Worksheet 1

1 Rate of strain tensor: examples

(1.a) Shear flow Take a velocity field with $u = y$ and $v, w = 0$.

1. Calculate its velocity gradient tensor.
2. What is its symmetric and anti-symmetric parts?
3. What is its divergence and vorticity vector?

(1.b) Straining flow Take a velocity field with $u = x$, $v = -y$, and $w = 0$.

1. Do the same calculation.
2. [Adapted from Aris 1962, Exercise 4.42.1] Take a line connecting the origin and a fluid parcel, show that if θ is the angle between the line and the x-axis, then the rate of change of $\log \tan \theta$ is constant along a particle path.

(1.c) [From Aris 1962, Exercise 4.45.1] Take the velocity field

$$\boldsymbol{v} = \alpha\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{x}. \tag{1}$$

1. Do the same calculation.
2. Interpret the motion of a fluid parcel in this flow.

(1.d) Rankine vortex Now we use the cylindrical coordinate with (r, θ, z). We have the flow field

$$u_\theta = \begin{cases} \Omega r, & r < a \\ \dfrac{\Omega a^2}{r}, & r < a \end{cases} \tag{2}$$

and $u_r = u_z = 0$.

1. Do the same calculation (by converting the velocity into the Cartesian coordinate).

We will come back to this example to practice polar coordinate and to study vorticity and circulation.

2 From divergence theorem to Green's theorem

(2.a) [From Aris 1962, Exercise 3.31.3] By taking $\boldsymbol{a}$ to be independent of x, and $a_3 = 0$, show using divergence theorem that if A is an area in the (x, y)-plane bounded by a curve C, then

$$\iint_A \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y}\, \mathrm{d}A = \oint_C a_1 t_2 - a_2 t_1\, \mathrm{d}s \tag{3}$$

where $\boldsymbol{t} = (t_1, t_2)$ is the unit tangent vector to C.

(2.b) [From Aris 1962, Exercise 3.31.4] Deduce from the previous question that

$$\iint_A \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \,\mathrm{d}A = \oint_C a_1 t_1 + a_2 t_2 \,\mathrm{d}s \tag{4}$$

The right hand side can be written as

$$\oint_C a_1 \mathrm{d}x + a_2 \mathrm{d}y \tag{5}$$

and we have derived the Green's theorem.

3 Green's identity

(3.a) We take a 3D scalar field ψ and a 3D vector field $\boldsymbol{\Gamma}$ with sufficient smoothness defined on some region $U \subset \mathbb{R}^3$. Show the identity

$$\iiint_U (\psi \, \nabla \cdot \boldsymbol{\Gamma} + \boldsymbol{\Gamma} \cdot \nabla \psi) \; dV = \oiint_{\partial U} \psi \, (\boldsymbol{\Gamma} \cdot \boldsymbol{n}) \; dS = \oiint_{\partial U} \psi \boldsymbol{\Gamma} \cdot d\boldsymbol{S}. \tag{6}$$

(3.b) Use (6) to show the Green's first identity. Take 3D scalar fields ψ and φ both with sufficient smoothness:

$$\iiint_U \left(\psi \, \nabla^2 \varphi + \nabla \psi \cdot \nabla \varphi\right) \; dV = \oiint_{\partial U} \psi \, (\nabla \varphi \cdot \boldsymbol{n}) \; dS = \oiint_{\partial U} \psi \, \nabla \varphi \cdot d\boldsymbol{S}. \tag{7}$$

(3.c) Show the Green's second identity:

$$\iiint_U \left(\psi \, \nabla^2 \varphi - \varphi \, \nabla^2 \psi\right) \; dV = \oiint_{\partial U} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot d\boldsymbol{S}. \tag{8}$$

This shows that the Laplacian is a self-adjoint operator for functions vanishing on the boundary so that the right hand side of the above identity is zero.

4 Expanding gas

(4.a) In a tube We have a velocity field $u = x$ and $v, w = 0$.

1. Write down the "differential form" of the mass conservation equation.
2. Use the method of characteristic to solve for the density $\rho(x, t)$.

(4.b) In free 3D space Now have a velocity field $u = x$, $v = y$, and $w = z$. Do the same calculation.

References

Aris, Rutherford. 1962. *Vectors, Tensors and the Basic Equations of Fluid Mechanics.* Courier Corporation. ISBN: 978-0-486-13489-5.