GFD exam

January 2021

1. (Rossby waves) The linear quasi-geostrophic shallow water equations in the presence of a constant zonal mean flow U can be written as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) q + \beta\frac{\partial\psi}{\partial x} = 0 \quad \text{with} \quad q = (\nabla^2 - \kappa_D^2)\psi. \tag{1}$$

Here κ_D is the reciprocal of the deformation radius.

(a) Find the dispersion relation for plane Rossby waves of the form $\psi = \hat{\psi}\exp(i[kx + \ell y - \omega t])$ and compute the group velocity in the y-direction.

(b) Consider waves generated near $y = 0$ by an undulating sidewall and propagating into the upper half plane $y \geq 0$. The undulations are steady and sinusoidal with a certain wavenumber k. Given (β, κ_D, k), find the range of U that allows for propagating waves. (Hint: the flow in $y \geq 0$ consists of a steady plane wave.). What happens if U falls outside this range?

(c) For U in the range for propagating waves find the y-wavenumber ℓ. (Hint: this involves a suitable radiation condition to be stated.) Sketch some phase lines of the Rossby wave in the upper half plane.

2. (Geostrophic adjustment) This question asks about the long term fate of an anomaly in sea height. The initial condition below will seem rather unphysical, but the idea is that an anomaly of scale L could be created by an atmospheric storm; here we simplify the geometry to make the mathematics more straightforward. And by "long term", we mean long relative to the time scale of the dynamic response of the ocean, but not so long as to require a treatment of dissipation.

   Suppose an inviscid and initially motionless ocean begins with the initial height condition, h_0, given by

$$h_0(x,y) = \begin{cases} H & -L/2 < x < L/2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

   You can assume the surface layer of the ocean has uniform density ρ and rests on top of a motionless abyss of density $\rho + \delta\rho$ (with $\delta\rho << \rho$) the is effectively infinite in depth $D >> H$. Thus the upper layer of the ocean can be modeled by the shallow water equations with reduced gravity $g' = g\delta\rho/\rho$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

   Starting at time $t = 0$, the system is allowed to evolve freely.

   (a) Characterise the final state of the system depending on the scale of L. In particular, when would you expect the region of anomalous sea height to persist indefinitely, i.e., $h(0,y) = H$ as $t \to \infty$, vs. when would you expect the anomaly at the center to vanish, $h(0,y) \to 0$ as $t \to \infty$? Justify your answer!

   (b) (bonus) Suppose the scale of the initial perturbation L is set by an atmospheric storm, and is thus on order of a thousand km. Make some rough assumptions about the depth of the thermocline H and the reduced gravity g' and comment on whether an anomaly will persist.

3. (Rayleigh-Kuo criterion for QGSW)

The QG Shallow Water equations are:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad q = \nabla^2\psi - k_d^2\psi + \beta y$$

where $J(f, g) = f_x g_y - f_y g_x$. The velocity field is related to the streamfunction by $u = -\psi_y$ and $v = \psi_x$. Assume the lateral boundary conditions to be periodic in both x and y.

(a) Linearize the equations about a background state given by $u = \overline{U}(y)$, $v = 0$. What is the background PV gradient $\overline{q}_y$?

(b) Consider a perturbation of the form

$$\psi'(x, y, t) = \tilde{\psi}(y) \exp[ik(x - ct)].$$

Derive the Rayleigh equation for this system. You should get a a second order ODE of the form:

$$\frac{d^2\tilde{\psi}}{\partial y^2} = F(\overline{U}, \overline{q}_y, c, k)\tilde{\psi}.$$

(c) Show that a necessary conditions for instability is the reversal of the background PV gradient. [There are multiple ways to do it. In class, we did it by analyzing the Rayleigh equations and by analyzing the wave activity. Pick one]