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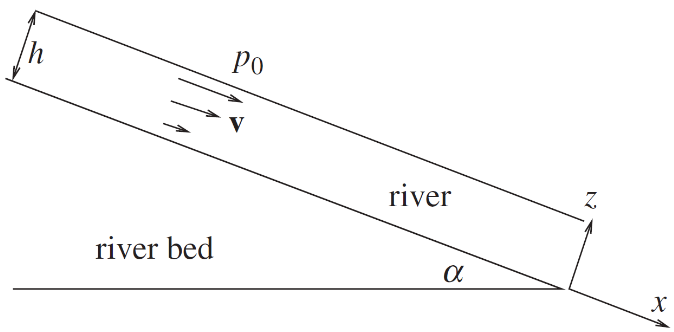
Worksheet 6

1 Reynolds number for simple model for river flow

[From Falkovich 2018, §1.4.3] We use a simple inclined plane as a model for river flow.



In lecture you have the obtained the solution of:

$$p(z) = p_0 + \rho g(h - z)\cos\alpha, \tag{1}$$

$$v(z) = \frac{\rho g \sin\alpha}{2\eta} z(2h - z). \tag{2}$$

(1.a) Take the kinematic viscosity of water to be $\nu = \eta/\rho = 10^{-2}$ cm^{2}s^{-1}. Calculate $\boldsymbol{v}$ at the surface for a rain puddle with thickness $h = 1$ mm on a slope $\alpha \sim 10^{-2}$.

(1.b) How about a slow plain rivers (like the Danube) with $h \sim 10$ m on a slope $\alpha \sim 10^{-4}$?

(1.c) Which speed is reasonable?

(1.d) The unrealistic high velocity for the river case above is because in reality rivers are turbulent. Calculate the Reynolds number for the two cases.

2 Vorticity equation derivation

Remember that we have the vector identity:

$$\boldsymbol{v}(\nabla \cdot \boldsymbol{v}) = \frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \times \boldsymbol{\omega}. \tag{3}$$

To obtain the vorticity equation, we need to take the curl of this.

(2.a) Derive the vector identity:

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b}. \tag{4}$$

Hint: remember we have

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{5}$$

(2.b) Show that the compressible vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{v} \tag{6}$$

The incompressible version is an straightforward corollary.

3 Vorticity equation for compressible flows

[From Vallis 2017, §4.2] Using mass-conservation for compressible flows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{7}$$

Obtain the alternative compressible vorticity equation:

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}} \cdot \nabla)\boldsymbol{v} \tag{8}$$

where

$$\tilde{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}}{\rho}. \tag{9}$$

4 Fundamental solution to the heat equation

For details, see §5.1 and 5.2 of Shearer and Levy 2015.

(4.a) Self-similar solution from dimensional analysis.

(4.b) Fundamental solution is the Green's function.

References

Falkovich, Gregory. 2018. *Fluid Mechanics.* Cambridge University Press, April 12, 2018. ISBN: 978-1-108-22820-6.

Shearer, Michael, and Rachel Levy. 2015. *Partial Differential Equations: An Introduction to Theory and Applications.* Princeton University Press, March 1, 2015. ISBN: 978-1-4008-6660-1.

Vallis, Geoffrey K. 2017. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation.* 2nd ed. Cambridge: Cambridge University Press. ISBN: 978-1-107-06550-5. https://doi.org/10.1017/9781107588417.