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Worksheet 8

1 Examples of vorticity and circulation

When we deal with vorticity and circulation, it is most convenient to work with polar coordinate. In (2D) polar coordinate, the vorticity (scalar) is

$$\omega = \frac{1}{r}\left(\frac{\partial(ru^\theta)}{\partial r} - \frac{\partial u^r}{\partial \theta}\right). \tag{1}$$

We will not derive this here, but a good reference is §5.1 and §5.6 of Brannon 2004.

(1.a) Rigid body rotation For a body in rigid body rotation, the velocity distribution is given by

$$u^\theta = \Omega r \quad \text{and} \quad u^r = 0 \tag{2}$$

where Ω is the angular velocity of the fluid (rigid body). Calculate the vorticity of this flow. How is it related to the angular velocity?

(1.b) Rankine vortex We have the flow field

$$u^\theta = \begin{cases} \Omega r, & r < a \\ \dfrac{\Omega a^2}{r}, & r \geq a \end{cases} \tag{3}$$

and $u^r = 0$.

- Calculate the vorticity of this flow.
- Calculate the circulation of this flow around a circle with radius R.

(1.c) Point vortex We take the $a \to 0$ limit of the Rankine vortex. For this, we define $K = \Omega a^2$ and keep this a constant as we take a to zero.

- How does the vorticity change as we take $a \to 0$.
- What is now the circulation of this flow around a circle with radius $R > a$.

2 Point vortex velocity as a Green's function

(2.a) Biot–Savart law We have the velocity field around a single point vortex. Use this and the idea of Green's function to write down the velocity field for a vortex field. This is just the 2D Biot–Savart law relating vorticity and velocity.

(2.b) Green's function for the Poisson equation Incompressible flow can be represented using a streamfunction ψ, where[1]

$$(u, v) = (-\psi_y, \psi_x). \tag{4}$$

- Show that the streamfunction is the solution to the Poisson equation with the vorticity on the RHS:

$$\nabla^2 \psi = \omega. \tag{5}$$

- Convert the Biot–Savart law to a streamfunction. This streamfunction of the Green's function for the Poisson equation. Now you could solve the Poisson equation in $\mathbb{R}^2$.

3 Inviscid, incompressible vortex dynamics near a wall

Now we study inviscid, incompressible flow in the upper half plane. At the wall we use the no penetration boundary condition:

$$v(0, y) = 0. \tag{6}$$

(3.a) Place a point vortex into the upper plane.

- How could we make sure the flow satisfies the boundary condition?
- Write this in the context of the Biot–Savart law.

Hint: try placing an imaginary point vortex in the lower half plane.

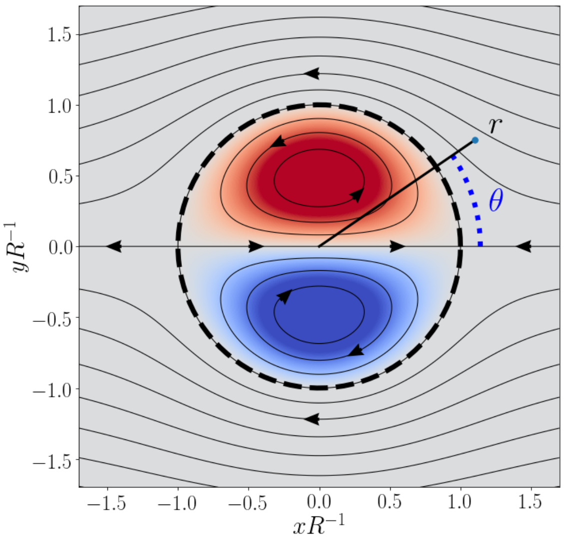
(3.b) Think about the flow using a streamfunction.

- What is the boundary condition on the streamfunction?
- Write this using the Green's function formula for the solution of the Poisson equation.

4 Two interesting and advanced vortices examples

(4.a) Kirchhoff Vortex Kirchhoff vortex is a compact elliptical patch of vorticity that rotate while keeping its shape. It is a generalization of the Rankine vortex and it is the basis of study of instability.

For details of the Kirchhoff Vortex, see `http://www.damtp.cam.ac.uk/user/hl278/KirchoffVortex.pdf`. For an example study of its stability, see Mitchell and Rossi 2008.



(4.b) Lamb–Chaplygin dipole The Lamb–Chaplygin dipole has also compact vorticity but it has positive and negative vorticity inside. It is a steady solution to the Euler equation. For more see Meleshko and Heijst 1994, image above from wikipedia.

References

Brannon, Rebecca M (2004). *Curvilinear Analysis in a Euclidean Space.*

Meleshko, V. V. and G. J. F. van Heijst (Aug. 1994). "On Chaplygin's Investigations of Two-Dimensional Vortex Structures in an Inviscid Fluid". In: *Journal of Fluid Mechanics* 272, pp. 157–182. ISSN: 0022-1120, 1469-7645. DOI: 10.1017/S0022112094004428.

Mitchell, T. B. and L. F. Rossi (May 2008). "The Evolution of Kirchhoff Elliptic Vortices". In: *Physics of Fluids* 20.5, p. 054103. ISSN: 1070-6631. DOI: 10.1063/1.2912991.

[1]sometimes it is defined with the opposite sign