Recitation Materials

for NYU Undergraduate Introduction To Fluid Dynamics

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Spring, 2023

Contents

Chapter 0

Before the Course Materials

0.1 READ ME

This document is a compilation of the worksheets used in the recitation sessions for NYU Undergraduate Introduction To Fluid Dynamics in Spring of 2023.

The LaTeX files of this document can be found at https://github.com/Empyreal092/UgradFluids_Worksheet.

Chapter 1

Flow Kinematics

1.1 Rate of strain tensor: examples

(1.a) Shear flow Take a velocity field with $u = y$ and $v, w = 0$.

1. Calculate its velocity gradient tensor.
2. What is its symmetric and anti-symmetric parts?
3. What is its divergence and vorticity vector?

(1.b) Straining flow Take a velocity field with $u = x$, $v = -y$, and $w = 0$.

1. Do the same calculation.
2. [Adapted from Aris 1962, Exercise 4.42.1] Take a line connecting the origin and a fluid parcel, show that if θ is the angle between the line and the x-axis, then the rate of change of $\log \tan \theta$ is constant along a particle path.

(1.c) [From Aris 1962, Exercise 4.45.1] Take the velocity field

$$\boldsymbol{v} = \alpha \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{x}. \tag{1.1}$$

1. Do the same calculation.
2. Interpret the motion of a fluid parcel in this flow.

(1.d) Rankine vortex Now we use the cylindrical coordinate with (r, θ, z). We have the flow field

$$u_\theta = \begin{cases} \Omega r, & r < a \\ \dfrac{\Omega a^2}{r}, & r \geq a \end{cases} \tag{1.2}$$

and $u_r = u_z = 0$.

1. Do the same calculation (by converting the velocity into the Cartesian coordinate).

We will come back to this example to practice polar coordinate and to study vorticity and circulation.

1.2 From divergence theorem to Green's theorem

(2.a) [From Aris 1962, Exercise 3.31.3] By taking $\boldsymbol{a}$ to be independent of x, and $a_3 = 0$, show using divergence theorem that if A is an area in the (x, y)-plane bounded by a curve C, then

$$\iint_A \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} \, \mathrm{d}A = \oint_C a_1 t_2 - a_2 t_1 \, \mathrm{d}s \tag{1.3}$$

where $\boldsymbol{t} = (t_1, t_2)$ is the unit tangent vector to C.

(2.b) [From Aris 1962, Exercise 3.31.4] Deduce from the previous question that

$$\iint_A \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \, \mathrm{d}A = \oint_C a_1 t_1 + a_2 t_2 \, \mathrm{d}s \tag{1.4}$$

The right hand side can be written as

$$\oint_C a_1 \mathrm{d}x + a_2 \mathrm{d}y \tag{1.5}$$

and we have derived the Green's theorem.

1.3 Green's identity

(3.a) We take a 3D scalar field ψ and a 3D vector field $\boldsymbol{\Gamma}$ with sufficient smoothness defined on some region $U \subset \mathbb{R}^3$. Show the identity

$$\iiint_U (\psi \, \nabla \cdot \boldsymbol{\Gamma} + \boldsymbol{\Gamma} \cdot \nabla \psi) \, dV = \oiint_{\partial U} \psi \, (\boldsymbol{\Gamma} \cdot \boldsymbol{n}) \, dS = \oiint_{\partial U} \psi \boldsymbol{\Gamma} \cdot d\boldsymbol{S}. \tag{1.6}$$

(3.b) Use (1.6) to show the Green's first identity. Take 3D scalar fields ψ and φ both with sufficient smoothness:

$$\iiint_U \left(\psi \, \nabla^2 \varphi + \nabla \psi \cdot \nabla \varphi\right) \, dV = \oiint_{\partial U} \psi \, (\nabla \varphi \cdot \boldsymbol{n}) \, dS = \oiint_{\partial U} \psi \, \nabla \varphi \cdot d\boldsymbol{S}. \tag{1.7}$$

(3.c) Show the Green's second identity:

$$\iiint_U \left(\psi \, \nabla^2 \varphi - \varphi \, \nabla^2 \psi\right) \, dV = \oiint_{\partial U} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot d\boldsymbol{S}. \tag{1.8}$$

This shows that the Laplacian is a self-adjoint operator for functions vanishing on the boundary so that the right hand side of the above identity is zero.

(3.d) In free 3D space Now have a velocity field $u = x$, $v = y$, and $w = z$. Do the same calculation.

1.4 Deformation tensor: examples

(4.a) Straining flow Take a velocity field with $u = x$, $v = -y$, and $w = 0$.

1. Calculate the deformation tensor $\boldsymbol{F}$.
2. Verify the Lagrangian equation of evolution for $\boldsymbol{F}$

$$\left.\frac{\partial \boldsymbol{F}}{\partial t}\right|_{\boldsymbol{\alpha}} = \nabla_{\boldsymbol{x}} \boldsymbol{v} \cdot \boldsymbol{F}. \tag{1.9}$$

(4.b) Take a velocity field with $u = 2x^{1/2}$, $v = 2y^{1/2}$, and $w = 2z^{1/2}$.

1. Do the same calculation.
2. Also verify the Eulerian equation of evolution for $\boldsymbol{F}$:

$$\left.\frac{\partial \boldsymbol{F}}{\partial t}\right|_{\boldsymbol{x}} + (\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{F} = \nabla_{\boldsymbol{x}} \boldsymbol{v} \cdot \boldsymbol{F}. \tag{1.10}$$

1.5 Evolution of an infinitesimal material line element

Take an infinitesimal material line element $\delta\boldsymbol{\ell}$, where it is the infinitesimal material element connecting $\boldsymbol{\ell}$ and $\boldsymbol{\ell} + \delta\boldsymbol{\ell}$. Show that its evolution equation follows the equation

$$\frac{D\delta\boldsymbol{\ell}}{Dt} = \nabla_{\boldsymbol{x}} \boldsymbol{v} \cdot \delta\boldsymbol{\ell}. \tag{1.11}$$

You will do this in the homework using (1.10). I want you to think about it from another perspective: use your physical intuition and argue using infinitesimal time.

Hint: local in time, the velocity at $\boldsymbol{\ell}$ is $\boldsymbol{v}$ and the velocity at $\boldsymbol{\ell} + \delta\boldsymbol{\ell}$ is $\boldsymbol{v} + \delta\boldsymbol{v}$. What is $\delta\boldsymbol{v}$?

Chapter 2

Euler Equations

2.1 Derivation of the Lamb vector

A useful vector identity in fluid mechanics is

$$\boldsymbol{v}(\nabla \cdot \boldsymbol{v}) = \frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \times \boldsymbol{\omega}. \tag{2.1}$$

For example, we use this identity in the derivation of the Bernoulli's principle. The Lamb vector is defined as

$$\boldsymbol{\ell} = \boldsymbol{v} \times \boldsymbol{\omega}. \tag{2.2}$$

We will derive the the vector identity (4.1).

(1.a) Show that the cross product can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk} a_i b_j \boldsymbol{e}_{(k)} \tag{2.3}$$

where ϵ_{ijk} is the permutation symbol:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{, if any two of } i, j, k \text{ are the same} \\ 1 & \text{, if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{, if } i, j, k \text{ is an odd permutation of } 1, 2, 3. \end{cases} \tag{2.4}$$

(1.b) [From Aris 1962, Exercise 2.32.1] Show by enumerating typical cases that

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{2.5}$$

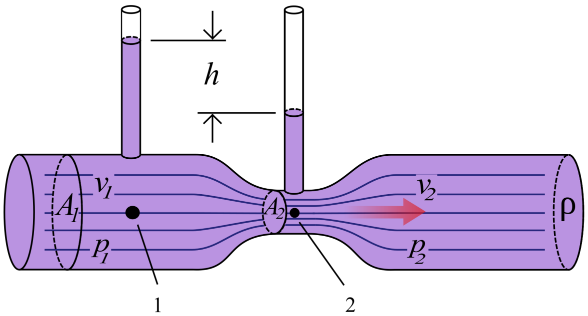
(1.c) Use (4.3) to show (4.1).

2.2 Bernoulli's principle: examples

(2.a) Flow out of a water tank Imagine a water tank with height h of water inside. At the bottom there is a small hole. What would be the speed of the water flowing out of the hole.

(2.b) Venturi effect Calculate the fluid velocity difference at point 1 and 2 as a function of h.

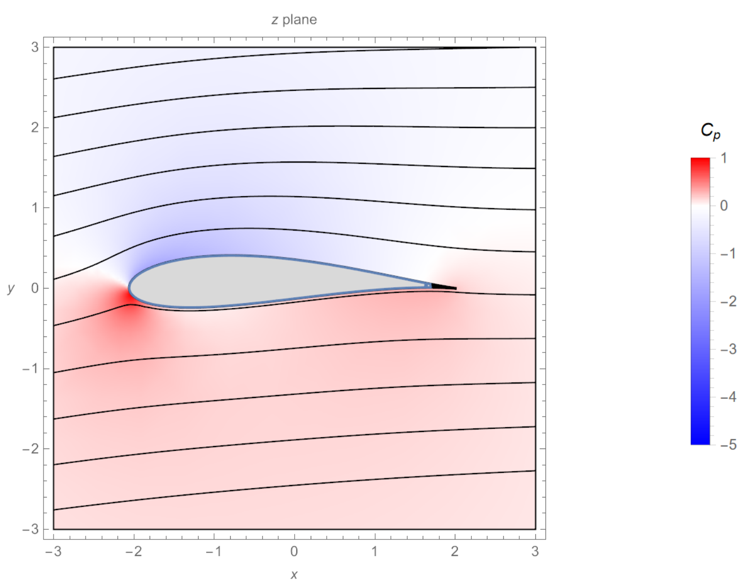
Image from Wikipedia for Venturi effect. It is best to extend the lower end of the two vertical tubes to the center of the horizontal tube so that they stop at point 1 and 2.



(2.c) Pitot's tube Using the idea of the above device, think of a device that measures the speed of the fluid flow.

(2.d) Lift on an airfoil: a first look Use Bernoulli's principle to explain how lift is generated by a plane's wing.

Image generated by Wolfram software from https://demonstrations.wolfram.com/JoukowskiAirfoilFlowField/. The lines are streamlines, the color is the pressure field.



2.3 The ABC flows

By using (4.1), we can write the incompressible Euler equation as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\nabla H, \tag{2.6}$$

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2.7}$$

where H is the energy

$$H = \frac{p}{\rho} + \frac{1}{2}\boldsymbol{v}^2. \tag{2.8}$$

The Arnold, Beltrami, Childress (ABC) flow is an interesting steady state solution of Euler. It has the velocity field on the 2π-periodic 3D domain

$$\begin{cases} u = A\sin z + C\cos y \\ v = B\sin x + A\cos z \\ w = C\sin y + B\cos x \end{cases} \tag{2.9}$$

where A, B, and C are constant parameters. Despite its simple appearance in the Eulerian frame, the Lagrangian behavior of this flow is presumably chaotic. To quote Dombre et al. 1986 which named this flow

> Three-dimensional steady flows with a simple Eulerian representation can have a chaotic Lagrangian structure. By this we mean that infinitesimally close fluid particles following the streamlines may separate exponentially in time, while remaining in a bounded domain, and that individual streamlines may appear to fill entire regions of space.

and

> From a fluid dynamical viewpoint flows with chaotic streamlines are interesting because they may considerably enhance transport without being turbulent in the usual sense - they only display what might be called 'Lagrangian turbulence'.

We will be less ambitious and show some basic properties of this flow.

(3.a) Show the ABC flows are incompressible.

(3.b) Show the ABC flows are Beltrami flows. That is

$$\boldsymbol{\omega} \times \boldsymbol{v} = 0. \tag{2.10}$$

Hint: Show $\boldsymbol{\omega} = \boldsymbol{v}$. Why does this identity shows $\boldsymbol{v}$ is Beltrami?

(3.c) Conclude the ABC flows are exact solution to the steady state incompressible Euler equation.

Chapter 3

Navier–Stokes Equations

3.1 Taylor–Green vortex

The Taylor–Green vortex is an exact solution to the 2D Navier-Stokes equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.3}$$

The first appearance of this flow is in Taylor 1923. The name of this flows comes from the joint work Taylor and Green 1937. A 3D extension of the Taylor–Green vortex appeared recently in Antuono 2020. The exact solutions to the fluid equations are important tools for benchmarking numerical solvers for fluids dynamics.

(1.a) The Taylor–Green vortex has velocity

$$u = \cos x \sin y F(t) \tag{3.4}$$

$$v = -\sin x \cos y F(t). \tag{3.5}$$

Show, by plugging the velocity into the momentum equation, that the pressure field is given by

$$p = -\frac{\rho}{4}(\cos 2x + \cos 2y)F^2(t). \tag{3.6}$$

(1.b) Now show that the time component is

$$F(t) = e^{-2\nu t}. \tag{3.7}$$

3.2 Alternative form for momentum dissipation

(2.a) [From Aris 1962, Exercise 3.24.5] Show the vector identity

$$\nabla^2 \boldsymbol{v} = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla \times (\nabla \times \boldsymbol{v}). \tag{3.8}$$

(2.b) [From Aris 1962, §6.11] Show that the incompressible Navier-Stokes can alternatively be written as

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p - \mu \nabla \times \boldsymbol{\omega}. \tag{3.9}$$

3.3 Energy dissipation in Navier-Stokes

Remember that the incompressible Navier-Stokes can alternatively be written as

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p - \mu \nabla \times \boldsymbol{\omega}. \tag{3.10}$$

(3.a) Show the vector identity

$$\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b}). \tag{3.11}$$

(3.b) [From Aris 1962, Exercise 6.14.4] For an incompressible Newtonian fluid moving within a fixed stationary boundary with no body forces, show that the total rate of dissipation of kinetic energy is

$$-\mu \iiint_V \boldsymbol{\omega}^2 \, \mathrm{d}V. \tag{3.12}$$

(3.c) To get another form of the total rate of dissipation of kinetic energy, we can work with the dissipation in Navier-Stokes directly. Dot it with $\boldsymbol{v}$ and integrate over the domain, we have

$$\mu \iiint_V \boldsymbol{v} \nabla^2 \boldsymbol{v} \, \mathrm{d}V = -\mu \iiint_V \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \, \mathrm{d}V. \tag{3.13}$$

(3.d) In lecture you learned the total rate of dissipation of kinetic energy is

$$-2\mu \iiint_V \boldsymbol{E} : \boldsymbol{E} \, \mathrm{d}V. \tag{3.14}$$

These three forms should all be equal. Make sense of this by assume

$$\iiint_V \boldsymbol{E} : \boldsymbol{E} \, \mathrm{d}V = \iiint_V \boldsymbol{W} : \boldsymbol{W} \, \mathrm{d}V. \tag{3.15}$$

(3.e) Now show

$$\iiint_V \nabla \boldsymbol{v} : \nabla \boldsymbol{v}^\top \, \mathrm{d}V = 0. \tag{3.16}$$

Show that this implies (3.15).

3.4 Vena contracta

[From Falkovich 2018, §1.1.4] We have derived the flow speed for the efflux from a small orifice under the action of gravity from Bernoulli's principle. It follows the Torricelli formula $v = \sqrt{2gh}$. We now will derive the rate of discharge. Suppose the opening size on the side-wall is S, in reality, the discharge rate will not be vS, because of the phenomenon called vena contracta. As a first understanding, I quote the explanation in Falkovich's book:

> Indeed, streamlines converge from all sides toward the orifice so that the jet continues to converge for a while after coming out (Figure below). Moreover, the converging motion makes the pressure in the interior of the jet somewhat greater than that at the surface (as is clear from the curvature of streamlines) so that the velocity in the interior is somewhat less than $\sqrt{2gh}$. The experiment shows that contraction ceases and the jet becomes cylindrical at a short distance beyond the orifice. This point is called "vena contracta" and the ratio of the jet area there to the orifice area is called the coefficient of contraction. The estimate for the discharge rate is $\sqrt{2gh}$ times the orifice area times the coefficient of contraction. For a round hole in a thin wall, the coefficient of contraction is experimentally found to be 0.62.

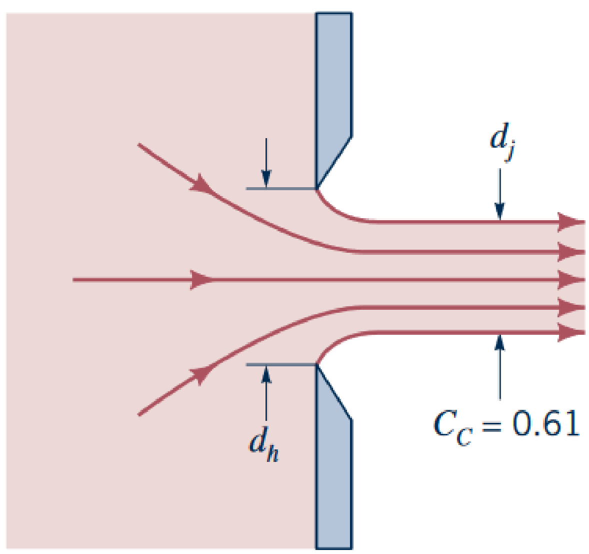
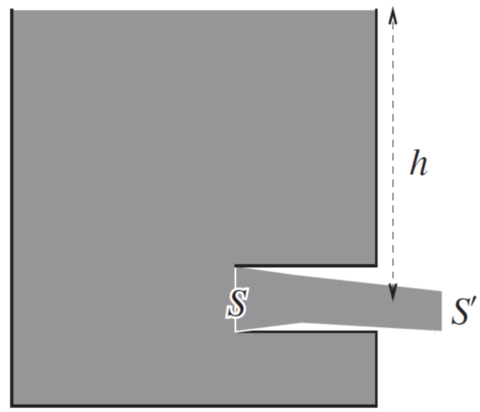


Figure 3.1: Picture from Gerhart, Gerhart, and Hochstein 2020.

We will work through a slightly different set-up called Borda mouthpiece where the coefficient of contraction could be obtained by hand. The Borda mouthpiece is a cylindrical tube, projecting inward to the center of the bucket.



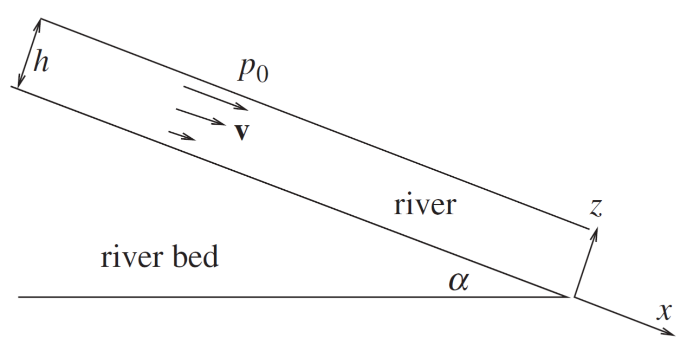
Name S' the size of the jet area so that the coefficient of contraction is S'/S.

(4.a) Assume S' given, what is the discharge rate. What is the momentum flux of this discharge.

(4.b) We know that the horizontal momentum flux should be equal to a surface integral of the force from the wall of the bucket. Use this relation, and $v = \sqrt{2gh}$ to obtain the coefficient of contraction is $S'/S = 0.5$.

3.5 Reynolds number for simple model for river flow

[From Falkovich 2018, §1.4.3] We use a simple inclined plane as a model for river flow.



In lecture you have the obtained the solution of:

$$p(z) = p_0 + \rho g(h - z)\cos\alpha, \tag{3.17}$$

$$v(z) = \frac{\rho g \sin\alpha}{2\eta} z(2h - z). \tag{3.18}$$

(5.a) Take the kinematic viscosity of water to be $\nu = \eta/\rho = 10^{-2}$ cm^{2}s^{-1}. Calculate $\boldsymbol{v}$ at the surface for a rain puddle with thickness $h = 1$ mm on a slope $\alpha \sim 10^{-2}$.

(5.b) How about a slow plain rivers (like the Danube) with $h \sim 10$ m on a slope $\alpha \sim 10^{-4}$?

(5.c) Which speed is reasonable?

(5.d) The unrealistic high velocity for the river case above is because in reality rivers are turbulent. Calculate the Reynolds number for the two cases.

Chapter 4

Vorticity Dynamics

4.1 Vorticity equation derivation

Remember that we have the vector identity:

$$\boldsymbol{v}(\nabla \cdot \boldsymbol{v}) = \frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \times \boldsymbol{\omega}. \tag{4.1}$$

To obtain the vorticity equation, we need to take the curl of this.

(1.a) Derive the vector identity:

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b}. \tag{4.2}$$

Hint: remember we have

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{4.3}$$

(1.b) Show that the compressible vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{v} \tag{4.4}$$

The incompressible version is an straightforward corollary.

4.2 Vorticity equation for compressible flows

[From Vallis 2017, §4.2] Using mass-conservation for compressible flows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{4.5}$$

Obtain the alternative compressible vorticity equation:

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\tilde{\boldsymbol{\omega}} = (\tilde{\boldsymbol{\omega}} \cdot \nabla)\boldsymbol{v} \tag{4.6}$$

where

$$\tilde{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}}{\rho}. \tag{4.7}$$

4.3 Fundamental solution to the heat equation

For details, see §5.1 and 5.2 of Shearer and Levy 2015.

(3.a) Self-similar solution from dimensional analysis.

(3.b) Fundamental solution is the Green's function.

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