1 Pressure is a Lagrange multiplier for incompressible flow

We have talked about how pressure is a Lagrange multiplier that enforces incompressibility. This is the easier to show for Stokes flow. Show that the Stokes equation on Ω

$$-\mu\nabla^2\boldsymbol{u} + \nabla p = f \tag{1}$$
$$\boldsymbol{u}|_{\partial\Omega} = 0 \tag{2}$$
$$\nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

is equivalent to a constrained energy minimization problem:

$$\min_{\boldsymbol{u}} \frac{\mu}{2} \int_\Omega \|\nabla \boldsymbol{u}\|^2 - f\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \tag{4}$$
$$\text{subject to } \nabla \cdot \boldsymbol{u} = 0. \tag{5}$$

2 Uniqueness of solution for Stokes

Show that the solution to the Stokes system is unique from the energy minimizing statement above. This is a similar but slightly different approach compared to the one in Acheson 1990, §7.4.

3 Life at low Reynolds number

If we have time left, we will take a look at the classic and highly readable paper Purcell 1977. Life at low Reynolds number is counter-intuitive and full of surprise.

References

Acheson, D. J. (Mar. 15, 1990). *Elementary Fluid Dynamics*. Clarendon Press. 408 pp. ISBN: 978-0-19-859679-0.

Purcell, E. M. (Jan. 1977). "Life at Low Reynolds Number". In: *American Journal of Physics* 45.1, pp. 3–11. ISSN: 0002-9505. DOI: 10.1119/1.10903.