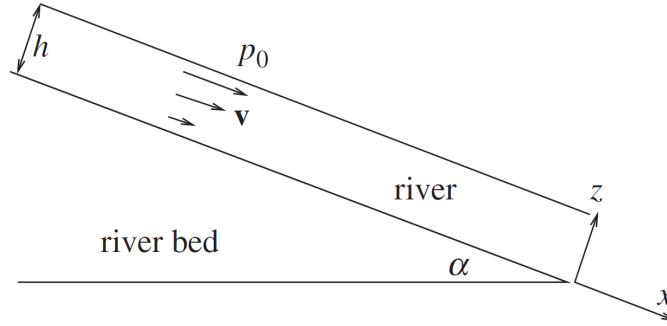


1 Reynolds number for simple model for river flow

[From Falkovich 2018, §1.4.3] We use a simple inclined plane as a model for river flow.



In lecture you have obtained the solution of:

$$p(z) = p_0 + \rho g(h - z) \cos \alpha, \quad (1)$$

$$v(z) = \frac{\rho g \sin \alpha}{2\eta} z(2h - z). \quad (2)$$

(1.a) Take the kinematic viscosity of water to be $\nu = \eta/\rho = 10^{-2} \text{ cm}^2\text{s}^{-1}$. Calculate v at the surface for a rain puddle with thickness $h = 1 \text{ mm}$ on a slope $\alpha \sim 10^{-2}$.

(1.b) How about a slow plain rivers (like the Danube) with $h \sim 10 \text{ m}$ on a slope $\alpha \sim 10^{-4}$?

(1.c) Which speed is reasonable?

(1.d) The unrealistic high velocity for the river case above is because in reality rivers are turbulent. Calculate the Reynolds number for the two cases.

2 Vorticity equation derivation

Remember that we have the vector identity:

$$\mathbf{v}(\nabla \cdot \mathbf{v}) = \frac{1}{2} \nabla v^2 - \mathbf{v} \times \boldsymbol{\omega}. \quad (3)$$

To obtain the vorticity equation, we need to take the curl of this.

(2.a) Derive the vector identity:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b}. \quad (4)$$

Hint: remember we have

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}. \quad (5)$$

(2.b) Show that the compressible vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{v} - \boldsymbol{\omega} \nabla \cdot \mathbf{v} \quad (6)$$

The incompressible version is an straightforward corollary.

3 Vorticity equation for compressible flows

[From Vallis 2017, §4.2] Using mass-conservation for compressible flows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (7)$$

Obtain the alternative compressible vorticity equation:

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\mathbf{v} \cdot \nabla) \tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}} \cdot \nabla) \mathbf{v} \quad (8)$$

where

$$\tilde{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}}{\rho}. \quad (9)$$

4 Fundamental solution to the heat equation

(4.a) Self-similar solution from dimensional analysis.

(4.b) Fundamental solution is the Green's function.

References

- Falkovich, Gregory. 2018. *Fluid Mechanics*. Cambridge University Press, April 12, 2018. ISBN: 978-1-108-22820-6.
- Vallis, Geoffrey K. 2017. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. 2nd ed. Cambridge: Cambridge University Press. ISBN: 978-1-107-06550-5. <https://doi.org/10.1017/9781107588417>.