1 Derivation of the Lamb vector

A useful vector identity in fluid mechanics is

$$\boldsymbol{v}(\nabla \cdot \boldsymbol{v}) = \frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \times \boldsymbol{\omega}. \tag{1}$$

For example, we use this identity in the derivation of the Bernoulli's principle. The Lamb vector is defined as

$$\boldsymbol{\ell} = \boldsymbol{v} \times \boldsymbol{\omega}. \tag{2}$$

We will derive the the vector identity (1).

(1.a) Show that the cross product can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk} a_i b_j \boldsymbol{e}_{(k)} \tag{3}$$

where ϵ_{ijk} is the permutation symbol:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{, if any two of } i, j, k \text{ are the same} \\ 1 & \text{, if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{, if } i, j, k \text{ is an odd permutation of } 1, 2, 3. \end{cases} \tag{4}$$

(1.b) [From Aris 1962, Exercise 2.32.1] Show by enumerating typical cases that

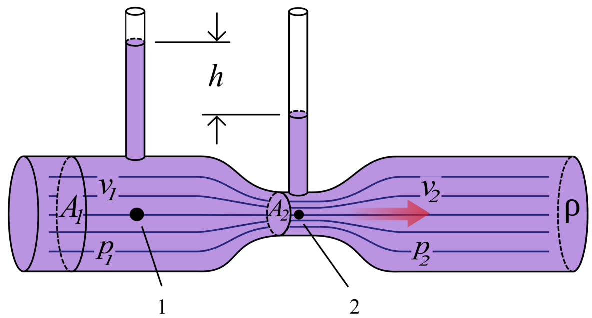
$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}. \tag{5}$$

(1.c) Use (5) to show (1).

2 Bernoulli's principle: examples

(2.a) Flow out of a water tank Imagine a water tank with height h of water inside. At the bottom there is a small hole. What would be the speed of the water flowing out of the hole.

(2.b) Venturi effect Calculate the fluid velocity difference at position 1 and 2



(2.c) Pitot's tube Using the idea of the above device, think of a device that measures the speed of the fluid flow.

3 The ABC flows

By using (1), we can write the incompressible Euler equation as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\nabla H, \tag{6}$$

$$\nabla \cdot \boldsymbol{v} = 0. \tag{7}$$

where H is the energy

$$H = \frac{p}{\rho} + \frac{1}{2}\boldsymbol{v}^2. \tag{8}$$

The Arnold, Beltrami, Childress (ABC) flow is an interesting steady state solution of Euler. It has the velocity field on the 2π-periodic 3D domain

$$\begin{cases} u = A \sin z + C \cos y \\ v = B \sin x + A \cos z \\ w = C \sin y + B \cos x \end{cases} \tag{9}$$

where A, B, and C are constant parameters. Despite its simple appearance in the Eulerian frame, the Lagrangian behavior of this flow is presumably chaotic. To quote Dombre et al. 1986 which named this flow

> Three-dimensional steady flows with a simple Eulerian representation can have a chaotic Lagrangian structure. By this we mean that infinitesimally close fluid particles following the streamlines may separate exponentially in time, while remaining in a bounded domain, and that individual streamlines may appear to fill entire regions of space.

and

> From a fluid dynamical viewpoint flows with chaotic streamlines are interesting because they may considerably enhance transport without being turbulent in the usual sense - they only display what might be called 'Lagrangian turbulence'.

We will be less ambitious and show some basic properties of this flow.

(3.a) Show the ABC flows are incompressible.

(3.b) Show the ABC flows are Beltrami flows. That is

$$\boldsymbol{\omega} \times \boldsymbol{v} = 0. \tag{10}$$

Hint: Show $\boldsymbol{\omega} = \boldsymbol{v}$. Why does this identity shows $\boldsymbol{v}$ is Beltrami?

(3.c) Conclude the ABC flows are exact solution to the steady state incompressible Euler equation.

References

Aris, Rutherford. 1962. *Vectors, Tensors and the Basic Equations of Fluid Mechanics.* Courier Corporation. ISBN: 978-0-486-13489-5.

Dombre, T., U. Frisch, J. M. Greene, M. Hénon, A. Mehr, and A. M. Soward. 1986. "Chaotic Streamlines in the ABC Flows." *Journal of Fluid Mechanics* 167 (June): 353–391. ISSN: 1469-7645, 0022-1120. https://doi.org/10.1017/S0022112086002859.