9.2.23}$$

and the integration over z on both sides yields

$$\int \frac{v+1}{v^2+2v-3}\, dv = -\int \frac{dz}{z} = -\log|z| + \log|C|. \tag{9.2.24}$$

The integral on the left-hand side—in accordance with the rules provided in the first part of this book (see Sect. 14.3)—can be found by decomposing the integrand function into simple fractions:

$$\int \frac{v+1}{v^2+2v-3}\,dv = \int \frac{v+1}{(v-1)(v+3)}\,dv = \int \left(\frac{1/2}{v-1} + \frac{1/2}{v+3}\right) dv$$
$$= \frac{1}{2}\big[\log|v-1| + \log|v+3|\big] = \frac{1}{2}\log|(v-1)(v+3)|. \tag{9.2.25}$$

The solution of the Eq. (9.2.22) has then the following implicit form:

$$\frac{1}{2}\log|(v-1)(v+3)| = -\log|z| + \log|C| \implies |(v-1)(v+3)|z^2 = C^2. \tag{9.2.26}$$

If the constant C^2 is denoted with D and is allowed to have any value (positive, negative, or zero), the modulus on the left-hand side may be omitted.

One still has to rewrite this equation in the initial variables y and x. To this end we first substitute $v = u/z$, then $u = y - 1/2$ and $z = x + 1/2$:

$$\left(\frac{y-1/2}{x+1/2} - 1\right)\left(\frac{y-1/2}{x+1/2} + 3\right)\left(x + \frac{1}{2}\right)^2 = D, \tag{9.2.27}$$

and after some simplification, we get

$$(y - x - 1)(y + 3x + 1) = D. \tag{9.2.28}$$

In order to establish the value of the constant D, the initial condition $y(0) = \sqrt{2}$ is used. After plugging $x = 0$ and $y = \sqrt{2}$ into (9.2.28), it turns out to be

$$D = (\sqrt{2} - 0 - 1)(\sqrt{2} + 3 \cdot 0 + 1) = (\sqrt{2} - 1)(\sqrt{2} + 1) = 1, \tag{9.2.29}$$

and the equation which constitutes the solution of (9.2.14) becomes

$$(y - x - 1)(y + 3x + 1) = 1. \tag{9.2.30}$$

Formally this equation can be unraveled and y found in an explicit form but (9.2.30) seems to be more clear. The curve that has been drawn (shown in Fig. 9.3) can, after all, be given some geometric interpretation if the following variables are defined:

$$\xi = x + y, \qquad \eta = x + \frac{1}{2}. \tag{9.2.31}$$

With these variables, (9.2.30) becomes a hyperbola equation:

$$\xi^2 - 4\eta^2 = 1.$$

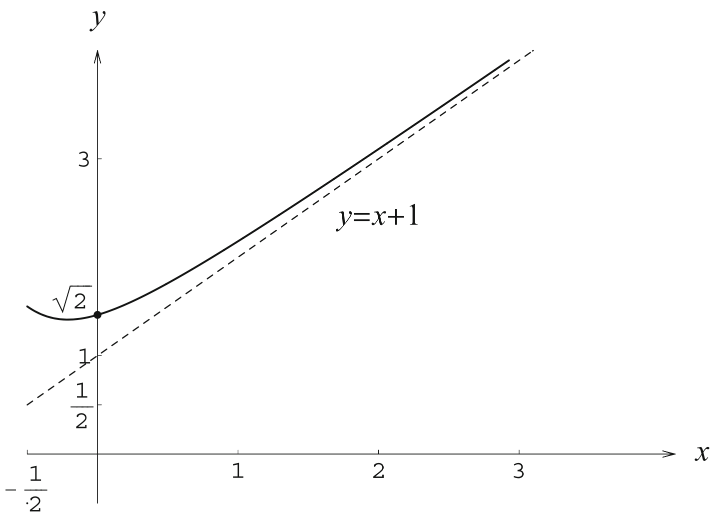

Fig. 9.3 The curve representing the solution of the Eq. (9.2.14) with the "initial" point marked

9.3 Solving Several Specific Equations

Problem 1

The general solution of the linear nonhomogeneous equation:

$$x(x+1)y' + (2x+1)y = x(3x+2) \tag{9.3.1}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}$.

Solution

A linear differential equation has in general the form:

$$f(x)y' + g(x)y = h(x), \tag{9.3.2}$$

where $f(x)$, $g(x)$, and $h(x)$ are assumed to be continuous in the domain in which the solutions are looked for (e.g., on a certain interval). As one can see in (9.3.2), the word "linear" refers to the variable y—which is an unknown here—and not to x. Equation (9.3.1) is exactly of this type with

$$f(x) = x(x+1), \quad g(x) = 2x+1 \quad \text{and} \quad h(x) = x(3x+2). \tag{9.3.3}$$

The process of solving linear nonhomogeneous equations is carried out in two steps. First, we deal with the homogeneous equation by putting $h(x) = 0$ and then applying the method of "variation of coefficients," where a special solution of the nonhomogeneous equation is found. Because of the linearity of the equation, the general solution (GSNHE) is the sum of the general solution of the homogeneous equation (GSHE) and a special solution of the nonhomogeneous equation (SSNHE), which can be symbolically expressed as

$$\text{GSNHE} = \text{GSHE} + \text{SSNHE}. \tag{9.3.4}$$

Below this routine is implemented to (9.3.1).

- *We solve the homogeneous equation.*
  After having put zero on the right-hand side, the Eq. (9.3.1) takes the form:

$$x(x+1)y' + (2x+1)y = 0, \tag{9.3.5}$$

  and hence it becomes a separable equation known from the previous section. Rewriting it as

$$\frac{y'}{y} = -\frac{2x+1}{x(x+1)}, \tag{9.3.6}$$

  one can see that both sides can be easily integrated over x. On the left, one gets the natural logarithm and on the right, again, the logarithm if one notices that the function in the numerator is a derivative of that in the denominator:

$$\log\left|\frac{y}{C}\right| = -\log|x^2+x| = \log\frac{1}{|x^2+x|} \implies |y| = |C|\,\frac{1}{|x^2+x|}. \tag{9.3.7}$$

  The modules on both sides can be omitted because the constant C is allowed to be of any sign. Finally,

$$y = C\,\frac{1}{x^2+x}. \tag{9.3.8}$$

- *We "vary" the constant.*
  Having regard to the fact that the right-hand side of (9.3.1) is actually different from zero, let us admit that C in expression (9.3.8) is not a true constant but a function of variable x, i.e., that the full solution can be postulated in the form:

$$y(x) = C(x)\,\frac{1}{x^2+x}. \tag{9.3.9}$$

By plugging this expression into (9.3.1) and taking into account that C now undergoes differentiation, one gets

$$x(x+1)\left[C'\,\frac{1}{x^2+x} - C\,\frac{2x+1}{(x^2+x)^2}\right] + C\,\frac{2x+1}{x^2+x} = x(3x+2)$$
$$\Longrightarrow \quad C' = 3x^2 + 2x. \tag{9.3.10}$$

The integration of the obtained equation leads, without problem, to the function $C(x)$:

$$C(x) = C_0 + x^3 + x^2. \tag{9.3.11}$$

Now, C_0 is a true constant.

After inserting the obtained expression into (9.3.9) the complete solution of the Eq. (9.3.1) is found as

$$\underbrace{y(x)}_{\text{GSNHE}} = \underbrace{C_0\frac{1}{x^2+x}}_{\text{GSHE}} + \underbrace{x}_{\text{SSNHE}}. \tag{9.3.12}$$

Problem 2

The general solution of the Riccati-type equation:

$$y' = xy^2 - \frac{y}{x} - \frac{2}{x^3}, \tag{9.3.13}$$

will be found for $x > 0$.

Solution

The Riccati equation has the general form:

$$y' = a(x)y^2 + b(x)y + c(x), \tag{9.3.14}$$

where functions $a(x)$, $b(x)$, and $c(x)$ are assumed to be continuous in the domain where solutions are looked for. Due to the presence of the y^2 term, it is a nonlinear equation. In the particular case $a(x) \equiv 0$ it boils down to a nonhomogeneous linear equation, and for $c(x) \equiv 0$, it becomes the so-called Bernoulli equation, studied in one of the following problems.

One is unable to formulate a universal method of solving (9.3.14) for arbitrary functions $a(x)$, $b(x)$, and $c(x)$. However, if they are relatively simple, one can try to guess some specific solution, which will further help to simplify the equation. In particular, in (9.3.13) the coefficients have power-law character, so one can try to find a specific solution of the same kind:

$$y_1(x) = x^\beta. \tag{9.3.15}$$

Plugging y_1 into (9.3.13), one finds

$$\beta x^{\beta-1} = x^{2\beta+1} - x^{\beta-1} - \frac{2}{x^3}. \tag{9.3.16}$$

For the obtained equation to have a chance of being satisfied (naturally it should be satisfied identically), the exponents of the variable x have to match as well as the coefficients, on both sides. In that way, one obtains the conditions: $\beta = -2$ and again $\beta = 1 - 1 - 2 = -2$. One can see that we are lucky to have relatively easily guessed a specific solution in the form of $y_1(x) = 1/x^2$. It should be stressed, however, that sometimes it can be very cumbersome or even impossible.

Having found $y_1(x)$, the Eq. (9.3.13) can be recast to the well-known nonhomogeneous linear equation. If a certain new function $z(x)$ is introduced with

$$y(x) = y_1(x) + \frac{1}{z(x)}, \quad \text{i.e.,} \quad y(x) = \frac{1}{x^2} + \frac{1}{z(x)}, \tag{9.3.17}$$

which gives

$$y' = -\frac{2}{x^3} - \frac{z'}{z^2}, \tag{9.3.18}$$

one obtains

$$-\frac{2}{x^3} - \frac{z'}{z^2} = x\left(\frac{1}{x^2} + \frac{1}{z}\right)^2 - \frac{1}{x}\left(\frac{1}{x^2} + \frac{1}{z}\right) - \frac{2}{x^3}. \tag{9.3.19}$$

Simplifying the above expression, one comes to the form:

$$z' + \frac{1}{x} z = -x. \tag{9.3.20}$$

Actually, a nonhomogeneous linear equation. has been obtained. As we know from the previous example, its solution can be found in two steps. First, one forgets about the right-hand side and solves the linear homogeneous equation—this is simply the separable equation:

$$z' + \frac{1}{x} z = 0 \implies \frac{z'}{z} = -\frac{1}{x}. \tag{9.3.21}$$

After having integrated both sides over x one finds

$$\log\left|\frac{z}{C}\right| = -\log|x| = \log\frac{1}{|x|} \implies z = \frac{C}{x}, \tag{9.3.22}$$

absolute value symbols having been omitted since the constant C is of an indefinite sign at this moment.

The second step for solving (9.3.20) is, as we know from the previous example, the "variation of coefficients." The full function $z(x)$ is postulated in the form:

$$z(x) = \frac{C(x)}{x}, \tag{9.3.23}$$

and upon inserting it into (9.3.20), an equation for the function ('constant') $C(x)$ is reached

$$\frac{C'}{x} - \frac{C}{x^2} + \frac{C}{x^2} = -x \implies C' = -x^2 \implies C = -\frac{1}{3}x^3 + C_0. \tag{9.3.24}$$

C_0 is here a true constant. From (9.3.23) one obtains

$$z(x) = \frac{C_0}{x} - \frac{1}{3}x^2, \tag{9.3.25}$$

which allows us to write the complete solution of the Eq. (9.3.13):

$$y(x) = \frac{1}{x^2} + \frac{1}{C_0/x - x^2/3} = \frac{1}{x^2} + \frac{3x}{3C_0 - x^3}. \tag{9.3.26}$$

As one sees, knowing one particular solution has allowed us to reduce the equation to an easily solvable form.

At the end, it is worth adding that if one had guessed at the beginning the *two* independent solutions (i.e., $y_1(x)$ and $y_2(x)$), the general formula for $y(x)$ could have been given without referring to any differential equation. This is because by manipulating three equations satisfied by the general and specific solutions:

$$\begin{cases} y' = a(x)y^2 + b(x)y + c(x), \\ y_1' = a(x)y_1^2 + b(x)y_1 + c(x), \\ y_2' = a(x)y_2^2 + b(x)y_2 + c(x), \end{cases} \tag{9.3.27}$$

one can first get rid of the function $c(x)$ (subtracting the second and the third equations from the first one):

$$\begin{aligned} y' - y_1' &= a(x)(y^2 - y_1^2) + b(x)(y - y_1), \\ y' - y_2' &= a(x)(y^2 - y_2^2) + b(x)(y - y_2), \end{aligned} \tag{9.3.28}$$

and then, dividing (9.3.28) correspondingly by $(y - y_1)$ and $(y - y_2)$ and subtracting them from each other, remove also $b(x)$:

$$\frac{y' - y_1'}{y - y_1} - \frac{y' - y_2'}{y - y_2} = a(x)(y_1 - y_2). \tag{9.3.29}$$

The left-hand side is the derivative of the expression

$$\log\left|\frac{y - y_1}{y - y_2}\right|,$$

so the integration on both sides gives

$$\log\left|\frac{y(x) - y_1(x)}{y(x) - y_2(x)}\right| - \log|B| = \int a(x)(y_1(x) - y_2(x))dx, \tag{9.3.30}$$

where B is a constant. The right-hand side does not contain any unknowns because $a(x)$ is given and $y_1(x)$ and $y_2(x)$ have already been found, so the explicit form of the function $y(x)$ can be easily obtained from (9.3.30).

In the case of the present exercise, one could easily check that the second of the specific solutions would have the form:

$$y_2(x) = -\frac{2}{x^2}.$$

The reader might ask at this point how we found $y_1(x)$ and $y_2(x)$. Well, one can determine them by plugging into the initial equation (9.3.13) a solution not in the form of x^β, but of αx^β. Then it still yields $\beta = -2$, but in addition, one gets the following condition for α: $-2\alpha = \alpha^2 - \alpha - 2$. This is the quadratic equation which has two solutions: $\alpha = 1$ and $\alpha = -2$, and hence one gets $y_1(x) = 1/x^2$ and $y_2(x) = -2/x^2$.

Since $a(x) = x$, the Eq. (9.3.30) entails

$$\left|\frac{y(x) - 1/x^2}{y(x) + 2/x^2}\right| = |B| \exp\left[\int x\left(\frac{1}{x^2} + \frac{2}{x^2}\right)dx\right] = |B| \exp\left[3\int \frac{1}{x}\,dx\right]$$
$$= |B| \exp\left[3\log|x|\right] = |B||x^3|, \tag{9.3.31}$$

and after omitting the modules and solving for y, we find

$$y(x) = \frac{2Bx + 1/x^2}{1 - Bx^3}. \tag{9.3.32}$$

The reader can easily be convinced that this expression is identical to the previously found solution (9.3.26) if the constant B is written as $B = 1/(3\,C_0)$.

Problem 3

The general solution of the Lagrange-type equation:

$$y = xy'^2 + y' \tag{9.3.33}$$

will be found.

Solution

The general form of the Lagrange equation for a function $y(x)$ is:

$$y = xf(y') + g(y'), \tag{9.3.34}$$

f and g being differentiable functions of one variable. Now the idea of solving it is to introduce a new function $z(x) := y'(x)$, and then to obtain the equation for the inverse function, i.e., for $x(z)$. As we will see, the new equation (still of the first order) is linear, and therefore, solvable using the method described in the first problem of this section.

In order to perform the substitution $y' \mapsto z$ in (9.3.33), it is necessary in the first instance to get rid of the function y on the left-hand side of the equation, for which one would not know what to insert (we strive for only variable x and z to remain). To this end, one has to differentiate both sides of (9.3.33) over x and in the second step introduce the new function. First one gets

$$y' = y'^2 + 2xy'y'' + y'', \tag{9.3.35}$$

and after plugging in $y' = z$, one comes to

$$z = z^2 + 2xzz' + z'. \tag{9.3.36}$$

At this point, in accordance with the above delineated program, one should move on from the equation for $z(x)$ to that for $x(z)$. However, using the inverse function is only possible if—at least locally—it exists. Irreversible solutions (when $z(x)$ is a constant) would be lost. If we wish to find all possible functions $y(x)$, first we have to make sure whether the Eq. (9.3.36) admits solutions in the form of $z(x) = a$, where a is a constant. To resolve it, let us plug z in this form into (9.3.36) getting (of course in this case $z' = 0$)

$$a = a^2 \implies a = 0 \;\vee\; a = 1. \tag{9.3.37}$$

This means that, among others, the Eq. (9.3.33) has the solutions of the form:

$$y(x) = A \quad \text{and} \quad y(x) = x + B, \tag{9.3.38}$$

where A and B are constants. These can be very easily determined by inserting the solution (9.3.38) into (9.3.33). In this way, one finds $A = 0$ and $B = 1$.

Keeping these solutions in mind, one should now look for the others, which, at least locally, are reversible and for which the procedure formulated above can be applied. Thanks to the inverse function theorem (see (8.0.5)) $dz/dx = (dx/dz)^{-1}$ and the equation for the function $x(z)$ has the form (the prime denotes now the differentiation over z):

$$z = z^2 + 2z\frac{x}{x'} + \frac{1}{x'}, \tag{9.3.39}$$

which leads to the linear equation:

$$x'z(1-z) - 2zx = 1. \tag{9.3.40}$$

Omitting first the unity on the right-hand side (i.e., considering the homogeneous equation), after separation of variables, one gets

$$\frac{1}{x}x' = \frac{2}{1-z}. \tag{9.3.41}$$

The integration over z on both sides in the manner used already several times in this chapter leads to

$$x(z) = C\,\frac{1}{(z-1)^2}. \tag{9.3.42}$$

In order to find the full solution of the Eq. (9.3.40), C must now be promoted to a function, i.e., postulate $x(z)$ in the form of:

$$x(z) = C(z)\,\frac{1}{(z-1)^2} \tag{9.3.43}$$

and derive an equation for $C(z)$. By inserting (9.3.43) into (9.3.40), one obtains

$$\left[C'\frac{1}{(z-1)^2} - 2C\,\frac{1}{(z-1)^3}\right]z(1-z) - 2zC\,\frac{1}{(z-1)^2} = 1, \tag{9.3.44}$$

which can be rewritten as

$$C' = \frac{1}{z} - 1. \tag{9.3.45}$$

Calculating the integral on the right-hand side does not pose any problem. Thereby, the function $C(z)$ is given by the formula:

$$C(z) = \log|z| - z + C_0, \tag{9.3.46}$$

with C_0 being now a "true" constant. The complete solution of the Eq. (9.3.40) is, therefore, as follows:

$$x(z) = \frac{\log|z| - z + C_0}{(z-1)^2}. \tag{9.3.47}$$

Now one should determine z, obtain the formula for dy/dx as a function of x, and then perform the integration of both sides. The problem is that the Eq. (9.3.47) is transcendental and cannot be explicitly solved for z. However, there exists another possibility. Equations (9.3.47) and (9.3.33) can be regarded as defining a parametric solution of our problem (this parameter will be the variable z):

$$\begin{cases} x(z) = \dfrac{\log|z| - z + C_0}{(z-1)^2}, \\ y(z) = \dfrac{z^2(\log|z| - z + C_0)}{(z-1)^2} + z. \end{cases} \tag{9.3.48}$$

The second equation has been obtained by inserting z into (9.3.33) in place of y' and the expression defined by the right-hand side of (9.3.47) instead of x. The set (9.3.48) is a parametric description of a curve in the plane xy. This curve, in addition to (9.3.38), constitutes the required solution of the Eq. (9.3.33). Its exemplary plot for $C_0 = 1$ is shown in Fig. 9.4. This solution should be treated as a local one. The curve as a whole does not constitute any function $y(x)$. Locally, however, beyond the point of coordinates $(1/2, -1/2)$, this function is well defined. The indicated point corresponds to the value of $z = -1$. It is easy to see, by calculating the derivative dx/dz with the use of the first formula of (9.3.48), that in a neighborhood of $z = -1$ the inverse function $z(x)$ does not exist. This is because one has $x'(-1) = 0$. Consequently, the procedure involving the (formal) calculation of z and substitution into the second equation (9.3.48), so as to obtain $y(x)$, could not be carried out. These issues will be dealt with in Sect. 8.2.

The second value of the parameter z, for which $x(z)$ and $y(z)$ are not defined, i.e., $z = 1$, does not preclude the existence of a differentiable function $y(x)$, unless the parametric definition of (9.3.48) is supplemented by the conditions: $x(1) = -1/2$ and $y(1) = 1/2$.

Problem 4

The solution of the Lagrange-type equation:

$$y + xy' \log y' + 1 = 0 \tag{9.3.49}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}$ and satisfying the condition $y(1) = -1$.

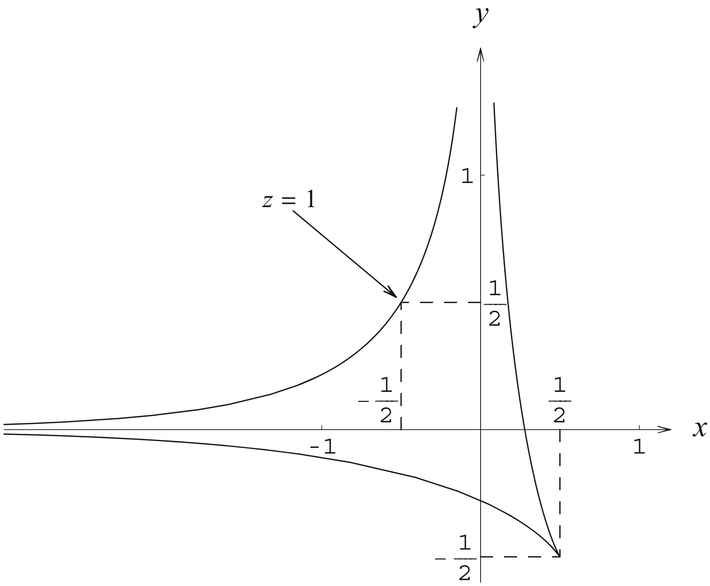

Fig. 9.4 An example of the curve (9.3.48) for the constant $C_0 = 1$

Solution

In order to solve the Eq. (9.3.49), the method described in the previous problem can be used. We start with the differentiation of both sides to get rid of the function y:

$$y' + y' \log y' + xy'' \log y' + xy'' = 0 \tag{9.3.50}$$

and with the substitution of the function $z(x)$ in place of $y'(x)$. In this way one obtains

$$z + z \log z + xz' \log z + xz' = 0. \tag{9.3.51}$$

Now one has to check whether this equation has a constant solution. For this purpose we plug in $z = a$ and try to find a:

$$a + a \log a = 0 \implies a(1 + \log a) = 0 \implies a = 0 \vee a = \frac{1}{e}. \tag{9.3.52}$$

This entails $y' = 0$ or $y' = 1/e$. Hence the two functions satisfying (9.3.49) have been found: $y = A$ and $y = x/e + B$. The values of the constants A and B can be easily established, by inserting the solutions into the initial equation. One obtains then: $A = -1$ and $B = -1$. It should be noted, however, that among these two functions only $y = -1$ meets the initial condition.

Now one should verify whether other solutions exist. As we know, it is convenient to derive the equation for the inverse function, i.e., $x(z)$. After the usage of $z' = 1/x'$, where the former prime denotes the differentiation over x and the latter one over z, it takes the form:

$$z + z\log z + \frac{x}{x'}\log z + \frac{x}{x'} = 0. \tag{9.3.53}$$

By multiplying both sides by x', one comes to the linear homogeneous equation:

$$z(1 + \log z)x' + (1 + \log z)x = 0, \quad \text{or} \quad (1 + \log z)(zx' + x) = 0. \tag{9.3.54}$$

The factor $(1 + \log z)$ vanishes for $z = 1/e$ but this solution has already been covered. Now we are looking for another one, so we require $zx' + x = 0$, arriving at the simple separable equation:

$$\frac{x'}{x} = -\frac{1}{z}. \tag{9.3.55}$$

The integration of both sides over z returns (C_1 is a certain constant)

$$\log\left|\frac{x}{C_1}\right| = -\log|z| = \log\frac{1}{|z|}, \quad \text{i.e.,} \quad x(z) = \frac{C_1}{z}. \tag{9.3.56}$$

This relation can be easily reversed, yielding $z(x) = C_1/x$. Since $z(x) = y'(x)$, one has

$$y' = \frac{C_1}{x} \tag{9.3.57}$$

and after integrating, one comes to

$$y(x) = C_1 \log|x| + C_2. \tag{9.3.58}$$

The initial condition allows us to relate the constants C_1 and C_2. They must satisfy the condition:

$$-1 = C_1 \log 1 + C_2 \implies C_2 = -1, \tag{9.3.59}$$

and the desired solution for $x > 0$ has the form:

$$y(x) = C_1 \log x - 1. \tag{9.3.60}$$

In contrast to the previous exercise, this time we succeeded in finding the function $y(x)$ explicitly. However, the reader noted for sure that even after having taken into account the initial condition, the function $y(x)$ still depends on an unknown constant C_1. This is a consequence of our procedure where the mutual differentiation of (9.3.49) was performed and instead the Eq. (9.3.50) was solved. This latter equation corresponds to the whole class of equations of the type (9.3.49), having an arbitrary constant on the right-hand side rather than zero. This means that the obtained expression for $y(x)$ should now be plugged into (9.3.49) in order to establish the value of C_1. In this way one finds

$$C_1 \log x - 1 + x\,\frac{C_1}{x}\log\frac{C_1}{x} + 1 = 0 \implies C - 1 \log C_1 = 0$$
$$\implies C_1 = 0 \;\vee\; C_1 = 1. \tag{9.3.61}$$

The first value leads to the function $y(x) = -1$ found at the beginning, and the second one yields

$$y(x) = \log x - 1. \tag{9.3.62}$$

Problem 5

The solution of the Bernoulli-type equation:

$$xy' - 3y = y^3x^2 \log x \tag{9.3.63}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying the condition $y(1) = 1$.

Solution

A differential equation which bears the name of Bernoulli has the general form:

$$y' + f(x)y = g(x)y^r. \tag{9.3.64}$$

As to $f(x)$ and $g(x)$, it is assumed that they are continuous in the considered interval and $r \in \mathbb{R}$. In the particular case when $r = 1$, a linear homogeneous equation is established and for $r = 0$ the equation is nonhomogeneous. The method for solving such equations is well known, and therefore, in this exercise we will handle another case ($r = 3$). It must be stressed that the procedure considered below is applicable to all values of parameter r, except 0 and 1.

The process consists of two steps: to bring the Bernoulli equation to a linear one and to solve the latter.

- *We obtain the linear equation.*
  One can easily convince oneself that using the function $z(x)$ defined as

$$z(x) = y(x)^{1-r}. \tag{9.3.65}$$

Equation (9.3.64) can be given the form of a linear equation for the function z. In the special case considered in this problem $r = 3$, so this is realized in the following way:

$$z(x) = y(x)^{1-3} = \frac{1}{y(x)^2} \implies y(x) = \frac{1}{\sqrt{z(x)}}. \tag{9.3.66}$$

The "+" sign in front of the square root is dictated by the text of the exercise ($y > 0$). Calculating the derivative $y' = -z'/(2\sqrt{z^3})$ and inserting it into (9.3.63), one finds

$$-x\frac{z'}{2\sqrt{z^3}} - \frac{3}{\sqrt{z}} = \left(\frac{1}{\sqrt{z}}\right)^3 x^2 \log x, \tag{9.3.67}$$

and next multiplying both sides by the factor $-2\sqrt{z^3}$ leads to the relatively simple nonhomogeneous linear equation:

$$xz' + 6z = -2x^2 \log x. \tag{9.3.68}$$

- *We solve the linear equation.*
  As we know from the former problems of this section, one first needs to find the general solution of the equation with zero on the right-hand side (i.e., of the homogeneous equation):

$$xz' + 6z = 0. \tag{9.3.69}$$

Separating variables and integrating of both sides over x, one finds

$$\frac{z'}{z} = -\frac{6}{x} \implies \log\left|\frac{z}{C}\right| = -6\log|x| = \log\frac{1}{x^6} \implies z(x) = \frac{C}{x^6}. \tag{9.3.70}$$

Now one has to "vary" the constant C, postulating

$$z(x) = \frac{C(x)}{x^6}, \tag{9.3.71}$$

and once this is plugged into (9.3.68), one comes to

$$x\frac{C'}{x^6} - x\frac{6C}{x^7} + \frac{6C}{x^6} = -2x^2 \log x \implies C' = -2x^7 \log x. \tag{9.3.72}$$

Calculating (by parts) the needed integral does not pose any difficulties. One gets

$$C(x) = -2\int x^7 \log x\, dx = -\frac{1}{4}\int \left[x^8\right]' \log x\, dx$$
$$= -\frac{1}{4}x^8 \log x + \frac{1}{4}\int x^8 \left[\log x\right]' dx = -\frac{1}{4}x^8 \log x + \frac{1}{4}\int x^7 dx$$
$$= -\frac{1}{4}x^8 \log x + \frac{1}{32}x^8 + C_0, \tag{9.3.73}$$

where C_0 is a constant. It remains to complete the formula for the function z according to (9.3.71):

$$z(x) = \frac{C_0}{x^6} - \frac{1}{4}x^2 \log x + \frac{1}{32}x^2 \tag{9.3.74}$$

and return to the function $y(x)$, by writing

$$y(x) = \left[\frac{C_0}{x^6} - \frac{1}{4}x^2 \log x + \frac{1}{32}x^2\right]^{-1/2}. \tag{9.3.75}$$

In this way the solution of the general equation (9.3.63) has been found and now the value of the constant C_0 can be fixed with the use of the initial condition. After placing $x = y = 1$, we get the equation:

$$1 = \left[C_0 + \frac{1}{32}\right]^{-1/2}, \tag{9.3.76}$$

from which it follows that $C_0 = 31/32$.

9.4 Solving Exact Equations

Problem 1

The general solution of the exact differential equation:

$$y'\left(2y - \frac{x}{x^2 + y^2}\right) + \frac{y}{x^2 + y^2} + 2x = 0 \tag{9.4.1}$$

will be found.

Solution

Let us consider the equation of the form $F(x, y) = C$, where C is a constant, and assume that it defines in a certain area an implicit function $y(x)$. Such functions were dealt with in Sect. 8.1. We noted there that if in place of the independent variable y one formally substitutes the solution, i.e., $y(x)$, then the equation:

$$F(x, y(x)) = C \tag{9.4.2}$$

becomes an identity and as a consequence the derivative of the left-hand side with respect to x must vanish. Since $F(x, y(x))$ depends on the variable x in both the explicit and implicit (i.e., through y) ways, one has

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{dy}{dx} \cdot \frac{\partial F}{\partial y} = 0. \tag{9.4.3}$$

Such a derivative is called "exact" and this is what the equation owes its name to (if the conditions specified below are met).

Equation (9.4.1), which is to be solved, is very similar in form to (9.4.3) when one could make the following identification:

$$\frac{\partial F}{\partial x} = \frac{y}{x^2 + y^2} + 2x \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y - \frac{x}{x^2 + y^2}. \tag{9.4.4}$$

If so, one could immediately write the general solution of (9.4.1), i.e., the implicit function $y(x)$ defined by the equation $F(x, y) = C$!

Thus the question arises whether one may always require that the equations (9.4.4) be met, and if so, how to determine explicitly the function F. The answer to the first question seems to be obvious. Given the equation in the form:

$$Q(x, y)\, y' + P(x, y) = 0, \tag{9.4.5}$$

with Q and P being arbitrary functions (about which it is only assumed that they are continuously differentiable in the considered domain), it is hard to believe that they are always partial derivatives of one common function F. Certainly it cannot be tacitly expected. Therefore, an appropriate condition for Q and P needs to be formulated. The key to solve the problem is the observation that if one really has $Q = \partial F/\partial y$ and $P = \partial F/\partial x$, then these functions must also be subject to the condition:

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial P}{\partial y}. \tag{9.4.6}$$

Hence it is a prerequisite for the existence of F. From the lecture of analysis the reader probably knows, however, that if the concerned area is simply connected (i.e., any closed curve can be contracted to a point without tearing apart), then the condition:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \tag{9.4.7}$$

is also sufficient, and the function F can be found from the prepared formula:

$$F(x, y) = \int_{x_0}^{x} P(x', y_0)dx' + \int_{y_0}^{y} Q(x, y')dx'. \tag{9.4.8}$$

As one can see, it is the integral over two sides of a rectangle whose opposite vertices are points of coordinates (x_0, y_0) and (x, y). If the domain is simply connected, such an integral over *any* curve that runs between these two points gives the same function F.

The reader might ask here about the role of the point (x_0, y_0) in formula (9.4.8). Well, the choice of it is irrelevant (if it is situated, together with (x, y), in the same simply connected set) and it contributes only to the constant C in the equation $F(x, y) = C$, the value of which is established later as long as a specific point lying on the curve is provided.

From the above, it appears that one should start solving this problem by verifying the condition (9.4.7), where in our case,

$$Q(x, y) = 2y - \frac{x}{x^2 + y^2} \quad \text{and} \quad P(x, y) = \frac{y}{x^2 + y^2} + 2x. \tag{9.4.9}$$

Therefore, let us calculate

$$\begin{aligned} \frac{\partial Q}{\partial x} &= -\frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \\ \frac{\partial P}{\partial y} &= \frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \end{aligned} \tag{9.4.10}$$

which entails, in fact, the condition (9.4.7). We should note, however, that the functions P and Q are not defined at the origin of the coordinate system due to their vanishing denominators. Their domain is the set $\mathbb{R}^2 \setminus \{(0, 0)\}$ which is not simply connected. A curve passing around the origin cannot be contracted to a point without tearing apart. On the other hand, if one cuts down the domain of required solutions for example to the upper half plane $\mathbb{R} \times \mathbb{R}_+$, the obtained function F would be unique (up to an additive constant).

In order to find $F(x, y)$, and thus solve the Eq. (9.4.1), one can now use formula (9.4.8), substituting P and Q in the form provided by (9.4.9). Thus, we have

$$\begin{aligned} F(x, y) &= \int_{x_0}^{x} \left(\frac{y_0}{x'^2 + y_0^2} + 2x' \right) dx' + \int_{y_0}^{y} \left(2y' - \frac{x}{x^2 + y'^2} \right) dy' \\ &= \left(\arctan \frac{x'}{y_0} + x'^2 \right) \Bigg|_{x_0}^{x} + \left(y'^2 - \arctan \frac{y'}{x} \right) \Bigg|_{y_0}^{y} \\ &= x^2 + y^2 - \arctan \frac{y}{x} - x_0^2 - y_0^2 + \arctan \frac{x}{y_0} + \arctan \frac{y_0}{x} - \arctan \frac{x_0}{y_0}. \end{aligned} \tag{9.4.11}$$

This expression apparently seems to be singular at $x = 0$. However, it should be noted that for $y > 0$ and $y_0 > 0$, which is assumed, the potentially "dangerous" expressions satisfy

$$\begin{aligned} &\lim_{x \to 0+} \left(\arctan \frac{y_0}{x} - \arctan \frac{y}{x} \right) = \frac{\pi}{2} - \frac{\pi}{2} = 0, \\ &\lim_{x \to 0-} \left(\arctan \frac{y_0}{x} - \arctan \frac{y}{x} \right) = -\frac{\pi}{2} + \frac{\pi}{2} = 0, \end{aligned} \tag{9.4.12}$$

which gives

$$\lim_{x \to 0} F(x, y) = y^2 - x_0^2 - y_0^2 - \arctan \frac{x_0}{y_0}. \tag{9.4.13}$$

On the other hand, one can calculate the $F(0, y)$ from the integral:

$$\begin{aligned} F(0, y) &= \int_{x_0}^{0} \left(\frac{y_0}{x'^2 + y_0^2} + 2x' \right) dx' + \int_{y_0}^{y} \left(2y' - \frac{0}{0^2 + y'^2} \right) dx' \\ &= y^2 - x_0^2 - y_0^2 - \arctan \frac{x_0}{y_0}, \end{aligned} \tag{9.4.14}$$

which is the same expression. Thereby, the definition of the function F can be easily supplemented for $x = 0$. Knowing that this is possible, we will strive to simplify expression (9.4.11) in order to get rid of all troublesome terms. For this purpose, the following identity is used twice:

$$\arctan a + \arctan b = \arctan \frac{a + b}{1 - ab}, \tag{9.4.15}$$

legitimate for a and b satisfying the condition $ab < 1$. First, selecting two terms out of (9.4.11), we transform the sum:

$$\arctan\frac{y_0}{x} - \arctan\frac{y}{x} = \arctan\frac{y_0}{x} + \arctan\frac{-y}{x} = \arctan\frac{x(y_0 - y)}{x^2 + y_0 y}, \tag{9.4.16}$$

plugging $a = y_0/x$ and $b = -y/x$ into (9.4.15). Naturally the necessary condition $ab < 1$ is met, since $ab = -y_0 y/x^2 < 0$. Then, again using (9.4.15), we add $\arctan(x/y_0)$ to the obtained result:

$$\begin{aligned}\arctan\frac{x(y_0 - y)}{x^2 + y_0 y} + \arctan\frac{x}{y_0} &= \arctan\frac{x(y_0 - y)/(x^2 + y_0 y) + x/y_0}{1 - x^2(y_0 - y)/(y_0 x^2 + y_0^2 y)} \\ &= \arctan\frac{x}{y}.\end{aligned} \tag{9.4.17}$$

This time different expressions for a and b are inserted into (9.4.15):

$$a = \frac{x(y_0 - y)}{x^2 + y_0 y} \quad \text{and} \quad b = \frac{x}{y_0} \tag{9.4.18}$$

since the needed condition will again be fulfilled:

$$ab = \frac{x(y_0 - y)}{x^2 + y_0 y} \cdot \frac{x}{y_0} = \frac{x^2 y_0 - x^2 y}{x^2 y_0 + y_0^2 y} = 1 - \frac{y(x^2 + y_0^2)}{x^2 y_0 + y_0^2 y} < 1. \tag{9.4.19}$$

All remaining terms in (9.4.11) exclusively dependent on x_0 and y_0 can be absorbed into a constant and finally the solution of the Eq. (9.4.1) is

$$x^2 + y^2 + \arctan\frac{x}{y} = C. \tag{9.4.20}$$

This expression is well defined in the whole upper half-plane, as we had wished. Differentiating them, one can get convinced that

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + y^2 + \arctan\frac{x}{y}\right) = 2x + \frac{y}{x^2 + y^2} = P(x, y), \\ \frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2 + \arctan\frac{x}{y}\right) = 2y - \frac{x}{x^2 + y^2} = Q(x, y).\end{aligned} \tag{9.4.21}$$

We leave to the reader the exercise to find and examine the solution of (9.4.1) if both y_0 and y belonged to the lower half-plane.

Problem 2

An integrating factor and the solution of the equation:

$$y'(2x^3 + 2x^2y - x) + y(2x^2 + 1) + 2x^3 = 0 \tag{9.4.22}$$

will be found.

Solution

Equation (9.4.22) has the form of (9.4.5), so we start with verifying whether it is exact. To this end one needs to find out if the condition (9.4.7) is satisfied. In the present exercise one has

$$Q(x, y) = 2x^3 + 2x^2y - x \quad \text{and} \quad P(x, y) = y(2x^2 + 1) + 2x^3. \tag{9.4.23}$$

The required derivatives have the form:

$$\frac{\partial Q}{\partial x} = 6x^2 + 4xy - 1, \quad \frac{\partial P}{\partial y} = 2x^2 + 1. \tag{9.4.24}$$

Their comparison shows that $\partial Q/\partial x \neq \partial P/\partial y$. Hence, the Eq. (9.4.22) is *not* exact! Does it mean then, that it cannot be solved using the method of this section? Fortunately it does not. If the equation is not *exact*, one can try to make it exact! This is because there is still a possibility that a certain factor has been reduced on both sides, due to the fact that the right-hand side of (9.4.3) vanishes. If so, a certain function $\mu(x, y)$ would exist—called the *integrating factor*—such that the equation

$$\underbrace{\mu(x, y)Q(x, y)}_{\tilde{Q}(x,y)}\, y' + \underbrace{\mu(x, y)P(x, y)}_{\tilde{P}(x,y)} = 0 \tag{9.4.25}$$

becomes exact. The problem is, therefore, to determine if a desired μ really exists. This question, if treated in a general way, is not easy. This is because if μ is a function of two independent variables, the condition

$$\frac{\partial \tilde{Q}}{\partial x} = \frac{\partial \tilde{P}}{\partial y} \tag{9.4.26}$$

leads to a nontrivial partial differential equation (containing both partial derivatives) for μ the solving of which is just as—if not more—difficult as solving the initial equation given in the text of this problem. So in order to follow this path, one needs to rely on some extraordinary simplification. For example, suppose that the unknown function μ actually depends not on both but on one variable only, leading to an ordinary differential equation instead of a partial one. So let us assume that $\mu = \mu(x)$ and verify if such a function can be found. Equation (9.4.26) takes the form:

$$\frac{\partial[(2x^3 + 2x^2y - x)\mu(x)]}{\partial x} = \frac{\partial[(2x^2y + y + 2x^3)\mu(x)]}{\partial y}, \tag{9.4.27}$$

or (the prime denotes here the derivative over x)

$$(2x^3 + 2x^2y - x)\mu'(x) + (6x^2 + 4xy - 1)\mu(x) = (2x^2 + 1)\mu(x). \tag{9.4.28}$$

If an ordinary differential equation for the function $\mu(x)$ is to be derived, it may not contain y. This variable must disappear from the Eq. (9.4.27), otherwise the assumption $\mu = \mu(x)$ would prove to be incorrect and one would be forced to look for another form of the integrating factor.

Let us transform (9.4.28) in the following way:

$$\begin{aligned} &(2x^3 + 2x^2y - x)\mu'(x) + (4x^2 + 4xy - 2)\mu(x) = 0 \\ &\quad\Longrightarrow\quad x(2x^2 + 2xy - 1)\mu'(x) + 2(2x^2 + 2xy - 1)\mu(x) = 0 \\ &\quad\Longrightarrow\quad (2x^2 + 2xy - 1)\left(x\mu'(x) + 2\mu(x)\right) = 0. \end{aligned} \tag{9.4.29}$$

Thereby, if we really find a function μ satisfying the equation (notice that y has in fact disappeared):

$$x\mu'(x) + 2\mu(x) = 0, \tag{9.4.30}$$

then automatically (9.4.26) will be met and (9.4.25) will become exact. The equation for μ, that has been found, can be easily solved by separating the variables:

$$\frac{\mu'}{\mu} = -\frac{2}{x} \quad\Longrightarrow\quad \mu(x) = \frac{A}{x^2}. \tag{9.4.31}$$

A constant A can be set equal to 1, as we are not looking for the most general form of the integrating factor, but it is enough to indicate a specific value for A.

After multiplying both sides of (9.4.22) by the factor $1/x^2$ found above, one gets the exact equation in the form:

$$\underbrace{\left(2x + 2y - \frac{1}{x}\right)}_{\tilde{Q}(x,y)} y' + \underbrace{2y + \frac{y}{x^2} + 2x}_{\tilde{P}(x,y)} = 0. \tag{9.4.32}$$

It is obvious that now one must have

$$\frac{\partial \tilde{Q}}{\partial x} = 2 + \frac{1}{x^2} = \frac{\partial \tilde{P}}{\partial y} \tag{9.4.33}$$

and formula (9.4.8) can be used:

$$\begin{aligned} F(x, y) &= \int_{x_0}^{x} \tilde{P}(x', y_0)dx' + \int_{y_0}^{y} \tilde{Q}(x, y')dy' \\ &= \int_{x_0}^{x} \left(2y_0 + \frac{y_0}{x'^2} + 2x'\right) dx' + \int_{y_0}^{y} \left(2x + 2y' - \frac{1}{x}\right) dy' \\ &= 2y_0(x - x_0) - \frac{y_0}{x} + \frac{y_0}{x_0} + x^2 - x_0^2 + 2x(y - y_0) + y^2 - y_0^2 - \frac{y}{x} + \frac{y_0}{x} \\ &= (x + y)^2 - \frac{y}{x} - (x_0 + y_0)^2 + \frac{y_0}{x_0}. \end{aligned} \tag{9.4.34}$$

The solution of the Eq. (9.4.22) has then the implicit form:

$$(x + y)^2 - \frac{y}{x} = C. \tag{9.4.35}$$

As it is a quadratic equation for y, one could also derive the explicit form of the function $y(x)$ from it. Because of the singularity of functions $\tilde{Q}(x, y)$ and $\tilde{P}(x, y)$ at $x = 0$, the solution (9.4.35) should be treated as local.

Problem 3

An integrating factor and the solution of the equation:

$$y'(y \cos y - \sin y + 2xy^2 - 2y^3) - 2xy^2 + 2y^3 = 0 \tag{9.4.36}$$

will be found. It is assumed that $y(0) = \pi$.

Solution

This equation has the form of (9.4.5), where

$$Q(x, y) = y \cos y - \sin y + 2xy^2 - 2y^3, \quad P(x, y) = -2xy^2 + 2y^3. \tag{9.4.37}$$

To make sure that this equation is exact, one has to check the condition (9.4.7). After having calculated the derivatives:

$$\frac{\partial Q}{\partial x} = 2y^2, \quad \frac{\partial P}{\partial y} = -4xy + 6y^2, \tag{9.4.38}$$

it becomes evident however that $\partial Q/\partial x \neq \partial P/\partial y$, i.e., the equation in this form is not exact. Based on the previous exercise, one should examine whether there exists an integrating factor μ which is relatively simple and easy to find. First of all, let us try $\mu = \mu(x)$. The differential equation would then assume the form:

$$y'(y\cos y - \sin y + 2xy^2 - 2y^3)\mu(x) + (-2xy^2 + 2y^3)\mu(x) = 0, \tag{9.4.39}$$

and the condition for exactness:

$$\frac{\partial}{\partial x}\Big[(y\cos y - \sin y + 2xy^2 - 2y^3)\mu(x)\Big] = \frac{\partial}{\partial y}\Big[(-2xy^2 + 2y^3)\mu(x)\Big] \tag{9.4.40}$$

would entail the following equation for the function μ:

$$(y\cos y - \sin y + 2xy^2 - 2y^3)\mu'(x) = 4y(y - x)\mu(x). \tag{9.4.41}$$

Unfortunately, it is not possible to entirely eliminate the variable y, which means that our first idea of $\mu = \mu(x)$ has proved to be inappropriate. An integrating factor *must* depend on the variable y. It is then worth it to continue and guess by using the second simplest possibility, namely that it depends on y only, i.e., $\mu = \mu(y)$. The requirement of exactness now gives

$$\frac{\partial}{\partial x}\Big[(y\cos y - \sin y + 2xy^2 - 2y^3)\mu(y)\Big] = \frac{\partial}{\partial y}\Big[(-2xy^2 + 2y^3)\mu(y)\Big] \tag{9.4.42}$$

and one gets the following equation for the function μ:

$$(-2xy^2 + 2y^3)\mu'(y) = (-4y^2 + 4xy)\mu(y) \tag{9.4.43}$$
$$\Longrightarrow \quad 2y^2(y - x)\mu'(y) = -4y(y - x)\mu(y) \quad \Longrightarrow \quad y\mu'(y) = -2\mu(y).$$

The prime denotes here the derivative over y. This time we are successful: the variable x completely disappeared from the equation and μ has actually proved to depend only on the variable y. The resulting equation can be easily solved by separating of variables:

$$\frac{\mu'}{\mu} = -\frac{2}{y} \quad \Longrightarrow \quad \mu(y) = \frac{1}{y^2}. \tag{9.4.44}$$

An integration constant has been omitted since, as we know, it is enough to indicate one specific freely chosen integrating factor. After taking it into account, the differential equation becomes exact and takes the form:

$$\underbrace{\left(\frac{\cos y}{y} - \frac{\sin y}{y^2} + 2x - 2y\right)}_{\tilde{Q}(x,y)} y' + \underbrace{(-2x + 2y)}_{\tilde{P}(x,y)} = 0. \tag{9.4.45}$$

Because of the singular character of $\tilde{Q}$ and the initial condition given in the exercise, we accept that the solution is looked for in the simply connected domain $\mathbb{R} \times \mathbb{R}_+$. Formula (9.4.8) gives then

$$\begin{aligned}
F(x, y) &= \int_{x_0}^{x} \tilde{P}(x', y_0)dx' + \int_{y_0}^{y} \tilde{Q}(x, y')dy' \\
&= \int_{x_0}^{x} (-2x' + 2y_0)dx' + \int_{y_0}^{y} \left(\frac{\cos y'}{y'} - \frac{\sin y'}{y'^2} + 2x - 2y'\right) dy' \\
&= (-x'^2 + 2x'y_0)\Big|_{x_0}^{x} + \left(\frac{\sin y'}{y'} + 2xy' - y'^2\right)\Big|_{y_0}^{y} \\
&= -x^2 + 2xy_0 + x_0^2 - 2x_0y_0 + \frac{\sin y}{y} + 2xy - y^2 - \frac{\sin y_0}{y_0} - 2xy_0 + y_0^2 \\
&= -(x - y)^2 + \frac{\sin y}{y} + (x_0 - y_0)^2 - \frac{\sin y_0}{y_0},
\end{aligned} \tag{9.4.46}$$

where, when calculating the integral, we used the fact that

$$\frac{\cos y}{y} - \frac{\sin y}{y^2} = \left[\frac{\sin y}{y}\right]'. \tag{9.4.47}$$

The solution of the Eq. (9.4.36) can, thereby, be written in the implicit form:

$$-(x - y)^2 + \frac{\sin y}{y} = C. \tag{9.4.48}$$

This is a transcendental equation with respect to y, so the explicit form of the function $y(x)$ cannot be derived (eventually it could be possible for its inverse, i.e., for the function $x(y)$). It remains to establish the value of the constant C using the initial condition:

$$C = -(0 - \pi)^2 + \frac{\sin \pi}{\pi} = -\pi^2.$$

Problem 4

An integrating factor and the solution of the differential equation:

$$y'\left(\frac{x}{y}-1\right)-\frac{y}{x}-3=0 \tag{9.4.49}$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}_+$. It is assumed that $y(1)=1$,

Solution

After having carefully analyzed the solutions of the preceding problems, the reader certainly already knows that in the first place one should check the condition (9.4.7). Regrettably one has

$$\frac{\partial}{\partial x}\left(\frac{x}{y}-1\right)=\frac{1}{y}, \quad \frac{\partial}{\partial y}\left(-\frac{y}{x}-3\right)=-\frac{1}{x} \tag{9.4.50}$$

upon multiplying by a certain integrating factor. First let us try $\mu=\mu(x)$. We obtain the condition:

$$\frac{\partial}{\partial x}\left[\mu(x)\left(\frac{x}{y}-1\right)\right]=\frac{\partial}{\partial y}\left[\mu(x)\left(-\frac{y}{x}-3\right)\right], \tag{9.4.51}$$

which leads to the equation impossible to satisfy:

$$\left(\frac{x}{y}-1\right)\mu'(x)=-\left(\frac{1}{y}+\frac{1}{x}\right)\mu(x). \tag{9.4.52}$$

There is no way to get rid of y, which means that our assumption was wrong. The integrating factor cannot be a function of variable x only. Let us, therefore, try $\mu=\mu(y)$. In this case, the following equation would have to be fulfilled:

$$\frac{\partial}{\partial x}\left[\mu(y)\left(\frac{x}{y}-1\right)\right]=\frac{\partial}{\partial y}\left[\mu(y)\left(-\frac{y}{x}-3\right)\right], \tag{9.4.53}$$

and consequently one would get (here $\mu' := d\mu/dy$)

$$\left(-\frac{y}{x}-3\right)\mu'(y)=\left(\frac{1}{y}+\frac{1}{x}\right)\mu(y). \tag{9.4.54}$$

It has a very similar form to (9.4.52) and it entails identical conclusions: the variable x cannot be removed, so μ certainly is not a function of y only. The lesson is that an integrating factor, if it exists, must be a function of *both* variables (x and y). However, trying to find such a factor in a general way leads us, as already mentioned in Exercise 2, to a partial differential equation for $\mu(x, y)$, which is just as or even more cumbersome to solve than the initial equation (9.4.49). Are we then doomed to failure and should we give up? Well, not necessarily. It is still possible that the function μ, formally depending on two variables, in fact, depends only on a certain combination of those variables. Then, denoting this combination by t, one would only have to deal with one variable and an ordinary equation for $\mu(t)$ would be obtained.

Now the question arises what combination to propose. There are an infinite number of them. Unfortunately, there is no good answer to this question. Naturally one begins by checking the simplest possibilities (unless the differential equation suggests its form), without any guarantee that our efforts will succeed. Our first attempt will be to choose an integrating factor as $\mu = \mu(t)$, where $t = x + y$, i.e., to make the assumption that it depends only on the sum of both variables. One gets then the following condition for the exactness:

$$\frac{\partial}{\partial x}\left[\mu(x+y)\left(\frac{x}{y}-1\right)\right] = \frac{\partial}{\partial y}\left[\mu(x+y)\left(-\frac{y}{x}-3\right)\right], \tag{9.4.55}$$

from which it follows (now $\mu' = d\mu/dt$):

$$\left(\frac{x}{y}+\frac{y}{x}+2\right)\mu' = -\left(\frac{1}{x}+\frac{1}{y}\right)\mu. \tag{9.4.56}$$

The condition of the success is the possibility of eliminating both variables x and y in favor of one new variable t. From (9.4.56), one obtains, upon bringing the factors on both sides to common denominators and simplifying,

$$\underbrace{(x^2+y^2+2xy)}_{=(x+y)^2}\mu' = -(x+y)\mu, \quad \text{or} \quad t^2\mu'(t) = -t\mu(t). \tag{9.4.57}$$

As it can be seen from this last equation, it contains—as we wished—only one independent variable t and the equation for μ is the ordinary differential equation. Separating variables, one obtains (an integration constant is skipped as usually)

$$\frac{\mu'}{\mu} = -\frac{1}{t} \implies \mu(t) = \frac{1}{t}, \quad \text{or} \quad \mu(x+y) = \frac{1}{x+y}. \tag{9.4.58}$$

Our exact equation has now the form:

$$y'\underbrace{\frac{x-y}{y(x+y)}}_{\tilde{Q}}\ \underbrace{-\frac{3x+y}{x(x+y)}}_{\tilde{P}} = 0, \tag{9.4.59}$$

and formula (9.4.8) can be applied:

$$\begin{aligned}
F(x, y) &= \int_{x_0}^{x} \tilde{P}(x', y_0)dx' + \int_{y_0}^{y} \tilde{Q}(x, y')dy' = -\int_{x_0}^{x} \frac{3x' + y_0}{x'(x' + y_0)}dx' \\
&+ \int_{y_0}^{y} \frac{x - y'}{y'(x + y')}dy' = -\int_{x_0}^{x} \left(\frac{1}{x'} + \frac{2}{x' + y_0}\right) dx' + \int_{y_0}^{y} \left(\frac{1}{y'} - \frac{2}{x + y'}\right) dy' \\
&= (-\log x' - 2\log(x' + y_0))\Big|_{x_0}^{x} + (\log y' - 2\log(x + y'))\Big|_{y_0}^{y} \\
&= \log x_0 - \log x + 2\log(x_0 + y_0) - 2\log(x + y_0) + \log y - \log y_0 \\
&- 2\log(x + y) + 2\log(x + y_0) = \log \frac{y}{x} - 2\log(x + y) - \log \frac{y_0}{x_0} \\
&+ 2\log(x_0 + y_0), \qquad (9.4.60)
\end{aligned}$$

where the assumption that $x, y, x_0, y_0 > 0$ has been used. The solution of (9.4.49) has then the final form:

$$\log \frac{y}{x} - 2\log(x + y) = C, \qquad (9.4.61)$$

the constant C being established from the initial condition $y(1) = 1$:

$$C = \log \frac{1}{1} - 2\log(1 + 1) = -\log 4.$$

Equation (9.4.49), solved above with the use of the integrating factor, is also a homogeneous equation. We suggest that the reader finds the solution using the procedure described in Sect. 9.2.

Problem 5

An integrating factor and the solution of the differential equation:

$$y'(2xy^2 + x^2y + x^3y - x) + y^2(2x^2 + x) + y(3x^3 - 1) = 0 \qquad (9.4.62)$$

will be found in the domain $\mathbb{R}_+ \times \mathbb{R}_+$. It is assumed that $y(1) = 1$.

Solution

This time Q and P are given in the form:

$$Q(x,y) = 2xy^2 + x^2y + x^3y - x, \quad P(x,y) = y^2(2x^2 + x) + y(3x^3 - 1). \tag{9.4.63}$$

After having calculated the derivatives:

$$\frac{\partial Q}{\partial x} = 2y^2 + 2xy + 3x^2y - 1, \quad \frac{\partial P}{\partial y} = 4x^2y + 2xy + 3x^3 - 1 \tag{9.4.64}$$

one sees that $\partial Q/\partial x \neq \partial P/\partial y$, i.e., the Eq. (9.4.62) is not exact. So, let us look for an integrating factor. First, assume that it has the form $\mu = \mu(x)$. If such a factor exists, the following equation should be met:

$$\frac{\partial}{\partial x}\left[\mu(x)(2xy^2 + x^2y + x^3y - x)\right] = \frac{\partial}{\partial y}\left[\mu(x)(y^2(2x^2 + x) + y(3x^3 - 1))\right], \tag{9.4.65}$$

which after taking derivatives can be given the form:

$$\mu'(x)(2xy^2 + x^2y + x^3y - x) + \mu(x)(-x^2y - 3x^3 + 2y^2) = 0. \tag{9.4.66}$$

It is easy to see that there is no way to fully get rid of the variable y and thus obtain an equation for the function $\mu(x)$. Our assumption that an integrating factor depends only on x was not correct.

In the same way, the reader can check that an integrating factor in the form of $\mu = \mu(y)$ cannot be found either. So let us then try the next possibility, $\mu = \mu(xy)$, and require

$$\frac{\partial}{\partial x}\left[\mu(xy)(2xy^2 + x^2y + x^3y - x)\right] = \frac{\partial}{\partial y}\left[\mu(xy)(y^2(2x^2 + x) + y(3x^3 - 1))\right]. \tag{9.4.67}$$

If one puts $t = xy$ and $\mu' = d\mu/dt$, this equation can be rewritten as

$$\begin{aligned}&\mu' y(2xy^2 {+}!x^2y + x^3y - x) + \mu(2y^2 + 2xy + 3x^2y - 1)\\ &\quad = \mu' x(y^2(2x^2 + x) + y(3x^3 - 1)) + \mu(4x^2y + 2xy + 3x^3 - 1).\end{aligned} \tag{9.4.68}$$

After some simplifications, it becomes

$$\begin{aligned}&\mu'(2xy^3 - x^3y^2 - 3x^4y) = \mu(x^2y + 3x^3 - 2y^2), \quad \text{or}\\ &\underbrace{xy}_{t} \cdot \mu'(2y^2 - x^2y - 3x^3) = -\mu(2y^2 - x^2y - 3x^3),\end{aligned} \tag{9.4.69}$$

and upon canceling identical factors on both sides, it is reduced to

$$t\mu'(t) = -\mu(t). \tag{9.4.70}$$

The solution of (9.4.70) can be found effortlessly: it is a separable equation well known from Sect. 9.1. As a result we find the integrating factor: $\mu = 1/t = 1/(xy)$. By multiplying both sides of (9.4.62) with μ, the exact equation is obtained. Now let us define

$$\tilde{Q}(x, y) = \frac{Q(x, y)}{xy} = 2y + x + x^2 - \frac{1}{y}, \tag{9.4.71}$$

$$\tilde{P}(x, y) = \frac{P(x, y)}{xy} = 2xy + y + 3x^2 - \frac{1}{x}, \tag{9.4.72}$$

and write the solution using (9.4.8):

$$\begin{aligned}
F(x, y) &= \int\limits_{x_0}^{x} \tilde{P}(x', y_0)dx' + \int\limits_{y_0}^{y} \tilde{Q}(x, y')dy' \\
&= \int\limits_{x_0}^{x} \left(2x'y_0 + y_0 + 3x'^2 - \frac{1}{x'}\right) dx' + \int\limits_{y_0}^{y} \left(2y' + x + x^2 - \frac{1}{y'}\right) dy' \\
&= \left(x'^2 y_0' + x'y_0 + x'^3 - \ln x'\right)\Big|_{x_0}^{x} + (y'^2 + xy' + x^2y' - \ln y')\Big|_{y_0}^{y} \\
&= x^2y_0 - x_0^2y_0 + xy_0 - x_0y_0 + x^3 - x_0^3 - \ln x + \ln x_0 \\
&\quad + y^2 - y_0^2 + xy - xy_0 + x^2y - x^2y_0 - \ln y + \ln y_0 \\
&= x^3 + y^2 + xy + x^2y - \ln(xy) - x_0^2y_0 - x_0y_0 - x_0^3 - y_0^2 + \ln(x_0y_0),
\end{aligned} \tag{9.4.73}$$

with the choice that $x, y, x_0, y_0 > 0$. The expressions containing only x_0 and y_0 can be absorbed into a constant to be determined later from the initial condition. As a result, the general solution of the Eq. (9.4.62) can be written in the implicit form:

$$x^3 + y^2 + xy + x^2y - \ln(xy) = C. \tag{9.4.74}$$

It remains only to find the value of the constant C by substituting $x = y = 1$:

$$C = 1 + 1 + 1 + 1 - \ln(1 \cdot 1) = 4. \tag{9.4.75}$$

9.5 Exercises for Independent Work

Exercise 1 Find general solutions of the following differential equations:

(a) $1°.\ y' \log^2 y = 3y\,x^2, \quad 2°.\ x^2\, y'\, y^2 = e^y,$
(b) $1°.\ y' = \cos(y - x), \quad 2°.\ y' = \dfrac{y}{x} \log \dfrac{y}{x},$
(c) $1°.\ x\, y' + y = x^2 + 1, \quad 2°.\ y' = -\dfrac{y^2}{2} - \dfrac{y}{2x},$
(d) $1°.\ y = 2x\, y' + 2y'^2 + 1, \quad 2°.\ y' + \dfrac{1}{x}\, y = y^2 x\, e^x,$
(e) $\left(2xy + \dfrac{x}{x+y}\right) y' + \dfrac{x}{x+y} + y^2 + \log(x+y) = 0,$
(f) $2x\,(xy+1)\, y' + (y^2 - 2)x^2 + 2(y^2 + y - 2)x + 2y = 0.$

Answers

(a) $1°.\ y(x) = e^{\sqrt[3]{3x^3 + C}}$, $2°.$ implicit solution: $y^2 + 2y + 2 = e^y(1/x + C)$,
(b) $1°.\ y(x) = 2\arctan(x + C) + x$, $2°.\ y(x) = x\, e^{1+Cx}$,
(c) $1°.\ y(x) = C/x + x^2/3 + 1$, $2°.\ y(x) = 1/(x + C\sqrt{x})$,
(d) $1°.$ implicit solution: $y(z) = -2z^3 + 2z^2 + 2C/z + 1$, $x(z) = C/z^2 - z^2$,
$2°.\ y(x) = 1/(-xe^x + Cx)$,
(e) Implicit solution: $xy^2 + x\log(x+y) = C$,
(f) Implicit solution: $(x^2y^2 + 2xy - 2x^2)e^x = C$.

Exercise 2 Solve the Eq. (9.4.49) using the method of Sect. 9.2.

Exercise 3 Find the solutions of the following differential equations with given additional conditions:

(a) $1°.\ y'y(y-1) = \dfrac{1}{x+1}\, e^{-y}$, $y(0) = 0$, $\quad 2°.\ y' = \left(\dfrac{x}{y}\right)^4$, $y(0) = 1$,
(b) $1°.\ y'\cos x + y\sin x = 1$, $y(\pi) = 1$, $\quad 2°.\ y' = xy^2 + y$, $y(0) = 1$,
(c) $y = 2xy' + y'^2$, the curve going through $(1, 3)$,
(d) $y' + y = 2\sqrt[3]{y}$, $y(0) = 8$,
(e) $(1 + 2xy)y' + y^2 + 2x = 0$, the curve going through $(1, 0)$,
(f) $(x + 2x^2y)y' + 3xy^2 + 2y + 2 = 0$, the curve going through $(1, 1)$.

Answers

(a) $1°.$ Implicit solution: $e^y(y^2 - 3y + 3) - 3 = \log(x+1)$, $2°.\ y(x) = \sqrt[5]{x^5 + 1}$,

(continued)

(b) 1°. $y(x) = \sin x - \cos x$, 2°. $y(x) = 1/(1-x)$,
(c) Parametric solution: $y(z) = (10/z - z^2)/3$, $x(z) = (5/z^2 - 2z)/3$,
(d) $y(x) = \sqrt{8\left(1 + e^{-2x/3}\right)^3}$,
(e) $y + xy^2 + x^2 = 1$,
(f) $x^3y^2 + x^2y + x^2 = 3$.

Chapter 10
Solving Differential Equations of Higher Orders

In this chapter, we deal with **differential equations of higher orders**, which means "second order or higher," i.e., of the general form:

$$F(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = 0, \tag{10.0.1}$$

where n denotes the order of the equation. The special and very important type of higher order differential equation constitutes the so-called **linear equations with constant coefficients**, either homogeneous or nonhomogeneous:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_1 y'(x) + a_0 y(x) = b(x). \tag{10.0.2}$$

The term "linear" refers to the first power of y and its derivatives. The coefficients a_i, where $i = 1, 2, \dots, n$, are constants and it is assumed that $a_n \neq 0$ (otherwise one could not call this an equation of "nth order"). If the function $b(x) \equiv 0$, the equation is called a **homogeneous linear equation**. For $b(x) \not\equiv 0$, i.e., in the case of a **nonhomogeneous linear equation**, two methods are explored in detail: the Lagrange method of the **variation of constants** and that of **predictions**.

Additionally, the Cauchy problem for the Eq. (10.0.2) is specified by supplying n values: $y(x_0), y'(x_0), \dots, y^{(n-1)}(x_0)$. This leads to a unique solution.

The other kinds of equations addressed in this chapter include some special types of (10.0.1) for $n = 2$ for which it is possible to reduce the order of the equation and also particular examples of the **linear equations with variant coefficients**:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = d(x). \tag{10.0.3}$$

The solution of the nonhomogeneous equation (10.0.3) can be found by varying coefficients C_1 and C_2 of the general solution of the corresponding homogeneous equation:

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$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0. \tag{10.0.4}$$

This equation has two independent solutions $y_1(x)$ and $y_2(x)$, so the general solution may be written in the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x). \tag{10.0.5}$$

Unfortunately, no general method of solving (10.0.4) exists (contrary to that of the form (10.0.2)).

The **Wronskian** is the determinant consisting of the solutions of the nth order linear equation in the following way:

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}. \tag{10.0.6}$$

It can be proved that if $W(x) \not\equiv 0$, the solutions $y_1, y_2, \ldots, y_n$ are independent.

10.1 Solving Linear Equations with Constant Coefficients

Problem 1

The general solution of the third-order equation:

$$2y''' - 7y'' + 4y' + 4y = 0 \tag{10.1.1}$$

will be found.

Solution

If one examines an equation of the type:

$$y' = \lambda \cdot y, \tag{10.1.2}$$

which, as a matter of fact means that the differentiation of the function y boils down to multiplying it by a constant, it is clear that the solution must have the form:

$$y(x) = \text{const} \cdot e^{\lambda x}. \tag{10.1.3}$$

Any linear equation with constant coefficients, such as (10.1.1), is in fact similar in nature and one can look for the function $y(x)$ again in the form (10.1.3). Let us then substitute such y into the Eq. (10.1.1), neglecting the inessential multiplicative constant. As it has already been told, the differentiation of expressions such as $e^{\lambda x}$ leads only to the emergence of appropriate powers of the constant λ as numerical factors:

$$\begin{aligned} &2\lambda^3 e^{\lambda x} - 7\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 4e^{\lambda x} = 0 \\ &\quad \Longrightarrow \quad 2\lambda^3 - 7\lambda^2 + 4\lambda + 4 = 0. \end{aligned} \tag{10.1.4}$$

The last equation is called the *characteristic equation* for the differential equation (10.1.1). In the following examples, it will be written out right away without showing the steps of substituting $e^{\lambda x}$ into (10.1.3). Solving (10.1.4) for λ, one finds the characteristic roots. They can be real or complex and this latter case, as one will see in subsequent examples, does not preclude finding the solution and only implies the appearance of the trigonometric functions sine and cosine in place of exponential ones.

Equation (10.1.4) can be given the form:

$$2(\lambda - 2)^2 \left(\lambda + \frac{1}{2}\right) = 0, \tag{10.1.5}$$

from which it is obvious that it has two characteristic roots:

$$\lambda_1 = 2, \; p_1 = 2 \quad \text{and} \quad \lambda_2 = -\frac{1}{2}, \; p_2 = 1. \tag{10.1.6}$$

The symbols $p_{1,2}$ denote their multiplicities. The functions e^{2x} and $e^{-x/2}$ with arbitrary coefficients must, therefore, constitute specific solutions of (10.1.1). However, as it is easy to verify by the direct insertion, there is an additional special solution in the form xe^{2x}, which is due to the fact that $\lambda_1 = 2$ was a double root. In general, among possible solutions for any λ of multiplicity p, all functions of the type

$$e^{\lambda x}, \; xe^{\lambda x}, \; \ldots, x^{p-1}e^{\lambda x} \tag{10.1.7}$$

can appear. Each root provides us with as many independent solutions as the amount of its multiplicity, and as a consequence, the considered differential equation has as many independent solutions as its degree.

There remains a question, how to build a *general* solution of (10.1.1) using these *special* solutions. The answer derives from the linear nature of our equation (i.e., the function y and its derivatives occur only in the first power). Thanks to this property, the equation has a pleasant feature that the sum of any of its solutions constitutes again a solution. As a result for the full solution, one can write

$$y(x) = (Ax + B)e^{2x} + Ce^{-x/2}. \tag{10.1.8}$$

The constants A, B, and C, if necessary, can be established from the initial conditions.

Problem 2

The solution of the differential equation:

$$y'' + y = \sin x + \cos x \tag{10.1.9}$$

satisfying the conditions: $y(0) = 1$ and $y'(0) = 0$, will be found.

Solution

Comparing the above equation to (10.1.1), one important difference is noted: the nonzero right-hand side of (10.1.9). Such an equation is called "linear nonhomogeneous," and we refer to the right-hand side as "nonhomogeneity." Similarly as in the equation in the previous exercise, all the coefficients for y and its derivatives are *constants* (i.e., numbers independent of x). Thereby, it is also an equation with constant coefficients. Its solution can be found in two steps. First we forget about the nonhomogeneity and find the general solution of this simplified equation (i.e., of the linear homogeneous equation) using the method of the previous example, and then we try to extend it so as to take into account the right-hand side of (10.1.9). There are here at our disposal two useful methods: relatively simple method of predictions, which unfortunately cannot be used for the arbitrary form of the right-hand side, the method of "variation of coefficients," which is more general but tedious and in the case of first-order equations was dealt with in Sect. 9.3.

1. *We solve linear homogeneous equation.*
   If in the Eq. (10.1.9) the right-hand side is omitted, it takes the form:

$$y'' + y = 0. \tag{10.1.10}$$

   The characteristic equation $\lambda^2 + 1 = 0$ has two single roots:

$$\lambda_1 = i \quad \text{and} \quad \lambda_2 = -i. \tag{10.1.11}$$

   In accordance to what we know from the previous problem, one can now write down the general solution (10.1.10) as

$$y_0(x) = Ae^{ix} + Be^{-ix}, \tag{10.1.12}$$

with certain constants A and B. Better yet, and in any case more "elegantly," we can use Euler's formulas:

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x \tag{10.1.13}$$

and rewrite $y_0(x)$ in the form of

$$\begin{aligned} y_0(x) &= A(\cos x + i \sin x) + B(\cos x - i \sin x) \\ &= \underbrace{(A + B)}_{C_1} \cos x + \underbrace{i(A - B)}_{C_2} \sin x = C_1 \cos x + C_2 \sin x. \end{aligned} \tag{10.1.14}$$

In place of exponentials, one now has trigonometric functions. This result should not be surprising. If the variable x had the meaning of time, then (10.1.10) would become the harmonic-oscillator equation (with $\omega = 1$), the evolution of which is given by sine or cosine functions. expression (10.1.14) with arbitrary constants C_1 and C_2 will be addressed as a starting point for the second step, which is finding the solution of the nonhomogeneous equation.

At this point it is worth mentioning, how one would proceed if characteristic roots were complex with nonvanishing real and imaginary parts, i.e., if $\lambda_{1,2} = \lambda_r \pm i\lambda_i$. Here it has been assumed $\bar{\lambda}_1 = \lambda_2$, since—as the reader probably knows from the lecture of algebra—a polynomial with real coefficients of the second or higher degree have complex roots always occurring in pairs: λ, $\bar{\lambda}$. In the considered case, we would write

$$\begin{aligned} Ae^{\lambda_1 x} + Be^{\lambda_2 x} &= e^{\lambda_r x} \left(Ae^{i\lambda_i x} + Be^{-i\lambda_i x} \right) \\ &= e^{\lambda_r x} \left[A(\cos \lambda_i x + i \sin \lambda_i x) + B(\cos \lambda_i x - i \sin \lambda_i x) \right] \\ &= e^{\lambda_r x} [\underbrace{(A + B)}_{C_1} \cos \lambda_i x + \underbrace{i(A - B)}_{C_2} \sin \lambda_i x] \\ &= e^{\lambda_r x} (C_1 \cos \lambda_i x + C_2 \sin \lambda_i x). \end{aligned} \tag{10.1.15}$$

For the negative values of λ_r, we would simply deal with a *dampened oscillator*.

2. *We "vary" the constants.*
   Let us now suppose that C_1 and C_2 are not true constants but functions of the variable x. As it will be seen, one can determine $C_1(x)$ and $C_2(x)$ in such a way that the function

$$y(x) = C_1(x) \cos x + C_2(x) \sin x \tag{10.1.16}$$

will constitute the solution of (10.1.9). In order to obtain the equations for unknown functions $C_1(x)$ and $C_2(x)$, let us plug (10.1.16) into (10.1.9). To this end, one has to calculate the derivatives of $y(x)$:

$$y'(x) = C_1'(x)\cos x + C_2'(x)\sin x - C_1(x)\sin x + C_2(x)\cos x. \tag{10.1.17}$$

Before explicitly writing the expression for $y''(x)$, let us impose the condition that the sum of the first two terms in (10.1.17) vanishes:

$$C_1'(x)\cos x + C_2'(x)\sin x = 0. \tag{10.1.18}$$

The effect to be achieved, thanks to this assumption, is that in the formula for y'', and thus in the equations for C_1 and C_2, the second derivatives of these "constants" will not appear. Only the first derivatives emerge, and the equations will be much easier to solve. The condition (10.1.18) can seem to the reader somewhat mysterious. Its full explanation is postponed to Sect. 11.2, and at this point we would like only to point out that if it does not lead to a contradiction (e.g., if one does not get too many equations with respect to the number of unknowns) and we are able to satisfy initial conditions, then there is no reason to deny it. This is because currently only a *special* solution of the nonhomogeneous equation is desired (the *general character* is contained in constants $C_{1,2}$ in formula (10.1.14)). Thus, one has

$$y'(x) = -C_1(x)\sin x + C_2(x)\cos x \tag{10.1.19}$$

and as a consequence

$$y''(x) = -C_1'(x)\sin x + C_2'(x)\cos x - C_1(x)\cos x - C_2(x)\sin x. \tag{10.1.20}$$

By inserting (10.1.20) and (10.1.16) into (10.1.9), one comes to the equation:

$$-C_1'(x)\sin x + C_2'(x)\cos x = \sin x + \cos x. \tag{10.1.21}$$

It should be noted that all terms not containing the derivatives of C_1 and C_2 are gone. This is not a coincidence but a consequence of the fact that (10.1.14) represents the solution of the equation without the right-hand side, i.e., (10.1.10).

Equations (10.1.18) and (10.1.21) constitute a system that will allow us to determine the unknowns C_1' and C_2':

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0, \\ -C_1'(x)\sin x + C_2'(x)\cos x = \sin x + \cos x. \end{cases} \tag{10.1.22}$$

This task is not difficult. After simple calculations, one finds

$$C_1'(x) = -\sin x(\sin x + \cos x),$$
$$C_2'(x) = \cos x(\sin x + \cos x), \tag{10.1.23}$$

and in order to find both functions only simple integrations are needed:

$$\begin{aligned}
C_1(x) &= -\int \sin x(\sin x + \cos x)dx = \int (-\sin^2 x - \sin x \cos x)dx \\
&= \frac{1}{2}\int (\cos 2x - 1 - \sin 2x)dx = -\frac{x}{2} + \frac{1}{4}(\sin 2x + \cos 2x) + \tilde{C}_1, \\
C_2(x) &= \int \cos x(\sin x + \cos x)dx = \int (\cos^2 x + \sin x \cos x)dx \\
&= \frac{1}{2}\int (\cos 2x + 1 + \sin 2x)dx = \frac{x}{2} + \frac{1}{4}(\sin 2x - \cos 2x) + \tilde{C}_2.
\end{aligned} \tag{10.1.24}$$

The symbols $\tilde{C}_1$ and $\tilde{C}_2$ refer here to some authentic constants.

The complete solution of the Eq. (10.1.9) is obtained upon inserting the expressions for $C_1(x)$ and $C_2(x)$ into (10.1.16):

$$\begin{aligned}
y(x) &= \tilde{C}_1 \cos x + \tilde{C}_2(x) \sin x + \left(-\frac{x}{2} + \frac{1}{4}(\sin 2x + \cos 2x)\right) \cos x \\
&+ \left(\frac{x}{2} + \frac{1}{4}(\sin 2x - \cos 2x)\right) \sin x \\
&= \underbrace{\tilde{C}_1 \cos x + \tilde{C}_2 \sin x}_{\text{GSHE}} + \underbrace{\frac{1}{4}(2x + 1) \sin x + \frac{1}{4}(-2x + 1) \cos x}_{\text{SSNHE}},
\end{aligned} \tag{10.1.25}$$

where, in order to simplify the result, the following formulas have been used:

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta,$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta, \tag{10.1.26}$$

for $\alpha = 2x$ and $\beta = x$. As one can see, the general solution of the Eq. (10.1.9) has the structure known from Sect. 9.3:

$$\text{GSNHE} = \text{GSHE} + \text{SSNHE}. \tag{10.1.27}$$

Notice that the "general" character of the solution of a nonhomogeneous equation comes from the "general" character of the solution of the homogeneous equation, as it has already been discussed.

The two so far arbitrary constants $\tilde{C}_1$ and $\tilde{C}_2$ will be determined with the use of the initial conditions: $y(0) = 1$ and $y'(0) = 0$. It should already be clear to the reader from the lecture that the fact of the presence of *two* constants and the need to exploit *two* initial conditions (for y and y') follows from the *second* order of the Eq. (10.1.9). Analogously, when one considers the motion of a point-like mass under the action of a certain known force in classical mechanics, it is well known that in order to find the position at a specific time, apart from the differential equation itself (i.e., equation of the motion), one also needs the *two* values: initial position of the mass and its initial velocity. Thus one has

$$\begin{aligned} y(0) = 1 &\Longrightarrow \tilde{C}_1 + \frac{1}{4} = 1 \quad \Longrightarrow \quad \tilde{C}_1 = \frac{3}{4}, \\ y'(0) = 0 &\Longrightarrow \tilde{C}_2 + \frac{1}{4} - \frac{1}{2} = 0 \quad \Longrightarrow \quad \tilde{C}_2 = \frac{1}{4}. \end{aligned} \tag{10.1.28}$$

As mentioned above, the full solution of a nonhomogeneous linear equation with constant coefficients can also be sometimes found using "predictions." These predictions refer to SSNHE, since GSHE is well known and given by the formula such as (10.1.14) with any constants (let us call them C_1 and C_2). This method will be described in detail in the following problem. Here, although it may seem somewhat mysterious, SSNHE, denoted with $y_s(x)$, will be postulated in the form of

$$y_s(x) = D_1 x \cos x + D_2 x \sin x. \tag{10.1.29}$$

It comes from analyzing the right-hand side of (10.1.9), where there are sine and cosine functions multiplied by constants (which may be regarded as polynomials of zero degree). We, therefore, predict y_s to be in the same form, except that it is necessary to raise the degrees of polynomials by 1 due to the "resonance"—sine and cosine on the right-hand side of (10.1.9) have the same "frequencies" as those occurring in y_0 (cf. (10.1.14)). We will return to this issue and to formula (10.1.29) in the following exercise, and now it will only be checked what one gets by inserting the "guessed" solution in the form of

$$y(x) = y_0(x) + y_s(x) \tag{10.1.30}$$

into the Eq. (10.1.9). After having performed the needed differentiations, one obtains

$$\begin{aligned} &-C_1 \cos x - C_2 \sin x - 2D_1 \sin x - D_1 x \cos x + 2D_2 \cos x - D_2 x \sin x \\ &+C_1 \cos x + C_2 \sin x + D_1 x \cos x + D_2 x \sin x = \sin x + \cos x \end{aligned}$$

$$\Longrightarrow \quad -2D_1 \sin x + 2D_2 \cos x = \sin x + \cos x$$
$$\Longrightarrow \quad D_1 = -\frac{1}{2} \quad \wedge \quad D_2 = \frac{1}{2}. \tag{10.1.31}$$

Plugging these values into (10.1.29) and next into (10.1.30), it is easy to see that the expression identical to (10.1.25) has been obtained. The differences between them are limited to the definitions of arbitrary constants and will completely disappear after exploiting the initial conditions.

Problem 3

The solution of the differential equation:

$$y''' - 4y'' + 4y' = e^{2x} + 1 + x \tag{10.1.32}$$

satisfying the conditions: $y(0) = 0$, $y'(0) = 1/2$, and $y'(0) = 0$, will be found.

Solution

As in the previous example, we are dealing here with a linear, nonhomogeneous equation with constant coefficients, so the familiar scheme of proceeding can be used. This equation, of the third order, can be easily brought to that of the second one by integrating both sides. This possibility is owed to the absence of the function y on the left-hand side where only its derivatives occur. However, below we are not going to take advantage of this simplification.

1. *We solve linear homogeneous equation.*
   First let us omit the right-hand side and look for the general solution of

$$y''' - 4y'' + 4y' = 0. \tag{10.1.33}$$

Its characteristic equation has the form:

$$\lambda^3 - 4\lambda^2 + 4\lambda = 0, \quad \text{or} \quad \lambda(\lambda - 2)^2 = 0. \tag{10.1.34}$$

It has two roots: $\lambda_1 = 0$ with multiplicity $p_1 = 1$ and $\lambda_2 = 2$ with multiplicity $p_2 = 2$. In such a situation—already discussed in the first problem of the present section—the general solution may be written as

$$y_0(x) = Ae^{0x} + (Bx + C)e^{2x} = A + (Bx + C)e^{2x}, \tag{10.1.35}$$

A, B, and C being constants.

2. *We "vary" the constants.*
Now we will seek the full solution of the Eq. (10.1.36), assuming that A, B, and C are, in fact, functions of the variable x:

$$y(x) = A(x) + (B(x)x + C(x))e^{2x}. \tag{10.1.36}$$

To this end, one has to calculate the three derivatives needed in (10.1.32). After having found the first one:

$$y' = A' + (B'x + C')e^{2x} + (2Bx + 2C + B)e^{2x}, \tag{10.1.37}$$

the supplementary condition is imposed:

$$A' + (B'x + C')e^{2x} = 0. \tag{10.1.38}$$

Consequently in the formula for the y'', the second derivatives of A, B, and C do not appear and one obtains

$$y''(x) = (2B'x + 2C' + B')e^{2x} + 4(Bx + C + B)e^{2x}. \tag{10.1.39}$$

The new complementary condition takes the form:

$$2B'x + 2C' + B' = 0. \tag{10.1.40}$$

The third (and last) derivative which we need is given by the expression:

$$y'''(x) = 4(B'x + C' + B')e^{2x} + 4(2Bx + 2C + 3B)e^{2x}. \tag{10.1.41}$$

Upon inserting y', y'', and y''' into Eq. (10.1.32), it can be observed that all terms not containing derivatives of A, B, and C disappear and we get, together with (10.1.38) and (10.1.40), the set of three equations for three unknowns A', B', and C':

$$\begin{cases} A' + B'xe^{2x} + C'e^{2x} = 0, \\ B'(2x+1) + 2C' = 0, \\ 4B'(x+1)e^{2x} + 4C'e^{2x} = e^{2x} + 1 + x. \end{cases} \tag{10.1.42}$$

Upon multiplying the second equation by $-2e^{2x}$ and adding it to the last one, one obtains

$$\begin{aligned} &2B'e^{2x} = e^{2x} + 1 + x \implies \\ &B' = \frac{1}{2}\,e^{-2x}\left(e^{2x} + 1 + x\right) = \frac{1}{2} + \frac{1}{2}(1+x)e^{-2x}. \end{aligned} \tag{10.1.43}$$

The simple integration leads to

$$B(x) = \frac{1}{2}x - \frac{1}{8}(3 + 2x)e^{-2x} + \tilde{B}, \tag{10.1.44}$$

where $\tilde{B}$ is an integration constant.

At this point, before proceeding with calculations, it is worthwhile to note some observations. In the Eq. (10.1.43), the factor e^{2x} on the left-hand side came from the solution of the homogeneous equation ("2" in the exponent is just one of the characteristic roots), and its right-hand side is simply the "nonhomogeneity" of the Eq. (10.1.32). While calculating B', after having multiplied both sides by e^{-2x}, the exponential factors in the first term canceled since the nonhomogeneity contained identical factor. Consequently, only a constant remained, i.e., a polynomial of zero degree. Integrating this polynomial, when passing from (10.1.43) to (10.1.44), this degree increases by 1. In turn, the second term on the right-hand side of (10.1.43) does not contain an exponential factor (it would be more precise to say that it contains e^{0x}). The exponential factors cannot cancel and the polynomial $1 + x$ has not raised its degree (although obviously it has changed). So, where the exponential factors were identical, there was an increase in the degree of a given polynomial and where they were different, such an effect did not appear. This observation will be useful to us later when formulating the method of "predictions."

After having found $B(x)$, the calculating of $C(x)$ and $A(x)$ does not pose any problem. The second of equations (10.1.42) gives

$$C' = -\frac{1}{2}(2x + 1)B' = -\frac{1}{4}\left[1 + 2x + (1 + 3x + 2x^2)e^{-2x}\right], \tag{10.1.45}$$

and the integration leads to the following result:

$$C(x) = \underline{-\frac{1}{4}x - \frac{1}{4}x^2} + \frac{1}{16}(7 + 10x + 4x^2)e^{-2x} + \tilde{C}. \tag{10.1.46}$$

It should be noted again that the degree of the polynomial, which constituted the factor in front of e^{2x} on the right-hand side of (10.1.32)—as we remember, it was a constant, i.e., a polynomial of zero degree—increased by 1 (as underlined in formula (10.1.46)). This is a consequence of the presence of $x \cdot B'(x)$ in the second of equations (10.1.42), and thereby it stems from the fact that the second root of the characteristic equation (10.1.34) has multiplicity 2. Let us summarize then: the polynomial multiplying the exponential factor e^{2x} in the Eq. (10.1.32) (in the nonhomogeneity), due to the coincidence with a characteristic root of multiplicity *two*, increased its degree also by *two*. Let us remember this observation.

The calculation of $A(x)$ is similar. One finds A' from the first of the equations (10.1.42), and an easy integration leads to

$$A(x) = \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{8}e^{2x} + \tilde{A}. \tag{10.1.47}$$

By plugging the obtained formulas into (10.1.36), one gets the general solution of the Eq. (10.1.36):

$$y(x) = \underbrace{\tilde{A} + (\tilde{B}x + \tilde{C})e^{2x}}_{\text{GSHE}} + \underbrace{\frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{4}x^2e^{2x}}_{\text{SSNHE}}. \tag{10.1.48}$$

In the SSNHE, the terms already contained in the GSHE have been omitted since their inclusion would only lead to redefining still undetermined constants $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$.

Let us stop for a moment at the SSNHE. It has the form derived from the right side of (10.1.32) with some modifications. One of them is clear: the raising of the polynomial degree by 2 (at e^{2x}). We have already signaled that this is due to the appearance of the characteristic root ($\lambda = 2$) of the homogeneous equation with multiplicity 2 in the exponent on the right-hand side. What may seem puzzling, however, is the second amendment: in the place of the first-degree polynomial $(1 + x)$, there appears now a second-degree polynomial $(x/2 + x^2/8)$. A moment of reflection, however, explains the riddle. There is also a second coincidence! The polynomial $1 + x$ is, after all, nothing other than $(1 + x)e^{0x}$, and 0 also appears in the list of characteristic roots. It has multiplicity 1, thus raising the polynomial degree again by 1.

In order to determine the constants $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, we are going to exploit the conditions: $y(0) = 0$, $y'(0) = 1/2$, and $y''(0) = 0$. The following set of equations is obtained:

$$\begin{cases} \tilde{A} + \tilde{C} = 0, \\ \tilde{B} + 2\tilde{C} + 1/2 = 1/2, \\ 4\tilde{B} + 4\tilde{C} + 3/4 = 0, \end{cases} \tag{10.1.49}$$

out of which one finds $\tilde{A} = -3/16$, $\tilde{B} = -3/8$, and $\tilde{C} = 3/16$.

Thereby, the problem has been solved, but in accordance with what was announced, we are going to discuss below the second way to find SSNHE: the so-called method of predictions. As mentioned in the previous problem, it cannot be applied to arbitrary nonhomogeneity in the equation, but fortunately it can for the right-hand side of (10.1.32). This method is based on the observation that the expression $w(x)e^{\rho x}$, where $w(x)$ is a certain polynomial and ρ is a constant, changes its nature neither under differentiation nor integration, although, of course, may be subject to a modification of the polynomial degree and its coefficients. So, if the right-hand side of a nonhomogeneous equation is just of that structure (and because of the linearity of the integration and the differentiation as well as of the differential equation itself, it will remain valid also in the case of several terms of this type with arbitrary polynomials and different constants ρ), then $y_s(x)$ (i.e., SSNHE) must have an identical structure too.

Now let us look at the right-hand side of (10.1.32), i.e., at the expression:

$$e^{2x} + 1 + x = 1 \cdot e^{2x} + (1 + x) \cdot e^{0x}. \tag{10.1.50}$$

The first term is a zero-degree polynomial multiplied by an exponential factor. If none of the characteristic roots of the Eq. (10.1.34) were equal to 2, then one would expect y_s to be again a polynomial of the zero degree (i.e., a constant) times the factor e^{2x}. However, 2 *is* the characteristic root—this situation can be called a "resonance," although using this term has physical justification only for imaginary roots—and, what is more, of multiplicity 2, which entails the increasing of the polynomial degree by the same value (it was noticed earlier, while applying the "varying coefficients" method). Thus, our prediction could formally be written as

$$(a_0 + a_1 x + a_2 x^2) \cdot e^{2x}, \tag{10.1.51}$$

but the first and the second terms in the brackets may be omitted since they are already present in $y_0(x)$ (cf. (10.1.35)).

One still needs to take into account the second term in (10.1.50). This time we have the polynomial of the first degree, but again there emerges a "resonance," giving rise to the increase of this degree. The multiplicity of $\lambda = 0$ equals 1, so one expects the following contribution to $y_s(x)$:

$$(b_0 + b_1 x + b_2 x^2) \cdot e^{0x}, \tag{10.1.52}$$

where the first term can be omitted as it has already been included in (10.1.35). As a result, we advocate $y_s(x)$ in the form of

$$y_s(x) = a_2 x^2 e^{2x} + b_1 x + b_2 x^2, \tag{10.1.53}$$

and only constants a_2, b_1, and b_2 remain to be determined. One can now find the solution by plugging $y = y_0 + y_s$ into (10.1.32) or even simply y_s, since y_0 as a solution of the homogeneous equation will fully disappear from the equation. Let us then calculate the set of needed derivatives:

$$\begin{aligned}
y_s'(x) &= 2a_2(x + x^2)e^{2x} + b_1 + 2b_2 x, \\
y_s''(x) &= 2a_2(1 + 4x + 2x^2)e^{2x} + 2b_2, \\
y_s'''(x) &= 4a_2(3 + 6x + 2x^2)e^{2x},
\end{aligned} \tag{10.1.54}$$

and we get an equality, which must be identically satisfied:

$$4a_2 e^{2x} + 4(b_1 - 2b_2 + 2b_2 x) = e^{2x} + 1 + x. \tag{10.1.55}$$

It leads to the equations:

$$\begin{cases} 4a_2 = 1, \\ 4(b_1 - 2b_2) = 1, \\ 8b_2 = 1. \end{cases} \tag{10.1.56}$$

As a result one finds: $a_2 = 1/4$, $b_2 = 1/8$, and $b_1 = 1/2$, which gives

$$y_s(x) = \frac{1}{4}x^2e^{2x} + \frac{1}{2}x + \frac{1}{8}x^2, \tag{10.1.57}$$

which is identical to the SSNHE obtained by the method of varying coefficients (cf. (10.1.48)).

In the previous exercise, we also promised the explanation of formula (10.1.29). Now we have already enough knowledge to justify it. It is sufficient to note that $\sin x$ and $\cos x$ on the right-hand side of (10.1.9) can be temporarily rewritten with the use of known formulas:

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right), \quad \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \tag{10.1.58}$$

and it turns out that the right-hand side has the form needed for predictions, i.e., $w(x)e^{\rho x}$, where $w(x)$ is a polynomial of the zero degree and $\rho = \pm i$. Both these values of ρ are already in the list of characteristic roots of the homogeneous equation (cf. (10.1.11))—and for equations with real coefficients they always appear *in pairs*, which was spoken of when writing out expression (10.1.15)—with multiplicities equal to 1. Therefore, by the same number the polynomial degree should be raised. After having predicted $y_s(x)$ in this way:

$$y_s(x) = (A_1x + B_1)e^{ix} + (A_2x + B_2)e^{-ix}, \tag{10.1.59}$$

one can come back to trigonometric functions using formulas (10.1.13), getting

$$\begin{aligned} y_s(x) &= (A_1x + B_1)(\cos x + i\sin x) + (A_2x + B_2)(\cos x - i\sin x) \\ &= \big[\underbrace{(A_1 + A_2)}_{D_1}x + \underbrace{B_1 + B_2}_{E_1}\big]\cos x + \big[\underbrace{i(A_1 - A_2)}_{D_2}x + \underbrace{i(B_1 - B_2)}_{E_2}\big]\sin x \\ &= (D_1x + E_1)\cos x + (D_2x + E_2)\sin x. \end{aligned} \tag{10.1.60}$$

The constants E_1 and E_2 can be now put to zero, since such terms have already been taken into account in the formula for y_0 (cf. (10.1.14)), and thus y_s becomes identical to (10.1.29). It should be noted, however, that there is no need to pass each time from trigonometric functions to exponentials and vice versa. The method of predictions can be applied directly to nonhomogeneities such as $w(x)\sin(\rho x)$ and $w(x)\cos(\rho x)$, if one remembers that the "resonance" occurs when in the list of characteristic roots not ρ but $\pm i\rho$ appears. This is what was done in the former problem, directly writing out the appropriate form of $y_s(x)$. However, it must be noted that if the nonhomogeneity contains only one of the trigonometric functions (either sine or cosine), one still has to predict SSNHE in general terms with both of them. For example, if the right-hand side of the Eq. (10.1.9) equaled simply $\sin x$, our prediction would still have the form (10.1.29). For, the differentiations and integrations convert the sine and cosine functions into each other.

10.2 Using Various Specific Methods

Problem 1

The solution of the equation:

$$9(x+1)y' = (y'')^2 \tag{10.2.1}$$

satisfying the conditions: $y(0) = 1/4$ and $y'(0) = 1$, will be found.

Solution

A characteristic feature of the equation above immediately noticeable is the presence of the first and the second derivatives of $y(x)$ and the absence of the function itself. We already met the same situation in the Eq. (10.1.32). As we saw in such a case, it is possible to reduce the order of the equation by one. In this problem, the easiest way to do that is to introduce—in place of y—a new dependent variable defined as:

$$z(x) = y'(x). \tag{10.2.2}$$

Equation (10.2.1) takes then the form:

$$9(x+1)z = (z')^2 \tag{10.2.3}$$

and becomes a first-order equation.

The right-hand side of (10.2.3) cannot be negative, so the same must be true for the left-hand one too. The expressions $x + 1$ and z are, therefore, of equal signs. Guided by the initial conditions as given above, we assume that we are interested only in the domain $x > -1$ (which also means that $z > 0$).

In order to separate variables, as it was done in Sect. 9.1, one needs to get rid of the square on the right-hand side. So there are two options: either $z' = 3\sqrt{(x+1)z}$ or $z' = -3\sqrt{(x+1)z}$. Considering first the former case, one can write

$$\frac{dz}{\sqrt{z}} = 3\sqrt{x+1}\,dx. \tag{10.2.4}$$

The integration of both sides gives

$$2\sqrt{z} = 2(x+1)^{3/2} + 2C, \tag{10.2.5}$$

a constant being denoted with the symbol $2C$ for the convenience below. Now it is easy to calculate z. It is expressed by the formula:

$$z = \left[(x+1)^{3/2} + C\right]^2. \tag{10.2.6}$$

To obtain an expression for y, one just has to recall that $z = y'$ and integrate the right-hand side of (10.2.6):

$$\begin{aligned} y(x) &= \int \left((x+1)^{3/2} + C\right)^2 dx = \int \left((x+1)^3 + 2C(x+1)^{3/2} + C^2\right) dx \\ &= \frac{1}{4}(x+1)^4 + \frac{4}{5}C(x+1)^{5/2} + C^2 x + D, \end{aligned} \tag{10.2.7}$$

D being a constant.

In the case when $z' = -3\sqrt{(x+1)z}$, in the Eqs. (10.2.4) and (10.2.5) there appears the additional "minus" on the right-hand sides. An integration constant C is however indefinite at this stage and equally well could have been called "$-C$," and then this "minus" might be factored out. Squaring both sides of (10.2.5) leads again to the results given by formulas (10.2.6) and (10.2.7).

Now we are going to fix the constants C and D using initial conditions. By inserting $x = 0$ into the formulas for $y(x)$ and $y'(x)$, i.e., for $z(x)$, the set of equations is obtained:

$$\begin{cases} \dfrac{4}{5}C + D + \dfrac{1}{4} = \dfrac{1}{4}, \\ (C+1)^2 = 1, \end{cases} \tag{10.2.8}$$

from which it follows that either $C = 0$ and $D = 0$, or $C = -2$ and $D = 8/5$. In the former case one has

$$2(x+1)^{3/2} + 2C > 0, \tag{10.2.9}$$

and therefore, it corresponds to the choice of $z' = 3\sqrt{(x+1)z}$. This is clear, since the left-hand side of the Eq. (10.2.5) is positive, and it only makes sense if the same can be said about the other. For these constants, the function $y(x)$ reduces to the simple form:

$$y(x) = \frac{1}{4}\,(x+1)^4,$$

about which one knows that it is convex. It is obvious that in this situation the inequality $z' = y'' > 0$ had to be satisfied.

For $C = -2$ and $D = 8/5$, the sign of expression (10.2.9) depends on the value of x and gets changed for $x = \sqrt[3]{4} - 1$, which is again consistent with the fact that the obtained function:

$$y(x) = \frac{1}{4}(x+1)^4 - \frac{8}{5}(x+1)^{5/2} + 4x + \frac{8}{5} \tag{10.2.10}$$

has a point of inflection here. However, to fulfill the equation (which is equivalent to (10.2.5)):

$$\sqrt{z} = 2 - (x+1)^{3/2}, \tag{10.2.11}$$

one needs to cut the domain to the interval $]-1, \sqrt[3]{4} - 1[$ in order to ensure the right-hand side remains positive.

Problem 2

The solution of the equation:

$$y\,y'' + y^3 y' = (y')^2 \tag{10.2.12}$$

satisfying the conditions: $y(1) = 1$ and $y'(1) = 3/2$, will be found.

Solution

What is striking in an equation similar to the above one is no apparent dependence on x. Formally x acts as an independent variable and y as dependent one, but not always the explicit calculation of $y(x)$ is possible. Sometimes—as in the problems of Sect. 9.4—one has to work with its implicit form, and sometimes it is desirable to find $x(y)$ instead of $y(x)$, without any guarantee that this relation can be inverted. As we will see in a moment in the equations of the type (10.2.12), it is worth doing so. The variable y will be chosen as the independent one and its derivative as dependent thanks to the substitution $y' = y'(x(y))$. At the same time, it is useful to denote y' with the symbol z, which entails the reduction of the order of the equation by one.

Before we proceed with our plan, we need to make sure that any solution will not be lost in this way. A constant function does not have its inverse anywhere, so this procedure cannot give us this kind of solution (if such a solution at all exists). Other functions—and we are talking only about differentiable ones—at least locally are reversible (see Sect. 8.2), so our procedure will be reasonable.

The first step is, therefore, to plug $y = \text{const}$ into (10.2.12) and verify whether the equation is satisfied. Since all derivatives of a constant equal zero, and consequently all the terms on the left- and right-hand sides vanish, the equation actually is met. The value of the constant must be equal to 1 upon exploiting the first initial condition. Unfortunately, one is not able to satisfy the requirement $y'(1) = 3/2$, so any constant solution must be rejected. Below we are going to look for other ones assuming that $y \neq \text{const}$, and, therefore, that at least in a certain domain its derivative does not vanish.

In order to obtain the closed equation for the function $z(y)$, one has yet to get rid of y''. For this purpose we shall use the formula for the derivative of the composite function:

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}z = \frac{dz}{dy} \cdot \frac{dy}{dx} = z' \cdot z. \tag{10.2.13}$$

This leads to the following form of the Eq. (10.2.12):

$$yzz' + y^3 z = z^2 \implies z(yz' - z + y^3) = 0. \tag{10.2.14}$$

It appears that either $z = 0$ (but this case has already been dealt with, as it means that $dy/dx = 0$, i.e., $y = \text{const}$), or

$$yz' - z = -y^3. \tag{10.2.15}$$

Let us recall again: z plays here the role of the dependent variable (the function), and the independent variable (the argument) is y. The prime denotes the differentiation over y.

Equation (10.2.15) is of a well-known form. It is the linear nonhomogeneous first-order equation, and the procedure of solving it was formulated in Problem 1 of Section (9.3). It consists of two steps: solving the homogeneous linear equation and "varying" a constant.

- *We solve the homogeneous equation.*
  After having omitted the right-hand side, one obtains the equation:

$$yz' - z = 0, \tag{10.2.16}$$

  which can be solved by separation of variables. Rewriting it as

$$\frac{z'}{z} = \frac{1}{y} \tag{10.2.17}$$

and integrating on both sides over y, one finds

$$\log\left|\frac{z}{C}\right| = \log|y|, \quad \text{i.e.,} \quad |z| = |Cy|. \tag{10.2.18}$$

Due to the indefinite sign of the constant C, one can omit the absolute value symbols on both sides and simply write $z = Cy$.

- *We "vary" the constant.*
  Now let us assume that

$$z(y) = C(y)y \tag{10.2.19}$$

and insert z in this form into (10.2.15), getting after some simplification

$$y^2 C'(y) = -y^3 \implies C'(y) = -y. \tag{10.2.20}$$

Solving this simple equation, one finds

$$C(y) = -\frac{y^2}{2} + \tilde{C}, \tag{10.2.21}$$

where $\tilde{C}$ is now a "true" constant.

This result indicates that the function $z(y)$ has the form:

$$z(y) = -\frac{y^3}{2} + \tilde{C}y, \tag{10.2.22}$$

or equivalently

$$y' = \frac{dy}{dx} = -\frac{y^3}{2} + \tilde{C}y. \tag{10.2.23}$$

Thus one can write

$$\int \frac{dy}{-y^3/2 + \tilde{C}y} = \int dx. \tag{10.2.24}$$

The integral on the left-hand side can be found for sure: rational functions are integrated in accordance with the rules formulated in Volume 1 (Sect. 14.3). First the denominator is factorized and then the whole function is expanded into partial fractions. However, this expansion is $\tilde{C}$ dependent, so it is a good idea to first establish its value, using the appropriate initial condition. Otherwise, one would have to separately deal here with all possible cases. Since $y'(1) = 3/2$ and $y(1) = 1$, from (10.2.23) we find $\tilde{C} = 2$. For this value, the expansion of the integrand expression into partial fractions proceeds as follows:

$$\frac{1}{-y^3/2+2y} = -\frac{2}{y(y^2-4)} = -\frac{2}{y(y-2)(y+2)} = \frac{1}{2y} - \frac{1}{4(y-2)} - \frac{1}{4(y+2)}. \tag{10.2.25}$$

Every single fraction is now integrated without any difficulty, getting in place of (10.2.24):

$$\frac{1}{2}\log\left|\frac{y}{D}\right| - \frac{1}{4}\log\left|y^2-4\right| = x, \tag{10.2.26}$$

where D is an integration constant. Incorporating the initial condition, one finds

$$\frac{1}{2}\log\frac{1}{|D|} - \frac{1}{4}\log|1^2-4| = 1 \implies |D| = \frac{\sqrt{3}}{3e^2} \tag{10.2.27}$$

and at the end

$$\log\frac{y^2}{|y^2-4|} = 4x - 4 - \log 3. \tag{10.2.28}$$

This equation can be easily inverted if necessary. First one gets

$$\frac{y^2}{y^2-4} = \pm\frac{1}{3}\,e^{4x-4}, \tag{10.2.29}$$

where the initial condition obliges us to select "−." Then y is obtained in the form of

$$y = 2\sqrt{\frac{1}{1+3e^{4-4x}}}, \tag{10.2.30}$$

where, this time, when calculating $\sqrt{y^2}$, "+" has been chosen, again due to the initial condition.

Problem 3

The general solution of the equation:

$$x^2y'' - (2x^2+x)y' + (x^2+x)y = x^3e^x, \tag{10.2.31}$$

will be found for $x > 0$.

Solution

Solving a linear differential equation of the second or higher order with coefficients which are variable dependent is generally a difficult task. Unfortunately there exists no universal algorithm—such as that we had at our disposal addressing equations with constant coefficients—which would always lead us to the result in a compact form (incidentally this result may not exist at all). However, there are some methods that can be applied in certain specific situations. For (10.2.31), such a possibility constitutes the so-called Liouville's method, spoken of below.

Let us imagine that we have to solve an equation of the form:

$$a(x)y'' + b(x)y' + c(x)y = d(x) \tag{10.2.32}$$

with appropriately smooth functions $a(x)$, $b(x)$, $c(x)$, and $d(x)$. Suppose that in some way one has managed to guess one of the solutions of the homogeneous equation, i.e., one knows a (nonzero) function $y_1(x)$ satisfying

$$a(x)y_1'' + b(x)y_1' + c(x)y_1 = 0. \tag{10.2.33}$$

Some other (independent and nonzero) solution $y_2(x)$ has to satisfy the identical equation:

$$a(x)y_2'' + b(x)y_2' + c(x)y_2 = 0. \tag{10.2.34}$$

Now, after multiplying the former by y_2, and the latter by y_1 and subtracting from each other, the terms without derivatives cancel and one obtains

$$a(x)(y_2'' \cdot y_1 - y_1'' \cdot y_2) + b(x)(y_2' \cdot y_1 - y_1' \cdot y_2) = 0. \tag{10.2.35}$$

It should be noted that the expression in the first brackets can be rewritten as

$$y_2'' \cdot y_1 - y_1'' \cdot y_2 = \left[y_2' \cdot y_1 - y_1' \cdot y_2\right]', \tag{10.2.36}$$

since the terms constituting the products of first derivatives cancel. Let us now denote

$$W(x) := y_2'(x) \cdot y_1(x) - y_1'(x) \cdot y_2(x). \tag{10.2.37}$$

This quantity is called a *Wronskian* and as it can be easily seen, it constitutes the determinant of the matrix:

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}. \tag{10.2.38}$$

After having used (10.2.36), the Eq. (10.2.35) becomes an easy-to-solve differential equation of the first order for $W(x)$:

$$a(x)W'(x) + b(x)W(x) = 0. \tag{10.2.39}$$

This type of equation was addressed in Sect. 9.1, so the reader should not be surprised revealing that the solution of (10.2.39) has the form:

$$W(x) = W_0 e^{-\int \frac{b(x)}{a(x} dx}, \tag{10.2.40}$$

where W_0 is a constant.

So now that we have already an explicit form of $W(x)$, (10.2.37) becomes a linear equation for the function $y_2(x)$:

$$y_1(x)y_2'(x) - y_1'(x)y_2(x) = W_0 e^{-\int \frac{b(x)}{a(x)} dx}, \tag{10.2.41}$$

which can be solved by first separating variables (homogeneous equation), and then "varying" a constant (nonhomogeneous equation).

Below this entire procedure is going to be applied to the Eq. (10.2.31). One has to start by guessing any solution of the equation:

$$x^2 y'' - (2x^2 + x)y' + (x^2 + x)y = 0. \tag{10.2.42}$$

Unfortunately, there is no universal prescription for it other than using one's own intuition and imagination. In our case, a guideline might be that the sum of the coefficient functions of y'', y', and y equals zero: $x^2 - (2x^2 + x) + x^2 + x = 0$. So if it occurs $y'' = y' = y$, we could factor them out and the Eq. (10.2.42) would be met. As one knows, there exists such a function that after having been differentiated does not change its form: it is the exponential function. Let us then insert $y_1(x) = e^x$, and we obtain

$$x^2 e^x - (2x^2 + x)e^x + (x^2 + x)e^x = 0. \tag{10.2.43}$$

Thereby we have succeeded in finding $y_1(x)$! Let us now plug it into (10.2.41), which yields

$$\begin{aligned} e^x y_2'(x) - e^x y_2(x) &= W_0 e^{-\int \frac{b(x)}{a(x)} dx} = W_0 e^{\int \frac{2x^2+x}{x^2} dx} = W_0 e^{\int \left(2+\frac{1}{x}\right) dx} \\ &= W_0 e^{2x+\log|x|+C} = C_1 x e^{2x}. \end{aligned} \tag{10.2.44}$$

The absolute value at x has been omitted in view of the condition that $x > 0$, given in the text of the problem, and the symbol of a new constant C_1 has been introduced to denote $W_0 e^C$. Canceling the exponential factors, one comes to the equation:

$$y_2'(x) - y_2(x) = C_1 x e^x. \tag{10.2.45}$$

The solution of the homogeneous part of this equation is obvious: it is simply Ce^x. "Varying" the constant C, we advocate y_2 in the form of

$$y_2(x) = C(x)e^x, \tag{10.2.46}$$

and after inserting it into (10.2.45), the following equation for $C(x)$ is got:

$$\begin{aligned} C'(x)e^x = C_1 x e^x &\Longrightarrow C'(x) = C_1 x \\ &\Longrightarrow C(x) = C_1 \frac{x^2}{2} + C_2. \end{aligned} \tag{10.2.47}$$

Now we know enough to write the general solution of the homogeneous equation (10.2.42):

$$y_0(x) = C_1 \frac{x^2}{2} e^x + C_2 e^x. \tag{10.2.48}$$

At this point, a question could be raised, why does one claim that y_0 given by the above formula is a *general* solution? After all, one has been looking only for y_2, i.e., for the second *special* solution. Well, it should be noted that, among the solutions of the Eq. (10.2.41), there appears also y_1, if the constant W_0 (and therefore, also C_1) is set equal to zero. Indeed by choosing $C_1 = 0$ and $C_2 = 1$, one has $y_0(x) = y_1(x)$. Therefore, the solution y_2 from the very beginning should have been called y_0 because it is not so much *the second* but the *full* solution of the homogeneous equation.

The final step which remains to be done is to include the right-hand side of (10.2.31) by "varying" the constants C_1 and C_2. This method applied in Sect. 10.1 to linear equations with *constant* coefficients shall also be effective for equations with *variable* ones—one needs only the equation to be linear. So one writes

$$y(x) = C_1(x) \frac{x^2}{2} e^x + C_2(x) e^x \tag{10.2.49}$$

and follow Sect. 10.1. We first calculate

$$y'(x) = C_1' \frac{x^2}{2} e^x + C_2' e^x + C_1 x e^x + C_1 \frac{x^2}{2} e^x + C_2 e^x \tag{10.2.50}$$

and require that

$$C_1' \frac{x^2}{2} e^x + C_2' e^x = 0. \tag{10.2.51}$$

The need to impose supplementary condition was pointed out earlier and its justification will be given in Problem 4 of Sect. 11.2. The next step consists in calculating the second derivative:

$$y''(x) = C_1' \left(x + \frac{x^2}{2} \right) e^x + C_2' e^x + C_1 \left(1 + 2x + \frac{x^2}{2} \right) e^x + C_2 e^x \tag{10.2.52}$$

and plugging y, y', and y'' into (10.2.31). As a result, one gets (apart from (10.2.51)) the second equation for C_1' and C_2':

$$x^2 \left[C_1' \left(x + \frac{x^2}{2} \right) e^x + C_2' e^x \right] = x^3 e^x. \tag{10.2.53}$$

Simplifying the expressions, one sees that the following equations are to be solved:

$$\begin{cases} C_1' \dfrac{x^2}{2} + C_2' = 0, \\ C_1' \left(x + \dfrac{x^2}{2} \right) + C_2' = x. \end{cases} \tag{10.2.54}$$

The easiest way to solve this system is to subtract both equations. As a result one finds

$$C_1' x = x \implies C_1' = 1 \implies C_1 = x + 2\tilde{C}_1. \tag{10.2.55}$$

The "2" in front of the constant $\tilde{C}_1$ is introduced only for convenience. The determination of C_2 is equally easy:

$$C_2' = -\frac{x^2}{2} \implies C_2 = -\frac{x^3}{6} + \tilde{C}_2. \tag{10.2.56}$$

Finally one has

$$y(x) = \tilde{C}_1 x^2 e^x + \tilde{C}_2 e^x + \frac{x^3}{2} e^x - \frac{x^3}{6} e^x = \tilde{C}_1 x^2 e^x + \tilde{C}_2 e^x + \frac{x^3}{3} e^x. \tag{10.2.57}$$

Problem 4

The general solution of the equation:

$$x^2 y'' + 3xy' + y = \log x + x \tag{10.2.58}$$

will be found for $x > 0$.

Solution

Based on the previous exercise, we first handle the homogeneous equation:

$$x^2y'' + 3xy' + y = 0. \tag{10.2.59}$$

Inspired by the structure of the expression on the left-hand side, we are going to seek a specific solution in the power-law form: $y_1(x) = x^\alpha$, where α is a constant. The reason for this idea is the observation that the derivative of the power expression lowers the exponent by 1 (except for the case $\alpha = 0$). However, each derivative in (10.2.59) is accompanied by a coefficient whose exponent is identical to the order of a given derivative. Consequently all terms should restore their initial form x^α. With the correct choice of α, these terms have a chance to cancel and one will get zero. Therefore, let us insert the proposed y_1 into (10.2.59):

$$x^2y_1'' + 3xy_1' + y_1 = [\alpha(\alpha-1) + 3\alpha + 1]\, x^\alpha = 0, \tag{10.2.60}$$

getting the condition:

$$\alpha(\alpha-1) + 3\alpha + 1 = \alpha^2 + 2\alpha + 1 = (\alpha+1)^2 = 0. \tag{10.2.61}$$

Obviously, it is satisfied for $\alpha = -1$, which means that a specific solution has just been found:

$$y_1(x) = \frac{1}{x}. \tag{10.2.62}$$

To find another solution (or rather, as we know from the previous problem, the general one), one can use formula (10.2.41) which is a differential equation for $y_2(x)$. However, as we would like to familiarize the reader with another variant of the Liouville's method, which can turn out to be useful too, we are proceeding further in some other way. Namely, we are going to look for the general solution in the form $y_0(x) = y_1(x)u(x)$, where $u(x)$ is a function to be determined. By plugging it into (10.2.59), one obtains the equation:

$$x^2(y_1''u + 2y_1'u' + y_1u'') + 3x(y_1'u + y_1u') + y_1u = 0, \tag{10.2.63}$$

which, after elementary transformations, can be written in the form:

$$x^2y_1u'' + (2x^2y_1' + 3xy_1)u' + \underbrace{(x^2y_1'' + 3xy_1' + y_1)}_{=0} = 0. \tag{10.2.64}$$

The expression in the last brackets equals zero, since $y_1(x) = 1/x$ satisfies (10.2.59). Using this fact and substituting the explicit form of $y_1(x)$, one comes to the very simple equation which can be solved immediately:

$$\begin{aligned} xu'' + u' = 0 \Longrightarrow [xu']' = 0 \quad \Longrightarrow \quad xu' = C_1 \\ \Longrightarrow u' = \frac{C_1}{x} \quad \Longrightarrow \quad u(x) = C_1 \log x + C_2. \end{aligned} \tag{10.2.65}$$

This means that the full solution of the homogeneous equation is

$$y_0(x) = C_1 \frac{\log x}{x} + C_2 \frac{1}{x}, \tag{10.2.66}$$

and that of the nonhomogeneous one, after having “varied” the constants C_1 and C_2 is

$$y(x) = C_1(x) \frac{\log x}{x} + C_2(x) \frac{1}{x}. \tag{10.2.67}$$

This expression should be now inserted into (10.2.58). For this purpose the first derivative of y is calculated:

$$y' = C_1' \frac{\log x}{x} + C_2' \frac{1}{x} + C_1 \frac{1 - \log x}{x^2} - C_2 \frac{1}{x^2}, \tag{10.2.68}$$

imposing the condition:

$$C_1' \frac{\log x}{x} + C_2' \frac{1}{x} = 0, \tag{10.2.69}$$

and then the second one:

$$y'' = C_1' \frac{1 - \log x}{x^2} - C_2' \frac{1}{x^2} + C_1 \frac{-3 + 2\log x}{x^3} + C_2 \frac{2}{x^3}. \tag{10.2.70}$$

Upon inserting y'', y', and y to the initial equation and after simplifying (10.2.69), one comes to the following system of equations for C_1' and C_2':

$$\begin{cases} C_1' \log x + C_2' = 0, \\ C_1'(1 - \log x) - C_2' = \log x + x. \end{cases} \tag{10.2.71}$$

Adding them to each other, one gets: $C_1' = \log x + x$ and the simple integration gives

$$C_1 = x(\log x - 1) + \frac{1}{2} x^2 + \tilde{C}_1. \tag{10.2.72}$$

In turn, the first of equations (10.2.71) allows to find C_2:

$$C_2' = -C_1' \log x = -\log^2 x - x \log x. \tag{10.2.73}$$

The calculation of integrals on the right-hand side is elementary (for a moment integration constants are neglected):

$$\begin{aligned}
\int \log^2 x\, dx &= \int [x]' \log^2 x\, dx = x \log^2 x - \int x[\log^2 x]' dx \\
&= x \log^2 x - 2 \int \log x\, dx = x \log^2 x - 2x(\log x - 1), \\
\int x \log x\, dx &= \int \left[\frac{x^2}{2}\right]' \log x\, dx = \frac{x^2}{2} \log x - \frac{1}{2} \int x^2 \left[\log x\right]' dx \\
&= \frac{x^2}{2} \log x - \frac{1}{2} \int x\, dx = \frac{x^2}{2} \log x - \frac{1}{4} x^2.
\end{aligned} \tag{10.2.74}$$

As a result, one has

$$\begin{aligned}
C_2(x) &= -x \log^2 x + 2x(\log x - 1) - \frac{x^2}{2} \log x + \frac{1}{4} x^2 + \tilde{C}_2 \\
&= -x \log^2 x + \left(-\frac{x^2}{2} + 2x\right) \log x + \frac{1}{4} x^2 - 2x + \tilde{C}_2
\end{aligned} \tag{10.2.75}$$

and upon using formula (10.2.67) and reducing terms, the final expression for the function can be written as

$$y(x) = \tilde{C}_1 \frac{\log x}{x} + \tilde{C}_2 \frac{1}{x} + \log x + \frac{1}{4} x - 2. \tag{10.2.76}$$

10.3 Exercises for Independent Work

Exercise 1 Find the general solutions of the equations:

(a) $y'' - 2y' + 10y = 0$,
(b) $y''' + 3y'' + 4y' + 12y = 0$,
(c) $y'' - 4y' + 4y = x + xe^{2x}$,
(d) $y''' - 3y'' + y' - 3y = 2\cos x - \sin x$.

Answers

(a) $y(x) = e^x(C_1 \sin 3x + C_2 \cos 3x)$,
(b) $y(x) = C_1 \sin 2x + C_2 \cos 2x + C_3 e^{-3x}$,
(c) $y(x) = e^{2x}(C_1 + C_2 x) + (2x^3 e^{2x} + 3x + 3)/12$,
(d) $y(x) = (C_1 - x/4) \sin x + (C_2 - x/4) \cos x + C_3 e^{3x}$.

Exercise 2 Solve the Eq. (10.1.32) by first breaking it down to a second order equation and then using the method of "variation" of constants or the method of predictions.

Exercise 3 Find the solutions of the equations satisfying the initial conditions:

(a) $y'' + 4y' + 3y = e^{-3x} + 3x$, $y(0) = -1$, $y'(0) = 0$,
(b) $y''' + 3y'' + 4y' + 2y = e^{-x} \sin x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$.

Answers

(a) $y(x) = e^{-3x}(1/12 - x/2) + e^{-x}/4 + x - 4/3$,
(b) $y(x) = -e^{-x}[\cos x + (x/2 - 1) \sin x - 2]$.

Exercise 4 Find the solutions of the equations:

(a) $y'^2 y'' = \dfrac{1}{3}$,
(b) $\dfrac{3}{2y}\, y'' + 3y^2 y' = y'^2$, satisfying the conditions $y(0) = 1$, $y'(0) = 12$.

Answers

(a) $y(x) = 3(x + C_1)^{4/3}/4 + C_2$,
(b) $y(x) = \sqrt{3} \tan(3\sqrt{3}x + \pi/6)$.

Exercise 5 Solve the Eq. (10.2.31) using the Liouville's method described in Problem 4 of Sect. 10.2, by predicting one of the solutions of the homogeneous equation in the form $y_1(x) = e^x$.

Chapter 11
Solving Systems of First-Order Differential Equations

This last chapter, in which we deal with ordinary differential equations, is concerned with the systems of equations of the first order. The majority of time will be devoted to the following **systems of linear equations with constant coefficients**:

$$\begin{aligned} \frac{dy_1}{dx} &= a_{11}\, y_1(x) + a_{12}\, y_2(x) + \ldots + a_{1n}\, y_n(x) + b_1(x), \\ \frac{dy_2}{dx} &= a_{21}\, y_1(x) + a_{22}\, y_2(x) + \ldots + a_{2n}\, y_n(x) + b_2(x), \\ &\ldots \\ \frac{dy_n}{dx} &= a_{n1}\, y_1(x) + a_{n2}\, y_2(x) + \ldots + a_{nn}\, y_n(x) + b_n(x), \end{aligned} \tag{11.0.1}$$

for some unknown functions $y_1(x), y_2(x), \ldots, y_n(x)$. If all functions $b_i(x)$ identically vanish, the system is called **homogeneous**, if not—**nonhomogeneous**.

In order to solve this system, the **method of elimination of variables** can be used, leading to one equation on the nth order, as that described by (10.0.2). This method may also be applied to systems of nonlinear equations as we will show in the following section.

The system (11.0.1) can be rewritten in the matrix form:

$$\frac{dy}{dx} = Ay(x) + B(x), \tag{11.0.2}$$

where

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ , \ldots \\ y_n(x) \end{bmatrix}, \quad B(x) = \begin{bmatrix} b_1(x) \\ b_2(x) \\ , \ldots \\ b_n(x) \end{bmatrix}, \tag{11.0.3}$$

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and the quadratic matrix A is made from the constant coefficients:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \tag{11.0.4}$$

The most effective method of solving large systems seems to be by exploiting the resolvent. The **resolvent** for the system (11.0.2) is called the matrix $R(x, x_0)$ satisfying

$$\frac{d}{dx}R(x, x_0) = A \cdot R(x, x_0) \quad \text{and} \quad R(x_0, x_0) = \mathbb{1}, \tag{11.0.5}$$

where $\mathbb{1}$ denotes the unit matrix $n \times n$. Then, in the case of $B(x) \equiv 0$, the solution matching the initial conditions specified by the set of values $y(x_0)$ can be written in the form

$$y(x) = R(x, x_0)y(x_0). \tag{11.0.6}$$

For the nonhomogeneous system, instead of (11.0.6), one has

$$y(y) = R(x, x_0)y(x_0) + \int_{x_0}^{x} R(x, x')B(x')dx'. \tag{11.0.7}$$

In Sect. 11.2, the reader will become acquainted in detail with the procedures of finding the resolvent.

11.1 Using the Method of Elimination of Variables

Problem 1

Using the method of the elimination of variables, the general solution of the set of differential equations:

$$\begin{cases} 2\dot{x} - \dot{y} - x = 0, \\ \dot{x} + 2\dot{y} - y = 0, \end{cases} \tag{11.1.1}$$

where $\cdot = d/dt$, will be found.

Solution

One of the possible ways of solving a set of differential equations for several unknown functions is to eliminate subsequent variables and obtain one equation for one unknown function. This equation, naturally, will be of higher order and solving it can eventually pose problems. However, in the previous section, we became acquainted with several methods that can facilitate this task, in so far as the equation can be classified into one of the known types. There are obviously many more methods than that have been chosen, but a complete presentation of this subject is beyond the scope of this textbook.

If one wished to completely eliminate the variable y from the system (11.1.1) and obtain an equation for $x(t)$, one would have to get rid of both y and $\dot{y}$. Surely, it is not sufficient to calculate y from the second equation in the form of

$$y = \dot{x} + 2\dot{y}, \tag{11.1.2}$$

since after differentiating both sides and inserting the result in place of $\dot{y}$ into the first equation, it will in turn contain the second derivative $\ddot{y}$. For the complete removal of y and $\dot{y}$ one must use both equations, writing

$$\begin{cases} \dot{y} = 2\dot{x} - x, \\ y = \dot{x} + 2\dot{y} = \dot{x} + 2(2\dot{x} - x) = 5\dot{x} - 2x, \end{cases} \tag{11.1.3}$$

the latter having been simplified with the use of the former. Having obtained the formulas for y and $\dot{y}$ (expressed only through x), one easily gets from the first equation (the second one would lead to the identity $0 = 0$):

$$5\ddot{x} - 4\dot{x} + x = 0. \tag{11.1.4}$$

As one can see, a well-known linear homogeneous equation with constant coefficients is to be solved. Its characteristic equation is quadratic:

$$5\lambda^2 - 4\lambda + 1 = 0, \tag{11.1.5}$$

with the negative discriminant equal to -4. The characteristic roots are then complex:

$$\lambda_1 = \frac{2}{5} + \frac{1}{5}\,i, \quad \lambda_2 = \frac{2}{5} - \frac{1}{5}\,i, \tag{11.1.6}$$

which means that one has to deal with the situation discussed in the previous chapter. In order to find the function $x(t)$, we can immediately make use of formula (10.1.15), substituting $\lambda_r = 2/5$ and $\lambda_i = 1/5$:

$$x(t) = e^{2t/5}\left[A\cos\left(\frac{1}{5}t\right) + B\sin\left(\frac{1}{5}t\right)\right]. \tag{11.1.7}$$

The second of the unknown functions, i.e., $y(t)$, can be obtained by plugging (11.1.7) into the second equation of (11.1.3):

$$\begin{aligned}
y(t) = 5\dot{x}(t) - 2x(t) &= 5\frac{d}{dt}\left\{e^{2t/5}\left[A\cos\left(\frac{1}{5}t\right) + B\sin\left(\frac{1}{5}t\right)\right]\right\} \\
&\quad -2e^{2t/5}\left[A\cos\left(\frac{1}{5}t\right) + B\sin\left(\frac{1}{5}t\right)\right] \\
&= 5\left\{\frac{2}{5}e^{2t/5}\left[A\cos\left(\frac{1}{5}t\right) + B\sin\left(\frac{1}{5}t\right)\right]\right. \\
&\quad \left. +\frac{1}{5}e^{2t/5}\left[-A\sin\left(\frac{1}{5}t\right) + B\cos\left(\frac{1}{5}t\right)\right]\right\} \\
&\quad -2e^{2t/5}\left[A\cos\left(\frac{1}{5}t\right) + B\sin\left(\frac{1}{5}t\right)\right] \\
&= e^{2t/5}\left[-A\sin\left(\frac{1}{5}t\right) + B\cos\left(\frac{1}{5}t\right)\right].
\end{aligned} \tag{11.1.8}$$

The general result, i.e., expressions (11.1.7) and (11.1.8), depends on two constants: A and B, due to the fact that (11.1.1) is the set of *two* equations of the *first* order $(2 \cdot 1 = 2)$.

Problem 2

Using the method of the elimination of variables, the solution of the set of differential equations:

$$\begin{cases} \dot{z} + \dot{x} = x - y, \\ 2\dot{x} + \dot{y} = -1, \\ \dot{y} + \dot{z} = x - t, \end{cases} \tag{11.1.9}$$

satisfying the condition: $x(0) = y(0) = z(0) = 0$, will be found.

Solution

Proceeding as in the previous exercise, we would like to get an equation for one of the functions: $x(t)$, $y(t)$, or $z(t)$, by eliminating the other two. It seems that the easiest way to do that is to determine x from the last equation (which luckily does not contain $\dot{x}$) and remove it from the other ones. In order to go on, one has to find the expression for $\dot{x}$. Thus, one has

$$x = \dot{y} + \dot{z} + t, \tag{11.1.10}$$

which implies that

$$\dot{x} = \ddot{y} + \ddot{z} + 1. \tag{11.1.11}$$

Eliminating x and $\dot{x}$ from the first and from the second equation of (11.1.9), one obtains

$$\dot{z} + \ddot{y} + \ddot{z} + 1 = \dot{y} + \dot{z} + t - y \implies \ddot{z} = -\ddot{y} + \dot{y} - y + t - 1, \tag{11.1.12}$$

and

$$2(\ddot{y} + \ddot{z} + 1) + \dot{y} = -1. \tag{11.1.13}$$

After having plugged into the former $\ddot{z}$ obtained from the latter, one comes to the equation for the function $y(t)$:

$$3\dot{y} - 2y = -2t - 1. \tag{11.1.14}$$

A nice feature is that it is an equation of the *first* order and not of the *third* one, which could have been expected on the basis of (11.1.9). Such a situation, however, should be treated as an exception and not as a rule.

Equation (11.1.14) is linear, nonhomogeneous, with constant coefficients, so the solving of it is carried out in the standard way. Omitting the right-hand side, one gets the extremely simple equation:

$$3\dot{y} - 2y = 0, \tag{11.1.15}$$

with the obvious solution:

$$y(t) = C\, e^{2t/3}. \tag{11.1.16}$$

The complete solution has the form:

$$y(t) = C(t)\, e^{2t/3}, \tag{11.1.17}$$

which should be inserted into (11.1.14). The equation for varied “constant” C:

$$\dot{C} = -\frac{1}{3}\, e^{-2t/3}(2t + 1) \tag{11.1.18}$$

leads to the result (after having integrated by parts):

$$C(t) = e^{-2t/3}(t+2) + \tilde{C}. \tag{11.1.19}$$

Now the full formula for $y(t)$ can be completed:

$$y(t) = \tilde{C}e^{2t/3} + t + 2, \tag{11.1.20}$$

and since $y(0) = 0$, one can immediately put $\tilde{C} = -2$.

The second equation of (11.1.9) allows us now to easily find $x(t)$:

$$\begin{aligned} &2\dot{x} + \dot{y} = -1 \implies \frac{d}{dt}[2x + y + t] = 0 \implies 2x(t) = -y(t) - t + D \\ &\implies x(t) = e^{2t/3} - t \underbrace{-1 + \frac{D}{2}}_{\tilde{D}} \implies x(t) = e^{2t/3} - t + \tilde{D}, \end{aligned} \tag{11.1.21}$$

and the requirement $x(0) = 0$ implies that $\tilde{D} = -1$.

Now it remains only to calculate $\dot{z}$, for example using the third of the equations (11.1.9):

$$\begin{aligned} \dot{z} &= -\dot{y} + x - t = -\frac{d}{dt}\left[-2e^{2t/3} + t + 2\right] + e^{2t/3} - t - 1 - t \\ &= \frac{4}{3}e^{2t/3} - 1 + e^{2t/3} - 2t - 1 = \frac{7}{3}e^{2t/3} - 2t - 2, \end{aligned} \tag{11.1.22}$$

and after having integrated, one comes to

$$z(t) = \frac{7}{2}e^{2t/3} - t^2 - 2t + \tilde{E}. \tag{11.1.23}$$

In order to fulfill the condition $z(0) = 0$ the constant $\tilde{E}$ has to be equal to $-7/2$. Expressions (11.1.20), (11.1.21), and (11.1.23) constitute the complete set of solutions for the system.

Problem 3

Using the method of the elimination of variables, the solution of the set of differential equations:

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = x^2, \end{cases} \tag{11.1.24}$$

satisfying $x(0) = -1$ and $y(0) = 0$, will be found for $t \in]-\pi/2, \pi/2[$.

Solution

The method of the elimination of variables is applicable to systems of nonlinear equations as well, although in the case of more complicated forms, we have to recognize the possibility that our effort may turn out futile. The system (11.1.24) is, however, simple enough to try to apply this method.

It seems that the simplest way to achieve the initial goal is to determine $x(t)$ from the second equation and insert it into the first one. We encourage the reader to be convinced that the opposite, i.e., finding $y(t)$ from the first equation and inserting it into the second one, leads to more complex calculations. Thus one has

$$x = -\sqrt{\dot{y}}. \tag{11.1.25}$$

The "minus" has been put here in order to avoid future conflict with the initial conditions. It is worth noting at this point that by using the initial conditions, together with the equations (11.1.24), the values of the derivatives $\dot{x}(0) = 0$ and $\dot{y}(0) = 1$ can immediately be obtained. They will be of use to us below.

In the first equation of (11.1.24), one will need the derivative of the function $x(t)$, so it must now be found:

$$\dot{x} = -\frac{1}{2\sqrt{\dot{y}}}\,\ddot{y}. \tag{11.1.26}$$

After eliminating it, the second order equation for $y(t)$ can be obtained, which is easily solved in two steps. One simply has to write

$$\ddot{y} = 2\dot{y}y \implies \frac{d}{dt}\left(\dot{y} - y^2\right) = 0 \implies \dot{y} - y^2 = C_1 \tag{11.1.27}$$

and determine the value of the constant C_1 by substituting $t = 0$ (thus $C_1 = 1$). Then the Eq. (11.1.27) will be solved by separating the variables:

$$\begin{aligned}\dot{y} = y^2 + 1 \implies \int \frac{dy}{y^2+1} = \int dt \implies \arctan y = t + C_2\\ \implies y(t) = \tan(t + C_2).\end{aligned} \tag{11.1.28}$$

Again one can use now the initial conditions, which yield $\tan C_2 = 0$ implying that $C_2 = k\pi$, where k is an integer. However, the presence of k has no meaning and one can safely set $k = 0$, since

$$y(t) = \tan(t + k\pi) = \tan t. \tag{11.1.29}$$

Solutions with $k \neq 0$ are due to the last step in (11.1.28). A unique result $C_2 = 0$ would be obtained when setting $t = 0$ and using the condition $y(0) = 0$ already in the equation $\arctan y = t + C_2$.

Determining the other function does not present any problem. One just uses (11.1.25) and finds

$$x(t) = -\sqrt{\dot{y}} = -\sqrt{\frac{1}{\cos^2 t}} = -\frac{1}{\cos t}. \tag{11.1.30}$$

The sign of the cosine is due to the fact that for $-\pi/2 < t < \pi/2$ it is positive (i.e., $\sqrt{\cos^2 t} = +\cos t$).

11.2 Solving Systems of Linear Equations with Constant Coefficients

Problem 1

The general solution of the set of differential equations:

$$\begin{cases} \dot{x} - x - 3y = 0, \\ \dot{y} - x + y = 0 \end{cases} \tag{11.2.1}$$

will be found.

Solution

One of the simplest differential equations for an unknown function u is the equation $\dot{u} = A \cdot u$, where $A = \text{const}$. Its solution is obviously well known and can be immediately written as

$$u(t) = e^{At} u_0. \tag{11.2.2}$$

The symbol u_0 denotes a constant: the value of the function $u(t)$ for $t = 0$ ($u_0 = u(0)$). Let us imagine now that the function $u(t)$ is in fact a system of functions, e.g.

$$u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{11.2.3}$$

The symbol A now stands for a matrix—in the present case a 2×2 matrix—whose elements are constant numbers. The question is whether one still can write the solution in the form of (11.2.2), surely keeping in mind that u_0 would now mean the vector initial conditions:

$$u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}. \tag{11.2.4}$$

The answer to the above question is affirmative. However, one could hardly consider formula (11.2.2) as the solution of the equation if one was not able to explicitly find the expression e^{At}, i.e., to calculate the exponent of a matrix. Then this is our main challenge. This quantity, referred to as the "resolvent," is usually denoted with the symbol R. As we know from the theoretical introduction, when using it, the solution can be given the form:

$$u(t) = R(t, t_0)u(t_0). \tag{11.2.5}$$

In general, the following equations hold:

$$\frac{d}{dt}R(t, t_0) = A \cdot R(t, t_0) \quad \text{and} \quad R(t_0, t_0) = 1, \tag{11.2.6}$$

where "1" denotes the unit matrix written sometimes as $\mathbb{1}$. They ensure the fulfillment of both the differential equation, since

$$\frac{d}{dt}u(t) = \frac{d}{dt}\left[R(t, t_0)u(t_0)\right] = \frac{d}{dt}R(t, t_0) \cdot u(t_0) = A \cdot \underbrace{R(t, t_0) \cdot u(t_0)}_{u(t)} = A \cdot u(t), \tag{11.2.7}$$

and the initial conditions, if such are imposed. In our case, we have $t_0 = 0$, and since A is an array of constant numbers, we simply have the $R(t, 0) = e^{At}$ spoken of above.

Finding this quantity is the main issue below. The initial system of equations (11.2.1) in fact has this form, as can be seen after rewriting it,

$$\dot{u} = \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}}_{A} u. \tag{11.2.8}$$

One could say that the problem has been resolved: we have the matrix A and the initial conditions are not required, so formula (11.2.2) constitutes the solution. However, as it has already been mentioned, one still should find the matrix e^{At} explicitly. There are several ways to do this. First of all, it must be realized that this quantity can be given a meaning through the Taylor series. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{11.2.9}$$

which means that our expression is defined as

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}. \tag{11.2.10}$$

In contrast to the former exponential, this formula does not require anything more than only products and sums of matrices. Moreover, no matter how many times one multiplied or added 2×2 matrices with each other, the result will still be a 2×2 matrix. Thus we have the first observation: the expression e^{At} in our example is a square matrix with two columns and two rows. It operates particularly easy on the eigenvectors of the matrix A. Notice that A has exactly two eigenvectors (let us denote them as v_1 and v_2), which is due to the fact that the characteristic polynomial:

$$\varphi(\lambda) = \det[A - \lambda \mathbb{1}] = \det \begin{bmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda-2)(\lambda+2) \tag{11.2.11}$$

has two distinct roots:

$$\lambda_1 = 2, \quad \lambda_2 = -2. \tag{11.2.12}$$

As the reader probably remembers from the lecture of algebra, a matrix does not have to possess, so many eigenvectors as its dimension. Sometimes—and this may occur in the case when there are multiple characteristic roots—eigenvectors are lacking, which might have a significant impact on the solution of the system (11.2.8). Such a matrix is called a Jordan or non-diagonalizable matrix. In this exercise, however, this kind of complication is not faced: we simply have $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$ and consequently $A^n v_1 = \lambda_1^n v_1$ and $A^n v_2 = \lambda_2^n v_2$ for any natural n. Acting with the "troublesome" operator e^{At} on v_1 and v_2, one thus finds

$$\begin{aligned} e^{At} v_1 &= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} v_1 = \sum_{n=0}^{\infty} \frac{\lambda_1^n t^n}{n!} v_1 = e^{\lambda_1 t} v_1 \\ e^{At} v_2 &= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} v_2 = \sum_{n=0}^{\infty} \frac{\lambda_2^n t^n}{n!} v_2 = e^{\lambda_2 t} v_2. \end{aligned} \tag{11.2.13}$$

At this point, the reader probably presumes what should be done next: the vector of initial values u_0 should be decomposed onto the base vectors v_1 and v_2:

$$u_0 = C_1 v_1 + C_2 v_2. \tag{11.2.14}$$

Because of this, the right-hand side of (11.2.2) may be given the clear form:

$$e^{At} u_0 = e^{At}(C_1 v_1 + C_2 v_2) = C_1 e^{At} v_1 + C_2 e^{At} v_2 = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2, \tag{11.2.15}$$

where the exponents do not contain matrices any more.

Below we are going to execute this plan, first determining v_1 and v_2. This procedure is probably well known to the reader from algebra. First one has to solve the equation:

$$(A - \lambda_1 \mathbb{1})\, v_1 = 0, \quad \text{i.e.,} \quad \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{11.2.16}$$

which corresponds to the set of equivalent equations:

$$\begin{cases} -v_{1x} + 3v_{1y} = 0, \\ v_{1x} - 3v_{1y} = 0. \end{cases} \tag{11.2.17}$$

This should not be surprising, as the matrix $A - \lambda_1 \mathbb{1}$ is singular (cf. (11.2.11) and (11.2.12)). The same is true, of course, for the matrix $A - \lambda_2 \mathbb{1}$. A singular matrix has linearly dependent rows, which is equivalent to the statement that equations (11.2.17) are dependent on each other. Assuming arbitrarily $v_{1x} = 3$, one finds $v_{1y} = 1$ and the first of the eigenvectors $v_1 = [3, 1]$ is obtained. Naturally, in place of "3," one could have taken any (nonzero) number, which would result only in the appropriate change of the value of the constant C_1 in (11.2.14) but without modifying the solution of the system (11.2.1).

Now we are looking for a solution of the equation:

$$(A - \lambda_2 \mathbb{1})\, v_2 = 0, \quad \text{i.e.,} \quad \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{11.2.18}$$

which this time leads to the system

$$\begin{cases} 3v_{2x} + 3v_{2y} = 0, \\ v_{2x} + v_{2y} = 0. \end{cases} \tag{11.2.19}$$

Setting $v_{2x} = 1$, one obtains $v_{2y} = -1$ and the second eigenvector turns out to have the form: $v_2 = [1, -1]$.

Now, using (11.2.14), one can express constants C_1 and C_2 by x_0 and y_0 defined by formula (11.2.4). The following equations must be satisfied:

$$\begin{cases} x_0 = 3C_1 + C_2, \\ y_0 = C_1 - C_2, \end{cases} \tag{11.2.20}$$

implying that

$$C_1 = \frac{1}{4}(x_0 + y_0) \quad \text{and} \quad C_2 = \frac{1}{4}(x_0 - 3y_0). \tag{11.2.21}$$

In order to explicitly write the solution of (11.2.1), formula (11.2.15) can be used:

$$u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{4} e^{2t}(x_0 + y_0) \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{1}{4} e^{-2t}(x_0 - 3y_0) \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{11.2.22}$$

which is equivalent to

$$\begin{aligned} x(t) &= \frac{3}{4} e^{2t}(x_0 + y_0) + \frac{1}{4} e^{-2t}(x_0 - 3y_0), \\ y(t) &= \frac{1}{4} e^{2t}(x_0 + y_0) - \frac{1}{4} e^{-2t}(x_0 - 3y_0). \end{aligned} \tag{11.2.23}$$

Problem 2

The general solution of the set of differential equations:

$$\begin{cases} 2\dot{x} - \dot{y} - x = 0, \\ \dot{x} + 2\dot{y} - y = 0 \end{cases} \tag{11.2.24}$$

will be found.

Solution

In the previous example, we learned how to proceed with an equation $\dot{u} = A \cdot u$, where A is a matrix of constant numbers. Now, setting about solving the system (11.2.24), one must first make sure that it has the desirable form and find the matrix A. In order to achieve this, the equations (11.2.24) will simply be treated as those for two unknowns $\dot{x}$ and $\dot{y}$ to be determined:

$$\begin{cases} \dot{x} = \dfrac{2}{5}\, x + \dfrac{1}{5}\, y, \\ \dot{y} = -\dfrac{1}{5}\, x + \dfrac{2}{5}\, y, \end{cases} \implies \dot{u} = \underbrace{\begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix}}_{A} u. \tag{11.2.25}$$

As we know, the next step is to find the eigenvalues and eigenvectors of the matrix A. Its characteristic polynomial:

$$\varphi(\lambda) = \det[A - \lambda \mathbb{1}] = \det \begin{bmatrix} 2/5 - \lambda & 1/5 \\ -1/5 & 2/5 - \lambda \end{bmatrix} = \left(\frac{2}{5} - \lambda\right)^2 + \frac{1}{25} \tag{11.2.26}$$

has two distinct roots:

$$\lambda_1 = \frac{2}{5} + i\,\frac{1}{5}, \qquad \lambda_2 = \frac{2}{5} - i\,\frac{1}{5}, \tag{11.2.27}$$

which means that the matrix is fully diagonalizable and there are two eigenvectors v_1 and v_2 to be found. First, we solve the equation:

$$(A - \lambda_1 \mathbb{1})\, v_1 = 0, \quad \text{so} \quad \begin{bmatrix} -i/5 & 1/5 \\ -1/5 & -i/5 \end{bmatrix} \cdot \begin{bmatrix} v_{1x} \\ v_{1y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11.2.28}$$

corresponding to the system:

$$\begin{cases} i v_{1x} = v_{1y}, \\ v_{1x} = -i v_{1y}. \end{cases} \tag{11.2.29}$$

These equations are equivalent to each other, which can easily be seen by multiplying one of them by i. Assuming now $v_{1x} = 1$, one gets $v_{1y} = i$ and the first eigenvector is: $v_1 = [1, i]$. Now we are looking for the solution of the equation:

$$(A - \lambda_2 \mathbb{1})\, v_2 = 0, \quad \text{so} \quad \begin{bmatrix} i/5 & 1/5 \\ -1/5 & i/5 \end{bmatrix} \cdot \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{11.2.30}$$

or equivalently of the set:

$$\begin{cases} i v_{2x} = -v_{2y}, \\ v_{2x} = i v_{2y}. \end{cases} \tag{11.2.31}$$

For arbitrarily chosen $v_{2x} = 1$, one has $v_{2y} = -i$ and the second eigenvector has the form: $v_2 = [1, -i]$.

Using (11.2.14), the constants C_1 and C_2 can be expressed by x_0 and y_0 defined in (11.2.4). The following conditions have to be met:

$$\begin{cases} x_0 = C_1 + C_2, \\ y_0 = iC_1 - iC_2, \end{cases} \tag{11.2.32}$$

implying that

$$C_1 = \frac{1}{2}(x_0 - iy_0) \quad \text{and} \quad C_2 = \frac{1}{2}(x_0 + iy_0). \tag{11.2.33}$$

Finally the explicit solution of the system (11.2.24) is obtained from formula (11.2.15):

$$u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2}\, e^{2t/5} \left\{ e^{it/5}(x_0 - iy_0) \begin{bmatrix} 1 \\ i \end{bmatrix} + e^{-it/5}(x_0 + iy_0) \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\} \tag{11.2.34}$$

and the functions $x(t)$ and $y(t)$ have the form:

$$x(t) = e^{2t/5}\left(x_0 \cos\frac{t}{5} + y_0 \sin\frac{t}{5}\right),$$
$$y(t) = e^{2t/5}\left(-x_0 \sin\frac{t}{5} + y_0 \cos\frac{t}{5}\right). \tag{11.2.35}$$

Problem 3

The general solution of the set of differential equations:

$$\begin{cases} \dot{x} - \dot{y} + x - 2z = 0, \\ -\dot{x} + 2\dot{y} + x + 2z = 0, \\ \dot{y} + \dot{z} - x + 2z = 0 \end{cases} \tag{11.2.36}$$

will be found.

Solution

Denoting

$$u(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \tag{11.2.37}$$

one can rewrite this system of equations in the form $\dot{u} = A \cdot u$. However, in order to find the matrix A, one first has to solve algebraically (11.2.36), treating $\dot{x}$, $\dot{y}$, and $\dot{z}$ as unknowns. This task is not difficult and as a result one gets

$$\begin{cases} \dot{x} = -3x + 2z, \\ \dot{y} = -2x, \\ \dot{z} = 3x - 2z, \end{cases} \implies \dot{u} = \underbrace{\begin{bmatrix} -3 & 0 & 2 \\ -2 & 0 & 0 \\ 3 & 0 & -2 \end{bmatrix}}_{A} u. \tag{11.2.38}$$

The function $u(t)$, as we know from (11.2.2), has the form:

$$u(t) = e^{At} u_0, \tag{11.2.39}$$

where

$$u_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}. \tag{11.2.40}$$

The characteristic polynomial of the matrix A is of the third degree due to the dimension of the array ($n = 3$):

$$\varphi(\lambda) = \det[A - \lambda \mathbb{1}] = \det \begin{bmatrix} -3-\lambda & 0 & 2 \\ -2 & -\lambda & 0 \\ 3 & 0 & -2-\lambda \end{bmatrix}$$
$$= (-3-\lambda)(-\lambda)(-2-\lambda) + 6\lambda = -\lambda^2(\lambda + 5), \qquad (11.2.41)$$

but it has only two roots (the symbol p_i denotes the multiplicity of the i-th root):

$$\lambda_1 = -5, \; p_1 = 1 \quad \text{and} \quad \lambda_2 = 0, \; p_2 = 2. \qquad (11.2.42)$$

Since $p_2 > 1$, it is worth checking the number of eigenvectors that matrix A possesses. It is known that there are at least two (since there are two different eigenvalues) and not more than three (because this is the dimension of space in which the matrix operates). We will try to find them all below. To do this first we solve the equation:

$$(A - \lambda_1 \mathbb{1})\, v_1 = 0, \quad \text{i.e.,} \quad \begin{bmatrix} 2 & 0 & 2 \\ -2 & 5 & 0 \\ 3 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (11.2.43)$$

The first and the last rows differ only by a multiplicative constant and, therefore, lead to the equivalent equations. We can skip one of them and obtain the set of two equations:

$$\begin{cases} 2v_{1x} + 2v_{1z} = 0, \\ -2v_{1x} + 5v_{1y} = 0. \end{cases} \qquad (11.2.44)$$

Assuming, as in the previous exercise, that $v_{1x} = 1$, one obtains $v_{1z} = -1$ and $v_{1y} = 2/5$, which leads to the first of the eigenvectors in the form: $v_1 = [1, 2/5, -1]$. Now this step is repeated in order to find v_2:

$$(A - \lambda_2 \mathbb{1})\, v_2 = 0, \quad \text{or} \quad \begin{bmatrix} -3 & 0 & 2 \\ -2 & 0 & 0 \\ 3 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{2x} \\ v_{2y} \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (11.2.45)$$

Once again the first and last rows lead to the equivalent equations (differing only by the sign), while the second row is obviously independent. A similar situation took place—and had to take place—in the case of λ_1, since it was a *single* root (i.e., that with $p_1 = 1$). However, λ_2 is a *double* root ($p_2 = 2$) and one could expect all rows to be dependent (i.e., the rank of $A - \lambda_2 \mathbb{1}$ to be equal to $n - p_2 = 1$). The fact that this is not the case constitutes the important information: the matrix A is of

Jordan character and has only two eigenvectors! To find this second one, we solve the system of equations resulting from the first two rows in (11.2.45):

$$\begin{cases} -3v_{2x} + 2v_{2z} = 0, \\ -2v_{2x} = 0, \end{cases} \tag{11.2.46}$$

which entails $v_{2x} = v_{2z} = 0$. If, in addition, the value of v_{2y} is fixed to 1, one gets: $v_2 = [0, 1, 0]$.

We have found v_1 and v_2, but two vectors do not form a basis in three-dimensional space. For this set, a third vector should be added. To this end, it is enough to choose v_3 so that it is linearly independent of the set $\{v_1, v_2\}$, but still more elegant would be to adopt v_3 in the form constituting (together with the vector v_2) the so-called Jordan chain, i.e.,

$$[A - \lambda_2 \mathbb{1}]v_3 = v_2. \tag{11.2.47}$$

It is easy to verify by a direct computation that this condition is satisfied by the vector $v_3 = [-1/2, 0, -3/4]$.

Now one has to expand u_0 in the basis composed of v_1, v_2, v_3, that is to say we are looking for constants $C_{1,2,3}$ for which one has

$$u_0 = C_1 v_1 + C_2 v_2 + C_3 v_3 \implies$$
$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2/5 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} -1/2 \\ 0 \\ -3/4 \end{bmatrix}. \tag{11.2.48}$$

The obtained set of three equations:

$$\begin{cases} C_1 - \frac{1}{2} C_3 = x_0, \\ \frac{2}{5} C_1 + C_2 = y_0, \\ -C_1 - \frac{3}{4} C_3 = z_0 \end{cases} \tag{11.2.49}$$

can be easily solved for $C_{1,2,3}$, yielding

$$\begin{cases} C_1 = \frac{1}{5}(3x_0 - 2z_0), \\ C_2 = -\frac{1}{25}(6x_0 - 25y_0 - 4z_0), \\ C_3 = -\frac{4}{5}(x_0 + z_0). \end{cases} \tag{11.2.50}$$

Knowing the values of the constants $C_{1,2,3}$, one can write the solution of the system (11.2.36) in the form of

$$\begin{aligned} u(t) &= e^{At}u_0 = e^{At}(C_1v_1 + C_2v_2 + C_3v_3) = C_1e^{At}v_1 + C_2e^{At}v_2 + C_3e^{At}v_3 \\ &= C_1e^{\lambda_1 t}v_1 + C_2e^{\lambda_2 t}v_2 + C_3e^{\lambda_2 t}e^{(A-\lambda_2 \mathbb{1})t}v_3 \\ &= C_1e^{\lambda_1 t}v_1 + C_2e^{\lambda_2 t}v_2 + C_3e^{\lambda_2 t}[\mathbb{1} + (A-\lambda_2\mathbb{1})t]v_3. \end{aligned} \tag{11.2.51}$$

It should be noted that having applied above the expansion:

$$e^{(A-\lambda_2\mathbb{1})t}v_3 = \sum_{k=0}^{\infty} \frac{(A-\lambda_2\mathbb{1})^k t^k}{k!}\, v_3, \tag{11.2.52}$$

all terms containing $(A-\lambda_2\mathbb{1})^k v_3$ for $k \geq 2$ have been omitted, since pursuant to (11.2.47) and to the fact that v_2 is the eigenvector corresponding to the eigenvalue λ_2, one gets

$$(A-\lambda_2\mathbb{1})^2 v_3 = (A-\lambda_2\mathbb{1})v_2 = 0. \tag{11.2.53}$$

Going back to formula (11.2.51), one can now rewrite it in the form:

$$u(t) = C_1e^{\lambda_1 t}v_1 + e^{\lambda_2 t}\left[C_2v_2 + C_3(v_3 + tv_2)\right], \tag{11.2.54}$$

or more explicitly

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} &= \frac{1}{5}(3x_0 - 2z_0)e^{-5t}\begin{bmatrix} 1 \\ 2/5 \\ -1 \end{bmatrix} - \frac{1}{25}(6x_0 - 25y_0 - 4z_0)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &\quad - \frac{4}{5}(x_0 + z_0)\begin{bmatrix} -1/2 \\ t \\ -3/4 \end{bmatrix}. \end{aligned} \tag{11.2.55}$$

The above equation leads to the following formulas for the functions $x(t)$, $y(t)$, and $z(t)$:

$$\begin{aligned} x(t) &= \frac{1}{5}\left[(3x_0 - 2z_0)e^{-5t} + 2x_0 + 2z_0\right], \\ y(t) &= \frac{1}{25}\left[(6x_0 - 4z_0)e^{-5t} - 6x_0 + 25y_0 + 4z_0\right] - \frac{4}{5}(x_0 + z_0)t, \\ z(t) &= \frac{1}{5}\left[(-3x_0 + 2z_0)e^{-5t} + 3x_0 + 3z_0\right]. \end{aligned} \tag{11.2.56}$$

Notice that the solution is not purely exponential: apart from e^{-5t}, one finds also a polynomial of the *first* degree in t (strictly speaking, this is an exponential factor $e^{0\,t}$ multiplied by this polynomial). This is a consequence of the nonvanishing expression $A - \lambda_2 \mathbb{1}$ when acting on v_3 (since v_3 is not an eigenvector!). We see, therefore, that polynomial factors (of degree at most equal to $p_i - 1$) may appear if any eigenvector is lacking, i.e., when the matrix A is of Jordan character.

The full set of functions has been found but we do not abandon this problem yet. Below we would like to familiarize the reader with another—and very elegant—way of finding the expression e^{At}, which could be called the method of "dividing a function by a polynomial."

Suppose that a polynomial $w_n(z)$ of the n-th degree with roots at $z_1, z_2, \ldots, z_k$ of multiplicities $p_1, p_2, \ldots, p_k$ on the plane is given as well as an analytical function F (i.e., expandable in the Taylor series) on an open set $\mathcal{O}$ containing all z_i. If this function vanished, together with the appropriate number of its derivatives at each of these points, i.e., if the following equations were fulfilled:

$$F(z_i) = 0, \quad F'(z_i) = 0, \ldots, F^{(p_i-1)}(z_i) = 0, \quad \text{for } i = 1, 2, \ldots, k, \tag{11.2.57}$$

then one could deduce that this function could be divided by the polynomial $w_n(x)$ without any remainder, i.e., that

$$F(z) = w_n(z) \cdot G(z). \tag{11.2.58}$$

The function G has to be *analytical* too, and relation (11.2.58) simply means that out of the function F one can extract the factors $(z - z_i)^{p_i}$. However, this case would be quite specific and in general it is hard to imagine that the equations (11.2.57) are satisfied for a casual function. In such a situation, one has to slightly modify the relation (11.2.58). It will be now applied not to the function $F(z)$ itself, but to the expression $F(z) - r(z)$, where $r(z)$ is a polynomial chosen so as to ensure that the conditions (11.2.57) are met. Because there are n in total (since $p_1 + p_2 + \ldots + p_k = n$), in order to satisfy them, one needs a polynomial r containing n arbitrary constants, i.e., a polynomial of the degree $n - 1$. It will be denoted $r_{n-1}(z)$, and instead of the Eq. (11.2.58), we will have

$$F(z) - r_{n-1}(z) = w_n(z) \cdot G(z), \quad \text{or} \quad F(z) = w_n(z) \cdot G(z) + r_{n-1}(z). \tag{11.2.59}$$

This formula can be called the formula for dividing a function F by a polynomial w_n with the remainder r_{n-1}.

This formula is true for any analytic function F and any polynomial w_n, and we are going to use it choosing $F(z) = e^{zt}$, and as $w_n(z)$ taking the characteristic polynomial of the matrix A (i.e., $w_n(z) = \varphi(z)$). Equation (11.2.59) will then have the form:

$$e^{zt} = \varphi(z) \cdot G(z) + r_{n-1}(z). \tag{11.2.60}$$

The function G is not known but it is irrelevant to us. It is enough to know that in the domain $\mathcal{O}$ it has no singularities.

The left- and right-hand sides above can be expanded in a Taylor series. The equality of both sides simply ensures the identical values of all Taylor coefficients, but this does not depend on what would be inserted in place of z. One could even insert the matrix A for z and the relation (11.2.60) would still be met, since the coefficients at any powers of this matrix would be identical on both sides:

$$e^{At} = \varphi(A) \cdot G(A) + r_{n-1}(A). \tag{11.2.61}$$

We come now to the heart of the matter: pursuant to Cayley–Hamilton theorem known to the reader from algebra, one has $\varphi(A) = 0$, and the analyticity of the function G ensures also that $\varphi(A) \cdot G(A) = 0$. As a result, the expression e^{At}, which we are looking for, (i.e., the resolvent) reduces to

$$e^{At} = r_{n-1}(A). \tag{11.2.62}$$

The importance of the above formula should be stressed: the exponential of a matrix boils down to a polynomial of the degree less by one than the dimension of the matrix! So, all one has to do is to find the coefficients of this polynomial (it can be called a polynomial-remainder), and thus one finds e^{At}. They can be established, using formula (11.2.60): one simply has to substitute for z a sufficient number of different values and get the equations for the coefficients. Naturally not the random values of the variable z are going to be plugged in, but only those for which one gets rid of the unknown function G. Such obviously are the eigenvalues of the matrix A which were previously found. By definition they are roots of the characteristic polynomial. In this way, k equations are found:

$$e^{\lambda_1 t} = r_{n-1}(\lambda_1),\ e^{\lambda_2 t} = r_{n-1}(\lambda_2),\ \ldots,\ e^{\lambda_k t} = r_{n-1}(\lambda_k). \tag{11.2.63}$$

If, however, some roots of the characteristic polynomial happen to be multiple—and so it is in this problem—the equations are too few to find all the coefficients (since $k < n$). But still there is a way: if λ_i is a root of the characteristic equation with multiplicity p_i, not only $\varphi(\lambda_i) = 0$ holds, but also $\varphi'(\lambda_i) = 0$, $\varphi''(\lambda_i) = 0$ etc., up to $\varphi^{(p_i-1)}(\lambda_i) = 0$. Apart from (11.2.63) for every i, one will also have

$$\begin{aligned} \frac{d}{dx}\, e^{xt}\Big|_{x=\lambda_i} &= t e^{\lambda_i t} = r'_{n-1}(\lambda_i), \\ \frac{d^2}{dx^2}\, e^{xt}\Big|_{x=\lambda_i} &= t^2 e^{\lambda_i t} = r''_{n-1}(\lambda_i), \\ \ldots \qquad & \ldots \qquad \ldots \end{aligned} \tag{11.2.64}$$

$$\left.\frac{d^{p_i-1}}{dx^{p_i-1}}e^{xt}\right|_{x=\lambda_i} = t^{p_i-1}e^{\lambda_i t} = r_{n-1}^{(p_i-1)}(\lambda_i).$$

i.e., in total as many equations as one needs (i.e., n). As a result of the differentiation over x (i.e., due to the presence of multiple eigenvalues) there emerge polynomials in the variable t.

Now let us apply the method described above to the present exercise. One has $n = 3$, and therefore, the polynomial remainder is at most of the second degree, i.e., $r_2(z) = az^2 + bz + c$. Our first job consists of determining a, b, c. The two eigenvalues of the matrix A have already been found (see (11.2.42)), the first being single and the second double. Thus we have

$$\begin{aligned} e^{-5t} &= r_2(-5) = 25 \cdot a - 5 \cdot b + c, \\ e^{0t} &= r_2(0) = 0 \cdot a + 0 \cdot b + c = c, \\ te^{0t} &= r_2'(0) = 2 \cdot 0 \cdot a + b = b, \end{aligned} \tag{11.2.65}$$

which implies that $a = (e^{-5t} + 5t - 1)/25$, $b = t$, and $c = 1$. Now one can complete the formula for e^{At}:

$$e^{At} = a \cdot A^2 + b \cdot A + c \cdot \mathbb{1}. \tag{11.2.66}$$

Since

$$A^2 = \begin{bmatrix} -3 & 0 & 2 \\ -2 & 0 & 0 \\ 3 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 & 2 \\ -2 & 0 & 0 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 0 & -10 \\ 6 & 0 & -4 \\ -15 & 0 & 10 \end{bmatrix}, \tag{11.2.67}$$

one gets the resolvent in the form of

$$\begin{aligned} e^{At} &= \frac{1}{25}\left(e^{-5t} + 5t - 1\right)\begin{bmatrix} 15 & 0 & -10 \\ 6 & 0 & -4 \\ -15 & 0 & 10 \end{bmatrix} + t\begin{bmatrix} -3 & 0 & 2 \\ -2 & 0 & 0 \\ 3 & 0 & -2 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3e^{-5t} + 2 & 0 & -2e^{-5t} + 2 \\ 6e^{-5t}/5 - 4t - 6/5 & 5 & -4e^{-5t}/5 - 4t + 4/5 \\ -3e^{-5t} + 3 & 0 & 2e^{-5t} + 3 \end{bmatrix}. \end{aligned} \tag{11.2.68}$$

Now, to obtain (11.2.55) and (11.2.56), it is sufficient to act with this matrix on the vector u_0 pursuant to formula (11.2.39).

Problem 4

The solution of the set of differential equations:

$$\begin{cases} \dot{x} + \dot{y} + x - 2y = e^t, \\ 2\dot{x} + \dot{y} + x - 2y = t \end{cases} \tag{11.2.69}$$

satisfying $x(0) = y(0) = 1$, will be found.

Solution

The set of equations handled in this exercise differs from the former ones by the presence of right-hand sides—it is then a *nonhomogeneous* set. While solving it, one can use the well-known method of "varying" constants dealt with in Sect. 10.1. As we will simultaneously be convinced, this method applied in a natural way to the system of the type (11.2.69) will provide an explanation of the mysterious supplementary conditions given by the Eqs. (10.1.18), (10.1.38), and (10.1.40), imposed on constants subject to "variation," and whose justification was postponed for the present time.

To begin, let us rewrite (11.2.69) in a convenient matrix form. To this end, we subtract equations, getting first

$$\dot{x} = t - e^t, \tag{11.2.70}$$

and then, inserting this result into any of the equations (11.2.69), one finds

$$\dot{y} + x - 2y = 2e^t - t. \tag{11.2.71}$$

The solution of the resultant algebraic equation by elimination of variables is left to the reader and leads to the following form of the differential equation:

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}}_{u(t)} - \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}}_{u(t)} = \underbrace{\begin{bmatrix} t - e^t \\ 2e^t - t \end{bmatrix}}_{B(t)}, \tag{11.2.72}$$

i.e.,

$$\dot{u} - Au = B. \tag{11.2.73}$$

One can now proceed to solve (11.2.72) with the method of "varying" constants.

1. *We solve the homogeneous system of equations.*
Omitting the right-hand side in (11.2.72), the homogeneous system is obtained:

$$\dot{u}(t) = \frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{11.2.74}$$

Two equations for two unknown functions are dealt with, so obviously such must also be the number of columns and rows of the matrix A, as well as the degree of the characteristic polynomial:

$$\varphi(\lambda) = \det[A - \lambda\, \mathbb{1}] = \det\begin{bmatrix} -\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix} = \lambda(\lambda - 2). \tag{11.2.75}$$

It has two single roots: $\lambda_1 = 0$ and $\lambda_2 = 2$, and thus the matrix A is not of Jordan nature (that is, as we know from algebra, it is fully diagonalizable). In the solution no polynomials of t appear, only exponential functions (as such is also treated $1 = e^{0\,t}$).

Now, one needs to find the resolvent matrix and for this the method of dividing the functions by the polynomial is going to be used (see (11.2.37) and (11.2.62)). Since the characteristic polynomial is of the second degree, a polynomial remainder is of the first degree and it has the form: $r_1(z) = az + b$. As a result, one can write

$$e^{At} = a \cdot A + b \cdot \mathbb{1}, \tag{11.2.76}$$

and the problem boils down to finding the two constants: a and b. Substituting successively both eigenvalues into the Eq. (11.2.60), one gets the system of equations:

$$\begin{cases} e^{0t} = 0\,a + b = b, \\ e^{2t} = 2\,a + b, \end{cases} \tag{11.2.77}$$

which entails $a = (e^{2t} - 1)/2$ i $b = 1$. Finally

$$e^{At} = \frac{1}{2}\left(e^{2t} - 1\right)\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\left(e^{2t} - 1\right)/2 & e^{2t} \end{bmatrix}. \tag{11.2.78}$$

The homogeneous solution involves the resolvent matrix acting on a certain vector of constants $[C_1, C_2]$. These constants in the former problems had the interpretation of initial values for the functions, i.e., $C_1 = x(0)$, $C_2 = y(0)$, but now one can use initial conditions only after having found the nonhomogeneous solution. So we have

$$\begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\left(e^{2t}-1\right)/2 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \tag{11.2.79}$$

where subscripts "0" have been added to emphasize that it is not yet the solution of the complete system (11.2.69).

2. *We "vary" constants.*
Now let us suppose that the full solution has the form:

$$u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\left(e^{2t}-1\right)/2 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} \tag{11.2.80}$$

with some unknown functions $C_1(t)$ and $C_2(t)$, and differentiate over the argument t:

$$\dot{u}(t) = \frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -e^{2t} & 2e^{2t} \end{bmatrix} \cdot \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 0 \\ -\left(e^{2t}-1\right)/2 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} \dot{C}_1 \\ \dot{C}_2 \end{bmatrix}. \tag{11.2.81}$$

It is easy to see that after plugging $u(t)$ and $\dot{u}(t)$ into (11.2.72) the terms that do not contain derivatives of C_1 and C_2 cancel one another thanks to the fact that the expression (11.2.80) for fixed values of C_1 and C_2 is the solution of the homogeneous system (such a situation has already been met in Sect. 10.1). As a result one obtains

$$\begin{bmatrix} 1 & 0 \\ -\left(e^{2t}-1\right)/2 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} \dot{C}_1 \\ \dot{C}_2 \end{bmatrix} = \begin{bmatrix} t-e^t \\ 2e^t - t \end{bmatrix}. \tag{11.2.82}$$

The matrix on the left-hand side is nonsingular (its determinant equals e^{2t} and does not vanish for any t), so one has the Cramer system, which means that it has a unique solution. We are not going to again bother the reader with solving the system of two equations of the first degree for two unknowns and we give only the result:

$$\dot{C}_1 = t - e^t, \quad \dot{C}_2 = -\frac{1}{2}e^t + \frac{5}{2}e^{-t} - \frac{3}{2}te^{-2t} + \frac{1}{2}t. \tag{11.2.83}$$

It remains to calculate integrals of the above expressions, which is elementary, and one obtains

$$C_1(t) = -e^t + \frac{1}{2}t^2 + \tilde{C}_1,$$
$$C_2(t) = -\frac{1}{2}e^t - \frac{5}{2}e^{-t} + \frac{3}{8}e^{-2t}(2t+1) + \frac{1}{4}t^2 + \tilde{C}_2, \tag{11.2.84}$$

where $\tilde{C}_{1,2}$ are now true constants. The formulas for $x(t)$ and $y(t)$ can now be easily found from (11.2.80):

$$x(t) = 1 \cdot C_1(t) = \underbrace{\tilde{C}_1}_{\text{GSHE}} \underbrace{-e^t + \frac{1}{2}t^2}_{\text{SSNHE}},$$

$$\begin{aligned} y(t) &= -\frac{1}{2}\left(e^{2t} - 1\right) C_1(t) + e^{2t} C_2(t) \\ &= -\frac{1}{2}\left(e^{2t} - 1\right)\left(-e^t + \frac{1}{2}t^2 + \tilde{C}_1\right) \\ &\quad + e^{2t}\left(-\frac{1}{2}e^t - \frac{5}{2}e^{-t} + \frac{3}{8}e^{-2t}(2t+1) + \frac{1}{4}t^2 + \tilde{C}_2\right) \\ &= \underbrace{-\frac{1}{2}\left(e^{2t} - 1\right)\tilde{C}_1 + \tilde{C}_2 e^{2t}}_{\text{GSHE}} \underbrace{-3e^t + \frac{1}{8}(2t^2 + 6t + 3)}_{\text{SSNHE}}. \end{aligned} \tag{11.2.85}$$

As one can see, the general solution, similarly as it was in Sect. 10.1, is again of the form GSNHE = GSHE + SSNHE. It should be also noted that the polynomial not accompanied by an exponential factor (or more precisely: accompanied by e^{0t}) has raised its degree by one. On the right-hand side of (11.2.69) in the second equation there was the polynomial of the first degree, and in (11.2.85) of the second degree. We know already the cause. It is the coincidence (the "resonance") between the characteristic root $\lambda_1 = 0$ of the matrix A and the factor (equal to zero) in the exponent of te^{0t} in the nonhomogeneous part of the system.

At the end, one has to determine the unknown constants using the initial conditions $x(0) = y(0) = 1$. One gets $\tilde{C}_1 = 2$ and $\tilde{C}_2 = 29/8$.

It should be noted that knowing the resolvent for the homogeneous system is enough to immediately write down the solution of the nonhomogeneous system with incorporated initial conditions. It allows to avoid, maybe troublesome for the reader, the procedure of "varying" constants. As we know, the resolvent binds the solution for two different arguments:

$$u(t) = R(t, t_0)u(t_0) \tag{11.2.86}$$

and satisfies the conditions:

$$R(t_0, t_0) = \mathbb{1} \quad \text{and} \quad \frac{d}{dt}R(t, t_0) = A \cdot R(t, t_0). \tag{11.2.87}$$

If its form is already known, the solution of the nonhomogeneous system can be written as

$$u(t) = R(t, t_0)u(t_0) + \int_{t_0}^{t} R(t, s)B(s)ds. \tag{11.2.88}$$

A simple computation allows us to check that $u(t)$ satisfies (11.2.73):

$$\begin{aligned}
\frac{d}{dt}u(t) &= \frac{d}{dt}\left[R(t, t_0)u(t_0) + \int_{t_0}^{t} R(t, s)B(s)ds\right] \\
&= \underbrace{\dot{R}(t, t_0)}_{=A\cdot R(t,t_0)} u(t_0) + \int_{t_0}^{t} \underbrace{\dot{R}(t, s)}_{=A\cdot R(t,s)} B(s)ds + \underbrace{R(t, t)}_{=\mathbb{1}} B(t) \\
&= A\left[R(t, t_0)u(t_0) + \int_{t_0}^{t} R(t, s)B(s)ds\right] + B(t) = A \cdot u(t) + B(t).
\end{aligned} \tag{11.2.89}$$

In the above transformations in the second line, we made use of the well-known fact that $\frac{d}{dy}\int_{y_0}^{y} F(x)dx = F(y)$. In order to find $u(t) = [x(t), y(t)]$ satisfying (11.2.69), one simply puts $t_0 = 0$ (since the initial condition for (11.2.69) is specified at zero) and plugs the resolvent in the form (11.2.78) into (11.2.88) and in the second term substituting $t \mapsto t - s$ (because $R(t, s) = e^{A(t-s)}$). Thus, the whole issue boils down to several matrix multiplications and to performing a couple of integrations:

$$\begin{aligned}
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -\left(e^{2t} - 1\right)/2 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\quad + \int_0^t \begin{bmatrix} 1 & 0 \\ -\left(e^{2(t-s)} - 1\right)/2 & e^{2(t-s)} \end{bmatrix} \cdot \begin{bmatrix} s - e^s \\ 2e^s - s \end{bmatrix} ds \\
&= \begin{bmatrix} 1 \\ \left(e^{2t} + 1\right)/2 \end{bmatrix} + \int_0^t \begin{bmatrix} s - e^s \\ -3se^{2(t-s)}/2 + 5e^{2t-s}/2 - e^s/2 + s/2 \end{bmatrix} ds \\
&= \begin{bmatrix} 1 \\ \left(e^{2t} + 1\right)/2 \end{bmatrix} + \begin{bmatrix} t^2/2 - e^t + 1 \\ 17e^{2t}/8 - 3e^t + t^2/4 + 3t/4 + 7/8 \end{bmatrix} \\
&= \begin{bmatrix} 2 - e^t + t^2/2 \\ 21e^{2t}/8 - 3e^t + t^2/4 + 3t/4 + 11/8 \end{bmatrix}.
\end{aligned} \tag{11.2.90}$$

Finally, in accordance with what has been stated, it is time to explain the puzzling additional conditions appearing for nonhomogeneous linear equations of higher order. We are going to do this using the Eq. (10.1.9) as an example:

$$y'' + y = \sin x + \cos x. \tag{11.2.91}$$

If one denotes $y'(x) = v(x)$, the problem of solving the second-order equation will be reduced to that of the system of two first-order equations for two functions ($y(x)$ and $v(x)$):

$$\begin{cases} y'(x) - v(x) = 0, \\ v'(x) + y(x) = \sin x + \cos x, \end{cases} \tag{11.2.92}$$

i.e.,

$$\frac{d}{dx}\underbrace{\begin{bmatrix} y(x) \\ v(x) \end{bmatrix}}_{u(x)} - \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} y(x) \\ v(x) \end{bmatrix}}_{u(x)} = \underbrace{\begin{bmatrix} 0 \\ \sin x + \cos x \end{bmatrix}}_{B(x)}. \tag{11.2.93}$$

It can be solved by methods of this section: first the homogeneous system and then "varying" constants. The former has a particularly simple form:

$$\frac{d}{dx}\begin{bmatrix} y(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y(x) \\ v(x) \end{bmatrix}, \tag{11.2.94}$$

and the characteristic polynomial of the matrix A is

$$\varphi(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i). \tag{11.2.95}$$

Both the eigenvalues ($\pm i$) are single. The resolvent matrix is given by the equation analogous to (11.2.76):

$$e^{Ax} = a \cdot A + b \cdot \mathbb{1}, \tag{11.2.96}$$

and constants a and b are determined from the equations:

$$\begin{cases} e^{ix} = ai + b, \\ e^{-ix} = -ai + b. \end{cases} \tag{11.2.97}$$

Simple computations lead to the result:

$$a = \sin x, \quad b = \cos x, \tag{11.2.98}$$

where Euler's formulas (10.1.13) have been used. Plugging these values into (11.2.96), one gets:

$$e^{Ax} = A\sin x + \mathbb{1}\cos x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sin x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos x = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}. \tag{11.2.99}$$

The solution of the homogeneous system can now be written in the form of

$$\begin{bmatrix} y_0(x) \\ v_0(x) \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{11.2.100}$$

Applying now the procedure of varying constants, we assume that C_1 and C_2 are functions of the variable x and the solution of the full system (11.2.93) is

$$\begin{bmatrix} y(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} C_1(x) \\ C_2(x) \end{bmatrix}. \tag{11.2.101}$$

After having differentiated the right-hand side over the variable x and inserting into (11.2.93), one gets the equations for derivatives C_1' and C_2':

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \sin x + \cos x \end{bmatrix}, \tag{11.2.102}$$

which are identical with (10.1.22). As one can see, the first of the conditions, the origin of which in Sect. 10.1 was unclear, has now found its rationale. It emerges automatically if a higher-order equation is rewritten in the form of the system of first-order equations.

Problem 5

The general solution of the system of differential equations:

$$\begin{cases} \dot{x} + \dot{y} + 2x + y - 2z = t, \\ \dot{x} - \dot{y} + \dot{z} + 2y = 2t, \\ \dot{x} + \dot{y} - \dot{z} = te^{2t} \end{cases} \tag{11.2.103}$$

will be found.

Solution

This example can serve to familiarize the reader with still another—and probably the most clear, although somewhat computationally tedious—way of solving systems of linear first-order differential equations with constant coefficients. It is partly based on the method of predictions known to us from Sect. 10.1 and hence it is only applicable when the right-hand side (the non-homogeneity) has the form of terms such as $w(t)e^{\alpha t}$, where $w(t)$ is a polynomial and α is a constant. This category includes also expressions, where sine or cosine appears rather than $e^{\alpha t}$ thanks to Euler's formulas (10.1.13). The exponential may also reduce to the unity when $\alpha = 0$.

As usual, let us rewrite the system (11.2.103) in the matrix form:

$$\dot{u}(t) - Au(t) = B(t), \tag{11.2.104}$$

and then find the eigenvalues of A. It is easiest to start calculating $\dot{x}$ from (11.2.103) by adding two last equations. As a result, one gets

$$\dot{x} + y = \frac{1}{2}te^{2t} + t. \tag{11.2.105}$$

Using this result to eliminate $\dot{x}$ from the first equation, one obtains

$$\dot{y} + 2x - 2z = -\frac{1}{2}te^{2t} \tag{11.2.106}$$

and then from the second one

$$\dot{z} + 2x + y - 2z = -te^{2t} + t. \tag{11.2.107}$$

These equations can be collectively written in the form of

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}}_{u(t)} - \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ -2 & 0 & 2 \\ -2 & -1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}}_{u(t)} = \underbrace{\begin{bmatrix} te^{2t}/2 + t \\ -te^{2t}/2 \\ -te^{2t} + t \end{bmatrix}}_{B(t)}. \tag{11.2.108}$$

So, we already know the matrix A and the characteristic polynomial can be written as:

$$\varphi(\lambda) = \det[A - \lambda\,\mathbb{1}] = \det\begin{bmatrix} -\lambda & -1 & 0 \\ -2 & -\lambda & 2 \\ -2 & -1 & 2-\lambda \end{bmatrix} = -\lambda^2(\lambda - 2). \tag{11.2.109}$$

As can be seen, there are two characteristic roots with multiplicities $p_{1,2}$:

$$\lambda_1 = 0, \quad p_1 = 2 \quad \text{and} \quad \lambda_2 = 2, \quad p_2 = 1. \tag{11.2.110}$$

In a moment the reader will be convinced that the above is sufficient to write the general form of the solution for the system. The function $x(t)$ satisfying (together with $y(t)$ and $z(t)$) the system (11.2.103) must have the form:

$$x(t) = \alpha_1 t + \beta_1 + \gamma_1 e^{2t} + t^2(\sigma_1 t + \rho_1) + t(\mu_1 t + \nu_1)e^{2t}, \tag{11.2.111}$$

where α_1, β_1, γ_1, σ_1, ρ_1, μ_1, and ν_1 are constants to be determined. Similar formulas for other functions are

$$y(t) = \alpha_2 t + \beta_2 + \gamma_2 e^{2t} + t^2(\sigma_2 t + \rho_2) + t(\mu_2 t + \nu_2)e^{2t}, \tag{11.2.112}$$

$$z(t) = \alpha_3 t + \beta_3 + \gamma_3 e^{2t} + t^2(\sigma_3 t + \rho_3) + t(\mu_3 t + \nu_3)e^{2t}. \tag{11.2.113}$$

Below, using $x(t)$ as an example we are going to explain all the terms. The arguments leading to (11.2.112) and (11.2.113) are identical.

- $\alpha_1 t + \beta_1$. If this term is written in the form of $(\alpha_1 t + \beta_1)e^{0t}$, it becomes obvious that it is simply a contribution to GSHE originating from the eigenvalue $\lambda_1 = 0$ of multiplicity 2.
- $\gamma_1 e^{2t}$. This is again the contribution to GSHE, but arising from the eigenvalue $\lambda_2 = 2$ of multiplicity 1.
- $t^2(\sigma_1 t + \rho_1)$. This expression arises from non-homogeneity and constitutes the contribution to SSNHE. It results from the presence of t (i.e., the first-degree polynomial) on the right-hand side of (11.2.108). If written in the form of te^{0t}, it becomes visible that there is a coincidence ("resonance") with the eigenvalue λ_1. Because of the multiplicity equal to 2 the degree of the resulting polynomial is increased up to the 3. This justifies the additional factor t^2. It is not necessary (although it would not be a mistake) to take into account the two other terms of the polynomial corresponding to the powers t^0 and t^1, because they have already been taken into account in GSHE.
- $t(\mu_1 t + \nu_1)e^{2t}$. This expression constitutes the contribution to SSNHE too and stems from the presence of terms of the type te^{2t} on the right-hand side of (11.2.108). The resonance with λ_2 led to the increase of the polynomial degree by 1.

If the structure of expressions for $x(t)$, $y(t)$, and $z(t)$ is now clear, one can simply insert them into (11.2.108), obtaining the system of linear equations for unknown constants and patiently solve it. It is clear that the constants contained in GSHE, i.e., $\alpha_{1,2,3}$, $\beta_{1,2,3}$ and $\gamma_{1,2,3}$ will not be completely determined in this way (although one can find relations between them), because their values do not result from matching the left- and right-hand sides of equations (11.2.108), but from the initial conditions which in this problem are not considered. In turn, for the remaining 12 constants

($\sigma_{1,2,3}$, $\rho_{1,2,3}$, $\mu_{1,2,3}$, and $\nu_{1,2,3}$), the system of 12 equations is obtained and can be solved.

After having differentiated (11.2.111) one gets

$$\dot{x}(t) = \alpha_1 + 2\gamma_1 e^{2t} + 3\sigma_1 t^2 + 2\rho_1 t + (2\mu_1 t^2 + 2\mu_1 t + 2\nu_1 t + \nu_1)e^{2t}. \tag{11.2.114}$$

Expressions for $\dot{y}(t)$ and $\dot{z}(t)$ are not worth being explicitly written because they differ from the above only by replacing subscripts of constants: $1 \mapsto 2$ and $1 \mapsto 3$. Upon plugging the expression for all three functions and their derivatives into the system (11.2.108) and sorting out all terms, one gets three identities:

$$\begin{cases} \sigma_2 t^3 + (3\sigma_1 + \rho_2)t^2 + (2\rho_1 + \alpha_2)t + \alpha_1 + \beta_2 \\ \quad +[(2\mu_1 + \mu_2)t^2 + (2\mu_1 + 2\nu_1 + \nu_2)t + 2\gamma_1 + \nu_1 + \gamma_2]e^{2t} = \dfrac{1}{2}te^{2t} + t, \\ 2(\sigma_1 - \sigma_3)t^3 + (2\rho_1 + 3\sigma_2 - 2\rho_3)t^2 + 2(\alpha_1 + \rho_2 - \alpha_3)t + 2\beta_1 + \alpha_2 - 2\beta_3 \\ \quad +[2(\mu_1 + \mu_2 - \mu_3)t^2 + 2(\nu_1 + \mu_2 + \nu_2 - \nu_3)t + 2\gamma_1 + 2\gamma_2 + \nu_2 - 2\gamma_3]e^{2t} \\ \quad = -\dfrac{1}{2}te^{2t}, \\ (2\sigma_1 + \sigma_2 - 2\sigma_3)t^3 + (2\rho_1 + \rho_2 - 2\rho_3 + 3\sigma_3)t^2 + (2\alpha_1 + \alpha_2 - 2\alpha_3 + 2\rho_3)t \\ \quad +2\beta_1 + \beta_2 - 2\beta_3 + \alpha_3 + [(2\mu_1 + \mu_2)t^2 + (2\nu_1 + \nu_2 + 2\mu_3)t \\ \quad +2\gamma_1 + \gamma_2 + \nu_3]e^{2t} = -te^{2t} + t. \end{cases} \tag{11.2.115}$$

They look very complicated and the reader might ask a reasonable question here, how do I efficiently solve such a huge and complex system? If any effective way to achieve this is not found, then using this method will turn out to be unviable. The good news is that such a way does exist. One simply should not simultaneously deal with a great number of equations and unknowns, but try to extract from (11.2.115) simpler subsystems for a smaller number of variables. For example, comparing the coefficients at t^3 on the left- and right-hand sides of the identities yields a simple subsystem, which can be even solved mentally:

$$\begin{cases} \sigma_2 = 0, \\ \sigma_1 - \sigma_3 = 0, \\ 2\sigma_1 + \sigma_2 - 2\sigma_3 = 0. \end{cases} \tag{11.2.116}$$

It entails that $\sigma_2 = 0$, and $\sigma_1 = \sigma_3 =: \sigma$ (here and below the simplified notation is introduced, omitting subscripts where possible).

Another easy subsystem can be extracted by comparing the coefficients accompanying t^2 (we have already put $\sigma_2 = 0$ and $\sigma_1 = \sigma_3 = \sigma$ and the same will be done with the successive variables below):

$$\begin{cases} 3\sigma + \rho_2 = 0, \\ \rho_1 - \rho_3 = 0, \\ 2\rho_1 + \rho_2 - 2\rho_3 + 3\sigma = 0. \end{cases} \tag{11.2.117}$$

Its solution gives $\rho_2 = -3\sigma$ and $\rho_1 = \rho_3 =: \rho$.

The terms containing t in the first power lead to the subsystem:

$$\begin{cases} 2\rho + \alpha_2 = 1, \\ \alpha_1 - 3\sigma - \alpha_3 = 0, \\ 2\alpha_1 + \alpha_2 - 2\alpha_3 + 2\rho = 1, \end{cases} \tag{11.2.118}$$

with the solution $\alpha_1 = \alpha_3 =: \alpha$, $\alpha_2 = 1 - 2\rho$, and $\sigma = 0$.

Finally, comparing remaining constants on both sides, one has

$$\begin{cases} \alpha + \beta_2 = 0, \\ 2\beta_1 + 1 - 2\rho - 2\beta_3 = 0, \\ 2\beta_1 + \beta_2 - 2\beta_3 + \alpha = 0 \end{cases} \tag{11.2.119}$$

and consequently $\beta_1 = \beta_3 =: \beta$, $\beta_2 = -\alpha$ i $\rho = 1/2$.

Let us summarize the results obtained so far:

$$\sigma_1 = \sigma_2 = \sigma_3 = 0,\ \rho_1 = \rho_3 = \frac{1}{2},\ \rho_2 = 0,$$
$$\alpha_1 = \alpha_3 = \alpha, \alpha_2 = 0,\ \beta_1 = \beta_3 = \beta,\ \beta_2 = -\alpha. \tag{11.2.120}$$

Now let us turn to the terms containing e^{2t}. The system of equations for unknowns $\mu_{1,2,3}$ and $\nu_{1,2,3}$ turns out to be completely independent of the former ones. This important property of the decomposition of the large system into subsystems has already been mentioned. Comparing the coefficients of t^2e^{2t}, one gets

$$\begin{cases} 2\mu_1 + \mu_2 = 0, \\ \mu_1 + \mu_2 - \mu_3 = 0, \\ 2\mu_1 + \mu_2 = 0 \end{cases} \tag{11.2.121}$$

which entails $\mu_1 =: \mu$, $\mu_2 = -2\mu$ and $\mu_3 = -\mu$. In turn the coefficients of te^{2t} lead to

$$\begin{cases} 2\mu + 2\nu_1 + \nu_2 = \dfrac{1}{2}, \\ 2(\nu_1 - 2\mu + \nu_2 - \nu_3) = -\dfrac{1}{2}, \\ 2\nu_1 + \nu_2 - 2\mu = -1. \end{cases} \tag{11.2.122}$$

The solution to this subsystem can be written in the form: $\nu_1 =: \nu$, $\nu_2 = -2\nu - 1/4$, $\nu_3 = -\nu - 3/4$, $\mu = 3/8$. There still remain the terms containing e^{2t}:

$$\begin{cases} 2\gamma_1 + \nu + \gamma_2 = 0, \\ 2\gamma_1 + 2\gamma_2 - 2\nu - \dfrac{1}{4} - 2\gamma_3 = 0, \\ 2\gamma_1 + \gamma_2 - \nu - \dfrac{3}{4} = 0, \end{cases} \tag{11.2.123}$$

yielding $\gamma_1 =: \gamma$, $\gamma_2 = -2\gamma + 3/8$, $\gamma_3 = -\gamma + 5/8$, $\nu = -3/8$.

Collecting all the obtained results, one has

$$\begin{aligned} &\mu_1 = \frac{3}{8}, \mu_2 = -\frac{3}{4}, \mu_3 = -\frac{3}{8}, \nu_1 = \nu_3 = -\frac{3}{8}, \nu_2 = \frac{1}{2}, \\ &\gamma_1 = \gamma, \gamma_2 = -2\gamma + \frac{3}{8}, \gamma_3 = -\gamma + \frac{5}{8}, \end{aligned} \tag{11.2.124}$$

and after having plugged all constants into (11.2.111), (11.2.112), and (11.2.113) the desired functions are obtained:

$$\begin{aligned} x(t) &= \underbrace{\alpha t + \beta + \gamma e^{2t}}_{\text{GSHE}} + \underbrace{\frac{1}{2}t^2 + \frac{3}{8}(t^2 - t)e^{2t}}_{\text{SSNHE}}, \\ y(t) &= \underbrace{-\alpha - 2\gamma e^{2t}}_{\text{GSHE}} + \underbrace{\frac{1}{8}\left(-6t^2 + 4t + 3\right)e^{2t}}_{\text{SSNHE}}, \\ z(t) &= \underbrace{\alpha t + \beta - \gamma e^{2t}}_{\text{GSHE}} + \underbrace{\frac{1}{2}t^2 + \frac{1}{8}\left(-3t^2 - 3t + 5\right)e^{2t}}_{\text{SSNHE}}. \end{aligned} \tag{11.2.125}$$

11.3 Exercises for Independent Work

Exercise 1 Find the general solutions of the equations:

(a) $\begin{cases} \dot{x} + 4\dot{y} - x - y = 0, \\ \dot{x} - 2\dot{y} + 3y = 0, \end{cases}$

(b) $\begin{cases} \dot{x} + 2\dot{y} + x - 2y - z = 0, \\ \dot{x} - \dot{y} + 2\dot{z} - 2y = 0, \\ \dot{x} + \dot{y} - 2\dot{z} - y = 0, \end{cases}$

(c) $\begin{cases} \dot{x} = 2x - 3y - te^t, \\ \dot{y} = x - 2y - 1. \end{cases}$

Answers

(a) $x(t) = e^{t/2}[3C_1 \cos(t/2) - (C_1 + 10C_2)\sin(t/2)]$, $y(t) = e^{t/2}[3C_2 \cos(t/2) + (C_1 + C_2)\sin(t/2)]$,
(b) $x(t) = e^{t/4}[(6C_1 + 3C_2 - 6C_3)\cos(3t/4) - (2C_1 - 9C_2 - 2C_3) \times \sin(3t/4)]/5 - (C_1 + 3C_2 - 6C_3)/5$, $y(t) = e^{t/4}[3C_2 \cos(3t/4) + 2(C_3 - C_1)\sin(3t/4)]/3$, $z(t) = e^{t/4}[(C_1 + 3C_2 - C_3)\cos(3t/4) - (2C_1 - 3C_2/2 - 2C_3)\sin(3t/4)]/5 - (C_1 + 3C_2 - 6C_3)/5$,
(c) $x(t) = e^t(-3t^2/4 + t/4 - 1/8 + 3C_1 - 3C_2) + e^{-t}(-C_1 + 3C_2) - 3$, $y(t) = e^t(-t^2/4 + t/4 - 1/8 + C_1 - C_2) + e^{-t}(-C_1 + 3C_2) - 2$.

Exercise 2 Find the solutions of the equations satisfying the given initial conditions:

(a) $\begin{cases} \dot{x} + \dot{y} - x - y = 0, \\ \dot{x} - 2\dot{y} + 2y = 0, \end{cases} \quad x(0) = 1,\ y(0) = 0,$

(b) $\begin{cases} \dot{x} + \dot{y} + x - y + z = 0, \\ \dot{x} + \dot{y} - 2\dot{z} - 2y = 0, \\ \dot{x} - \dot{y} - 2\dot{z} + y = 0, \end{cases} \quad x(0) = 1,\ y(0) = 1,\ z(0) = -1,$

(c) $\begin{cases} \dot{x} = 2x + y - z + 6t, \\ \dot{y} = -x + z - 2e^{-t}, \\ \dot{z} = -y - z, \end{cases} \quad x(0) = 1,\ y(0) = -2,\ z(0) = 0.$

Answers

(a) $x(t) = e^{2t/3}$, $y(t) = -e^{2t/3} + e^t$,
(b) $x(t) = (2e^{-3t/2} - e^{3t/2} + 8)/9$, $y(t) = e^{3t/2}$, $z(t) = (e^{-3t/2} - 2e^{3t/2} - 8)/9$,
(c) $x(t) = -t^3 - 6t^2 - 11t - 12 + 12e^t + e^{-t}$, $y(t) = t^3 + 6t^2 + 5t + 6 - 8e^t$, $z(t) = -t^3 - 3t^2 + t - 7 + 4e^t + 3e^{-t}$.

Exercise 3 Solve the Eqs. (11.2.36) and (11.2.69) by eliminating variables.

Exercise 4 Find the resolvent for the system (11.2.103), and then using formula (11.2.88), obtain the complete solution, assuming $x(0) = 2$, $y(0) = -2$, $z(0) = 5/8$.

Answers

$$x(t) = t^2/2 + 3t/8 + 1 + e^{2t}\left(3t^2 - 3t + 8\right)/8,$$
$$y(t) = -3/8 + e^{2t}\left(-6t^2 + 4t - 13\right)/8,$$
$$z(t) = t^2/2 + 3t/8 + 1 - 3e^{2t}(t^2 + t - 8)/8.$$

Chapter 12
Integrating in Many Dimensions

The present chapter covers the problems connected with the integration in more than one dimension. Our concern involves the case of $\mathbb{R}^N$ and the integral is understood in the Riemann sense, which most often appears in physics or engineering. Its construction is a simple generalization of that formulated in Chap. 1: in $\mathbb{R}^2$, instead of intervals one has to divide the domain into rectangles, in $\mathbb{R}^3$ into cuboids, and so on. Then one calculates the upper and lower sums just as in Problem 1 in Sect. 1.1 and compare their limits as the appropriate largest diagonal tends to zero. Such integral is called the **Riemann** or **Darboux integral**. The question of the Lebesgue integral remains beyond the scope of the present problem set.

The appropriate **integrability theorem** states:

If the function $f : \mathbb{R}^N \supset D \to \mathbb{R}$ is continuous and the set D is compact, then the integral

$$\int_D d^N x\, f(\vec{x}) \tag{12.0.1}$$

exists. The requirement of the continuity may sometimes be softened, but in the practical calculations of this chapter, the theorem in the form stated above will be of use to us.

The **iterated integral** is an integral in many dimensions that has the form of subsequent one-dimensional integrals. For instance in $\mathbb{R}^3$ an iterated integral over a cuboid is

$$\int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz\, f(x, y, z), \tag{12.0.2}$$

differ by the order in which the integration is performed. These six iterated integrals in general do not have to be equal to one another.

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Any compact set D can be contained in a cuboid (C) with its edges parallel to the axes of the coordinate system. If the definition of the integrand function $f(\vec{x})$ is supplemented as follows:

$$g(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in D, \\ 0 & \text{if } \vec{x} \in C \setminus D, \end{cases} \tag{12.0.3}$$

then the integral of the function $f(\vec{x})$ over the set D can always be replaced with the integral of the function $g(\vec{x})$ over the cuboid C, i.e., such as (12.0.2).

The following **Fubini theorem** holds: for a continuous function all iterated integrals exist and are equal. Therefore, if the function is integrable, the value of the integral (12.0.1) can be obtained by replacing it with any of its iterated integrals. In the exemplary case of $N = 2$, the appropriate formula (one of possible two) is

$$\int\limits_D d^2x\, f(x, y) = \int\limits_{a_1}^{a_2} dx \int\limits_{b_1}^{b_2} dy\, g(x, y). \tag{12.0.4}$$

The set $D \subset \mathbb{R}^2$ is called **regular to the x-axis** if it can be defined by the inequalities $a_1 \le x \le a_2$ and $\phi(x) \le y \le \psi(x)$ for some fixed a_1 and a_2 and certain functions $\phi(x)$ and $\psi(x)$ continuous on the interval $[a_1, a_2]$. Then the iterated integral (12.0.4) can be written as

$$I = \int\limits_{a_1}^{a_2} dx \int\limits_{\phi(x)}^{\psi(x)} dy\, f(x, y). \tag{12.0.5}$$

In an analogous way, the set regular to the y-axis may be defined. Then instead of (12.0.5), one can write

$$I = \int\limits_{b_1}^{b_2} dy \int\limits_{\tilde{\phi}(y)}^{\tilde{\psi}(y)} dx\, f(x, y), \tag{12.0.6}$$

with the obvious replacements of symbols.

An exemplary regular set is provided in Fig. 12.2 in Sect. 12.2. It can be in a straightforward way generalized to more than two dimensions. For more complicated sets, the decomposition into the sums of regular sets is possible.

12.1 Examining the Integrability of Functions

Problem 1

The integrability of the function:

$$f(x, y) = \begin{cases} \dfrac{x-y}{(x+y)^3} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0) \end{cases} \tag{12.1.1}$$

on the set $A = [0, 2] \times [0, 1]$ will be examined.

Solution

As we know from the theoretical introduction, if a function f is continuous and a set D is compact (in $\mathbb{R}^N$ this statement simply means that it is closed and bounded), then there exists the integral:

$$\int_D d^N x\, f(\vec{x}),$$

or in other words, the function f is integrable on D. In our exercise, the set (A) is the closed rectangle with vertices at $(0, 0)$, $(2, 0)$, $(0, 1)$, and $(2, 1)$, so there is no doubt that it is compact. Then if the function turned out to be continuous, the problem would be solved. Unfortunately, as it is easy to see, this is not the case. Indeed, for iterated limits one gets

$$\lim_{x\to 0^+} \lim_{y\to 0^+} f(x, y) = \lim_{x\to 0^+} \frac{x}{x^3} = \infty, \tag{12.1.2}$$

$$\lim_{y\to 0^+} \lim_{x\to 0^+} f(x, y) = \lim_{y\to 0^+} \frac{-y}{y^3} = -\infty. \tag{12.1.3}$$

Inspecting the results (12.1.2) and (12.1.3), one can see that the function f is unbounded on the set A. A bounded function could not have infinite values of limits. All this means that it is not possible to directly apply Riemann's construction to calculate the integral. The upper sum can be made arbitrarily large for each partition of the set A into rectangles. Similar problems appear while dealing with the lower sum (it will "escape" to $-\infty$). The integral of the function (12.1.1) on the set A—if it exists at all—has a character of the two-dimensional improper integral. This issue will be addressed in Exercise 3.

Going back to our problem, we see on the basis of (12.1.2) and (12.1.3) that the limit of the function f for $(x, y) \to (0, 0)$ does not exist, and thereby this function cannot be continuous on the set A. These issues were dealt with in Sect. 4.2. The reader might now propose to modify the set A by removing the origin which is the source of the problem. It would then become the rectangle without one of the vertices. Probably on such a set the function would become continuous; however, in this case, the set would cease to be compact. The point $(0, 0)$ is a limit point for the set A and must belong to it unless the set is to be closed (see Chap. 6 in Part I). And a continuous function on the non-compact set does not have to be integrable.

If so, can one conclude that the function is not integrable? Of course not! The mathematical theorem we referred to states what happens when an assumption *is satisfied*. But one still does not know whether the integral (12.1.1) exists if it *is not satisfied*, i.e., if the function is not continuous. To resolve this problem, let us examine the so-called iterative integrals (that is to say, take single integrals over variables x and y in turn). These integrals must be understood as improper ones too. Again, it would be appropriate to recall the theorem (the Fubini theorem for improper integrals) which states that for an integrable function all iterated integrals are equal. Let us see if this is true in our case. If it is found that they have different values, the non-integrability would be proved. Let us first calculate

$$\int_0^1 dy \int_0^2 dx \frac{x-y}{(x+y)^3} = \int_0^1 dy \int_0^2 dx \frac{x+y-2y}{(x+y)^3}$$
$$= \int_0^1 dy \int_0^2 dx \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right] = \int_0^1 dy \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2}\right]\Bigg|_0^2$$
$$= \int_0^1 dy \left[-\frac{1}{y+2} + \frac{y}{(y+2)^2} + \frac{1}{y} - \frac{1}{y}\right] = -2\int_0^1 dy \frac{1}{(y+2)^2} = -\frac{1}{3}, \tag{12.1.4}$$

and then

$$\int_0^2 dx \int_0^1 dy \frac{x-y}{(x+y)^3} = \int_0^2 dx \int_0^1 dy \frac{2x-(x+y)}{(x+y)^3}$$
$$= \int_0^2 dx \int_0^1 dy \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right] = \int_0^2 dx \left[-\frac{x}{(x+y)^2} + \frac{1}{x+y}\right]\Bigg|_0^1$$
$$= \int_0^2 dx \left[-\frac{x}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x}\right] = \int_0^2 dx \frac{1}{(x+1)^2} = \frac{2}{3}. \tag{12.1.5}$$

As we see, by swapping the order of integrations two different results are got, which means that the double integral

$$\iint\limits_A f(x, y)\, dx\, dy$$

does not exist.

Problem 2

The integrability of the function:

$$f(x, y) = \frac{x - y}{(x + y + 1)^3} \tag{12.1.6}$$

on the set $A = [0, 2] \times [0, 1]$ will be examined.

Solution

In this exercise, the integrability can be ruled on very easily. The set A is compact—as in the previous problem—and the continuity of the function (cut down to this set) can be justified since it is a rational function (i.e., a quotient of two polynomials) and as such it is continuous everywhere apart from the zeros of the denominator. These, however, are located beyond A because for positive x and y the condition $x + y + 1 = 0$ cannot be satisfied. As one can see the presence of the unity in the denominator plays a key role!

The continuity can also be strictly shown with the methods of Sect. 4.2 by selecting the sequence of points $(x_n, y_n) \in A$ convergent to a certain point $(\tilde{x}, \tilde{y}) \in A$, i.e., satisfying $\lim_{n\to\infty}(x_n, y_n) = (\tilde{x}, \tilde{y})$ and by examining the behavior of the corresponding sequence of values. Then we have at our disposal the following estimate:

$$\begin{aligned}
|f(x_n, y_n) - f(\tilde{x}, \tilde{y})| &= \left|\frac{x_n - y_n}{(x_n + y_n + 1)^3} - \frac{\tilde{x} - \tilde{y}}{(\tilde{x} + \tilde{y} + 1)^3}\right| \\
&= \left|\frac{(x_n - y_n)(\tilde{x} + \tilde{y} + 1)^3 - (\tilde{x} - \tilde{y})(x_n + y_n + 1)^3}{(x_n + y_n + 1)^3(\tilde{x} + \tilde{y} + 1)^3}\right| \\
&\le \frac{\left|(x_n - y_n)(\tilde{x} + \tilde{y} + 1)^3 - (\tilde{x} - \tilde{y})(x_n + y_n + 1)^3\right|}{1 \cdot 1} \\
&= \left|(x_n - y_n)(\tilde{x} + \tilde{y} + 1)^3 - (\tilde{x} - \tilde{y})(x_n + y_n + 1)^3\right|
\end{aligned} \tag{12.1.7}$$

thanks to the fact that for nonnegative x_n, y_n, $\tilde{x}$, and $\tilde{y}$ the following obvious inequalities are met:

$$x_n + y_n + 1 \geq 1 \quad \text{and} \quad \tilde{x} + \tilde{y} + 1 \geq 1. \tag{12.1.8}$$

Even without performing further transformations, it is visible that when $(x_n, y_n) \underset{n\to\infty}{\longrightarrow} (\tilde{x}, \tilde{y})$, the above expression has to converge to 0, since both terms become identical. Thus

$$\lim_{n\to\infty} |f(x_n, y_n) - f(\tilde{x}, \tilde{y})| = 0, \tag{12.1.9}$$

which means that the function $f(x, y)$ is continuous (on the given set). The continuous function on a compact set is integrable and, therefore, the double integral does exist:

$$\iint_A f(x, y)\, dx\, dy.$$

In principle the problem might be concluded here. Yet it is worth it to calculate both iterated integrals and be convinced that they are actually equal. One needs to remember however—to which we come back while solving the next problem—that their equality does not prejudge the existence of the double integral. First we find

$$\begin{aligned}
&\int_0^1 dy \int_0^2 dx\, \frac{x-y}{(x+y+1)^3} = \int_0^1 dy \int_0^2 dx\, \frac{x+y+1-2y-1}{(x+y+1)^3} \\
&= \int_0^1 dy \int_0^2 dx \left[\frac{1}{(x+y+1)^2} - \frac{2y+1}{(x+y+1)^3}\right] \\
&= \int_0^1 dy \left[-\frac{1}{x+y+1} + \frac{y+1/2}{(x+y+1)^2}\right]\Bigg|_0^2 \\
&= \int_0^1 dy \left[-\frac{1}{y+3} + \frac{y+1/2}{(y+3)^2} + \frac{1}{y+1} - \frac{y+1/2}{(y+1)^2}\right] \qquad (12.1.10) \\
&= \int_0^1 dy \left[-\frac{5/2}{(y+3)^2} + \frac{1/2}{(y+1)^2}\right] = \left[\frac{5/2}{y+3} - \frac{1/2}{y+1}\right]\Bigg|_0^1 = -\frac{5}{24} + \frac{1}{4} = \frac{1}{24},
\end{aligned}$$

and then

$$\int_0^2 dx \int_0^1 dy\, \frac{x-y}{(x+y+1)^3} = \int_0^2 dx \int_0^1 dy\, \frac{2x+1-(x+y+1)}{(x+y+1)^3}$$
$$= \int_0^2 dx \int_0^1 dy \left[\frac{2x+1}{(x+y+1)^3} - \frac{1}{(x+y+1)^2}\right]$$
$$= \int_0^2 dx \left[-\frac{x+1/2}{(x+y+1)^2} + \frac{1}{x+y+1}\right]\Bigg|_0^1$$
$$= \int_0^2 dx \left[-\frac{x+1/2}{(x+2)^2} + \frac{x+1/2}{(x+1)^2} + \frac{1}{x+2} - \frac{1}{x+1}\right]$$
$$= \int_0^2 dx \left[\frac{3/2}{(x+2)^2} - \frac{1/2}{(x+1)^2}\right] = -\frac{3/2}{x+2} + \frac{1/2}{x+1}\Bigg|_0^2$$
$$= -\frac{3}{2}\left(\frac{1}{4} - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3} - 1\right) = \frac{1}{24}. \tag{12.1.11}$$

Problem 3

The integrability of the function:

$$f(x, y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^{3/2}} & \text{for } (x, y) \neq (0, 0), \\ 1 & \text{for } (x, y) = (0, 0) \end{cases} \tag{12.1.12}$$

on the set $D = [0, 1] \times [0, 1]$ will be examined.

Solution

Solving this problem will begin in an unusual way, namely by calculating both iterated integrals. With some luck—if these turned out to be different—the non-integrability of the function would be proved. First, one finds

$$\int_0^1 dx \int_0^1 dy \, \frac{xy}{(x^2+y^2)^{3/2}} = -\int_0^1 dx \, x \int_0^1 dy \, \frac{d}{dy}\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$
$$= -\int_0^1 dx \; \frac{x}{\sqrt{x^2+y^2}}\bigg|_0^1 = -\int_0^1 dx \left(\frac{x}{\sqrt{x^2+1}} - 1\right)$$
$$= \left(x - \sqrt{x^2+1}\right)\Big|_0^1 = 1 - \sqrt{2} + 1 = 2 - \sqrt{2}. \tag{12.1.13}$$

Now in principle the same calculations for $\int_0^1 dy \int_0^1 dx f(x, y)$ should be carried out, but one does not have to do it because the result is known in advance. Due to the symmetry of the function f when swapping the arguments ($x \leftrightarrow y$) and the similar symmetry of the integration domain one has

$$\int_0^1 dy \int_0^1 dx f(x, y) = \int_0^1 dy \int_0^1 dx f(y, x) = \int_0^1 dx \int_0^1 dy f(x, y) = 2 - \sqrt{2}. \tag{12.1.14}$$

The former equality is a consequence of the symmetry of the function itself and the latter, the independence of the integral of the integration variable symbol used.

Both iterated integrals are then equal. The question arises whether this means that the function $f(x, y)$ is integrable. The answer already known to the reader is clear: the equality of iterated integrals constitutes only the necessary condition of integrability but not the sufficient one. To be certain of the integrability, one needs to know more about the behavior of the function. If it was found for example that it is continuous as in Exercise 2, the issue would be resolved. However, as it is easy to see, the function has no limit at the origin. Let us just approach it along the straight line $y = x$ and it appears that

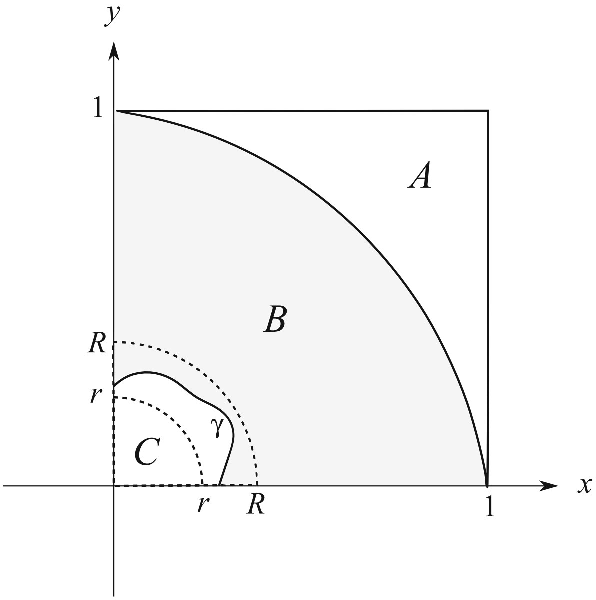
$$\lim_{x\to 0^+} f(x, x) = \lim_{x\to 0^+} \frac{x^2}{(2x^2)^{3/2}} = \frac{1}{2\sqrt{2}} \lim_{x\to 0^+} \frac{1}{x} = \infty. \tag{12.1.15}$$

It is clear that the function cannot be continuous at the point where it has no limit. The problem of integrability still remains unclear.

Pursuant to (12.1.15), it is evident that, as in Exercise 1, we are dealing with an unbounded function for which it is not possible to directly apply the Riemann construction. For the improper integral, one has to adopt the appropriate limiting procedure. Its idea is shown in Fig. 12.1. Let us separate the singular point, i.e., $(0, 0)$, from the integration domain with the use of a certain curve γ and examine the convergence of the integral

$$\iint_{B\cup A} f(x, y)\, dxdy,$$

Fig. 12.1 The way of calculating the two-dimensional improper integral



when "tightening" the curve around singularity. Since the integral $\iint_A f(x, y)\, dxdy$ does exist (it is an integral of a continuous function on a compact set), one needs only to handle the integral (the set B is drawn in gray):

$$I_\gamma = \iint\limits_B f(x, y)\, dxdy. \tag{12.1.16}$$

We still need two auxiliary integrals:

- on the set defined by the inequalities:

$$R^2 \leq x^2 + y^2 \leq 1 \quad \text{and} \quad x, y \geq 0,$$

- on the set defined by the inequalities:

$$r^2 \leq x^2 + y^2 \leq 1 \quad \text{and} \quad x, y \geq 0,$$

the parameters R and r being defined in the figure. The former integral will be denoted with the symbol I_R and the latter with I_r. Both of them are now well defined since one integrates a continuous function over compact sets. The reader probably noted at this point that if one wants to use this argument (of compactness) with respect to both sets A and B, one had to classify the splitting edge as belonging to both of them (otherwise the sets would not be compact). However, this is irrelevant for our reasoning, as the integral over the edge equals zero. Anyway, the separating of the set A had a purely technical reason of facilitating the integration (12.1.18) by the choice of the appropriate variables. The problem of hypothetical non-integrability of the function f is concerned with the neighborhood of the origin and not with the set A.

Because the function $f(x, y)$ is positive in the integration domain, the following inequalities hold:

$$I_R \le I_\gamma \le I_r. \tag{12.1.17}$$

The integral I_R is easy to calculate in polar variables:

$$I_R = \int_0^{\pi/2} d\varphi \int_R^1 d\rho\, \rho\, \frac{\rho^2 \cos\varphi \sin\varphi}{\rho^3} = \frac{1-R}{2} \int_0^{\pi/2} \sin 2\varphi\, d\varphi$$
$$= -\frac{1-R}{4} \cos 2\varphi \Big|_0^{\pi/2} = \frac{1-R}{4}, \tag{12.1.18}$$

where, in order to avoid a collision of symbols, the name of the radial variable from r in (6.2.2) has been changed to ρ. Likewise

$$I_r = \int_0^{\pi/2} d\varphi \int_r^1 d\rho\, \rho\, \frac{\rho^2 \cos\varphi \sin\varphi}{\rho^3} = \frac{1-r}{4}. \tag{12.1.19}$$

When the curve γ is being tightened to the origin, then $R, r \to 0$. With this, the integrals I_R and I_r are converging to the common limit of $1/4$, and pursuant to the inequality (12.1.17) we also have $I_\gamma \to 1/4$. One can conclude that the integral $\iint_B f(x, y)\, dxdy$ does exist and simultaneously so does $\iint_D f(x, y)\, dxdy$. The value of the latter is equal to $2 - \sqrt{2}$, since such was the value of the iterated integrals.

12.2 Calculating Integrals on Given Domains

Problem 1

The integral:

$$\iint_A dxdy\, e^{-x^2-y^2}, \tag{12.2.1}$$

where $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1 \ \wedge \ y \ge 0\}$, will be found.

Solution

In this section, we are going to become familiar with practical calculations of integrals in many dimensions, assuming that we are dealing with integrable functions. As we already know, for such functions all iterated integrals do exist and are equal. Our main concern will be which method of integration to use and what the guiding principle when choosing it should be. To this topic, the entire Chap. 13 is also dedicated, where various integrals important for physical or geometric applications are calculated.

Let us start with the integral (12.2.1), which can be easily found by the appropriate choice of variables. However, the question arises how one can know which variables to use. Any universal prescription which could be applied for various integrations is not possible to formulate, but we will do our best to give the reader at least a few tips.

Let us assume that we have to find an integral of a certain integrable function $f(x, y)$ on a set Z lying on the plane, i.e.,

$$I = \iint\limits_{Z} f(x, y)\, dxdy. \tag{12.2.2}$$

The first possibility which should be considered is the calculation of the integral in the Cartesian coordinates x and y as iterated integrals (or as a sum of them). If the set Z is regular to the x-axis (this concept appeared in the theoretical introduction at the beginning of this chapter, an illustration is shown in Fig. 12.2), i.e., it can be defined by the inequalities $a \le x \le b$ and $\phi(x) \le y \le \psi(x)$ for some fixed a and b and certain functions $\phi(x)$ and $\psi(x)$ continuous on the interval $[a, b]$, the corresponding iterated integral becomes

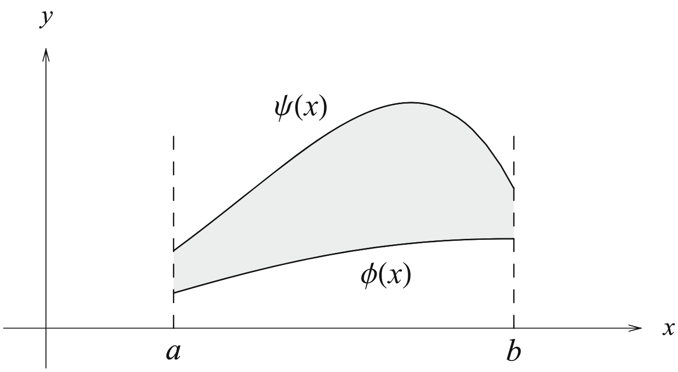


Fig. 12.2 An example of the set regular to the x-axis

$$I = \int\limits_a^b dx \int\limits_{\phi(x)}^{\psi(x)} dy\, f(x, y). \tag{12.2.3}$$

So, let us figure out whether the set A is regular to the x-axis. It is the top semi-circle centered at the origin and of unit radius, so it can certainly be defined by inequalities:

$$-1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \sqrt{1 - x^2}, \tag{12.2.4}$$

which entails that $a = -1$, $b = 1$, $\phi(x) = 0$, and $\psi(x) = \sqrt{1 - x^2}$. Consequently, formula (12.2.3) will take the specific form:

$$I = \int\limits_{-1}^{1} dx \int\limits_{0}^{\sqrt{1-x^2}} dy\, e^{-x^2-y^2} = \int\limits_{-1}^{1} dx\, e^{-x^2} \int\limits_{0}^{\sqrt{1-x^2}} dy\, e^{-y^2}. \tag{12.2.5}$$

Now, if the integration over y could be performed, the problem would boil down to the calculation of a one-dimensional integral. Often this will be the case but in the present exercise one is facing a serious problem: the integral $\int e^{-y^2} dy$ cannot be found in the explicit form (up to the numerical factor it is called the "error function" and the reader can be acquainted with it studying the so-called special functions). So something else must be tried. Let us check if the set A is not by coincidence regular also to the y-axis. If so, in place of (12.2.3) one could use the formula:

$$I = \int\limits_c^d dy \int\limits_{\alpha(y)}^{\beta(y)} dx\, f(x, y) \tag{12.2.6}$$

with certain constants $c < d$ and some continuous functions satisfying the condition $\alpha(y) \le \beta(y)$ on $[c, d]$. It turns out that the set A, i.e., the semi-circle, is regular to the y-axis too, because it can be described by the system of inequalities: $0 \le y \le 1$ and $-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$. Thus one can calculate the integral (12.2.1) as

$$I = \int\limits_0^1 dy \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, e^{-x^2-y^2} = \int\limits_0^1 dy\, e^{-y^2} \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, e^{-x^2}. \tag{12.2.7}$$

As we can see, the same problem has been encountered. This time the integration over variable x cannot be explicitly done. This means that it would be difficult to perform this integral in the Cartesian coordinates, although we must also be aware that there is no guarantee that it can be done in any other.

When selecting new integration variables, two elements should be taken into account: the dependence of the function f on x and y and the shape of the integration area. To one of them (and sometimes to both if one is lucky), the new variables should be fitted. In the case of the function $f(x, y) = e^{-x^2-y^2}$, it stands out that it depends only on the distance from the origin: $r = \sqrt{x^2 + y^2}$. This suggests that the polar coordinates might be the right choice, where

$$x = r\cos\varphi, \qquad y = r\sin\varphi \tag{12.2.8}$$

and $x^2 + y^2 = r^2$. In this case, the function f will not depend on the angle φ at all. The belief that the selection made is correct is further reinforced by the fact that the shape of the integration area in these variables is recorded in a particularly simple way:

$$0 \le r \le 1 \quad 0 \le \varphi \le \pi. \tag{12.2.9}$$

Since the Jacobian determinant occurring when passing from Cartesian to polar coordinates equals r, the integral (12.2.1) can be given the form:

$$\iint_A dxdy\, e^{-x^2-y^2} = \int_0^\pi d\varphi \int_0^1 dr\, r\, e^{-r^2}. \tag{12.2.10}$$

The integral over φ does not present any difficulties: it boils down to the length of the integration interval, i.e., π. The expression under the second integral is in turn the derivative of a certain function:

$$r\, e^{-r^2} = -\frac{1}{2} \cdot \frac{d}{dr} e^{-r^2}, \tag{12.2.11}$$

and therefore, it may be executed in the trivial way. The final result is

$$\iint_A dxdy\, e^{-x^2-y^2} = -\frac{\pi}{2}\, e^{-r^2}\Big|_0^1 = \frac{\pi}{2}\left(1 - \frac{1}{e}\right). \tag{12.2.12}$$

Problem 2

The integral:

$$I = \iiint_A \frac{dxdydz}{(x+y+z)^2}, \tag{12.2.13}$$

where $A = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le x + y \ \wedge\ 0 \le x, y \le 1\}$, will be found.

Solution

The integral in this problem should be naturally understood as a three-dimensional improper integral. The set A—the domain of the integration—is very simply defined in Cartesian coordinates. The integrand function also does not show any specific structure that would suggest to choose some other variables. For these reasons, for the integration in (12.2.13), one can choose the Cartesian variables and try to calculate the iterated integral:

$$I = \int_0^1 dx \int_0^1 dy \int_0^{x+y} dz \, \frac{1}{(x+y+z)^2}. \tag{12.2.14}$$

The limits of integration result directly from the definition of the set A given in the text of the problem. Of course one has

$$\int_0^{x+y} dz \, \frac{1}{(x+y+z)^2} = -\left.\frac{1}{x+y+z}\right|_0^{x+y} = -\frac{1}{2(x+y)} + \frac{1}{x+y} = \frac{1}{2(x+y)}, \tag{12.2.15}$$

so only the relatively easy double integral is left:

$$I = \int_0^1 dx \int_0^1 dy \, \frac{1}{2(x+y)} = \frac{1}{2}\int_0^1 dx \log(x+y)\Big|_0^1 = \frac{1}{2}\int_0^1 dx \left[\log(x+1) - \log x\right]. \tag{12.2.16}$$

The indefinite integral for the function $\log x$ is well known:

$$\int dx \log x = x \log x - x + C_1. \tag{12.2.17}$$

Likewise

$$\int dx \log(x+1) = (x+1)\log(x+1) - x + C_2. \tag{12.2.18}$$

Plugging both these formulas into (12.2.16), one finds

$$I = \frac{1}{2}\left[(x+1)\log(x+1) - x\log x\right]\Big|_0^1 = \log 2. \tag{12.2.19}$$

The integral in the lower limit is naturally treated as an improper one. This kind of integrations was already encountered in Exercise 2 of Sect. 2.1, so one can now simply use that result.

Problem 3

The integral:

$$I = \iiint\limits_A \frac{dxdydz}{(x^2 + y^2 + z^2)^\beta}, \tag{12.2.20}$$

where $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ and $\beta \in \mathbb{R}$, will be calculated.

Solution

For positive values of β, (12.2.20) has a meaning of a three-dimensional improper integral. A characteristic feature of the integrand is its dependence on variables x, y, and z only through the combination $x^2 + y^2 + z^2$. This suggests using the spherical variables r, θ, and φ defined as

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta. \tag{12.2.21}$$

These definitions were graphically presented previously in Fig. 6.1 on page 150. The integrand function depends only on one of them (i.e., r). This is a very significant simplification since the other two lead to trivial integrations. However, one needs to see, how the integration domain is expressed in these coordinates. It is possible that for the simplification of the integrand one will pay the price of very complicated integration limits. The set A is, however, a sphere centered at the origin and as such is described by the inequality $r \leq 1$ without referring to either φ or θ. Thus, the transition to spherical variables might be highly profitable. Bearing in mind that the Jacobian determinant in this case equals $r^2 \sin\theta$, we see that the triple integral fully factorizes into three separate integrals:

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\theta \int\limits_0^{1} dr\, r^2 \sin\theta\, \frac{1}{r^{2\beta}} = \underbrace{\left(\int\limits_0^{2\pi} d\varphi\right)}_{=2\pi} \cdot \left(\int\limits_0^{\pi} d\theta \sin\theta\right) \cdot \left(\int\limits_0^{1} dr\, r^{2(1-\beta)}\right). \tag{12.2.22}$$

The integral over φ reduces to 2π, and that over θ gives

$$\int\limits_0^{\pi} d\theta \sin\theta = -\cos\theta\Big|_0^{\pi} = -(-1) + 1 = 2. \tag{12.2.23}$$

The situations in which one integrates in the spherical variables a function dependent only on r are common. Therefore, it is worth remembering and using the fact that angular integral gives always the full solid angle:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta = 4\pi. \tag{12.2.24}$$

Thereby, only the integral over r is left. However, if $1-\beta < 0$, the function $r^{2(1-\beta)}$ diverges at zero and one has to work with an improper integral. As we know from Chap. 2, the condition for convergence of such an integral has the form (see formulas (2.1.7), (2.1.8), and (2.1.9)):

$$2(1-\beta) > -1,$$

i.e., $\beta < 3/2$. If it is satisfied, then in an easy way one finds

$$I = 4\pi \left. \frac{r^{3-2\beta}}{3-2\beta} \right|_0^1 = \frac{4\pi}{3-2\beta}. \tag{12.2.25}$$

12.3 Changing the Order of Integration

Problem 1

Assuming the integrability of the function f, the order of integrations in

$$I = \int_{-\pi/2}^{\pi/2} dx \int_{-1}^{\sin x} dy\, f(x, y) \tag{12.3.1}$$

will be reversed.

Solution

For an integrable function the iterated integral such as (12.3.1) is equal to the double integral on A. It is, therefore, reasonable to start solving this type of exercise with the detailed look at this set. This will enable us to write the desired second iterated integral. The inequalities which define A can be determined from the limits of integration in (12.3.1). Thus, this set is defined as follows:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid -\frac{\pi}{2} \le x \le \frac{\pi}{2} \;\wedge\; -1 \le y \le \sin x\right\}, \tag{12.3.2}$$

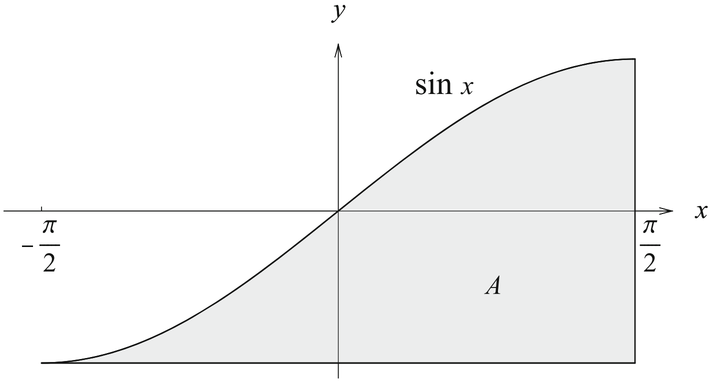


Fig. 12.3 The set A

and its graphical representation is shown in Fig. 12.3. On this basis, it is easy to observe that the same set could be described in the following way:

$$A = \left\{(x, y) \in \mathbb{R}^2 \,|\, -1 \le y \le 1 \quad \wedge \quad \arcsin y \le x \le \frac{\pi}{2}\right\}. \tag{12.3.3}$$

One can come to the same conclusion, rewriting the inequalities of the definition (12.3.2) in the form:

$$x \ge -\frac{\pi}{2} \quad \wedge \quad x \le \frac{\pi}{2} \quad \wedge \quad x \ge \arcsin y \quad \wedge \quad y \ge -1. \tag{12.3.4}$$

Since $\arcsin y \ge -\pi/2$, the first three of them imply that

$$\arcsin y \le x \le \frac{\pi}{2}. \tag{12.3.5}$$

In turn, the condition $y \le \sin x$ obviously entails $y \le 1$ and having regard to the fourth of the inequalities (12.3.4), one has

$$-1 \le y \le 1. \tag{12.3.6}$$

Collecting these results, we obtain the second iterated integral in the form:

$$I = \int\limits_{-1}^{1} dy \int\limits_{\arcsin y}^{\pi/2} dx f(x, y). \tag{12.3.7}$$

Problem 2

Assuming the integrability of the function f, the order of iterated integrations:

$$I = \int_0^2 dx \int_{-\sqrt{4x-x^2}}^{x} dy f(x, y) + \int_2^4 dx \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} dy f(x, y) \tag{12.3.8}$$

will be changed.

Solution

In expression (12.3.8), one is dealing with the sum of two integrals, so the integration set (denoted again by A) is a sum of the two subsets. Let us start with the first of them. It is defined with the inequalities:

$$0 \le x \le 2 \quad \text{and} \quad -\sqrt{4x-x^2} \le y \le x. \tag{12.3.9}$$

Note that for $x \in [0, 2]$ the inequality $y \le x$ guarantees satisfying the inequality $y \le \sqrt{4x-x^2}$. It stems from the following reasoning:

$$\begin{aligned} 2x(x-2) \le 0 \implies 2x^2 \le 4x \implies x^2 \le 4x - x^2 \\ \underset{0\le x\le 2}{\implies} x \le \sqrt{4x-x^2}. \end{aligned} \tag{12.3.10}$$

Thus one has

$$-\sqrt{4x-x^2} \le y \le \sqrt{4x-x^2}, \tag{12.3.11}$$

which implies that

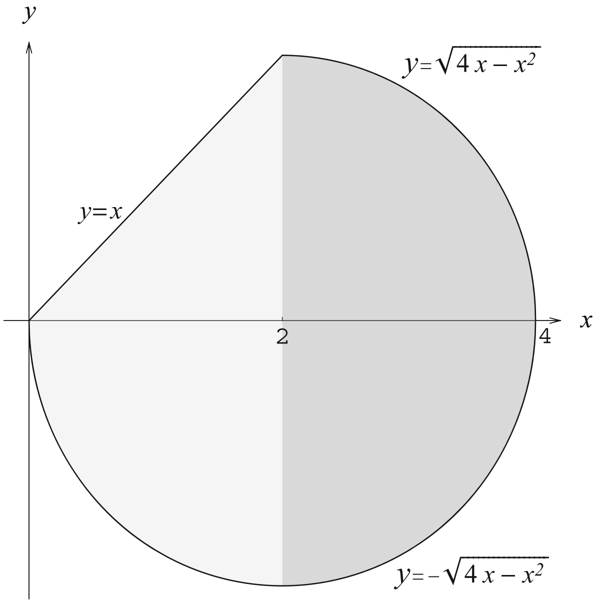
$$y^2 \le 4x - x^2, \quad \text{i.e.,} \quad (x-2)^2 + y^2 \le 4. \tag{12.3.12}$$

We have a circle centered at the point $(2, 0)$ and radius 2 and consequently the integration set constitutes its intersection with the strip $[0, 2] \times \mathbb{R}$ and the half-plane $y \le x$. It is shown in Fig. 12.4 marked with the light gray color.

The second subset is defined with the inequalities:

$$2 \le x \le 4 \quad \text{and} \quad -\sqrt{4x-x^2} \le y \le \sqrt{4x-x^2}. \tag{12.3.13}$$

Fig. 12.4 The set A



This time no transformations are needed in order to determine its shape: it is the intersection of the circle (12.3.12) and the strip $[2, 4] \times \mathbb{R}$ drawn in the figure with the darker color.

When solving the inequality (12.3.12) for x, it can easily be seen that the set A—the sum of both parts—can also be described with the inequalities:

$$2 - \sqrt{4 - y^2} \le x \le 2 + \sqrt{4 - y^2}, \quad \text{for } y \in [-2, 0] \tag{12.3.14}$$

and

$$y \le x \le 2 + \sqrt{4 - y^2}, \quad \text{for } y \in]0, 2]. \tag{12.3.15}$$

Consequently expression (12.3.8) may be rewritten in the form of:

$$I = \int\limits_{-2}^{0} dy \int\limits_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx f(x, y) + \int\limits_{0}^{2} dy \int\limits_{y}^{2+\sqrt{4-y^2}} dx f(x, y). \tag{12.3.16}$$

Problem 3

Assuming the integrability of the function f, the order of iterated integrations:

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{2-2y-2x} dz\, f(x, y, z) \tag{12.3.17}$$

will be changed into $\int dz \int dy \int dx$.

Solution

When looking at the integrations in (12.3.17), one can easily read out the inequalities, which define the area of integration. This system is as follows:

$$0 \le x \le 1, \quad 0 \le y \le 1 - x, \quad 0 \le z \le 2 - 2y - 2x. \tag{12.3.18}$$

Since all three variables are positive, it extends only in the first octant of the coordinate system and forms a pyramid (a tetrahedron) marked in Fig. 12.5 with bold lines.

The area is designated with four planes: $x = 0$, $y = 0$, $z = 0$, and $z = 2-2y-2x$. By placing subsequently one of the variables x, y, z equal to zero, we can determine,

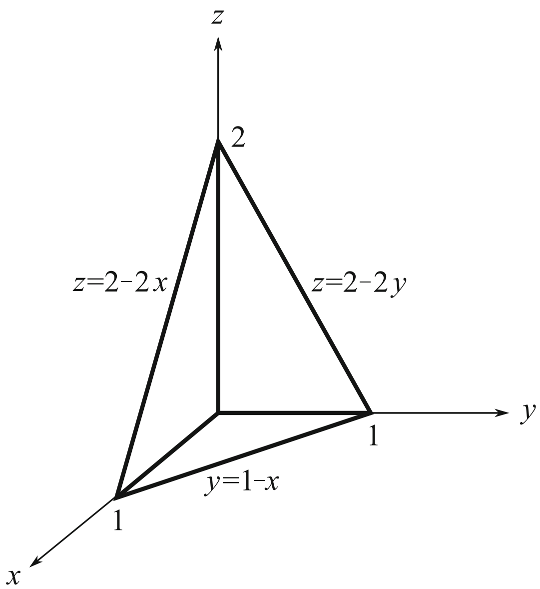


Fig. 12.5 The integration area in (12.3.17)

from this last equation, the lines along which this plane intersects the planes defined by the axes of coordinate system (i.e., obtain the equations of inclined edges). They are as follows:

$$\begin{cases} x = 0, \\ z = 2 - 2y, \end{cases} \qquad \begin{cases} y = 0, \\ z = 2 - 2x, \end{cases} \qquad \begin{cases} z = 0, \\ y = 1 - x, \end{cases} \tag{12.3.19}$$

which has been marked in the figure.

Now one must figure out how to describe the same set using another system of inequalities. Since the most external integral (i.e., the last integral to be calculated) is to be carried out over the variable z, one obtains immediately

$$0 \le z \le 2. \tag{12.3.20}$$

It is simply the widest range of variation of z corresponding to the situations where $x = y = 0$. The reader might here call this reasoning into question, arguing that when integrating the variables x and y other values will be acquired as well. Note, however, that they will be adjusted to the value of z and not the other way around. When z approaches 2, the admissible values of the other variables will get closer to zero, which is clear from the last inequality of (12.3.18) rewritten in the form:

$$0 \le x + y \le 1 - \frac{z}{2}. \tag{12.3.21}$$

Arbitrary numbers satisfying (12.3.20) may be substituted for z on the right-hand side but it will at most lead to narrowing down the integration intervals for x and y.

The integration limits for the following variable, that is to be y, can similarly be determined from the triangle (i.e., the wall of the pyramid) lying in the plane $x = 0$:

$$0 \le y \le 1 - \frac{z}{2}. \tag{12.3.22}$$

This is nothing other than the inequality (12.3.21) in which it has been set $x = 0$. Still, from (12.3.21), one has to establish the integration limits for x. One gets

$$0 \le x \le 1 - y - \frac{z}{2}. \tag{12.3.23}$$

The obtained results allow us to rewrite the iterated integral (12.3.17) in the alternative form:

$$I = \int\limits_0^2 dz \int\limits_0^{1-z/2} dy \int\limits_0^{1-y-z/2} dx\, f(x, y, z). \tag{12.3.24}$$

12.4 Exercises for Independent Work

Exercise 1 Examine the integrability of functions:

(a) $f(x, y) = \dfrac{xy}{xy+1}$ on the set $[0, 1] \times [0, 1]$,

(b) $f(x, y) = \begin{cases} \dfrac{x-y}{(x^2+y^2)^{3/2}} & \text{for } (x, y) \neq (0, 0), \\ 1 & \text{for } (x, y) = (0, 0), \end{cases}$
on the set $[0, 2] \times [0, 2]$.

> *Answers*
>
> (a) Integrable function,
> (b) Non-integrable function.

Exercise 2 Calculate the integrals over given domains:

(a) $\displaystyle\iint\limits_A (x+y)\log(x^2+y^2)\,dxdy$, where $A = [0, 1] \times [0, 1]$,

(b) $\displaystyle\iint\limits_A xy\sin(x^2+y^2)\,dxdy$,
where $A = \{(x, y) \mid x^2 + y^2 \leq \pi \ \wedge\ x, y \geq 0\}$.

> *Answers*
>
> (a) $(4\log 2 - 7 + \pi)/3$,
> (b) $\pi/4$.

Exercise 3 Calculate the integral (12.2.1) over the whole plane xy and obtain from it the known value of the Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

> *Answer*: $\sqrt{\pi}$.

Exercise 4 Assuming the integrability of the function f, change the order of integration in the iterated integrals:

(a) $\int\limits_0^1 dx \int\limits_0^x dy\ f(x, y) + \int\limits_1^2 dx \int\limits_0^{2-x} dy\ f(x, y),$

(b) $\int\limits_0^1 dy \int\limits_{-1}^1 dx\ f(x, y) + \int\limits_1^{\sqrt{2}} dy \int\limits_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} dx\ f(x, y),$

(c) $\int\limits_0^2 dx \int\limits_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} dy \int\limits_{x^2+y^2}^{2x} dz\ f(x, y, z)$ to $\int dz \int dy \int dx\ f(x, y, z).$

Answers

(a) $\int\limits_0^1 dy \int\limits_y^{2-y} dx\ f(x, y),$

(b) $\int\limits_{-1}^1 dx \int\limits_0^{\sqrt{2-x^2}} dy\ f(x, y),$

(c) $\int\limits_0^4 dz \int\limits_{-\sqrt{z(1-z/4)}}^{\sqrt{z(1-z/4)}} dy \int\limits_{z/2}^{\sqrt{z-y^2}} dx\ f(x, y).$

Chapter 13
Applying Multidimensional Integrals to Geometry and Physics

This last chapter of the second part of this book series is devoted solely to the applications of the multidimensional integrals to practical calculations of geometrical and physical quantities. We learn how to find surface areas of flat figures, volumes of solids, center of mass locations for certain mass distributions, moments of inertia of various solids, and quantities referring to the gravitational or electric fields.

The following **change-of-variables theorem** will especially be exploited:

Given a continuous function $f : \mathbb{R}^N \supset D \to \mathbb{R}$ and regular sets $\tilde{D}$ and D. Assume also that the mapping $\Phi : \mathbb{R}^N \supset \tilde{D} \to D \subset \mathbb{R}^N$ is bijective and is of the class C^1 on a certain open set containing $\tilde{D}$. Then

$$\int\limits_D d^N x\, f(\vec{x}) = \int\limits_{\tilde{D}} d^N \xi\, f(\vec{x}(\vec{\xi}))|J|, \tag{13.0.1}$$

where J denotes the determinant of the $N \times N$ Jacobian matrix (see (5.0.2)):

$$J = \frac{\partial(x_1, \ldots, x_N)}{\partial(\xi_1, \ldots, \xi_N)} = \det \begin{bmatrix} \partial x_1/\partial \xi_1 & \partial x_1/\partial \xi_2 & \cdots & \partial x_1/\partial \xi_N \\ \partial x_2/\partial \xi_1 & \partial x_2/\partial \xi_2 & \cdots & \partial x_2/\partial x i_N \\ \cdots & \cdots & \cdots & \cdots \\ \partial x_N/\partial \xi_1 & \partial x_N/\partial \xi_2 & \cdots & \partial x_N/\partial \xi_N \end{bmatrix}. \tag{13.0.2}$$

Assuming that the initial integral is formulated in the Cartesian coordinates, for the most common variables, the values of J are

- **polar coordinates** ($x = \rho \cos\varphi$, $y = \rho \sin\varphi$): $J = \rho$,
- **cylindrical coordinates** ($x = \rho \cos\varphi$, $y = \rho \sin\varphi$, $z = z$): $J = \rho$,
- **spherical coordinates** ($x = \rho \sin\theta \cos\varphi$, $y = \rho \sin\theta \sin\varphi$, $z = \rho \cos\theta$): $J = \rho^2 \sin\theta$.

No other theory involving integrals other than that found in Chap. 12 will be needed.

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13.1 Finding Areas of Flat Surfaces

Problem 1

The area of the figure defined as $A \cap B$ will be found, where

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 4x \le 0\},$$
$$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 4y \le 0\}. \tag{13.1.1}$$

Solution

Both sets A and B are circles, which can easily be established by rewriting the inequalities in the form of

$$A: \quad (x+2)^2 + y^2 \le 4, \qquad B: \quad x^2 + (y-2)^2 \le 4. \tag{13.1.2}$$

In the case of integrations on sets that are circles the use of polar variables is imposed. A problem one encounters is that the circles A and B are not concentric, so it is not clear where to place the origin of the coordinate system: at the point $(2, 0)$, $(0, 2)$, or any other. Fortunately, and it can be a surprise to the reader, the equation of a circle written in polar variables has a relatively simple form even if the origin is not located at the center of the circle. For example, the equation $x^2 + y^2 + 4x = 0$ expressed in variables r and φ defined with (12.2.8) takes the form:

$$r^2 + 4r\cos\varphi = 0. \tag{13.1.3}$$

When $\cos\varphi \le 0$, which is the case for $\varphi \in [\pi/2, 3\pi/2]$, then beside $r = 0$ this equation has also the second solution (which we are interested in here):

$$r = -4\cos\varphi. \tag{13.1.4}$$

The fact that $\pi/2 \le \varphi \le 3\pi/2$ is not surprising. Ultimately the circle is tangential to the y-axis at the origin and it is entirely contained between the two vertical tangential lines: this pointing "up" ($\varphi = \pi/2$) and that pointing "down" ($\varphi = 3\pi/2$).

Working similarly with the circle $x^2 + y^2 - 4y = 0$, which in turn is tangential to the x-axis, one will come to the conclusion that it can be described by

$$r = 4\sin\varphi, \tag{13.1.5}$$

for $\varphi \in [0, \pi]$. Both equations are relatively simple, and one will not have any difficulties in carrying out the integrations.

The area of $A \cap B$ marked in Fig. 13.1 in gray is given by the double integral:

$$P = \iint\limits_{A \cap B} dxdy. \tag{13.1.6}$$

In polar variables, this set is comprised of, in a natural way, two subsets separated by the straight line $y = -x$. They are defined by the inequalities:

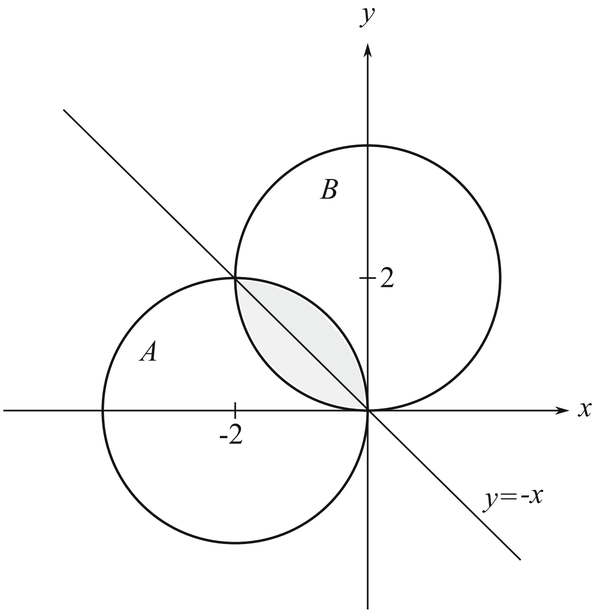
$$\frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{4} \quad \text{and} \quad 0 \leq r \leq -4\cos\varphi \tag{13.1.7}$$

for the first part and

$$\frac{3\pi}{4} < \varphi \leq \pi \quad \text{and} \quad 0 \leq r \leq 4\sin\varphi \tag{13.1.8}$$

for the second one. Bearing in mind that the Jacobian determinant associated with the transition to polar coordinates is equal to r, formula (13.1.6) can be rewritten in the form of the sum of integrations:

Fig. 13.1 The set $A \cap B$



$$P = \int\limits_{\pi/2}^{3\pi/4} d\varphi \int\limits_{0}^{-4\cos\varphi} dr\, r + \int\limits_{3\pi/4}^{\pi} d\varphi \int\limits_{0}^{4\sin\varphi} dr\, r. \tag{13.1.9}$$

Performing both of them and finding the surface area do not show any problem now. One gets

$$P = \int\limits_{\pi/2}^{3\pi/4} d\varphi\, \frac{r^2}{2}\bigg|_0^{-4\cos\varphi} + \int\limits_{3\pi/4}^{\pi} d\varphi\, \frac{r^2}{2}\bigg|_0^{4\sin\varphi} = 8\int\limits_{\pi/2}^{3\pi/4} d\varphi \cos^2\varphi + 8\int\limits_{3\pi/4}^{\pi} d\varphi \sin^2\varphi$$
$$= 8\int\limits_{\pi/2}^{3\pi/4} d\varphi\, \frac{1}{2}(1+\cos 2\varphi) + 8\int\limits_{3\pi/4}^{\pi} d\varphi\, \frac{1}{2}(1-\cos 2\varphi)$$
$$= 4\left(\frac{\pi}{4} + \frac{1}{2}\sin 2\varphi\bigg|_{\pi/2}^{3\pi/4}\right) + 4\left(\frac{\pi}{4} - \frac{1}{2}\sin 2\varphi\bigg|_{3\pi/4}^{\pi}\right) = 2\pi - 4. \tag{13.1.10}$$

We encourage the reader to check how this area could be found in the Cartesian coordinates.

Problem 2

The surface of the curvilinear quadrilateral confined within curves:

$$y = \frac{1}{x}, \quad y = \frac{27}{x}, \quad y = x^2, \quad y = 8x^2, \tag{13.1.11}$$

and located in the first quadrant of the coordinate system, will be found.

Solution

In this example we will discover to what extent the right choice of variables can simplify the integrations and help to find the surface area (of course this applies to other quantities as well). To this end we are going to solve this exercise in two ways and we will be able to compare the amount of work needed in both cases.

First, let us try to find the area of the quadrilateral shown in Fig. 13.2 (denoted with the symbol F), using Cartesian variables, i.e., to calculate integral:

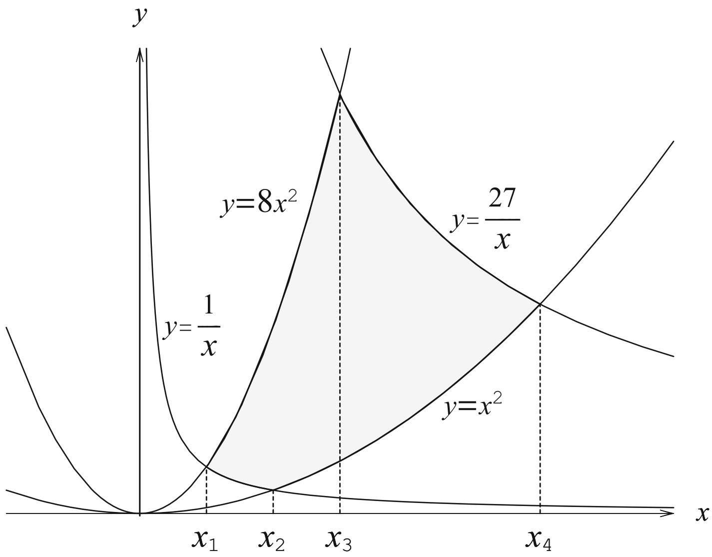


Fig. 13.2 The quadrilateral limited with the curves (13.1.11)

$$P = \iint_F dxdy. \tag{13.1.12}$$

One is dealing with the iterated integral so it is necessary to determine the coordinates of all vertices one-by-one looking for the points of intersection of the corresponding curves. First the following system of equations has to be solved:

$$\begin{cases} y = \dfrac{1}{x}, \\ y = 8x^2. \end{cases} \tag{13.1.13}$$

Eliminating y (this coordinate is not needed), one gets the simple equation for x:

$$8x^3 = 1 \implies x = \frac{1}{2}. \tag{13.1.14}$$

Hence, the x coordinate of the first vertex, i.e., x_1, is found. Now we are going to find x_2, by solving the system:

$$\begin{cases} y = \dfrac{1}{x}, \\ y = x^2. \end{cases} \tag{13.1.15}$$

Similarly as before one obtains the result: $x_2 = 1$. The remaining two systems of equations give

$$\begin{cases} y = \dfrac{27}{x}, \\ y = 8x^2, \end{cases} \quad \Longrightarrow \quad x_3 = \frac{3}{2} \tag{13.1.16}$$

and

$$\begin{cases} y = \dfrac{27}{x}, \\ y = x^2, \end{cases} \quad \Longrightarrow \quad x_4 = 3. \tag{13.1.17}$$

The equations (13.1.11) provide us with the integration limits for y, so one can write (13.1.12) in the form of

$$P = \int\limits_{x_1}^{x_2} dx \int\limits_{1/x}^{8x^2} dy + \int\limits_{x_2}^{x_3} dx \int\limits_{x^2}^{8x^2} dy + \int\limits_{x_3}^{x_4} dx \int\limits_{x^2}^{27/x} dy. \tag{13.1.18}$$

The y integrations are easily performed, leading to the result:

$$P = \int\limits_{1/2}^{1} \left(8x^2 - \frac{1}{x}\right) dx + \int\limits_{1}^{3/2} \left(8x^2 - x^2\right) dx + \int\limits_{3/2}^{3} \left(\frac{27}{x} - x^2\right) dx. \tag{13.1.19}$$

When calculating subsequent integrals over x, the needed surface area is found:

$$P = \left[\frac{8}{3}x^3 - \log x\right]\Bigg|_{1/2}^{1} + \frac{7}{3}x^3\Bigg|_{1}^{3/2} + \left[27\log x - \frac{1}{3}x^3\right]\Bigg|_{3/2}^{3} = 26\log 2. \tag{13.1.20}$$

Now let us move to the second method consisting of the appropriate choice of variables (ξ and η) suitable for a given problem. In our case, they are defined with formulas:

$$y = \frac{\xi}{x}, \quad y = \eta x^2, \quad \text{or} \quad \xi = xy, \quad \eta = \frac{y}{x^2}. \tag{13.1.21}$$

In these variables, the curvilinear quadrangle of Fig. 13.2 is expressed in an especially simple way:

$$1 \le \xi \le 27 \quad \text{and} \quad 1 \le \eta \le 8, \tag{13.1.22}$$

and the integral (13.1.12) takes the form:

$$P = \int_1^{27} d\xi \int_1^{8} d\eta \left| \underbrace{\frac{\partial(x, y)}{\partial(\xi, \eta)}}_{\text{Jacob. det.}} \right|. \tag{13.1.23}$$

One still needs the Jacobian determinant connected with the transition to new coordinates, but it is easier to find its inverse. The necessity of solving (13.1.21) for x and y will then be avoided. Obviously one has

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi\ \eta)}{\partial(x, y)} = 1, \tag{13.1.24}$$

which stems from the fact that Jacobian matrices for inverse functions are also inverse to each other. These issues were dealt with in Sect. 8.2 (see (8.0.5)). Therefore, let us calculate

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \det \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \det \begin{bmatrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{bmatrix} = \frac{3y}{x^2} = 3\eta. \tag{13.1.25}$$

It should be made clear that we had a lot of luck when expressing the Jacobian determinant in new variables. In general, (13.1.25) could depend on x and y in a nontrivial way. Passing to ξ and η would, in such a case, require us to unravel the equations (13.1.21).

By inserting this expression into (13.1.23), the result (13.1.20) is again obtained in a straightforward way:

$$P = \int_1^{27} d\xi \int_1^{8} d\eta \, \frac{1}{3\eta} = 26\, \frac{1}{3} \log \eta \Big|_1^8 = 26 \log 2. \tag{13.1.26}$$

Problem 3

The surface area of the lemniscate:

$$(x^2 + y^2)^2 = 2R^2(x^2 - y^2), \tag{13.1.27}$$

where $R > 0$, will be calculated.

Solution

Particularly convenient for calculating the area of the lemniscate are polar coordinates r and φ related to Cartesian ones with formulas already used several times:

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{13.1.28}$$

After plugging x and y in this form into (13.1.27) and using the Pythagorean trigonometric identity together with the formula for the cosine of a double angle:

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi, \tag{13.1.29}$$

it turns out that the equation of the lemniscate becomes (Fig. 13.3)

$$r^4 = 2R^2r^2\cos 2\varphi, \quad \text{i.e.,} \quad r^2(r^2 - 2R^2\cos 2\varphi) = 0. \tag{13.1.30}$$

If $\cos 2\varphi > 0$, i.e., for $-\pi/4 < \varphi < \pi/4$ or $3\pi/4 < \varphi < 5\pi/4$, beside $r = 0$ it has also the second solution (remember that r is nonnegative by definition):

$$r = R\sqrt{2\cos 2\varphi}. \tag{13.1.31}$$

This dependency, i.e., $r(\varphi)$, describes the lemniscate curve. Since the Jacobian determinant for the transition to new variables equals simply r, the area of the right "leaf" (i.e., the right half) is given by

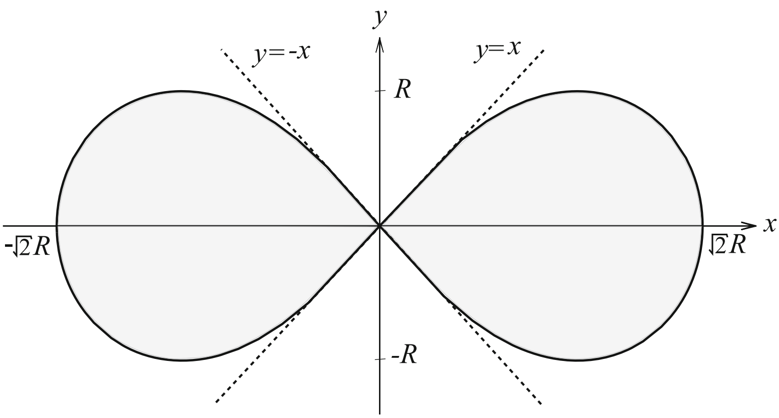


Fig. 13.3 The lemniscate (13.1.27)

$$\begin{aligned} P_1 &= \int\limits_{-\pi/4}^{\pi/4} d\varphi \int\limits_{0}^{R\sqrt{2\cos 2\varphi}} dr\, r = \frac{1}{2}\int\limits_{-\pi/4}^{\pi/4} d\varphi\, r^2\Big|_0^{R\sqrt{2\cos 2\varphi}} \\ &= R^2 \int\limits_{-\pi/4}^{\pi/4} d\varphi \cos 2\varphi = \frac{R^2}{2}\sin 2\varphi\Big|_{-\pi/4}^{\pi/4} = \frac{R^2}{2}[1-(-1)] = R^2. \end{aligned} \tag{13.1.32}$$

The full lemniscate, due to the symmetry of formula (13.1.27) when substituting $x \mapsto -x$, has the area equal to $P = 2P_1 = 2R^2$.

At the end, we would like to draw the reader's attention to the formula obtained after the r integration:

$$P_1 = \frac{1}{2}\int\limits_{-\pi/4}^{\pi/4} d\varphi\, r^2\Big|_0^{R\sqrt{2\cos 2\varphi}} = \frac{1}{2}\int\limits_{-\pi/4}^{\pi/4} d\varphi \left(R\sqrt{2\cos 2\varphi}\right)^2. \tag{13.1.33}$$

This is identical to (3.2.9) used in Sect. 3.2 when calculating the surface areas as a sum of areas of certain triangles (see Fig. 3.5) and constitutes its autonomous justification.

13.2 Calculating Volumes of Solids

Problem 1

The volume of the solid bounded with surfaces:

$$x^2 + y^2 = 1, \quad z = 0, \quad z = -\frac{x^2}{2} - y^2 + 2 \tag{13.2.1}$$

will be found.

Solution

The first step, without which it would be difficult to solve this task, is to imagine, or better to draw, the solid (let us denote it with B), whose volume is to be found. Strangely enough, it is not especially difficult. It is sufficient to observe that we are dealing with the cylinder, whose symmetry axis is simply that of the coordinate z (the surface of the cylinder is described with the equation $x^2 + y^2 = 1$), and cut-off from the bottom by the plane $z = 0$, and from the top by the elliptic paraboloid $z = -x^2/2 - y^2 + 2$. This latter surface owes its name to the fact that each cut with a vertical plane (i.e., parallel to the z-axis) is a parabola, and with a horizontal plane (i.e., $z = \text{const} < 2$)—an ellipse. It constitutes a "lid" of our solid as is shown in Fig. 13.4.

If one already has an idea about our solid, one needs now to decide in which variables the calculation of its volume would be the easiest. In Cartesian coordinates, one would have

$$V = \iiint\limits_B dxdydz, \tag{13.2.2}$$

but they are not adapted to the shape of B so this integral can be troublesome and time-consuming (though still possible). Which variables should be chosen, then? As it was mentioned earlier, the solid is cut out of a cylinder so the most natural variables appear to be cylindrical ones for which

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z; \tag{13.2.3}$$

the last equation simply means that the Cartesian variable z is left unchanged. Naturally, one does not have any guarantee that in the variables r, φ, and z the calculation will be easily done, but certainly they are worth considering in the first instance. The limiting surfaces of B will then be defined by the equations:

$$r = 1, \quad z = 0, \quad z = 2 - r^2\left(\frac{\cos^2\varphi}{2} + \sin^2\varphi\right). \tag{13.2.4}$$

This entails the following limits of integration for r and z:

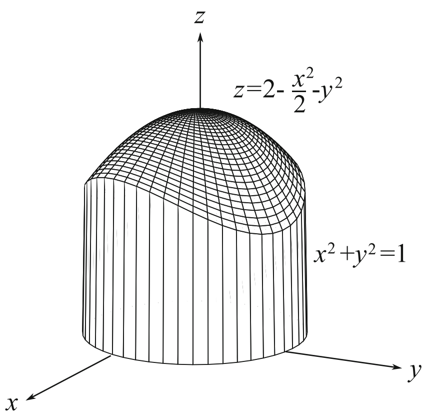


Fig. 13.4 The solid bounded with surfaces (13.2.1)

$$0 \le r \le 1, \quad 0 \le z \le 2 - r^2 \left(\frac{\cos^2 \varphi}{2} + \sin^2 \varphi \right) \tag{13.2.5}$$

and, surely, $0 \le \varphi < 2\pi$. Since for these variables the Jacobian determinant is identical as for the polar ones (and amounts to r), the volume can be written in the form of the following integral:

$$V = \int_0^{2\pi} d\varphi \int_0^1 dr \int_0^{z_0} dz\, r, \quad \text{where} \quad z_0 = 2 - r^2 \left(\cos^2 \varphi/2 + \sin^2 \varphi \right). \tag{13.2.6}$$

By performing an easy integration over z and next over r, we come to

$$\begin{aligned} V &= \int_0^{2\pi} d\varphi \int_0^1 dr\, rz \Big|_0^{2-r^2(\cos^2 \varphi/2+\sin^2 \varphi)} \\ &= \int_0^{2\pi} d\varphi \int_0^1 dr\, r \left[2 - r^2 \left(\frac{\cos^2 \varphi}{2} + \sin^2 \varphi \right) \right] \\ &= \int_0^{2\pi} d\varphi \left[r^2 - \frac{1}{4} r^4 \left(\frac{\cos^2 \varphi}{2} + \sin^2 \varphi \right) \right] \Bigg|_0^1 \\ &= \int_0^{2\pi} d\varphi \left[1 - \frac{\cos^2 \varphi}{8} - \frac{\sin^2 \varphi}{4} \right]. \end{aligned} \tag{13.2.7}$$

The squares of trigonometric functions have already been integrated a couple of times (see, e.g., (3.3.12)), so it is not surprising that

$$\int_0^{2\pi} d\varphi \sin^2 \varphi = \int_0^{2\pi} d\varphi \cos^2 \varphi = \pi. \tag{13.2.8}$$

By inserting this into (13.2.7), one finally gets

$$V = \pi \left(2 - \frac{1}{8} - \frac{1}{4} \right) = \frac{13\pi}{8}. \tag{13.2.9}$$

Problem 2

The volume of the solid limited by the surface:

$$(x^2 + y^2 + z^2)^{5/2} = R^3(x^2 + y^2 - z^2), \tag{13.2.10}$$

where $R > 0$, will be found.

Solution

The solid referred to in this problem is shown in Fig. 13.5 and again denoted with B. It is axially symmetric (with respect to the z-axis), which is due to the fact that the Eq. (13.2.10) depends on x and y only through the combination $x^2 + y^2$. If rewritten in spherical variables (12.2.21), it proves that the azimuthal angle φ disappears from (13.2.10) and only variables r and θ play a role:

$$r^5 = R^3 r^2 (\underbrace{\sin^2\theta - \cos^2\theta}_{=-\cos 2\theta}). \tag{13.2.11}$$

It is convenient to write this equation in the form of

$$r^2(r^3 + R^3 \cos 2\theta) = 0. \tag{13.2.12}$$

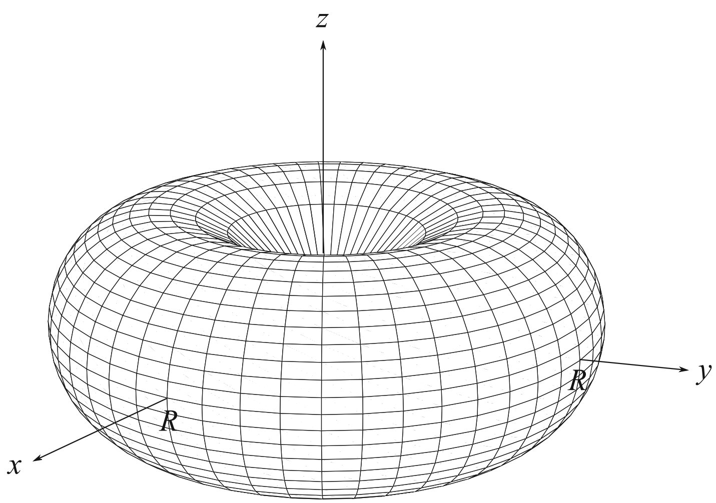


Fig. 13.5 The solid limited with the surface defined in Eq. (13.2.10)

It is clear that when $\cos 2\theta < 0$ the expression in brackets can vanish. This happens only for $\pi/4 < \theta < 3\pi/4$. For, we remember from the definition of spherical variables that the angle θ (i.e., the "latitude" measured from the north pole) lies in the interval $[0, \pi]$, and not $[0, 2\pi]$. In addition to $r = 0$, there appears, therefore, also the second solution of (13.2.12):

$$r = R\sqrt[3]{-\cos 2\theta}, \tag{13.2.13}$$

and the solid B extends for

$$0 \leq r \leq R\sqrt[3]{-\cos 2\theta}, \quad \frac{\pi}{4} < \theta < \frac{3\pi}{4}, \quad 0 \leq \varphi < 2\pi. \tag{13.2.14}$$

In order to make its visualization more transparent, in Fig. 13.6 the cross-section with the plane $y = 0$ is also provided. In view of the aforementioned axial symmetry, this cross-section is identical for each plane containing the z-axis.

One is now able to write down the integral enabling the calculation of the volume. Bearing in mind that the Jacobian determinant for spherical variables equals $r^2 \sin\theta$, one has

$$\begin{aligned} V &= \iiint_B dxdydz = \int_0^{2\pi} d\varphi \int_{\pi/4}^{3\pi/4} d\theta \int_0^{R\sqrt[3]{-\cos 2\theta}} dr\, r^2 \sin\theta \\ &= 2\pi \int_{\pi/4}^{3\pi/4} d\theta \sin\theta \left.\frac{r^3}{3}\right|_0^{R\sqrt[3]{-\cos 2\theta}} = -\frac{2\pi R^3}{3} \int_{\pi/4}^{3\pi/4} d\theta \sin\theta \cos 2\theta. \end{aligned} \tag{13.2.15}$$

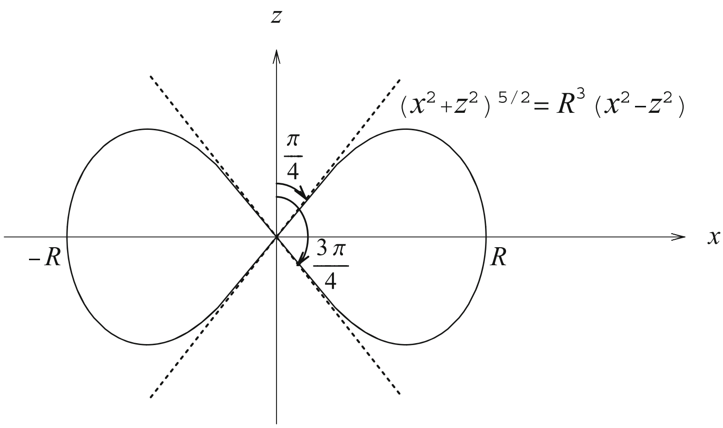


Fig. 13.6 The cut of the solid B with the plane $y = 0$

This last integral can easily be calculated as follows:

$$\int d\theta \sin\theta \cos 2\theta = \int d\theta \sin\theta (2\cos^2\theta - 1) \underset{t=\cos\theta}{=} \int dt (1 - 2t^2)$$
$$= t - \frac{2}{3} t^3 = \cos\theta - \frac{2}{3}\cos^3\theta, \tag{13.2.16}$$

and an integration constant is omitted. The result is:

$$V = -\frac{2\pi R^3}{3}\left(\cos\theta - \frac{2}{3}\cos^3\theta\right)\Big|_{\pi/4}^{3\pi/4}$$
$$= -\frac{2\pi R^3}{3}\, 2\left[-\frac{\sqrt{2}}{2} + \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] = \frac{4\sqrt{2}\pi}{9} R^3. \tag{13.2.17}$$

Problem 3

The volume of the torus T limited with the surface:

$$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2, \tag{13.2.18}$$

where $0 < r < R$, will be calculated.

Solution

The volume of the torus (13.2.18) was found in Exercise 2 of Sect. 3.3 with the use of formula (3.3.7) applicable for solids of revolution. This time we are going to find it, referring to multiple integrals—as we have been proceeding in other problems of this section—i.e., using the formula:

$$V = \iiint\limits_T dxdydz. \tag{13.2.19}$$

Naturally using Cartesian variables in this integral would be perplexing, mainly due to the integration limits, which would be expressed in a complicated way. A much better strategy would be to use variables that match the torus shape. Let us denote them with symbols ρ, φ, and θ. Both angles, φ and θ, have already been defined in Fig. 3.10 on page 94 and allow us to describe the surface (13.2.18) itself. However, in order to "enter" into the torus, one needs the additional variable ρ.

Ranges of variation for ρ, φ, and θ (and therefore, limits the integrations) are natural:

$$0 \le \rho \le r, \quad 0 \le \varphi < 2\pi, \quad 0 \le \theta < 2\pi, \tag{13.2.20}$$

and their relation to x, y, and z is as follows:

$$x = (R + \rho\cos\theta)\cos\varphi, \quad y = (R + \rho\cos\theta)\sin\varphi, \quad z = \rho\sin\theta. \tag{13.2.21}$$

In order to rewrite the integral (13.2.19) in these new variables, one first calculates the Jacobian determinant $\partial(x, y, z)/\partial(\rho, \varphi, \theta)$:

$$\begin{aligned}
J := \frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} &= \det\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\varphi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\varphi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\varphi & \partial z/\partial\theta \end{bmatrix} \\
&= \det\begin{bmatrix} \cos\theta\cos\varphi & -(R+\rho\cos\theta)\sin\varphi & -\rho\sin\theta\cos\varphi \\ \cos\theta\sin\varphi & (R+\rho\cos\theta)\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta & 0 & \rho\cos\theta \end{bmatrix} \\
&= (R+\rho\cos\theta)\rho\left[\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + \sin^2\theta(\cos^2\varphi + \sin^2\varphi)\right] \\
&= (R+\rho\cos\theta)\rho\left[\cos^2\theta + \sin^2\theta\right] = (R+\rho\cos\theta)\rho.
\end{aligned} \tag{13.2.22}$$

The expression for the volume (13.2.19) can now be given the form:

$$V = \int\limits_0^{2\pi} d\varphi \int\limits_0^r d\rho \int\limits_0^{2\pi} d\theta |(R+\rho\cos\theta)\rho| = 2\pi \int\limits_0^r d\rho\rho \int\limits_0^{2\pi} d\theta (R+\rho\cos\theta). \tag{13.2.23}$$

The integral of $\cos\theta$ over the entire period vanishes, so the second term in brackets disappears and one gets

$$V = (2\pi)^2 R \int\limits_0^r d\rho\, \rho = 4\pi^2 R \left.\frac{\rho^2}{2}\right|_0^r = 2\pi^2 R r^2, \tag{13.2.24}$$

i.e., the result known from formula (3.3.13).

Problem 4

The volume of the four-dimensional cylinder:

$$x^2 + y^2 + z^2 \le r^2, \quad 0 \le w \le h \tag{13.2.25}$$

will be found, where x, y, z, w are Cartesian coordinates in $\mathbb{R}^n$, and $r, h > 0$.

Solution

When using the words "volume of the cylinder" in the text of the exercise, we mean the generalized volume, which in Cartesian variables $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^n$ is given by the integral:

$$V = \int \ldots \int_W dx_1 dx_1 \ldots dx_n, \tag{13.2.26}$$

a solid being denoted with the symbol W. So defined volume corresponds to the "normal" volume, when $n = 3$, and the surface area for $n = 2$. In turn when $n = 1$ and a "solid" becomes simply an interval (or a sum of them), its volume is simply its length (or the sum of lengths). All of these volumes can be considered as a certain generalized volume or one can use in relation to all of them a universal concept of a "measure." We are interested below in the case $n = 4$, for which one has

$$V = \iiiint_W dx dy dz dw. \tag{13.2.27}$$

First we have to ask ourselves, in which variables is the integral (13.2.27) preferably calculated? One is dealing with the cylinder, so it seems to be clear: the best will probably be cylindrical coordinates, but they have to be introduced in four-dimensional space. Below we are going to describe how to do it. First of all, note that the first inequality of (13.2.25), i.e.,

$$x^2 + y^2 + z^2 \leq r^2, \tag{13.2.28}$$

describes a sphere in the three-dimensional space. Therefore, one should first introduce spherical variables in this space, and then attach the fourth variable w (it corresponds to the coordinate z in three dimensions). Thus, one has as usual:

$$x = \rho \sin\theta \cos\varphi, \quad y = \rho \sin\theta \sin\varphi, \quad z = \rho \cos\theta \tag{13.2.29}$$

together with the additional variable w. The angles φ and θ satisfy the inequalities:

$$0 \leq \varphi < 2\pi, \quad 0 \leq \theta \leq \pi, \tag{13.2.30}$$

and from (13.2.25) it stems that

$$\rho^2 \leq r^2 \quad \text{and} \quad 0 \leq w \leq h. \tag{13.2.31}$$

Rewriting formula (13.2.27) for the volume in the form of the iterated integral, one has

$$V = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{r} d\rho \int_0^{h} dw\, |J|. \tag{13.2.32}$$

The needed Jacobian determinant J is very easy to find. Just note that the problem to calculate the determinant of the 4×4 matrix reduces to the well-known determinant for the matrix 3×3. This is achieved in the following way:

$$J := \frac{\partial(x, y, z, w)}{\partial(\rho, \theta, \varphi, w)} = \det \begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\varphi & \partial x/\partial w \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\varphi & \partial y/\partial w \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\varphi & \partial z/\partial w \\ \partial w/\partial\rho & \partial w/\partial\theta & \partial w/\partial\varphi & \partial w/\partial w \end{bmatrix}$$

$$= \det \begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\varphi & 0 \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\varphi & 0 \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\varphi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\varphi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\varphi \end{bmatrix} \cdot \det[1] = \rho^2 \sin\theta. \tag{13.2.33}$$

First we used the known formula for the determinant of a block matrix:

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det A \cdot \det B, \tag{13.2.34}$$

and then the known expression for the Jacobian determinant was exploited.

It remains now to plug the obtained result into (13.2.32) and to perform the quadruple integration. The integrand expression depends neither on φ nor on w, so we immediately get

$$V = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{r} d\rho \int_0^{h} dw\, \rho^2 \sin\theta = 2\pi h \int_0^{\pi} d\theta \sin\theta \int_0^{r} d\rho\, \rho^2$$

$$= 2\pi h \int_0^{\pi} d\theta \sin\theta\, \frac{\rho^3}{3}\bigg|_0^r = \frac{2\pi}{3} hr^3(-\cos\theta)\bigg|_0^{\pi} = \frac{4\pi}{3} hr^3. \tag{13.2.35}$$

It is worth noticing that this result could have been predicted at the beginning. The volume of an "ordinary" cylinder is:

$$\text{measure of the base} \times \text{height}, \quad \text{i.e.,} \quad \pi r^2 \times h. \tag{13.2.36}$$

It is the same in our case with the difference that the "base" of a cylinder in four dimensions is not a circle, but a sphere and thus:

$$\frac{4}{3}\pi r^3 \times h.$$

This can be seen clearly in formula (13.2.35), which can be given the form:

$$V = \underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{r} d\rho\, \rho^2 \sin\theta}_{\text{volume of the sphere}} \quad \times \quad \underbrace{\int_0^{h} dw}_{\text{height of the cylinder}}. \tag{13.2.37}$$

13.3 Finding Center-of-Mass Locations

Problem 1

The center-of-mass location of the homogeneous figure limited by the lemniscate:

$$(x^2 + y^2)^2 = 2R^2(x^2 - y^2) \tag{13.3.1}$$

will be found, where $x > 0$ and $R > 0$.

Solution

As we know from physics, the center-of-mass of N point-like masses $m_1, m_2, \ldots, m_N$ at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ is located at (see (3.4.20))

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_N\vec{r}_N}{m_1 + m_2 + \ldots + m_N}. \tag{13.3.2}$$

In the case of a continuous mass distribution with a density $\rho(\vec{r})$ in a domain A, the sum gets converted into the volume integral:

$$\vec{r}_{CM} = \frac{\iiint\limits_A d^3r \rho(\vec{r})\vec{r}}{M}, \tag{13.3.3}$$

where M denotes the total mass, i.e.,

$$M = \iiint\limits_A d^3r\rho(\vec{r}). \tag{13.3.4}$$

In this exercise, we have to make a small modification of these formulas since we are concerned with the surface and not volume distribution of the mass. In such a situation, the integral must also be taken over the surface, and not over the volume:

$$\vec{r}_{CM} = \frac{\iint\limits_A d^2r\sigma(\vec{r})\vec{r}}{M}, \quad \text{where } M = \iint\limits_A d^2r\sigma(\vec{r}). \tag{13.3.5}$$

The symbol σ refers now to the surface mass density. When the body is homogeneous, that is if the density ρ (or σ) does not depend on coordinates, it can be extracted from the integrations and canceled in the numerator and denominator. The formulas are then simplified to

$$\vec{r}_{CM} = \frac{\iiint\limits_A d^3r\rho\,\vec{r}}{M} = \frac{\rho\iiint\limits_A d^3r\,\vec{r}}{\rho V} = \frac{\iiint\limits_A d^3r\,\vec{r}}{V} \tag{13.3.6}$$

in the volume case and

$$\vec{r}_{CM} = \frac{\iint\limits_A d^2r\sigma\,\vec{r}}{M} = \frac{\sigma\iint\limits_A d^2r\,\vec{r}}{\sigma S} = \frac{\iint\limits_A d^2r\,\vec{r}}{S} \tag{13.3.7}$$

for the surface case. V and S naturally denote the volume and the area occupied by the mass. Formula (13.3.7) will be particularly useful below since one is just dealing with a homogeneous flat figure.

In the first step, the area S should be found, but in this case one may use the result already obtained in Exercise 3 in Sect. 13.1 (see formula (13.1.32)) and substitute $S = R^2$.

The vector $\vec{r}_{CM}$ has two components: x_{CM} and y_{CM}. The value of the second can be established right away. Because of the symmetry of the figure under the reflection $y \mapsto -y$ (see Fig. 13.3), the coordinate y of the center of mass has to vanish. Otherwise, this symmetry would be violated. We are, therefore, left to find x_{CM}, in accordance with the formula:

$$x_{CM} = \frac{1}{R^2}\iint dxdy\,x. \tag{13.3.8}$$

Using polar coordinates in the same way as it was done when calculating the area in the aforementioned problem in Sect. 13.1 leads to

$$x_{CM} = \frac{1}{R^2}\int_{-\pi/4}^{\pi/4} d\varphi \int_0^{R\sqrt{2\cos 2\varphi}} dr\, r\, r\cos\varphi = \frac{1}{R^2}\int_{-\pi/4}^{\pi/4} d\varphi \cos\varphi \left.\frac{r^3}{3}\right|_0^{R\sqrt{2\cos 2\varphi}}$$

$$= \frac{2\sqrt{2}R}{3}\int_{-\pi/4}^{\pi/4} d\varphi \cos\varphi (\cos 2\varphi)^{3/2} = \frac{2\sqrt{2}R}{3}\int_{-\pi/4}^{\pi/4} d\varphi \cos\varphi (1-2\sin^2\varphi)^{3/2}. \tag{13.3.9}$$

By introducing a new integration variable in the form of $t = \sqrt{2}\sin\varphi$ and recalling that $\sin(\pm\pi/4) = \pm\sqrt{2}/2$, one can give the above expression in the form:

$$x_{CM} = \frac{2\sqrt{2}R}{3}\cdot\frac{1}{\sqrt{2}}\int_{-1}^{1} dt (1-t^2)^{3/2}. \tag{13.3.10}$$

The subsequent substitution, $t = \sin\alpha$, allows us to make use of the Pythagorean trigonometric identity and to get rid of the irrationality:

$$x_{CM} = \frac{2R}{3}\int_{-\pi/2}^{\pi/2} d\alpha \cos^4\alpha = \frac{2R}{3}\int_{-\pi/2}^{\pi/2} d\alpha \left[\frac{1}{2}(1+\cos 2\alpha)\right]^2$$

$$= \frac{R}{6}\int_{-\pi/2}^{\pi/2} d\alpha (1 + 2\cos 2\alpha + \cos^2 2\alpha). \tag{13.3.11}$$

While making these transformations, we have exploited the fact that for arguments belonging to the interval $[-\pi/2, \pi/2]$ the cosine is nonnegative, and therefore, one can write

$$(\cos^2\alpha)^{3/2} = |\cos\alpha|^3 = \cos^3\alpha,$$

and then the formula: $\cos^2\alpha = (1+\cos 2\alpha)/2$. It will be used once again below in order to get rid of the cosine squared in (13.3.11). Furthermore, we know that the integral of the cosine function over the entire period or its multiples vanishes. Thanks to it neither $\cos 2\alpha$ nor $\cos 4\alpha$ contributes and one finds

$$x_{CM} = \frac{R}{6}\int_{-\pi/2}^{\pi/2} d\alpha \left[1 + 2\cos 2\alpha + \frac{1}{2}(1+\cos 4\alpha)\right] = \frac{R}{6}\int_{-\pi/2}^{\pi/2} d\alpha\, \frac{3}{2} = \frac{\pi R}{4}. \tag{13.3.12}$$

From our previous considerations about the symmetry, it is known that y_{CM} must be equal to zero, but it is worth seeing how this result might be obtained from the calculations. When computing y_{CM}, after the substitution $\cos\varphi \mapsto \sin\varphi$ in the integrand expression (13.3.9), one would get the symmetric interval $[-\pi/4, \pi/4]$ over which an odd function φ would have to be integrated:

$$\sin\varphi(\cos 2\varphi)^{3/2}.$$

It is obvious that the result of such an integration vanishes.

Problem 2

The center-of-mass location of the homogeneous solid limited with the surface:

$$(x^2 + y^2 + z^2)^4 = z^7 \tag{13.3.13}$$

will be found.

Solution

The solid limited with the surface described by the Eq. (13.3.13) is the solid of revolution. This is due to the fact that this equation depends on variables x and y only through the combination $x^2 + y^2$. The rotary nature is particularly well visible in the spherical variables r, θ, and φ (as well as in the cylindrical ones) introduced by (6.2.12). Then in place of (13.3.13) one gets the equation:

$$r^8 = r^7\cos^7\theta, \quad \text{i.e.,} \quad r^7(r - \cos^7\theta) = 0, \tag{13.3.14}$$

which does not refer to the angle φ at all. The cross-section of any surface containing the z-axis is then identical. This means that the center-of-mass has to be situated on the z-axis, and thus without performing any calculation we get $x_{CM} = y_{CM} = 0$.

From the Eq. (13.3.14), it appears that for those angles θ, for which $\cos\theta > 0$, apart from the solution $r = 0$ the second possibility emerges: $r = \cos^7\theta$. This relation defines the surface of the solid, which consequently extends only to the values of the angle θ from the interval $[0, \pi/2]$. Its shape is demonstrated in Fig. 13.7. Since the mass density, according to the text of the exercise, is constant, one can use formula (13.3.6) for the third component of $\vec{r}_{CM}$ (the sole one, which still remains unknown), rewriting it directly in spherical variables:

$$z_{CM} = \frac{1}{V}\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^{\cos^7\theta} dr\, \underbrace{r^2\sin\theta}_{|J|}\, \underbrace{r\cos\theta}_{z}, \tag{13.3.15}$$

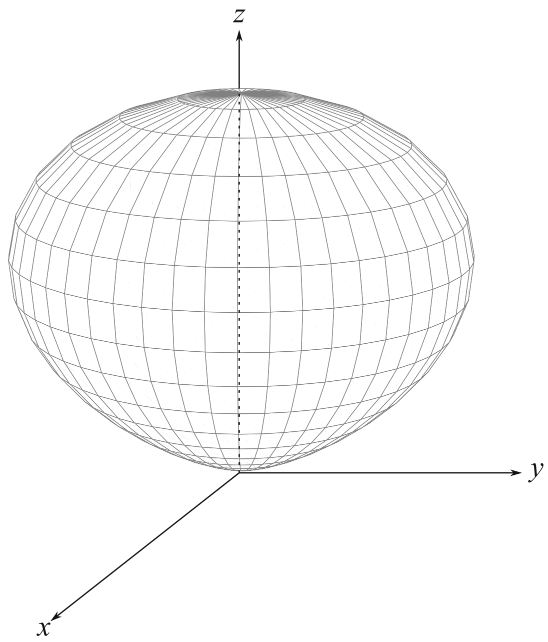


Fig. 13.7 The surface described with the Eq. (13.3.13)

where J denotes the Jacobian determinant. Integrating over φ and r, we find

$$z_{CM} = \frac{2\pi}{V}\int\limits_0^{\pi/2} d\theta \sin\theta\cos\theta \left.\frac{r^4}{4}\right|_0^{\cos^7\theta} = \frac{\pi}{2V}\int\limits_0^{\pi/2} d\theta \sin\theta\cos^{29}\theta. \tag{13.3.16}$$

The standard substitution $t = \cos\theta$ leads to

$$z_{CM} = \frac{\pi}{2V}\int\limits_0^1 dt\, t^{29} = \frac{\pi}{60V}\left. t^{30}\right|_0^1 = \frac{\pi}{60V}. \tag{13.3.17}$$

In order to find explicitly the value of z_{CM}, one has to obtain yet the volume V of the solid, calculating the similar integral:

$$V = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi/2} d\theta \int\limits_0^{\cos^7\theta} dr\, r^2 \sin\theta = 2\pi \int\limits_0^{\pi/2} d\theta \sin\theta \left.\frac{r^3}{3}\right|_0^{\cos^7\theta}$$

$$= \frac{2\pi}{3}\int\limits_0^{\pi/2} d\theta \sin\theta\cos^{21}\theta = \frac{2\pi}{3}\int\limits_0^1 dt\, t^{21} = \frac{\pi}{33}. \tag{13.3.18}$$

Plugging this result into (13.3.17), we finally get

$$z_{CM} = \frac{\pi}{60} \cdot \frac{33}{\pi} = \frac{11}{20}. \tag{13.3.19}$$

Problem 3

The center-of-mass of the half of the homogeneous ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1, \tag{13.3.20}$$

for $x \ge 0$, will be found.

Solution

Owing to the symmetry of the ellipsoid (and also its half) when reflected with respect to the plane defined by the axes x and z, i.e., when substituting $y \mapsto -y$, the center-of-mass has to be located on the plane $y = 0$. Likewise, the presence of symmetry when converting $z \mapsto -z$ leads to the conclusion that it must lie on the plane $z = 0$ too. One has, therefore,

$$y_{CM} = z_{CM} = 0, \tag{13.3.21}$$

and it remains only to find x_{CM}. The coordinate x does not exhibit any analogous symmetry due to the condition $x \ge 0$.

We are going to use formula (13.3.6), substituting for V the well-known expression for the volume of a semi-ellipsoid with semi-axes a, b, c:

$$V = \frac{1}{2} \cdot \frac{4}{3}\pi abc = \frac{2}{3}\pi abc. \tag{13.3.22}$$

This result could be easily reproduced by calculating the integral (13.2.2) in the variables defined by the equations:

$$x = ar \sin\theta \cos\varphi, \quad y = br \sin\theta \sin\varphi, \quad z = cr \cos\theta. \tag{13.3.23}$$

After plugging x, y, z in this form into (13.3.20) one gets for the ellipsoid a particularly simple inequality:

$$r^2 \le 1. \tag{13.3.24}$$

The angles θ and φ play an identical role as the spherical variables (see Fig. 6.1 on the page 150), so for the entire ellipsoid one has $0 \le \theta \le \pi$ and $-\pi \le \varphi \le \pi$

(of course one could also assume traditional $0 \leq \varphi \leq 2\pi$). In order to execute the integral (13.2.2), the Jacobian determinant for the transition from Cartesian coordinates to r, θ, φ will be needed. Its value can be easily determined, knowing that for the spherical variables it equals $r^2 \sin\theta$. One should note that the only difference between the Eqs. (13.3.23) and (6.2.12) is the presence of a, b, and c in the former. They appear in the Jacobian matrix as constants multiplying respectively the first, the second, and the third rows. However, as we know from algebra, the expression for the determinant contains only products of factors, each of which comes from another row. The only change in relation to the Jacobian determinant calculated for spherical variables is, therefore, the appearance of the additional factor abc in each term. As a result we obtain:

$$J = abcr^2 \sin\theta. \tag{13.3.25}$$

In these new variables, the volume of the semi-ellipsoid is expressed by the formula:

$$\begin{aligned} V &= \frac{1}{2}\int\limits_{-\pi}^{\pi} d\varphi \int\limits_0^{\pi} d\theta \int\limits_0^1 dr\, abcr^2 \sin\theta = \pi abc \int\limits_0^{\pi} d\theta \sin\theta \int\limits_0^1 dr\, r^2 \\ &= \pi abc(-\cos\theta)\Big|_0^{\pi} \frac{1}{3} r^3\Big|_0^1 = \frac{2}{3}\pi abc, \end{aligned} \tag{13.3.26}$$

which is identical to (13.3.22).

Now we calculate x_{CM}, reusing the above variables and noting that for the half of the ellipsoid, satisfying the condition $x \geq 0$, one has $-\pi/2 \leq \varphi \leq \pi/2$:

$$\begin{aligned} x_{CM} &= \frac{3}{2\pi abc}\int\limits_{-\pi/2}^{\pi/2} d\varphi \int\limits_0^{\pi} d\theta \int\limits_0^1 dr\, abcr^2 \sin\theta\, \underbrace{ar\sin\theta\cos\varphi}_{x} \\ &= \frac{3a}{2\pi}\int\limits_{-\pi/2}^{\pi/2} d\varphi\, cos\varphi \int\limits_0^{\pi} d\theta \sin^2\theta \int\limits_0^1 dr\, r^3 \\ &= \frac{3a}{2\pi}\sin\varphi\Big|_{-\pi/2}^{\pi/2} \cdot \int\limits_0^{\pi} d\theta\, \frac{1}{2}(1-\cos 2\theta)\, \frac{1}{4} r^4\Big|_0^1 = \frac{3a}{2\pi}\, 2\, \frac{\pi}{2} \cdot \frac{1}{4} = \frac{3}{8}a. \end{aligned} \tag{13.3.27}$$

Similarly as in the previous exercises, we used here the fact that the integral of the cosine function over the whole period vanishes:

$$\int\limits_0^{\pi} d\theta\, \cos 2\theta = 0.$$

13.4 Calculating Moments of Inertia

Problem 1

The moment of inertia of the homogeneous truncated cylinder with mass M described by the inequality:

$$x^2 + y^2 \le R^2, \tag{13.4.1}$$

with respect to the z-axis will be found. The base of the cylinder lies in the plane $z = 0$ and its top is cut by the plane $z = x + y + 2R$.

Solution

The definition of the moment of inertia for a set of material points with respect to the given axis was provided in Problem 3 in Sect. 1.2 (see formula (1.2.19)). In the case of a continuous mass distribution, the similar expression has the form:

$$I = \iiint\limits_{\text{solid}} dxdydz\, \rho\, (x^2 + y^2), \tag{13.4.2}$$

where for definiteness the z-axis has been chosen to play the role of the axis of rotation, and ρ denotes the mass density. In the case of a homogeneous solid, ρ is a constant and can be removed from under the integral.

In order to calculate the moment of inertia of a truncated cylinder, as shown in Fig. 13.8, it is convenient to proceed (which is natural) in the cylindrical coordinates r, φ, and z, defined by the relations (13.2.3). Because the cylinder is cut from the top by the plane:

$$z = x + y + 2R, \tag{13.4.3}$$

their range of variability is as follows:

$$0 \le r \le R, \quad 0 \le \varphi < 2\pi, \quad 0 \le z \le 2R + \underbrace{r(\cos\varphi + \sin\varphi)}_{x+y}. \tag{13.4.4}$$

Bearing in mind that the Jacobian determinant in this case equals r, in place of (13.4.2), one has to calculate the triple integral:

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^{R} dr \int\limits_0^{z_0} dz\, \rho\, r^3, \quad \text{where} \;\; z_0 = 2R + r(\cos\varphi + \sin\varphi). \tag{13.4.5}$$

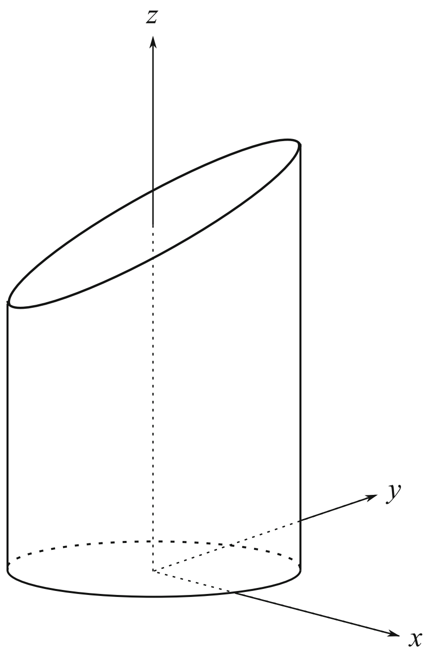


Fig. 13.8 The truncated cylinder cut with the plane (13.4.3)

Executing now the integral over z, which is trivial, and then using the fact that

$$\int_0^{2\pi} d\varphi\, \sin\varphi = \int_0^{2\pi} d\varphi\, \cos\varphi = 0, \tag{13.4.6}$$

one can easily find

$$I = \int_0^{2\pi} d\varphi \int_0^R dr\, \rho r^3 [2R + r(\cos\varphi + \sin\varphi)] = \int_0^{2\pi} d\varphi \int_0^R dr\, \rho r^3\, 2R$$
$$= 4\pi R\rho \int_0^R dr\, r^3 = 4\pi R\rho \frac{1}{4} r^4 \Big|_0^R = \pi\rho R^5. \tag{13.4.7}$$

The explicit expression for I has been obtained, but it would be helpful to get rid of the density ρ and express it through the mass of the cylinder. For this purpose from formula (13.2.2), V should be calculated, preferably in the cylindrical coordinates, and then substituted into $\rho = M/V$:

$$V = \int_0^{2\pi} d\varphi \int_0^R dr \int_0^{z_0} dz\, r. \tag{13.4.8}$$

The triple integral can be found in exactly the same way as above, yielding

$$V = \int_0^{2\pi} d\varphi \int_0^R dr\, r[2R + r(\cos\varphi + \sin\varphi)] = 4\pi R \int_0^R dr\, r = 4\pi R\, \frac{1}{2} r^2 \Big|_0^R = 2\pi R^3. \tag{13.4.9}$$

It should be noted that this value corresponds to the volume of a normal (i.e., uncut) cylinder of the height $2R$: $V = 2\pi R^3 = \pi R^2 \cdot 2R$. Upon inserting $\rho = M/V = M/(2\pi R^3)$ into (13.4.7), one gets the following value for the moment of inertia:

$$I = \pi R^5 \frac{M}{2\pi R^3} = \frac{1}{2} MR^2. \tag{13.4.10}$$

This result is known from the classical mechanics as the moment of inertia of an ordinary cylinder. This is not a coincidence, as seen from Fig. 13.9. If a piece of the truncated cylinder marked in light gray is moved into the place marked in dark gray, one gets an ordinary cylinder. With this operation the solid does not change its volume, which justifies the result (13.4.9). What is more, the mass distribution with respect to the rotation axis indicated in the figure does not change either, and therefore, I for both solids (before and after the transformation) must remain the same.

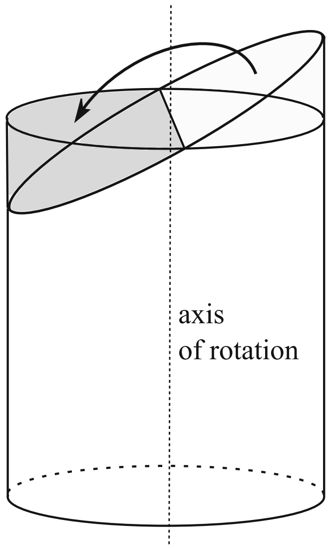


Fig. 13.9 The transformation of the truncated cylinder into a normal one

Problem 2

The moment of inertia of the nonhomogeneous cone:

$$\begin{cases} h^2(x^2+y^2) \leq R^2z^2, \\ 0 \leq z \leq h \end{cases} \tag{13.4.11}$$

with respect to the axis of symmetry of the cone will be found. The density of the solid is given by

$$\rho(x, y, z) = \frac{c}{x^2+y^2+R^2}. \tag{13.4.12}$$

R, h, c are positive constants.

Solution

The cross-section of the surface given by the equation $h^2(x^2+y^2) = R^2z^2$ with the plane $z = \text{const}$ is a circle of the equation $x^2+y^2 = (Rz/h)^2$. The radius of the circle equals Rz/h and, as one can see, it is linearly increasing with z, starting from 0 for $z = 0$ to R for $z = h$. So one is actually dealing with a cone which is standing on its apex.

In order to integrate over the cone volume, it is convenient to use, as in the previous problem, the cylindrical coordinates r, φ, and z (cf. (13.2.3)). For calculating the moment of inertia, we are going to exploit formula (13.4.2), rewriting it for the chosen variables. Aimed at determining the limits of integration, one needs to choose which integral will initially be calculated: that over r or over z (the φ integration does not pose any difficulties, since the cone has axial symmetry and the integrand expression does not depend on this angle). If one decides to execute first the integral over r—as it is done in this exercise so the reader can try to work in the opposite order—then for z, the maximal possible range may be taken, i.e., $0 \leq z \leq h$. The limits for the radial variable r will be adjusted to the radius of the circle corresponding to a given value of z:

$$0 \leq r \leq \frac{Rz}{h}. \tag{13.4.13}$$

For the azimuthal angle φ, one has, of course, $0 \leq \varphi < 2\pi$. Using these facts, one can write

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^h dz \int\limits_0^{Rz/h} dr\, r\, \rho\, r^2 = \int\limits_0^{2\pi} d\varphi \int\limits_0^h dz \int\limits_0^{Rz/h} dr\, \frac{cr^3}{r^2+R^2}, \tag{13.4.14}$$

the additional factor r arising from the Jacobian matrix. The integral over the angle φ is easily performed, giving just the factor of 2π, and for this over r it is the most convenient to choose a new variable t defined as $t = r^2$. Then we obtain

$$
\begin{aligned}
I &= 2\pi c\,\frac{1}{2}\int_0^h dz \int_0^{(Rz/h)^2} dt\,\frac{t}{t+R^2} = \pi c \int_0^h dz \int_0^{(Rz/h)^2} dt\left(1-\frac{R^2}{t+R^2}\right)\\
&= \pi c \int_0^h dz\left[t - R^2\log(t+R^2)\right]\Big|_0^{(Rz/h)^2}\\
&= \pi c \int_0^h dz\left[\frac{R^2}{h^2}z^2 - R^2\log\left(\frac{z^2}{h^2}+1\right)\right],
\end{aligned}
\tag{13.4.15}
$$

where the following transformation:

$$
\begin{aligned}
&\log R^2 - \log\left(\frac{R^2z^2}{h^2}+R^2\right) = \log R^2 - \log\left[R^2\left(\frac{z^2}{h^2}+1\right)\right]\\
&= \log R^2 - \log R^2 - \log\left(\frac{z^2}{h^2}+1\right) = -\log\left(\frac{z^2}{h^2}+1\right)
\end{aligned}
\tag{13.4.16}
$$

has been used.

In order to fully carry out (13.4.15), one has to calculate the integral of the form $\int dz\,\log(\alpha z^2+\beta)$, where α and β are certain positive constants. One can find it, integrating “by parts”:

$$
\begin{aligned}
&\int \log(\alpha z^2+\beta)\,dz\\
&= \int [z]'\log(\alpha z^2+\beta)\,dz = z\log(\alpha z^2+\beta) - \int z\,[\log(\alpha z^2+\beta)]'dz\\
&= z\log(\alpha z^2+\beta) - 2\alpha\int\frac{z^2}{\alpha z^2+\beta}\,dz = z\log(\alpha z^2+\beta) - 2\int\frac{z^2}{z^2+\beta/\alpha}\,dz\\
&= z\log(\alpha z^2+\beta) - 2\int\left(1-\frac{\beta}{\alpha}\cdot\frac{1}{z^2+\beta/\alpha}\right)dz\\
&= z\log(\alpha z^2+\beta) - 2z + 2\sqrt{\frac{\beta}{\alpha}}\arctan\left(z\sqrt{\frac{\alpha}{\beta}}\right),
\end{aligned}
\tag{13.4.17}
$$

where the known integral (for nonzero value of a):

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a},$$

has been used. This result allows us to calculate the integral (13.4.15). After plugging in $\alpha = 1/h^2$ and $\beta = 1$ and applying certain simplifications we get

$$I = \pi c R^2 h\left(\frac{7}{3} - \log 2 - \frac{\pi}{2}\right). \tag{13.4.18}$$

Now one has to find the mass M, calculating

$$M = \int_0^{2\pi} d\varphi \int_0^h dz \int_0^{Rz/h} dr\, r\,\rho = \int_0^{2\pi} d\varphi \int_0^h dz \int_0^{Rz/h} dr\, \frac{cr}{r^2+R^2}$$

$$= 2\pi c \int_0^h dz \int_0^{Rz/h} dr \left[\frac{1}{2}\log(r^2+R^2)\right]' = \pi c \int_0^h dz \log\left(\frac{z^2}{h^2}+1\right). \tag{13.4.19}$$

Again (13.4.17) is going to be used, leading to

$$M = \pi c h\left(\log 2 + \frac{\pi}{2} - 2\right), \quad \text{i.e.,} \quad c = \frac{M}{\pi h(\log 2 + \pi/2 - 2)}. \tag{13.4.20}$$

After having eliminated the parameter c from (13.4.18), one obtains the moment of inertia in the form in which it is expressed only through the mass and dimensions of the solid:

$$I = \frac{7/3 - \log 2 - \pi/2}{\log 2 + \pi/2 - 2}\, M R^2. \tag{13.4.21}$$

Note that the result does not depend on the cone height (h). This is understandable from the physical point of view, as stretching and compressing the cone vertically do not influence the mass distribution with respect to the rotation axis.

Problem 3

The moment of inertia of the homogeneous torus of the mass of M, limited with the surface:

$$(\sqrt{x^2+y^2} - R)^2 + z^2 = r^2, \tag{13.4.22}$$

with respect to the axis of symmetry, will be found. It is assumed that $0 < r < R$.

Solution

The variables ρ, φ, and θ convenient to perform the integrations over the torus volume were defined in Exercise 3 of Sect. 13.2. Their relations to the Cartesian ones are recalled below:

$$x = (R + \rho\cos\theta)\cos\varphi, \quad y = (R + \rho\cos\theta)\sin\varphi, \quad z = \rho\sin\theta. \tag{13.4.23}$$

The meaning of the angles θ and φ is explained in Fig. 3.10 on page 94. In that section, the Jacobian determinant associated with the transition from x, y, z to ρ, φ, θ was found too (see formula (13.2.22)):

$$\frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} = \rho\,(R + \rho\cos\theta). \tag{13.4.24}$$

In the chosen variables, expression (13.4.2) for the moment of inertia takes the form:

$$\begin{aligned} I &= \int_0^{2\pi} d\varphi \int_0^{r} d\rho \int_0^{2\pi} d\theta\, \rho\,(R + \rho\cos\theta)\rho_M \underbrace{(R + \rho\cos\theta)^2}_{x^2+y^2} \\ &= 2\pi\rho_M \int_0^{r} d\rho\, \rho \int_0^{2\pi} d\theta (R + \rho\cos\theta)^3, \end{aligned} \tag{13.4.25}$$

and this time the mass density has been denoted with the symbol ρ_M to avoid conflict with the radial coordinate ρ. The θ integration is performed over the entire period of the cosine function, and therefore, in the expansion of the expression $(R+\rho\cos\theta)^3$, the terms with odd powers of $\cos\theta$ do not contribute:

$$\int_0^{2\pi} d\theta\ \cos\theta = \int_0^{2\pi} d\theta\ \cos^3\theta = 0. \tag{13.4.26}$$

One also has

$$\int_0^{2\pi} d\theta\ \cos^2\theta = \frac{1}{2}\int_0^{2\pi} d\theta (1 + \cos 2\theta) = \pi. \tag{13.4.27}$$

Using these results, (13.4.25) can be rewritten in the form of

$$I = 2\pi\rho_M \int_0^r d\rho\, \rho \int_0^{2\pi} d\theta \left(R^3 + 3R^2\rho\cos\theta + 3R\rho^2\cos^2\theta + \rho^3\cos^3\theta\right)$$
$$= 2\pi^2\rho_M \int_0^r d\rho\, (2R^3\rho + 3R\rho^3) = 2\pi^2\rho_M \left(R^3\rho^2 + \frac{3}{4}\, R\rho^4\right)\Big|_0^r$$
$$= \pi^2\rho_M \left(2R^3r^2 + \frac{3}{2}\, Rr^4\right). \tag{13.4.28}$$

Then, ρ_M can be eliminated, thanks to the equation:

$$\rho_M = \frac{M}{V} = \frac{M}{2\pi^2 Rr^2}. \tag{13.4.29}$$

In lieu of V, we have plugged in the previously calculated volume of the torus (see (13.2.24)). After having used (13.4.29), the moment of inertia takes the simple form:

$$I = \frac{M}{2\pi^2 Rr^2}\, \pi^2 \left(2R^3r^2 + \frac{3}{2}\, Rr^4\right) = M\left(R^2 + \frac{3}{4}\, r^2\right). \tag{13.4.30}$$

In the case of $r \to 0$, the torus becomes a thin hoop of radius R and formula (13.4.30) reduces to the known result $I = MR^2$.

13.5 Finding Various Physical Quantities

Problem 1

The gravitational potential of the homogeneous sphere of mass M and radius R will be found at any point in the space outside the sphere.

Solution

As we know from physics, the gravitational potential generated by a point-like mass m at the distance r from it is given by the formula:

$$\phi = -\frac{Gm}{r}, \tag{13.5.1}$$

where G is the gravitational constant. In the case when the mass is extended—as in the current problem—one needs to partition it into points and, using the superposition principle, find the total potential energy as a scalar sum of potentials originating from these point sources. Consider, for example, a cell with infinitesimal dimensions $dx \times dy \times dz$ lying inside the mass, such as that shown in Fig. 13.10. At the observation point, marked with A, it will produce the following contribution to the potential:

$$-G\underbrace{\rho\, dxdydz}_{\text{solid}}\frac{1}{|\vec{r}-\vec{x}|}. \tag{13.5.2}$$

The vectors $\vec{r}$ and $\vec{x}$ are defined in the figure. As a result, the full potential is expressed in the form of the integral:

$$\phi(\vec{r}) = -\iiint\limits_{\text{mass}} d^3x\,\frac{G\rho}{|\vec{r}-\vec{x}|}. \tag{13.5.3}$$

In this exercise, the Cartesian coordinates of the points belonging to the solid (i.e., the source of the gravitational field) are denoted with x_1, x_2, and x_3, and the coordinates of the observation point analogously: r_1, r_2, and r_3. In the figure (and for further calculations), the observation point is chosen to be located on the positive semi-axis x_3 (i.e., z), so it has the coordinates $(0, 0, r)$, where $r = |\vec{r}|$. Due to the spherical symmetry of the solid and consequently of the gravitational field too, this special choice does not constitute any restriction, and will only simplify the calculations. All points located at the same distance $|\vec{r}|$ from the center of the sphere will have the same gravitational potential.

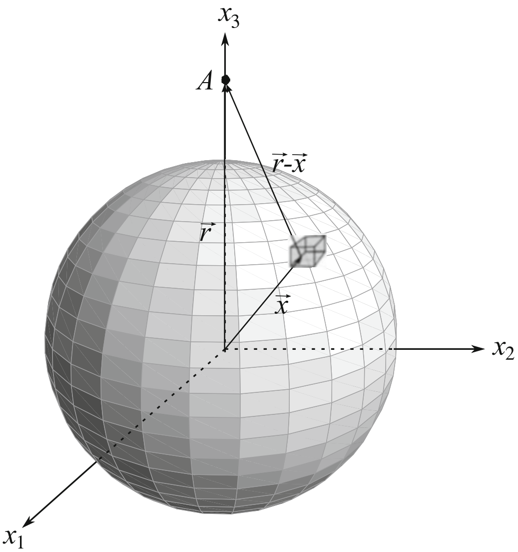


Fig. 13.10 Definition of vectors $\vec{r}$ and $\vec{x}$, occurring in formula (13.5.3)

The integral (13.5.3) will be naturally found in spherical variables x, θ, φ (because the symbol r in the current exercise plays another role, for the spherical radial variable the symbol x is used here), which were implemented by the equations (6.2.12):

$$x_1 = x\sin\theta\cos\varphi, \quad x_2 = x\sin\theta\sin\varphi, \quad x_3 = x\cos\theta. \tag{13.5.4}$$

Thereby one has to calculate

$$\phi(\vec{r}) = -\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{R} dx\, \underbrace{x^2\sin\theta}_{|J|}\, \frac{G\rho}{\sqrt{r^2+x^2-2xr\cos\theta}}, \tag{13.5.5}$$

where it was denoted $x := \sqrt{x_1^2+x_2^2+x_3^2}$. The benefit of placing the observation point (A) at the z-axis can be clearly seen: with this choice the integrand expression does not depend on the variable φ. This is because

$$\begin{aligned}
|\vec{r}-\vec{x}| &= \sqrt{(r_1-x_1)^2+(r_2-x_2)^2+(r_3-x_3)^2} \\
&= \sqrt{(x\sin\theta\cos\varphi)^2+(x\sin\theta\sin\varphi)^2+(r-x\cos\theta)^2} \\
&= \sqrt{x^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi)+x^2\cos^2\theta-2xr\cos\theta+r^2} \\
&= \sqrt{x^2\sin^2\theta+x^2\cos^2\theta-2xr\cos\theta+r^2} \\
&= \sqrt{r^2+x^2-2xr\cos\theta}.
\end{aligned} \tag{13.5.6}$$

The integral over φ boils down to multiplying the entire expression by 2π and obtaining (cf. Exercise 3 in Sect. 1.4)

$$\phi(\vec{r}) = -2\pi G\rho \int_0^{\pi} d\theta \int_0^{R} dx\, x^2\, \frac{\sin\theta}{\sqrt{r^2+x^2-2xr\cos\theta}}. \tag{13.5.7}$$

The remaining integrations can be performed in any order, but it is much easier to start with that over the angle θ. The key observation is that the integrand expression has the form $f(\cos\theta)\cdot\sin\theta$, which is dreamy for the introduction of the new variable $t = \cos\theta$. Then $d\theta\ \sin\theta$ becomes simply $-dt$ and one obtains

$$\phi(\vec{r}) = -2\pi G\rho \int_0^{R} dx\, x^2 \int_{-1}^{1} dt\, \frac{1}{\sqrt{r^2+x^2-2xrt}}$$

$$= 2\pi G\rho \int_0^R dx\, x^2\, \frac{1}{xr}\, \sqrt{r^2 + x^2 - 2xrt}\,\Big|_{-1}^{1}$$

$$= \frac{2\pi G\rho}{r} \int_0^R dx\, x \left(\sqrt{(x-r)^2} - \sqrt{(x+r)^2}\right). \tag{13.5.8}$$

The "minus" at dt in the first line has disappeared due to the reversal of the integration limits according to

$$\int_0^{\pi} d\theta\, \sin\theta\,(\ldots) \;\longmapsto\; -\int_1^{-1} dt\,(\ldots) = \int_{-1}^{1} dt\,(\ldots).$$

Now, one needs to simplify the integrand expression in (13.5.8). We know that for any real number a one has

$$\sqrt{a^2} = |a|,$$

and therefore,

$$\sqrt{(x+r)^2} = |x+r| = x + r, \tag{13.5.9}$$

since both r and x are nonnegative, and

$$\sqrt{(x-r)^2} = |x-r| = r - x, \tag{13.5.10}$$

as the observation point A is located outside the sphere and consequently $r > x$. By inserting these expressions into (13.5.8), one finds

$$\phi(\vec{r}) = \frac{2\pi G\rho}{r} \int_0^R dx\, x\,(r - x - (x+r)) = -\frac{4\pi G\rho}{r} \int_0^R dx\, x^2$$

$$= -\frac{4\pi G\rho}{r}\, \frac{1}{3}\, x^3\Big|_0^R = -\frac{G}{r}\, \underbrace{\frac{4}{3}\pi R^3}_{V}\, \rho = -\frac{GM}{r}. \tag{13.5.11}$$

A very simple result has been obtained, which should not be surprising and is well known for physicists: the gravitational field outside a spherically symmetric mass distribution is identical with that originating from a point mass located at the center of symmetry.

Problem 2

It will be proved that the electrostatic field originating from a homogeneously charged ball of radius R at a point located in its interior at the distance r from the center only comes from the charge accumulated in the small concentric ball of radius r.

Solution

The electrostatic field at the point $\vec{r}$ generated by a point-like charge q located at the origin of the coordinate system is oriented radially and given by the formula:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \cdot \frac{\vec{r}}{r^3}, \tag{13.5.12}$$

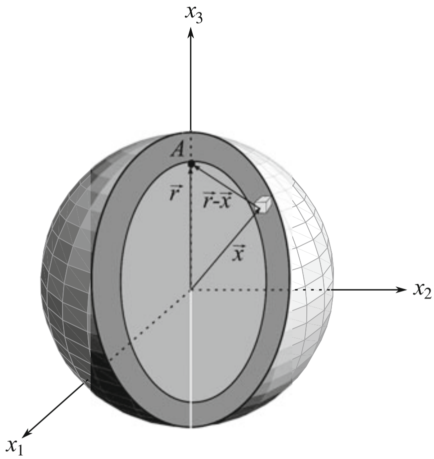
where ε_0 denotes the vacuum permittivity. If the charge is not point-like but distributed continuously in a certain region $\mathcal{O}$ with volume density ρ, the field $\vec{E}$ becomes

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{\mathcal{O}} d^3x\, \rho(\vec{x})\, \frac{\vec{r}-\vec{x}}{|\vec{r}-\vec{x}|^3}. \tag{13.5.13}$$

In the case of a spherically symmetrical distribution, the result of the integration, i.e., the vector $\vec{E}$, is again pointing radially and its value is identical at all points equidistant from the center of symmetry.

Back to the exercise, let us choose a certain point A inside the sphere. Due to the above-mentioned symmetry, it may—without weakening the generality of our considerations—be located on the third axis of the coordinate system, as shown in Fig. 13.11. This is what was done in the previous problem, to some extent simplifying the calculations. The electric field originating from the layer marked with dark gray color, denoted below with the symbol $\mathcal{O}$, will be directed along the x_3-axis (i.e., radially as spoken of above). This is because if the vector $\vec{E}$ had a certain nonzero component E_1 (i.e., parallel to the x_1-axis), then by turning this layer by 180° around the x_3-axis it would have to change into $-E_1$. But, on the other hand, the electrical field may not change when rotating a spherically symmetrical charge distribution as in the given layer. We must, therefore, have $E_1 = -E_1$, implying $E_1 = 0$. The analogous reasoning can be carried out as to the component E_2. This means that it is sufficient to calculate the component E_3 in compliance with the formula:

Fig. 13.11 The charged layer $\mathcal{O}$, which does not contribute to the electric field at the point A



$$E_3(r) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{\mathcal{O}} d^3x\, \rho\, \frac{r-x_3}{|\vec{r}-\vec{x}|^3} = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{\mathcal{O}} d^3x\, \frac{\rho\,(r-x_3)}{(x_1^2+x_2^2+(r-x_3)^2)^{3/2}}, \tag{13.5.14}$$

using the fact that $\vec{r} = [0, 0, r]$.

In the spherical variables defined with (13.5.4), the denominator of the integrand expression equals (see (13.5.6))

$$(x_1^2+x_2^2+(r-x_3)^2)^{3/2} = (r^2+x^2-2xr\cos\theta)^{3/2}. \tag{13.5.15}$$

Since the charge density is constant inside the ball, one gets

$$\begin{aligned} E_3(r) &= \frac{\rho}{4\pi\varepsilon_0}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\theta \int\limits_r^R dx\, \underbrace{x^2\sin\theta}_{|J|}\, \frac{r-x\cos\theta}{(r^2+x^2-2xr\cos\theta)^{3/2}} \\ &= \frac{\rho}{2\varepsilon_0}\int\limits_0^{\pi} d\theta \int\limits_r^R dx\, x^2\sin\theta\, \frac{r-x\cos\theta}{(r^2+x^2-2xr\cos\theta)^{3/2}}. \end{aligned} \tag{13.5.16}$$

In the previous exercise, we noticed that in a similar integral it was convenient to introduce the variable $t = \cos\theta$. The same can be done here, and (13.5.16) will take the form:

$$E_3(r) = \frac{\rho}{2\varepsilon_0}\int\limits_{-1}^{1} dt \int\limits_r^R dx\, x^2\, \frac{r-xt}{(r^2+x^2-2xrt)^{3/2}}. \tag{13.5.17}$$

When changing the order of integrations, one is going to perform ("by parts") the integral over t:

$$\int_{-1}^{1} dt\, \frac{r-xt}{(r^2+x^2-2xrt)^{3/2}} = \int_{-1}^{1} dt\,(r-xt)\left[\frac{1}{xr}\cdot\frac{1}{\sqrt{r^2+x^2-2xrt}}\right]'$$

$$= \frac{r-xt}{xr}\cdot\frac{1}{\sqrt{r^2+x^2-2xrt}}\Bigg|_{-1}^{1} + \frac{1}{r}\int_{-1}^{1} dt\,\frac{1}{\sqrt{r^2+x^2-2xrt}}$$

$$= \frac{r-x}{xr\sqrt{(r-x)^2}} - \frac{r+x}{xr\sqrt{(r+x)^2}} - \frac{1}{xr^2}\sqrt{r^2+x^2-2xrt}\Bigg|_{-1}^{1}$$

$$= \frac{r-x}{xr|r-x|} - \frac{r+x}{xr|r+x|} - \frac{1}{xr^2}|r-x| + \frac{1}{xr^2}|r+x|$$

$$= -\frac{1}{xr} - \frac{1}{xr} - \frac{x-r}{xr^2} + \frac{x+r}{xr^2} = -\frac{2}{xr} - \frac{1}{r^2} + \frac{1}{xr} + \frac{1}{r^2} + \frac{1}{xr} = 0, \tag{13.5.18}$$

having used the fact that $x > r$, i.e., $|r-x| = x-r$.

As a result of the integration over t, we have got zero, so this is also the value of $E_3(r)$. As a consequence, the electric field coming from the charge distributed in the outer layer disappears, and the only source of this field at A is the charge accumulated inside the sphere of radius r. As we know, this result can be obtained in electrodynamics with the use of the Gauss law.

It should be also noted that (13.5.16) disappears as a result of the integration over the angle θ, and not over x. This means that *any* spherically symmetric charge distribution (i.e., if $\rho = \rho(x)$) will give the same result.

Problem 3

The integral:

$$I_n = \int e^{-x^T M x} d^n x \tag{13.5.19}$$

will be calculated, where M is an $n \times n$ square positive-definite matrix and T denotes the transposition.

Solution

A physicist involved in field theory or statistical physics, sooner or later will come across calculating an integral of the form (13.5.19), so some time should be devoted to this problem. It should be pointed out that physicists often do not explicitly write the integration limits if it is taken over the entire space (that is, when each of variables x_i runs from $-\infty$ to $+\infty$) and use a single symbol $\int$ for denoting a multiple integral. Therefore, this is how the expression above ought to be understood.

In the case when $n = 1$, M becomes simply a positive number and the integral boils down to the Gaussian form, well known to the reader from the lecture of analysis:

$$I_1 = \int_{-\infty}^{\infty} dx\, e^{-Mx^2} = \sqrt{\frac{\pi}{M}}. \tag{13.5.20}$$

Our goal is to generalize this result to many variables.

First it should be noted that the matrix M is symmetrical, i.e., $M_{ij} = M_{ji}$. It is because any matrix C can be decomposed into symmetrical (C^A) and antisymmetrical (C^A) parts according to the pattern:

$$C_{ij} = \frac{1}{2}\left(C_{ij} + C_{ji}\right) + \frac{1}{2}\left(C_{ij} - C_{ji}\right) = C^S_{ij} + C^A_{ij}. \tag{13.5.21}$$

As for the antisymmetrical part, one has $C^A_{ij} = -C^A_{ji}$, so

$$\begin{aligned} x^T C^A x \;&=\; \sum_{i,j=1}^{n} x_i C^A_{ij} x_j = -\sum_{i,j=1}^{n} x_i C^A_{ji} x_j = -\sum_{i,j=1}^{n} x_j C^A_{ji} x_i \\ &\underset{i\leftrightarrow j}{=} -\sum_{i,j=1}^{n} x_i C^A_{ij} x_j = -x^T C^A x \quad\Longrightarrow\quad x^T C^A x = 0 \end{aligned} \tag{13.5.22}$$

which entails

$$x^T(C^S + C^A)x = x^T C^S x. \tag{13.5.23}$$

Thereby, it is henceforth assumed that M is a symmetric matrix. As we know from algebra, the real symmetrical $n \times n$ matrix is fully diagonalizable, i.e., it has exactly n eigenvectors. They will be denoted with the symbols $v_1, v_2, \dots, v_n$, and for the corresponding eigenvalues we reserve the symbols $\lambda_1, \lambda_2, \dots, \lambda_n$. Thus one can write

$$Mv_i = \lambda_i v_i \ , \quad \text{for } \ i = 1, 2, \ldots, n. \tag{13.5.24}$$

Not all the eigenvalues must be different, regardless it will not interfere with our reasoning in any way. However, from the assumption given in the text as to the positive definiteness of M, it asserts that all of them are positive.

The eigenvectors v_l and v_m corresponding to different eigenvalues are orthogonal (i.e., they fulfill the condition $v_l^T v_m = 0$—one can say that we have just defined the scalar product of vectors as the product of the single-row matrix by the single-column matrix). In turn, those corresponding to the identical eigenvalues—insofar as they are multiple—can be orthogonalized. For this reason, hereafter it is assumed that

$$v_l^T v_m = 0, \quad \text{for } \ l \neq m. \tag{13.5.25}$$

It is also convenient to introduce the abbreviation $\|v\|$ standing for $\sqrt{v^T v}$ (i.e., the norm of a vector). The vectors v_i for $i = 1, 2, \ldots, n$ constitute a base in the space $\mathbb{R}^n$, and therefore, any vector x has a decomposition:

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n, \tag{13.5.26}$$

the coefficients c_i being real numbers. Using the above facts, the expression in the exponent of (13.5.19) can be rewritten as follows:

$$\begin{aligned} x^T M x &= (c_1 v_1^T + c_2 v_2^T + \ldots + c_n v_n^T) M (c_1 v_1 + c_2 v_2 + \ldots + c_n v_n) \\ &= (c_1 v_1^T + c_2 v_2^T + \ldots + c_n v_n^T)(c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n) \\ &= c_1^2 \lambda_1 \|v_1\|^2 + c_2^2 \lambda_2 \|v_2\|^2 + \ldots + c_n^2 \lambda_n \|v_n\|^2. \end{aligned} \tag{13.5.27}$$

The idea that guides the calculation of the integral (13.5.19) is to change the integration variables from the components of the vector x to the coefficients c_i defined by (13.5.26). In order to achieve this, one needs to know the value of the Jacobian determinant related to this change of variables:

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(c_1, c_2, \ldots, c_n)}.$$

The derivative of the k-th component of the vector x over the l-th coefficient c can be found immediately from (13.5.26). It is simply the k-th component of the vector v_l:

$$\frac{\partial x_k}{\partial c_l} = (v_l)_k =: v_{l,k}. \tag{13.5.28}$$

This means that the Jacobian determinant simply equals

$$J = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(c_1, c_2, \ldots, c_n)} = \det \begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{n,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ v_{1,n} & v_{2,n} & \cdots & v_{n,n} \end{bmatrix} = \det\left[v_1, v_2, \ldots, v_n\right], \tag{13.5.29}$$

where the latter expression is understood as the determinant of the matrix whose columns are vectors v_i. By introducing the vectors e_i, constituting the orthonormal basis:

$$e_i := \frac{v_i}{\|v_i\|}, \quad i = 1, 2, \ldots, n, \tag{13.5.30}$$

one can write

$$\begin{aligned} J &= \det\left[\|v_1\|e_1, \|v_2\|e_2, \ldots, \|v_n\|e_n\right] = \|v_1\|\|v_2\| \cdots \|v_n\| \det\left[e_1, e_2, \ldots, e_n\right] \\ &= \pm\|v_1\|\,\|v_2\| \cdots \|v_n\|, \end{aligned} \tag{13.5.31}$$

since the determinant of a matrix built of e_i equals $+1$ or -1, depending on how they are sorted. When changing variables under the integral, not J is plugged in, but rather $|J|$, so this sign is inessential. It is worth noting, however, that the result is the product of factors, each of which depends only on one of the vectors v_i. Since due to the form (13.5.27), the exponential function also breaks up into the product, then in place of an n-fold integral (13.5.19) we will be left with n single (and identical) integrations of the form of (13.5.20). Consequently, the final result can be easily obtained

$$\begin{aligned} I_n \;&=\; \left[\int_{-\infty}^{\infty} dc_1\, \|v_1\|\, e^{-\lambda_1 \|v_1\|^2 c_1^2}\right] \left[\int_{-\infty}^{\infty} dc_2\, \|v_2\|\, e^{-\lambda_2 \|v_2\|^2 c_2^2}\right] \\ &\cdots \left[\int_{-\infty}^{\infty} dc_n\, \|v_n\|\, e^{-\lambda_1 \|v_n\|^2 c_n^2}\right] = \left[\|v_1\|\sqrt{\frac{\pi}{\lambda_1\|v_1\|^2}}\right] \left[\|v_2\|\sqrt{\frac{\pi}{\lambda_2\|v_2\|^2}}\right] \\ &\cdots \left[\|v_n\|\sqrt{\frac{\pi}{\lambda_n\|v_n\|^2}}\right] = \sqrt{\frac{\pi^n}{\lambda_1\,\lambda_2 \cdots \lambda_n}} = \frac{\pi^{n/2}}{\sqrt{\det M}}. \end{aligned} \tag{13.5.32}$$

As we know from algebra, the determinant of a matrix does not depend on the choice of the basis. In the case when this basis leads to the diagonal form of the matrix, its calculation is particularly simple: it is just the product of all eigenvalues (with their multiplicities taken into account). In a different basis, its calculation is more cumbersome, but the result will still be the same.

13.6 Exercises for Independent Work

Exercise 1 Find the surface area of the figure described with the equations (13.1.1) by performing the integral in the Cartesian coordinates.

Exercise 2 Find the surface area of the figures defined with the inequalities:

(a) $y \le 2x$, $y \ge \frac{1}{2}x$ and $y \le 2 - x$,
(b) $y \ge x^2$, $y \ge x^2 - 6x + 8$ and $y \le 4$,
(c) $y \ge \frac{1}{x}$ i $y \le 4 - x$,
(d) $(x^2 + y^2)^{5/2} \le R(x^4 + y^4)$, where $R > 0$.

Answers

(a) $2/3$,
(b) $10\sqrt{5}/3 - 6$,
(c) $4\sqrt{3} - 2\log(2 - \sqrt{3})$,
(d) $19\pi R^2/32$.

Exercise 3 Find the volume of solids defined with the inequalities:

(a) $z \ge x^2 + y^2 - 2$ i $z \le -x^2 - y^2 + 16$,
(b) $(x^2 + y^2 + z^2)^3 \le R^2(x^2 + y^2)^2$, where $R > 0$,
(c) $x^2 + y^2 + z^2 \le R\sqrt{z^2 - x^2 - y^2}$, where $R > 0$.

Answers

(a) 81π,
(b) $64\pi R^3/105$,
(c) $\pi R^3[3\sqrt{2}\log(1 + \sqrt{2}) - 2]/12$.

Exercise 4 Find the location of the center of mass of

(a) the homogeneous semi-circle $x^2 + y^2 = R^2$, $x \ge 0$,
(b) the homogeneous solid limited with the surfaces $z = 3(x^2 + y^2)$ and $z = x^2 + y^2 + 2$.

Answers

(a) $x_{CM} = 4R/3\pi$, $y_{CM} = 0$,
(b) $x_{CM} = 0$, $y_{CM} = 0$, $z_{CM} = 5/3$.

Exercise 5 Find the moments of inertia with respect to the axis of symmetry for

(a) the ball of mass M, radius R, and mass density $\rho(r) = ce^{-r/R}$, where $c > 0$, and r denotes the distance from the center of the ball,
(b) the cylinder of mass M, radius R, and mass density $\rho(r) = c \sin\left(\frac{\pi r}{2R}\right)$, where $c > 0$, and r denotes the distance from the cylinder axis,
(c) the cone of mass M, radius of the base R, and mass density $\rho(h) = ch$, where $c > 0$, and h denotes the distance from the base of the cone.

Answers

(a) $2(24e - 65)/(6e - 15)\, MR^2$,
(b) $3(1 - 8/\pi^2)\, MR^2$,
(c) $1/5\, MR^2$.

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Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-36848-7