New York University

Courant Institute of Mathematical Sciences

Department of Mathematics

Ph.D. Program in Atmosphere-Ocean Science & Mathematics

Written Comprehensive Exam

Geophysical Fluid Dynamics

Fall 2002 – Spring 2019

Atmosphere Ocean Written Examination

August 2002

1 Thermodynamics

An equation that is often useful in both the atmosphere and ocean is known as the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -\rho g$$

where p is tbe pressure in the fluid and ρ is it's density while g is the acceleration due to gravity.

1. Write down the equation for an ideal gas involving ρ p and T the fluids temperature. This is commonly referred to as the ideal gas law.

2. Assuming that the fluid has uniform temperature (isothermal) derive an equation for the variation of pressure with height using the ideal gas law and the hydrostatic approximation.

3. If a parcel of air is moved adiabatically what procedure is used? If it is moved isentropically what is used?

4. For isentropic parcel movement we have the equation

   $$\frac{\partial T}{\partial p} = \frac{\alpha T}{\rho c_p}$$

   where for an ideal gas we may take the thermal expansion coefficient $\alpha = 1/T$ and c_p is approximately constant (specific heat at constant pressure). Use these relations and the hydrostatic relation to show how temperature in a parcel moved isentropically varies in the vertical.

5. If the isentropic parcel discussed above is moved upwards in an isothermal atmosphere is the moved parcel heavier or lighter than its environment? What does this imply about the stability of an isothermal atmosphere?

2 Wave Motion

1. What do the term dispersion relation, group and phase velocity mean for a wave disturbance?

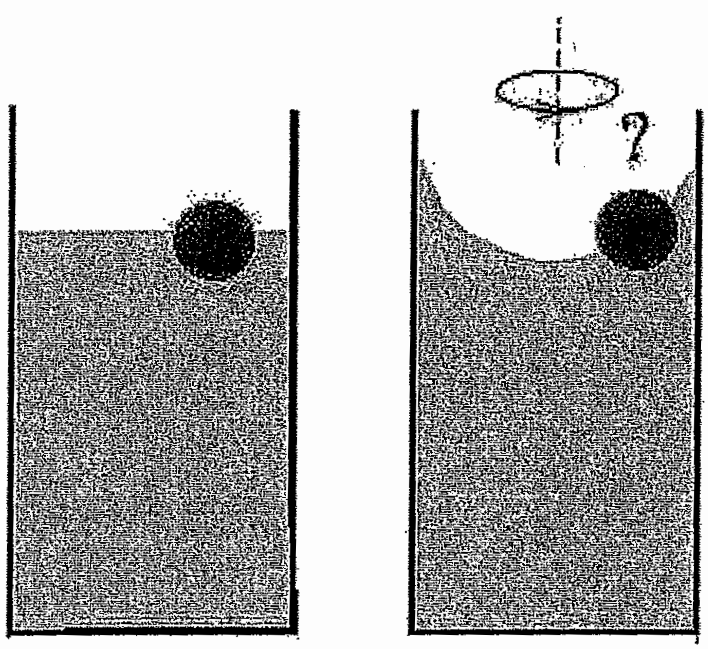
2. The equations for small disturbances in a domain of constant rotation rate which is uniform in the vertical are

$$\begin{aligned} u_t - fv &= -p_x \\ v_t + fu &= -p_y \\ p_t + c^2(u_x + v_y) &= 0 \end{aligned}$$

   where f is the constant Coriolis parameter. Derive the dispersion relation and group velocities for such disturbances. What is the minimum frequency of these disturbances and when this minimum occurs what is the group velocity of the corresponding waves?

3 Fluid Dynamics

A sphere of uniform density floating in a glass of water (left figure) can be at any distance from the axis of the glass. Where will the sphere be located if the glass is rotating with a constant angular velocity Ω (right figure)?



4 Geophysical Fluid Dynamics

Starting with the point-wise continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

and the point-wise momentum equations

$$-fv=-g\frac{\partial \eta}{\partial x}+\frac{1}{\rho_o}\frac{\partial \tau^x}{\partial z}$$

$$fu=-g\frac{\partial \eta}{\partial y}+\frac{1}{\rho_o}\frac{\partial \tau^y}{\partial z},$$

derive the depth-integrated continuity and momentum equations.

For generality, consider a steady-state ocean fluid in which the surface is denoted $\eta(x,y)$ and the bottom $H(x,y)$. It will also be useful to introduce definitions of the depth-integrated volume transports

$$M_x=\int_{-H}^{\eta} u\,dz, \quad M_y=\int_{-H}^{\eta} v\,dz.$$

5 Physics of Fluids

Stokes' Law gives the force needed to move a small sphere through a continuous, quiescent fluid at a certain velocity. The law is $F=6\pi R\nu V$, where R is the radius of the sphere, ν is the viscosity, and V is the velocity of the sphere moving through a continuous fluid.

Clouds are made of water droplets, and water is about 800 times denser than air. Why don't clouds fall like a rock to the ground? Provide calculations to back up your answer.

AOS Written Examination

28 August 2003

1. Kelvin's Cirulation Theorem

   Inviscid, constant density flow is described by the Euler equation,

   $$\frac{Du}{Dt} = -\frac{\nabla p}{\rho},$$

   where d/dt is the advective, or material derivative, u is the velocity field, p is the pressure and ρ is the density. The circulation around a material loop is defined as

   $$\Gamma = \oint u \cdot d\ell.$$

   (a) Relate the circulation to the vorticity $\omega \equiv \nabla \times u$.

   (b) Prove Kelvin's Circulation theorem:

   $$\frac{d}{dt}\Gamma = 0.$$

2. Ekman Layers

   Near the upper and lower boundaries of the ocean (and the lower boundary of the atmosphere), boundary layers form. These boundary layers, called Ekman layers, are typically modelled as geostrophic balance in the presence of an eddy viscosity. The model equation is

   $$f\hat{z} \times u = -\frac{\nabla p}{\rho_0} + \nu \frac{\partial^2 u}{\partial z^2}$$

   where ν is the eddy viscosity, and we take ρ_0 and f to be constant. We further assume that $u = 0$ at the boundaries and that $u = u_g$ in the interior, where

   $$u_g \equiv \frac{1}{f\rho_0}\hat{z} \times \nabla p$$

   is the geostrophic velocity.

   Now consider specifically a bottom boundary with a surface stress $\tau/\rho_0 = \nu \partial u/\partial z$.

   (a) Show that the transport in the Ekman layer is

   $$U_e \equiv \int_{-H}^{-H+\delta} u_e \, dz = \hat{z} \times \left(\frac{\tau}{f\rho_0}\right),$$

   where $u_e = u - u_g$, H is the depth of the ocean and δ is the thickness of the layer. Remember that the stress only exists at the boundary, and that at the velocity is purely geostrophic in the interior (*i.e.*, the ageostrophic velocity induced by the surface stress vanishes at the edge of the Ekman layer, $z = -H + \delta$).

(b) Using the continuity equation and remembering that vertical velocity is identically zero at the bottom surface, show that the divergence of the *total* transport in the Ekman layer is

$$\nabla \cdot U = -w_e,$$

where w_e is the vertical velocity at the upper edge of the Ekman layer.

(c) Relate the vertical velocity to the divergence of the geostrophic transport in the Ekman layer and the applied stress.

(d) Give a scaling estimate for the depth δ of the Ekman layer.

(e) It is well known that sediment in the bottom of a cup of fluid (say, tea leaves in a cup of tea) congregates toward the center of the cup when the fluid is stirred, regardless of the direction of stirring. Qualitatively modify the above Ekman-layer theory to explain this phenomenon.

3. **Geostrophic adjustment**

The two-dimensional shallow-water equations in a rotating frame are

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0$$

$$\frac{D\boldsymbol{u}}{Dt} + f\hat{z} \times \boldsymbol{u} + g\nabla h = 0,$$

where the velocity $\boldsymbol{u} = (u, v)$ and the layer depth h are functions of (x, y, t).

(a) Briefly describe the physical meaning of each term in these equations.

(b) Define 'potential vorticity' and state its conservation law.

(c) Assuming that $\boldsymbol{u} = \epsilon \boldsymbol{u}'$ and $h = H + \epsilon h'$ with constant H and $\epsilon \ll 1$, derive the linearized shallow-water equations at $O(\epsilon)$ and show that

$$q' \equiv \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \frac{f}{H}h' \quad \text{satisfies} \quad \frac{\partial q'}{\partial t} = 0.$$

Now consider the y-independent linear adjustment problem defined by the following initial conditions at $t = 0$:

$$u_0' = v_0' = 0, \quad h_0' = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}.$$

(d) Show that as $t \to +\infty$ the final steady state satisfies

$$u' = 0, \quad g\frac{dh'}{dx} = fv', \quad \frac{dv'}{dx} - \frac{f}{H}h' = -\frac{f}{H}h_0'.$$

Hence derive that

$$h' = \begin{cases} -\exp(-x/L_d) + 1 & x > 0 \\ +\exp(+x/L_d) - 1 & x < 0 \end{cases}$$

with $L_d = \sqrt{gH}/f$. Find v' and sketch h_0', h' and v'.

(e) Compute the disturbance energy contained in $x \in [-L, L]$

$$\mathcal{E}_L \equiv \frac{1}{2}\int_{-L}^{+L} \left(u'^2 + v'^2 + gh'^2\right)\, dx$$

for both the initial and final state. Show that the difference between these energies has a finite limit as $L \to \infty$. Where is the missing energy?

4. Shallow water waves

(a) Using the shallow water equations from problem 3, linearize the equations about a state of rest, substitute a wave solution and derive a dispersion relation.

(b) Consider the limit in which the wavenumber is much smaller the inverse deformation scale ($k \ll L_d^{-1}$, where L_d is given in problem 3). These are inertial oscillations. What path does a particle trace out in the presence of these oscillations?

(c) Now consider the opposite limit in which $k \gg L_d^{-1}$. What is the phase speed in this case?

(d) Consider a solution in which $u = 0$ everywhere, due to the presence of a boundary at $x = 0$ (this is will yield a Kelvin wave). Derive the structure of the resulting wave. What is the trapping scale of the wave? What direction does it propagate?

(e) At the equator, assuming $f = \beta y$ and $v = 0$, what is the form of the resulting wave? What is the trapping scale of the wave?

5. Dry atmospheric thermodynamics

The first law of thermodynamics is

$$T\mathrm{d}s = \mathrm{d}\epsilon + p\mathrm{d}\left(\frac{1}{\rho}\right) \tag{1}$$

with temperature T, specific entropy s, specific internal energy ϵ, pressure p, and density ρ.

(a) Which terms are zero for adiabatic changes of state?

(b) For an ideal gas $p = \rho RT$ with gas constant R. Show that in this case ϵ must be a function of T only. Compute the entropy function $s(T, p)$ for an ideal gas with $\epsilon = c_v T$.

(c) In a hydrostatic atmosphere

$$\frac{\partial p}{\partial z} = -\rho g \tag{2}$$

where g is taken as constant. Compute the adiabatic lapse rate of temperature, which corresponds to a hydrostatic atmosphere of uniform entropy. Does entropy increase or decrease with altitude in a stably stratified atmosphere?

(d) Consider a vertical coordinate $\theta(x, y, t)$ that replaces the use of altitude z such that, for instance, (2) becomes

$$\frac{\partial p}{\partial \theta} = -\rho g \frac{\partial z}{\partial \theta}. \tag{3}$$

Show that the zonal pressure force in a hydrostatic atmosphere becomes

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_\theta + \left(\frac{\partial g z}{\partial x}\right)_\theta \tag{4}$$

in the new coordinate system. (Subscripts denote the variable held constant in the derivative.)

(e) In isentropic coordinates θ is a function of entropy s and derivatives at constant θ correspond to adiabatic changes of state. Show that in this case the zonal pressure force is equal to $(\partial M/\partial x)_\theta$, where M is the Montgomery potential defined as

$$M = (c_v + R)T + gz. \tag{5}$$

Dept. of Mathematics, Written Comprehensive Examination, January 2004

Physics

Attempt all five problems, justifying your answers.

1) A rocket with an initial mass of 20000 kg. leaves Earth with zero initial velocity, and constantly ejects 0.1 kg. of burned fuel per second, at a speed of 1 meter per second relative to the rocket. How far will it have gone in one day? (Neglect relativistic effects).

2) Write down the nonlinear shallow water equations in a rotating environment (f-plane), and derive from them the (Lagrangian) conservation of potential vorticity.

3) Consider a one dimensional set of particles, so densely packed that we may assume it forms a continuum, yet sparse enough that we may neglect any interaction among them. Given an initial smooth velocity distribution $u(x, 0)$, derive a partial differential equation ruling the evolution of $u(x, t)$. Will it be valid for all times? Why? Can you also derive an evolution equation for the density of particles $\rho(x, t)$?

4) Will a pot of water placed on a furnace boil earlier if the pot is covered? Discuss all the elements at play.

5) Explain the basic physical mechanisms behind the Hadley cells and the Trade winds.

DEPARTMENT OF MATHEMATICS

September 2004

AOS Written Examination in Physics

Attempt all questions. Provide as much detail as possible in derivations so that appropriate credit may be given.

1 Thermodynamics

- Provide a definition for potential temperature θ. If one moves along a surface of equal potential temperature what happens to entropy η? Why?
- Derive the following relationship for an ideal gas between entropy and potential temperature:
$$\eta = c_p ln\theta + const \tag{1}$$
where c_p is the is the specific heat at constant pressure which you may assume does not depend on temperature or pressure for an ideal gas.

- Using the relationship between infinitesimal heat and entropy change derive a relationship for the time variation of entropy due to heating. Use equation (1) to derive an expression for the time rate of change of potential temperature due to this heating.
- The equation of state for an ideal gas may be written as
$$\rho = \rho(p, q, \theta)$$
where p is pressure and q is (specific) moisture. Use this relation and the relation derived in the previous question to derive an equation for the time variation of density due to heating.

2 Dynamics

- Linearize the equation for the time variation of density due to heating about a state of rest with constant mean <u>vertical</u> density variation. Use the hydrostatic relation and a two layer fluid model to convert this into an (approximate) equation for the time rate of change of pressure in both layers due to heating.
- Linearize in the same fashion the horizontal momentum equations in a rotating frame and so derive the following three equations in three unknowns for the (lower layer) response of a constantly stratified fluid to heating:
$$\begin{aligned} u_t - fv &= -\tfrac{1}{\bar{\rho}} p_x \\ v_t + fu &= -\tfrac{1}{\bar{\rho}} p_y \\ p_t + G(u_x + v_y) &= -KQ \end{aligned}$$

where $\bar{\rho}$ is the mean fluid density; Q is the heating rate at the interface of the layers; f is the Coriolis parameter and G and K are approximate constants depending on the stratification as well as thermodynamical constants. What are these equations commonly referred to as?

- Assume that linear frictional processes allow us to replace $\frac{\partial}{\partial t}$ by ϵ in these equations to get a steady state solution and that the Coriolis parameter is constant. Derive a single equation for pressure in response to heating. What is this equation commonly referred to? Solve this equation and the auxilliary equations for u and v to derive the complete fluid response to heating of the form:

$$Q = Q_0 sin(kx)$$

Provide a rough sketch of your solution.

Written Comprehensive Exam
January 2005
Geophysical Fluid Dynamics — CAOS

Please answer all four questions.

1. Consider the two-dimensional incompressible β-plane dynamics

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + P_x = 0, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + fu + P_y = 0, \quad u_x + v_y = 0 \tag{1}$$

where $f = f_0 + \beta y$ with positive constants f_0 and β. Assume that the flow is periodic in x and denote x-averaging by $\overline{(\dots)}$ such that $u = u' + \overline{u}$ with $\overline{u'} = 0$ for all fields. Show that

$$\overline{u}_t + \overline{(u'v')}_y = 0 \tag{2}$$

holds exactly provided the flow is bounded by a solid wall at some y.

Show that for linear waves relative to a state of rest the linear vorticity disturbance

$$q' = v'_x - u'_y \quad \text{satisfies} \quad q'_t + \beta v' = 0 \tag{3}$$

and derive the conservation law for a suitable pseudomomentum p. Show that $\overline{u}_t = \mathrm{p}_t$ holds at leading order and indicate the sign of $\overline{u}_t$ for arriving or departing waves.

2. Consider the generic two-dimensional vortex–stream function dynamics

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0 \quad \text{and} \quad \mathcal{L}\psi = q \tag{4}$$

in a domain D with $\psi = 0$ on the boundary of D. Here $\mathcal{L}$ is a linear self-adjoint operator. Show that the energy and enstrophy

$$\mathcal{E} = -\frac{1}{2}\int_D \psi q \,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad \mathcal{Z} = +\frac{1}{2}\int_D q^2 \,\mathrm{d}x\mathrm{d}y \tag{5}$$

are conserved. If the right-hand side of (4a) is changed to $\nu\mathcal{L}q$ with $\nu > 0$ constant, show that $\mathrm{d}\mathcal{E}/\mathrm{dt} = -\nu\mathcal{Z}$ and hence energy decreases. Under what condition on $\mathcal{L}$ does enstrophy decrease as well?

Let D be the entire plane and restrict $\mathcal{L}$ to have constant real coefficients. Show that if the spectral energy density is $E(\kappa, t)$ then the spectral enstrophy density is $\hat{\mathcal{L}}E$, where $\hat{\mathcal{L}}$ is the symbol of $\mathcal{L}$ in Fourier space and κ is the wavenumber magnitude.

On planet Alpha the operator

$$\mathcal{L} = \nabla^2 - \alpha^2\nabla^4 \tag{6}$$

with a fixed length scale $\alpha > 0$. Do you expect a forward or inverse cascade of energy? Consider a forced–dissipative equilibrium in which energy is injected with rate ϵ_1 at wavenumber κ_1 and extracted at two wavenumbers κ_2 and κ_3 at rates $\epsilon_{2,3}$ such that $\epsilon_1 = \epsilon_2 + \epsilon_3$. Assume $\kappa_2 < \kappa_1 < \alpha^{-1} < \kappa_3$ and make a sketch of the situation in κ-space. Find $\epsilon_{2,3}$ and consider the limit of ϵ_2/ϵ_1 as $\kappa_3 \to \infty$ with $\{\kappa_1, \kappa_2, \alpha\kappa_3\}$ fixed.

3. The rotating unstratified incompressible equations in the two-dimensional xz-plane are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + P_x = 0, \quad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = 0, \quad \frac{\mathrm{D}w}{\mathrm{D}t} + P_z = 0, \quad u_x + w_z = 0 \tag{7}$$

with constant $f > 0$. Briefly describe the physical meaning of all terms. What are the "Taylor columns" that occur for very slow and nearly steady flows?

In a plane wave all fields are given by $w' = \hat{w}\exp(i[kx + mz - \omega t])$ etc., with real parts understood. Show that a *single* plane wave is an exact solution of the fully nonlinear equations.

For a linear plane wave around a state of rest derive the dispersion relation

$$\omega(k, m) = \pm f\frac{m}{\kappa}, \quad \kappa = \sqrt{k^2 + m^2} \tag{8}$$

and compute the group velocity components $c_g = (u_g, w_g)$. Show that the *horizontal* group and phase velocities have opposite signs. Show that upward radiation of waves from a boundary at $z = 0$ implies $\mathrm{sgn}(m) = \mathrm{sgn}(\omega)$.

What wavenumbers (k, m) correspond to Taylor columns? Find a simple link between (7) and the non-rotating stratified Boussinesq equations in the same domain. Hence, describe the analogue of Taylor columns in the Boussinesq system, which is called "blocking".

4. Now let f be a slowly varying function of x in (7). Define what is meant by a "slowly varying wavetrain" and write down the ray-tracing equations for $k(t)$ and $m(t)$ based on a dispersion function $\Omega(k, m, x)$ for upward propagating waves. What is the ray-tracing equation for $\omega(t)$?

From (7) linearized around a state of rest deduce the conservation law for the linear kinetic energy

$$\frac{\partial}{\partial t}\left(\frac{|\mathbf{u}'|^2}{2}\right) + \frac{\partial}{\partial x}(P'u') + \frac{\partial}{\partial z}(P'w') = 0. \tag{9}$$

Show that for a slowly varying wavetrain (9) averaged over the rapidly varying wave phase becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(Eu_g) + \frac{\partial}{\partial z}(Ew_g) = 0, \tag{10}$$

where $E = \overline{|\mathbf{u}'|^2}/2$. Hint: derive first that

$$E = \overline{u'^2}f^2/\omega^2 \quad \text{and} \quad -\kappa^2 P' = f^2\frac{k}{\omega}u'. \tag{11}$$

For a steady and z-independent wavetrain find $E(x)$ for $x > 0$ subject to given $\{E_0, m_0, \omega_0\}$ at $x = 0$ (assume $\omega_0^2 < f_0^2$) and show that it equals

$$E(x) = E_0\sqrt{\frac{f_0^2 - \omega_0^2}{f^2 - \omega_0^2}}. \tag{12}$$

Comment on what happens to E at a location where $f = \omega_0$. Is this physically correct? Are the waves reflected or absorbed?

Written Comprehensive Exam
January 2006
Geophysical Fluid Dynamics — CAOS

Please answer all questions.

1. The two-dimensional shallow-water equations in a rotating frame are

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0$$

$$\frac{D\boldsymbol{u}}{Dt} + f\hat{z} \times \boldsymbol{u} + g\nabla h = 0,$$

where the velocity $\boldsymbol{u} = (u, v)$ and the layer depth h are functions of (x, y, t). Briefly describe the physical meaning of each term in these equations. Define 'potential vorticity' and state its conservation law.

Assuming that $\boldsymbol{u} = \epsilon \boldsymbol{u}'$ and $h = H + \epsilon h'$ with constant H and $\epsilon \ll 1$, derive the linearized shallow-water equations at $O(\epsilon)$ and show that

$$q' \equiv \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \frac{f}{H}h' \quad \text{satisfies} \quad \frac{\partial q'}{\partial t} = 0.$$

Consider a y-independent linear adjustment problem defined by initial conditions $\{u'_0, v'_0, h'_0\}$ as functions of $x \in R$ at $t = 0$. Show that the final steady state (u', v', h') that remains in the limit $t \to +\infty$ can be found from solving

$$\frac{\mathrm{d}^2 h'}{\mathrm{d}x^2} - \kappa_R^2 h' = \frac{f}{g} q'_0 \tag{1}$$

where $\kappa_R^2 = f^2/gH$. Find and sketch the Green's function for (1) in an unbounded domain and show that in the case of initial conditions $u'_0 = 0, v'_0 = \delta(x), h'_0 = 0$ the final

$$h' = \frac{f}{2g}\exp(-|x|\kappa_R)\mathrm{sgn}(x).$$

In general, are the energies of the initial and final states equal? Explain this physically.

2. Consider the two-layer quasi-geostrophic equations

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \quad q_i = \nabla^2\psi_i \pm F(\psi_2 - \psi_1), \tag{2}$$

where $i = \{1, 2\}$, the upper sign applies for the upper layer $i = 1$, and $F = f^2/g'H$ is $0.5\kappa_R^2$ from the lectures. Assume that $q_2 = 0$ everywhere at the initial time, and hence at all times. This is a model for a forced upper ocean on top of an unforced abyss. Clearly, q_1 is now the only active prognostic variable and ψ_2 can be eliminated from the diagnostic relations.

Show that in Fourier space the relationship between $\hat{\psi}_1$ and $\hat{q}_1$ is

$$\left(-\kappa^2 - F + \frac{F^2}{F + \kappa^2}\right)\hat{\psi}_1 = \hat{q}_1, \tag{3}$$

where κ is the wavenumber magnitude. Find the analogous relation between $\hat{\psi}_2$ and $\hat{q}_1$ and discuss how the stream functions in the two layers react to the upper layer PV distribution in the limits $\kappa^2 \ll F$ and $\kappa^2 \gg F$. Give a physical interpretation of these two limits.

This model with $q_2 = 0$ has conserved energy $\mathcal{E}$ and enstrophy $\mathcal{Z}$

$$\mathcal{E} = -\frac{1}{2}\int \psi_1 q_1 \mathrm{d}x\mathrm{d}y, \quad \mathcal{Z} = +\frac{1}{2}\int q_1^2 \mathrm{d}x\mathrm{d}y, \tag{4}$$

which do not involve the lower layer explicitly. Formulate the corresponding $\mathcal{E}$-$\mathcal{Z}$ constraints on the energy spectrum $E(\kappa, t)$ and investigate whether this model has an inverse or forward energy cascade if forcing is applied at a wavenumber $\kappa_f > \sqrt{F}$ and dissipation is applied at $\kappa_1 \ll \sqrt{F}$ and $\kappa_2 \gg \kappa_f$.

3. The rotating three-dimensional Boussinesq system linearized around a state of rest such that $u = u' + O(a^2)$ etc. with small wave amplitude $a \ll 1$ is

$$u'_t - fv' + P'_x = 0, \quad v'_t + fu' + P'_y = 0, \quad w'_t + P'_z = b' \tag{5}$$

$$b'_t + N^2 w' = 0, \quad u'_x + v'_y + w'_z = 0 \tag{6}$$

with $f = f_0 + \beta y$ and constant N. Show that the PV disturbance

$$q' \equiv v'_x - u'_y + b'_z f/N^2 \quad \text{satisfies} \quad q'_t + \beta v' = 0. \tag{7}$$

Assuming that the flow is x-periodic and that $\overline{(\ldots)}$ denotes x-averaging, show that the pseudomomentum conservation law

$$(\mathsf{p}_G + \mathsf{p}_R)_t + \nabla \cdot \boldsymbol{F} = 0 \tag{8}$$

holds where

$$\mathsf{p}_G \equiv \frac{1}{N^2}\overline{b'(u'_z - w'_x)} \quad \text{and} \quad \mathsf{p}_R \equiv -\frac{1}{2\beta}\overline{q'^2} \tag{9}$$

are the pseudomomentum densities for gravity and Rossby waves, respectively, and

$$\boldsymbol{F} \equiv (0, \overline{u'v'}, \overline{u'w'} - \frac{f}{N^2}\overline{b'v'}) \tag{10}$$

is the Eliassen–Palm flux.

GFD Written Examination

January 2008

1. **Thermodynamics**

   (a) Define potential temperature for a fluid.

   (b) Express the potential temperature of an ideal gas as function of its temperature and pressure.

   (c) Derive the adiabatic lapse rate Γ_{ad} for an ideal gas.

   (d) Discuss the behavior of the fluid when the actual lapse rate $\Gamma > \Gamma_{\mathrm{ad}}$ and $\Gamma < \Gamma_{\mathrm{ad}}$.

2. **Potential Vorticity**

   The rotating shallow water equations are

$$\begin{aligned} u_t + \boldsymbol{u} \cdot \nabla u - fv &= -gh_x \\ v_t + \boldsymbol{u} \cdot \nabla v + fu &= -gh_y \\ h_t + \nabla \cdot (\boldsymbol{u}h) &= 0 \end{aligned}$$

   where $\boldsymbol{u} = (u, v)$ is the horizontal velocity, $h = h(x, y, t)$ is the depth of the fluid, g is gravitational acceleration and f is the Coriolis parameter.

   (a) Define the potential vorticity for the rotating shallow water equations (SWPV).

   (b) Show that the SWPV is conserved.

   (c) A column of air is at rest at a latitude of 30°N and has a thickness of 10 km and a radius of 500 km. What will be the zonal velocity at the edge of the column when it goes over a 2 km thick mountain? What will be its relative vorticity after moving to a latitude of 60°N with the same initial thickness?

   (d) Show that for a parcel initially at rest in the Northern hemisphere, there is an absolute lower bound on its relative vorticity if it remains in the same hemisphere, but no upper bound. Give a physical interpretation for this lower bound and how it can be achieved.

3. **Quasi-geostrophy and Rossby waves**

   (a) Use physical scaling arguments to derive (informally) the shallow water quasigeostrophic equation (i.e. the conservation statement for the quasigeostrophic potential vorticity, QGPV). Assume a β-plane.

   (b) Linearize this equation about a state of rest, assume a wave solution, and derive a dispersion relation for the Rossby waves on a β-plane.

   (c) Determine the phase and group velocities for the waves.

   (d) Show that the zonally averaged wave momentum transport $\overline{u'v'}$ is in the opposite direction of the group velocity of the waves. [The overline indicates a zonal average, i.e. in the x-direction.]

4. **Baroclinic instability**

The QG equations for a two-layer fluid are:

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0$$
$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0$$

where $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$. The potential vorticity in the two layer q_1 an q_2 are

$$q_1 = \nabla^2\psi_1 + \frac{k_d^2}{2}(\psi_2 - \psi_1) + \beta y$$
$$q_2 = \nabla^2\psi_2 + \frac{k_d^2}{2}(\psi_1 - \psi_2) + \beta y$$

with ψ_1 and ψ_2 the streamfunction in each layer. We assume the system is in a zonal channel (periodic in x) with rigid walls at y_1 to the south and y_2 to the north.

(a) Linearize the governing equations for a small disturbance on a background state with $-\partial\psi_1/\partial y = u_1 = U$ and $-\partial\psi_2/\partial y = u_2 = -U$.

(b) The wave activity density in each layer is

$$\mathcal{A}_i = \frac{\overline{q_i'^2}}{2d\overline{q_i}/dy}$$

where again the overline indicates an average in x. Derive the governing equation for $\mathcal{A}_1$ and $\mathcal{A}_2$.

(c) Show that the tendency for the sum $\mathcal{A}_1 + \mathcal{A}_2$ can be written as the divergence of a flux:

$$\frac{\partial(\mathcal{A}_1 + \mathcal{A}_2)}{\partial t} + \frac{\partial F}{\partial y} = 0.$$

What is F?

(d) Show that the total wave activity

$$\int_{y_1}^{y_2} (\mathcal{A}_1 + \mathcal{A}_2)\, dy$$

is conserved.

(e) Use the conservation of wave activity to obtain a necessary condition for instability. For what value of U is the flow stable?

GFD Written Examination

August 2008

1. **Isothermal atmosphere**

   Consider a hydrostatic atmosphere with a constant temperature T_0.

   (a) Determine the vertical profile for density ρ, pressure p and potential temperature θ.

   (b) Define the Brunt-Vaisala frequency. How is it related to convective instability?

   (c) Determine the Brunt-Vaisala frequency for an isothermal atmosphere.

   (d) What is the highest frequency for an internal gravity wave in a stratified atmosphere? Justify your answer.

2. **Gravity waves and Rossby waves**

   The two-dimensional shallow-water equations in a rotating frame are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + g\nabla h = 0,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0$$

   where the velocity $\boldsymbol{u} = (u, v)$ and the layer depth h are functions of (x, y, t).

   (a) Linearize the shallow-water equations about a state of rest; i.e. assume $\boldsymbol{u} = \boldsymbol{u}'$ and $h = H + h'$ with constant H and drop terms involving squares of primed quantities. Assuming a wave solution of the form $(u, v, h) = (u_0, v_0, h_0)e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$, derive the dispersion relation for inertia-gravity waves

$$\omega_g = 0 \text{ or } \pm c_g\sqrt{K^2 + L_d^{-2}},$$

   where $K^2 = k^2 + \ell^2$, $c_g = \sqrt{gH}$ and $L_d = c/f$.

   (b) Informally derive the shallow water quasigeostrophic equations, and state the basis for your approximations (three scaling assumptions are necessary). Linearize these about a state of rest, assume a wave solution, and thereby obtain the dispersion relation for Rossby waves

$$\omega_r = \frac{-k\beta}{K^2 + L_d^{-2}}$$

   (c) Sketch both dispersion relationships on the same axes, nondimensionalizing time with f and space with L_d.

   (d) Comment on the conditions necessary for a resonant interaction between these two waves. At what latitudes and/or spatial scales might this occur? You may find it useful to also define the equatorial deformation scale, $L_e = \sqrt{c_g/\beta}$.

3. **Geostrophic adjustment**

   The conservation law for the the shallow water potential vorticity (SWPV) is

$$\frac{Dq}{Dt} = 0, \quad \text{where} \quad q = \frac{\zeta + f}{h}, \quad \zeta = \hat{z} \cdot \nabla \times \boldsymbol{u}$$

(a) Show that when the flow is linearized about a state of rest, as in problem 2, the linear SWPV

$$q' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \frac{f}{H}h' \quad \text{satisfies} \quad \frac{\partial q'}{\partial t} = 0.$$

Now consider the y-independent linear adjustment problem defined by the following initial conditions at $t = 0$:

$$u_0' = v_0' = 0, \quad h_0' = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

(b) Show that as $t \to +\infty$ the final steady state satisfies

$$u' = 0, \quad g\frac{dh'}{dx} = fv', \quad \frac{dv'}{dx} - \frac{f}{H}h' = -\frac{f}{H}h_0'.$$

Hence derive that

$$h' = \begin{cases} -\exp(-x/L_d) + 1, & x > 0 \\ +\exp(+x/L_d) - 1, & x < 0 \end{cases}$$

Find v' and sketch h_0', h' and v'.

(c) Compute the disturbance energy contained in $x \in [-L, L]$

$$\mathcal{E}_L \equiv \frac{1}{2}\int_{-L}^{+L} \left(u'^2 + v'^2 + gh'^2\right) dx$$

for both the initial and final state. Show that the difference between these energies has a finite limit as $L \to \infty$. Where is the missing energy?

4. Shear instability

Assuming a constant depth $h = H$, the shallow water potential vorticity equation becomes the two-dimensional vorticity equation

$$\frac{D}{Dt}(\zeta + f) = 0.$$

What happens to the velocity field? Relate the vorticity and velocity field to a streamfunction ψ.

(a) Taking $f = f_o + \beta y$, linearize the vorticity equation about a mean zonal shear $U(y)$. Assuming a channel flow (periodic in x, walls at $y = 0$ and $y = 1$) and a wave solution of the form $\psi = \hat{\psi}(y)e^{ik(x-ct)}$, derive the the Rayleigh-Kuo equation,

$$(U - c)(\hat{\psi}_{yy} - k^2\hat{\psi}) + (\beta - U_{yy})\hat{\psi} = 0.$$

(b) Noting that $\hat{\psi}$ may be complex, derive an amplitude equation involving terms like $|\hat{\psi}|^2$. Integrating over the domain in y and separating the real and imaginary parts, derive a *necessary* condition for *instability.*

(c) Comment on the stability of eastward versus westward jets in this barotropic system: which are more stable?

GFD Written Examination

January 2010

1. Consider the shallow water quasigeostrophic (SWQG) system

$$\frac{\partial q}{\partial t} + J(\psi, q) + \beta \frac{\partial \psi}{\partial x} = 0, \qquad q = \nabla^2 \psi - F\psi$$

in a zonal channel.

(a) Show that $\psi = -Uy + A\cos(kx + \ell y - \omega t)$ is an exact solution to the *nonlinear* equations, so long as the appropriate dispersion relationship is satisfied. What is this dispersion relationship?

(b) Compute the westward phase speed of these (Rossby) waves. Note that the phase speed is not simply a Doppler shift of Rossby waves for a resting mean state — explain what is going on (consider the mean PV gradient and the mechanism of Rossby waves).

(c) For a given value of β (i.e. a given latitude) and a given mean flow speed, what wavelength will be stationary? Estimate the dimensional value of this wavelength and comment on its relevance to the observed structure of the jet stream.

2. Consider again the SWQG system in a zonal channel, but now linearized about a mean flow $U = U(y)$ (i.e. take $\psi = \psi' - yU$). Assuming a solution of the form $\psi'(x, y, t) = \text{Re}\ \hat{\psi}(y) \exp[ik(x - ct)]$, derive the amplitude equation

$$\hat{\psi}_{yy} - (k^2 + F)\hat{\psi} + \frac{\Pi}{U - c}\hat{\psi} = 0,$$

where $\Pi \equiv \beta + FU - U_{yy}$ is the mean meridional PV gradient (β has been included in the mean). Derive a necessary condition for shear instability in this flow. Comment on the physical interpretation. What does the term FU represent?

3. The first law of thermodynamics for a gas with constant composition is

$$\mathrm{d}Q = \mathrm{d}I + p\,\mathrm{d}\alpha,$$

where I is the internal energy per unit mass, $\mathrm{d}Q$ is the heat input per unit mass, p is the pressure and α is the specific volume (the inverse of the density, ρ).

(a) Show that for an ideal gas, the first law may be written

$$\mathrm{d}Q = c_p\,\mathrm{d}T - \alpha\,\mathrm{d}p, \tag{1}$$

where c_p is the specific heat at constant pressure. [Recall that for an ideal gas, $I \propto T$ and $p = \rho RT$, where R is the gas constant, $c_p = R + c_v$ and $c_v = (\partial I/\partial T)_\alpha$ is the specific heat at constant volume.]

(b) Using equation (??), show that in a hydrostatic atmosphere, the temperature at height z is

$$T(z) = T_s - \Gamma_d z,$$

where T_s is the surface temperature and $\Gamma_d = g/c_p$ ($c_p = 1004$ J kg^{-1} K^{-1} for Earth's atmosphere).

(c) Show that the pressure at height z in an adiabatic, hydrostatic atmosphere is

$$p(z) = p_s \left(\frac{T - \Gamma_d z}{T_s} \right)^{c_p/R}$$

where p_s is the surface pressure.

4. Air over the Sahara subsides continuously from above and is thus very dry and warm (this is the descending branch of the Hadley cell). The environmental lapse rate $\Gamma_e = \mathrm{d}T_e/\mathrm{d}z$ is less than the adiabatic lapse rate Γ_d, implying that energy is lost as parcels descend. Given a subsidence velocity w, derive an expression for the energy flux to space over the Sahara (in W/m^2). [Hint: note that the heat loss per unit *volume* is $\mathrm{d}H = \rho\,\mathrm{d}Q$, and use (??) to show that $\mathrm{d}H/\mathrm{d}t = \rho(c_p\Gamma + g)w$.]

   If the observed lapse rate is $\Gamma \approx 7$ K/km and the observed flux of heat to space is $J \approx 20$ W/m^2, what is the subsidence rate w?

5. Starting from the rotating shallow water equations, and assuming two inviscid layers overlying a flat bottom, but with a free upper surface, derive the two-layer shallow water system, and define the kinetic and available potential energies for it. Assuming small Rossby number, derive thermal-wind balance for this system.

GFD Written Examination

September 2010

Please attempt all four problems.

1. Shallow-water Hadley cell

The shallow water equations on a sphere can be written as

$$\frac{Du}{Dt} - \frac{uv}{a}\tan\phi - 2\Omega v \sin\phi = -\frac{g}{a}\frac{\partial h}{\partial \lambda}$$
$$\frac{Dv}{Dt} + \frac{u^2}{a}\tan\phi + 2\Omega u \sin\phi = -\frac{g}{a}\frac{\partial h}{\partial \phi}$$
$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\phi}\left(\frac{\partial(hu)}{\partial \lambda} + \frac{\partial(hv\cos\phi)}{\partial\phi}\right) = 0,$$

where

$$u = a\cos\phi\,\frac{D\lambda}{Dt} \quad \text{and} \quad v = a\,\frac{D\phi}{Dt}$$

are the zonal and meridional velocities, respectively, h is the layer thickness, a is the radius of the sphere, g is the gravitational acceleration, ϕ is the latitude, λ is the longitude, Ω is the rotation rate, and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{1}{a\cos\phi}u\frac{\partial}{\partial\lambda} + \frac{1}{a}v\frac{\partial}{\partial\phi}$$

is the material derivative.

(a) What is the expression for the angular momentum M with respect to the axis of rotation?

(b) Show that for a flow independent of longitude, the angular momentum M is conserved, i.e. $DM/Dt = 0$.

(c) What is the expression for the geostrophic balance in spherical coordinates?

(d) Consider now a zonally uniform flow, with constant angular momentum M_0 corresponding to solid body rotation at a latitude ϕ_0. Determine the zonal wind and thickness distribution, assuming a flow in geostrophic balance. Sketch the solution for $\phi_0 = 0°$ and $\phi_0 = 30°$. Discuss the qualitative behavior of these solutions from a physical point of view.

(e) In the atmosphere, the Hadley circulation can be approximated as an angular momentum conserving flow in the upper troposphere. Based on the results above, discuss what physical mechanisms can prevent the Hadley cell from expanding all the way to the polar regions.

2. **Inertial Oscillations**

   Consider the momentum equations for inviscid, two-dimensional flow in a rotating frame of reference on the f-plane. Linearize the equations about a state of rest. Neglect the pressure terms and determine the general solution to the resulting equations.

   (a) Show that the speed of fluid parcels is constant.

   (b) Show that the trajectory of the fluid parcels is a circle with radius $|U|/f$, where $|U|$ is the fluid speed.

   (c) Do the parcels circle clockwise or counterclockwise?

   (d) What is the period of oscillation of the fluid parcels? How does this compare with the period of rotation of the frame of reference, $2\pi/\Omega = 4\pi/f$?

3. **Baroclinic instability**

   The QG equations for a two-layer fluid are:

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0, \qquad q_1 = \nabla^2\psi_1 + \frac{k_d^2}{2}(\psi_2 - \psi_1)$$
$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0, \qquad q_2 = \nabla^2\psi_2 + \frac{k_d^2}{2}(\psi_1 - \psi_2)$$

   where $J(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$. Assume the lateral boundary conditions to be periodic in both x and y.

   (a) Linearize the governing equations about the background state with $-\partial\psi_1/\partial y = u_1 = U$ and $-\partial\psi_2/\partial y = u_2 = -U$. What are the mean PV gradients in each layer?

   (b) Write the linearized equations in terms of the barotropic and baroclinic streamfunctions, $\psi \equiv (\psi_1 + \psi_2)/2$ and $\tau = (\psi_1 - \psi_2)/2$, respectively. Then assume a plane-wave solution of the form $\psi' = \text{Re}\,\hat{\psi}\exp[i(kx + ly - \omega t)]$ (and likewise for τ), and compute the dispersion relationship.

   (c) For what wavenumbers is the flow stable and unstable? Relate the stability to the PV gradients, recalling the Charney-Stern theorem.

   (d) In the stable regime, the dispersion relation gives travelling waves. What are these waves, and what are their phase speeds? What mechanism allows their existence?

   (e) In the unstable regime, what is the energy source for the growing disturbance? If you can, compute the linear energy budget for the system and compare the relative amounts of available potential energy (APE) and kinetic energy (KE).

4. **Sverdrup balance**

Consider a two-dimensional, non-divergent, constant-density fluid on the β-plane, in a domain bounded by rigid walls with dimensions $[0, \pi] \times [0, \pi]$. The flow is initially at rest, but is forced by a stress of the form $\boldsymbol{\tau} = \hat{\boldsymbol{x}}\tau_0 \cos(y)$, and dissipated by linear drag. The momentum and mass equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + (f + \beta y)\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p + \boldsymbol{\tau} - r\boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

where f, β and r are positive constants, and $\boldsymbol{u} = u(x, y, t)\hat{\boldsymbol{x}} + v(x, y, t)\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{z}}$.

(a) Derive the advection equation for the vertical vorticity, $\zeta = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}$. [Hint: First convert the advection term using the identity $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \nabla |\boldsymbol{u}|^2/2$, then take the curl of the equation and use the identity $\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \boldsymbol{b} + \boldsymbol{a} \nabla \cdot \boldsymbol{b} - \boldsymbol{b} \nabla \cdot \boldsymbol{a}$.]

(b) Defining integration over the domain $\langle \cdot \rangle \equiv \pi^{-2} \int_0^\pi \int_0^\pi \cdot \; dx\, dy$, find the exact form of the integrated vorticity, $\langle \zeta \rangle$ as a function of time. What is it's value in the limit $t \to 0$? [Hint: integrate the vorticity equation, and remember that there is no flow through the boundaries, and that because mass is conserved, $\langle v \rangle = 0$. The solve then resulting ODE.]

(c) Assuming the drag r is very small, that the flow is in steady state, and that the nonlinear term in the vorticity equation may be neglected, determine the functional form of the velocity field, $\boldsymbol{u}(x, y)$. Sketch contours if possible. Explain the flow in terms of the Lagrangian conservation of total vorticity $\zeta + f + \beta y$. Can this velocity field be correct near the boundaries? What must happen there?

GFD Written Examination

December 2010

Please attempt all four problems.

1. **Atmospheric thermodynamics**

   Recall that for a dry ideal gas undergoing changes in temperature dT and specific volume $d\alpha$ (where $\alpha = 1/\rho$), the change in specific entropy dS is

   $$T\,dS = c_v\,dT + p\,d\alpha$$

   where p is the pressure and c_v is the specific heat at constant volume. An ideal gas obeys $p = \rho RT$, where R is the gas constant.

   (a) *Potential temperature* is the temperature a parcel of air would have if transported adiabatically from a reference pressure p_0 to a pressure p. Show that the potential temperature for dry air is

   $$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

   where $c_p = R + c_v$ is the specific heat at constant pressure.

   (b) Derive the functional form of the pressure p for a dry, hydrostatic atmosphere with constant potential temperature $\theta = T_0$. Show that such an atmosphere will have a discrete top at some height z_T, such that $p(z_T) = 0$. Give the expression for z_T.

2. **Internal waves**

Consider the rotating Boussinesq equations

$$u_t + \boldsymbol{v}\cdot\nabla u - fv = -\phi_x \tag{1}$$
$$v_t + \boldsymbol{v}\cdot\nabla v + fu = -\phi_y \tag{2}$$
$$w_t + \boldsymbol{v}\cdot\nabla w \quad = -\phi_z + b \tag{3}$$
$$u_x + v_y + w_z = 0 \tag{4}$$
$$b_t + \boldsymbol{v}\cdot\nabla b + N^2 w = 0 \tag{5}$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$ is the gradient operator, f is the (constant) Coriolis parameter and N is the (constant) buoyancy frequency. The total density is $\rho = \rho_0 + \hat{\rho}(z) + \rho'(x,y,z,t)$ and the pressure is $p = \hat{p}(z) + p'(x,y,z,t)$, with $d\hat{p}/dz = -g(\rho_0 + \hat{\rho})$. In terms of these variables, the buoyancy and pressure potential are $b = -g\rho'/\rho_0$ and $\phi = p'/\rho_0$, respectively, and the squared buoyancy frequency is $N^2 = -(g/\rho_0)d\hat{\rho}/dz$.

(a) Linearize these equations about a state of rest and derive a single wave equation for w,

$$\nabla^2 w_{tt} + f^2 w_{zz} + N^2 \nabla_H^2 w = 0. \tag{6}$$

Assuming a plane-wave solution $\propto \exp[i(kx+ly+mz-\omega t)]$, derive the dispersion relationship

$$\omega^2 = \frac{N^2(k^2+l^2) + f^2 m^2}{k^2+l^2+m^2}. \tag{7}$$

Notice that these waves are isotropic in the horizontal plane, so **for the rest of the problems, consider the 2D case with $l = 0$.** [Note: Defining an angle θ such that $k = \kappa\cos\theta$ and $m = \kappa\sin\theta$, we may also write the dispersion relation as $\omega^2 = N^2\cos^2\theta + f^2\sin^2\theta$.]

(b) Compute the group velocity $\boldsymbol{c} = c_x\hat{\boldsymbol{x}} + c_z\hat{\boldsymbol{z}}$. It's quickest to use implicit differentiation, and leave the answers in terms of ω. Show that *the group velocity is perpendicular to the phase velocity*, and that *the group velocity vanishes when $f = N$.*

(c) How are the horizontal and vertical phase and group velocities related in the cases $f < N$ and $f > N$? (i.e. in which cases are their signs are the same or opposite?). Which of these is more relevant to the atmosphere or ocean, by the way?

(d) Describe the structure of the non-rotating wave ($f = 0$) when (i) $k = 0$ and (ii) $m = 0$.

(e) How does the dispersion relation change if hydrostatic balance is assumed from the beginning? Comparing to the full dispersion relationship above, when is hydrostatic balance a good approximation? Consider this question both with $f \neq 0$ and with $f = 0$.

3. Lee waves

Using the 2D non-rotating Boussinesq equations in the x-z plane (just as in problem 2 but with $f = 0$ and $l = 0$), consider a mean zonal current U flowing over topography $z_b(x) = h\cos(kx)$, where U, h and k are all fixed, positive constants. Instead of re-linearizing about this mean flow, use the same linearization as in problem 2, but transform to a coordinate system moving with the flow, which yields time-dependent topography $z_b(x + Ut) = h\cos[k(x + Ut)]$. Thus internal waves of zonal wavenumber k and frequency $\omega = -Uk$ are continuously forced by the boundary.

(a) Show that in the limit of small topographic height h, the bottom boundary condition becomes $w|_{z=0} = Ukh\sin(kx - \omega t)$.

(b) Remembering that ω and k are fixed by the forcing, show that the dispersion relation then becomes

$$m = \pm\sqrt{\frac{N^2}{U^2} - k^2}.$$

What sign must m have to be a physical solution? [Hint: consider the vertical group velocity computed in the previous problem.]

(c) Describe the structure of the wave in the two cases $k > N/U$ and $k < N/U$, and explain these results in physical terms.

* In the latter case, at what rate is momentum transferred upward by the waves? Supposing this momentum is dissipated somewhere, what will this do to the mean flow? [This part is harder than the others — skip it unless you have plenty of time.]

4. **Barotropic stability of an axisymmetric vortex**

The two-dimensional vorticity conservation law is

$$\zeta_t + J(\psi, \zeta) = 0, \qquad \zeta = \nabla^2\psi.$$

Working in polar coordinates (r, θ), consider perturbations $\psi'(r, \theta, t)$ about an axisymmetric basic state $\overline{\psi}(r)$, so that the full streamfunction is $\psi(r, \theta, t) = \overline{\psi}(r) + \psi'(r, \theta, t)$, with $|\psi'| \ll |\overline{\psi}|$.

Show that a necessary condition for instability is that $\partial_r\overline{\zeta}$ must change sign somewhere in the domain.

Note that in cylindrical coordinates, the relevant operators are

$$J(A, B) = r^{-1}(A_r B_\theta - A_\theta B_r),$$
$$\nabla^2 A = r^{-1}\partial_r(rA_r) + r^{-2}A_{\theta\theta},$$

the velocities are $u = \boldsymbol{u}\cdot\hat{\theta} = \psi_r$ and $v = \boldsymbol{u}\cdot\hat{r} = -r^{-1}\psi_\theta$, and the boundary conditions are that $u|_{r=0} = u|_{r\to\infty} = 0$.

[Hint: Assume a normal mode solution of the form $\psi' = \mathcal{R}\{\hat{\psi}(r)e^{i(m\theta - \omega t)}\}$, then compute an amplitude equation by multiplying by $r\hat{\psi}^*$ and integrating in r. This is another example of the Rayleigh inflection point criterion, but now for an axisymmetric vortex rather than a zonal jet.]

GFD Written Examination

January 2012

Please attempt all four problems.

1. **Coriolis Acceleration**

   In a motionless, non-rotating atmosphere, an airplane stays aloft because the lift generated by the wings balances gravity. In this case, its wings are level — parallel to the ground. On Earth, in order for a jet to fly along a line of constant latitude, its wings must be slightly tilted to counteract the deflection due to the Coriolis acceleration. Assuming a speed $U > 0$ (eastward) and latitude $\theta > 0$ (northern hemisphere), derive an expression for the angle γ the plane must be tilted from upright (the $\hat{z}$ direction) to maintain its latitude. [*Hint:* Remember that 'up,' the $\hat{z}$ direction, already includes the centrifugal term. A force diagram may be helpful.]

2. **Generalized Taylor-Proudman Theorem**

   The momentum equation for an inviscid fluid of constant density ρ_0 on a rotating planet is $Dv/Dt + 2\Omega \times v = -\nabla\phi$, where Ω is the (constant) planetary rotation vector and $\phi = p/\rho_0$. In the limit of small Rossby number, show that the fluid will tend to move in columns *aligned with the rotation axis.*

3. **Instability of stratified shear flow**

   Show that a sufficiently stratified fluid can suppress shear instability. Consider the Boussinesq equations in the x-z plane,

$$u_t + uu_x + wu_z = -\phi_x \tag{1}$$

$$w_t + uw_x + ww_z = -\phi_z + b \tag{2}$$

$$u_x + w_z = 0 \tag{3}$$

$$b_t + ub_x + wb_z + wN^2 = 0 \tag{4}$$

   where $b = -g\rho/\rho_0$ and $\phi = p/\rho_0$, and $\bar{b}_z = N^2$ is the mean stratification. Linearize these about a mean shear $\bar{u}(z)$, introduce a streamfunction such that $u = \psi_z$ and $w = -\psi_x$, seek a plane wave solution $\propto \exp[ik(x-ct)]$, and derive the Taylor-Goldstein equation,

$$(\bar{u}-c)(\psi_{zz} - k^2\psi) + \left(\frac{N^2}{\bar{u}-c} - \bar{u}_{zz}\right)\psi = 0.$$

   Take the boundary conditions to be $\psi(0) = \psi(H) = 0$. By deriving an amplitude equation, show that a necessary condition for instability is that the Richardson number

$$\mathrm{Ri} < \frac{1}{4}, \quad \text{where} \quad \mathrm{Ri} = \frac{N^2}{(\bar{u}_z)^2}.$$

   [*Hint:* You may find it useful to introduce $\Psi = \sqrt{\bar{u}-c}\,\psi$, multiply by Ψ^*, then derive a condition under which the imaginary part of the wavespeed may be non-zero.]

4. (a) **Surface QG:** Consider the quasigeostrophic equations in a horizontally periodic, vertically semi-infinite domain, with a flat boundary at $z = 0$ and constant stratification N:

$$
\begin{aligned}
q_t + J(\psi, q) &= 0, \quad z > 0, \\
b_t + J(\psi, b) &= 0, \quad z = 0, \\
\psi &\to 0 \ \text{ as } \ z \to \infty,
\end{aligned}
$$

where $q = \nabla^2\psi + s^2\psi_{zz}$ is the QG potential vorticity, $b = f\psi_z|_{z=0}$ is the surface buyoancy, and $s = f/N$. We've assumed $\beta = 0$. Show that if q is identically 0, then the interior flow is determined by b, with a vertical structure that decays exponentially away from the surface with a decay scale that decreases with horizontal scale, so that small horizontal scales are trapped near the surface. What is the decay scale? [Use a Fourier expansion in the horizontal and express the decay scale in terms of the wavenumber modulus K.]

(b) **Baroclinic instability:** Now assume β is non-zero and impose a mean shear $-\Psi_y = U(z) = \Lambda z + \Gamma z^2$, where Λ, Γ and β are all positive. Compute the mean buoyancy and PV. Under what conditions on might the flow be baroclinically unstable? You may use the Charney-Stern theorem (i.e. you need not derive it, but you may if you need to). Comment on the role of β here — how does it contrast with its role in the two-layer baroclinic instability problem? [Recall that in the two-layer problem, a sufficiently large β will shut off the instability entirely].

GFD Written Examination

August 2012

Please attempt all four problems.

1. **Geostrophic Adjustment in a Reduced Gravity model**

   Consider a simplified model for the ocean, consisting of a layer of moving fluid with density ρ, bounded above by a rigid lid, and overlying a deep abyss of motionless water with density $\rho + \Delta\rho$, where $\Delta\rho \ll \rho$.

   (a) Assuming no variation in Coriolis parameter f (i.e. the f-plane), argue that the equations describing this system are the reduced gravity shallow water equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

   where (u, v) are the horizontal velocities, h is the depth of the layer, and $g' = g\Delta\rho/\rho$ is the reduced gravity. Note that h may be 0 if there is an outcrop.

   (b) Show that this system conserves the following form of potential vorticity (PV),

$$Q = \frac{f + \partial v/\partial x - \partial u/\partial y}{h}.$$

   (c) Consider now an idealization of an extreme warming event over half ($x < 0$) of a resting ocean. The initial state has $h = H$ (constant) for $x < 0$, and $h = 0$ for $x > 0$ (thus the lower layer 'outcrops' at $x = 0$), *with no variation in* y, and $u = v = 0$. The front will slump, and after a long time will evolve to a new steady state. Assuming $u, v \to 0$ and $h \to H$ as $x \to -\infty$, use conservation of PV to derive steady, long-time solutions for v and h, denoting the new outcrop position at $x = a$. Sketch the solutions. What is the length-scale that characterizes the final state? Use volume conservation $\int_{-\infty}^{0}(H - h)\,dx = \int_0^a h\,dx$ to find a.

   (d) Compute the initial and final state available potential energies and final state kinetic energy. Did the adjustment conserve energy? If not, where did the extra energy go?

2. Reduced gravity SWQG model

(a) Starting from the above shallow water equations, but now assuming $f = f_0 + \beta y$, the β-plane shallow water quasigeostrophic (SWQG) system is

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad \text{where} \quad q = \nabla^2\psi - R^{-2}\psi + \beta y$$

and $R = \sqrt{g'H}/f$ is the internal deformation scale. What three scaling assumptions are needed to derive these equations?

(b) Show that $\psi = -Uy + A\cos(kx + \ell y - \omega t)$ is an exact solution to the *nonlinear* equations, so long as the appropriate dispersion relationship is satisfied. What is this dispersion relationship for these Rossby waves?

(c) Compute the westward phase speed of the Rossby waves. Note that the phase speed is not a simple Doppler shift of Rossby waves in a resting mean state — explain what is going on (consider the mean PV gradient and the mechanism of Rossby waves).

(d) Show that any localized disturbance will move westward at the long-Rossby-wave speed, i.e. show that

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = -\beta R^2 \quad \text{where} \quad \boldsymbol{X} = \frac{\iint x\psi \, dx}{\iint \psi \, \mathrm{d}x}$$

3. Lamb Wave

Consider an isothermal atmosphere in hydrostatic balance.

(a) Determine the vertical profile for density and pressure.

(b) Write the linearized equations of motion governing the evolution of a small perturbation.

(c) Obtain the vertical structure and dispersion relationship associated with the eigenmodes that have no vertical velocity ($w = 0$), and offer a physical interpretation of these modes.

4. **Hurricane Inflow**

Consider an axisymmetric atmosphere on the f-plane. The equations of motions in cylindrical coordinates are

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + f v_\theta$$
$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} - f v_r$$

where v_θ is the tangential velocity component and v_r is the radial velocity component.

(a) Show that the angular momentum

$$M = v_\theta r + \frac{f r^2}{2}$$

is conserved following a parcel trajectory.

(b) Determine the tangential velocity of an air parcel initially at rest at $r = r_0$ as it moves toward the center, conserving its angular momentum.

(c) If the angular momentum of the flow is uniform, density is uniform and the flow is in gradient wind balance (i.e. neglect the advection and time-derivative terms in the equation for v_r), determine the pressure field as function of the radius.

(d) Take $f = 10^{-4}\ \mathrm{s}^{-1}$, $\rho = 1\ \mathrm{kg\ m}^{-3}$, $r_0 = 10^6$ m and $p(r_0) = 10^5$ Pa. Determine the distances at which the pressure reaches 90000 Pa and 0 Pa. Comment on the possibility of $p = 0$.

GFD - Written exam - Janaury 2013

1. Discuss briefly (1 short paragraph) the following concepts:
   (a) potential temperature
   (b) potential vorticity
   (c) barotropic instability
   (d) baroclinic instability
   (e) geostrophic adjustment

2. Consider the inviscid, adiabatic Boussinesq equations on an f-plane,

$$\begin{aligned}\frac{D\vec{v}}{Dt} + \vec{f_0} \times \vec{v} &= -\nabla\phi + b\hat{k} \\ \nabla \cdot \vec{v} &= 0 \\ \frac{Db}{Dt} + N^2 w &= 0,\end{aligned}$$

where N^2 captures the stratification of the background state.

(a1) Linearize the equations around a background state at rest.

(a2) Consider wave-like solutions to the linearized equations, where, for example, the perturbation vertical velocity w' can be written

$$w' = \mathrm{Re}\, \tilde{w} \exp[i(kx + mz - \omega t)].$$

Compute the dispersion relation for these waves.

(b1) Discuss the stationary solutions ($\omega = 0$ or $\partial/\partial t = 0$) of the linearized equations. What are the key properties of such a flow and what are the relationships between the velocity, pressure and buoyancy fields.

(c1) Consider the case when $N^2 > 0$. What type of motions are these? What is the structure associated with these, and what are the restoring mechanisms?

(c2) Still for $N^2 > 0$, what is the range of value that ω can take? (Hint, does the frequency depend on f_0 and N?) What would happen if the system is forced by oscillatory motions at a frequency outside this range (discuss qualitatively)?

(d1) Consider noes the case when $N^2 < 0$. Describe the non-stationary solutions. Where in the atmosphere and ocean would you expect to find unstable cases (that is, what physical processes could lead to unstable conditions)?

(e1) What is hydrostatic balance (HSB)? If we assume that the system is in HSB, how will it affect the equations and the dispersion relation?

(e2) For what scale waves does HSB appear to have a very signficant impact on the behavior of the waves. From a physical standpoint, do you think that HSB is valid in this limit? If not, why?

3. Consider the hydrostatic Bousinessq equations

$$\begin{aligned}
\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} &= -\nabla\phi \\
\frac{\partial\phi}{\partial z} &= b \\
\nabla \cdot \vec{v} &= 0 \\
\frac{Db}{Dt} + N^2 w &= 0,
\end{aligned}$$

on an β-plane, so that $\vec{f} = (f_0 + \beta y)\hat{k}$. For simplicity, assume N^2 is constant.

(a) One of the goals of this question will be to derive the "Quasi-Geostrophic" (QG) equations for this system. Assuming that $\vec{u} \sim U\hat{u}$, $(x, y) \sim L(\hat{x}, \hat{y})$, and $z \sim H\hat{z}$, determine the scaling for w, ϕ, and b. For w, just use mass conservation for now. Assume that βy scales with the relative vorticity.

(b) Nondimenionalize the equations, and show that a natural small parameter, the Rossby number Ro, comes about. This will allow you to consider an expansion of our variables in order Ro, i.e.

$$\hat{u} = \hat{u}_0 + Ro\, \hat{u}_1 + ...$$

You will also need to assume something about the length scales of the system to make things work. How should L scale in terms of the system parameters?

(c) What does the zeroth order horizontal momentum equation tell you? What does the zeroth order thermodynamic (buoyancy) equation tell you?

(d) To derive a close system for the zeroth order flow, consider the equations at first order. Take advantage of the fact that the zeroth order velocity is nondivergent to derive a closed system in terms of just the zeroth order variables.
Hint: The fact that the zeroth order flow is non-divergent allows you to express everything in terms of a single variable, the streamfunction, ψ.

(e) Explain the physical principles that allow you to described the QG flow in terms of a single equation? (*Hint:* What is conserved?)

(f) The zeroth order flow of the QG system is non-divergent, so that there is no vertical velocity at leading order. Derive an elliptic equation for the O(Ro) vertical velocity w in terms of the zeroth order flow.
Hint: In the asymptotic expansion used to solve for the zeroth order terms, we had to eliminate w_1 from the vorticity and buoyancy equations. Use these to get the first order vertical velocity.

(g) The equation for the vertical velocity in pressure coordinates is denoted by the symbol ω (not to be confused with frequency or the rotation of the Earth!), so that the vertical velocity equation is known as the "omega-equation." Early weather forecast models used the QG equations. Why would meterologist want to be able to know the vertical velocity (besides just pure academic interests)?

GFD Written Examination

August 2013

Please attempt all four problems.

1. (Gravity waves) Consider a fluid governed by the Boussinesq equation:

$$\frac{d\mathbf{V}}{dt} = -\nabla p + b\mathbf{k} \tag{1}$$

$$\frac{db}{dt} = 0 \tag{2}$$

$$\nabla \cdot \mathbf{V} = 0 \tag{3}$$

Here, $\mathbf{V}$ is the three dimensional velocity field, p is the pressure, b is the buoyancy and $\mathbf{k}$ is the unit vector in the vertical direction.
a. Linearize the equation of motions with respect to a reference state at rest with buoyancy $\overline{b}(z) = N_0^2 z$, where N_0 is a constant.
b. Determine the dispersion relation for small perturbation on this reference state.
c. Determine the spatial structure of theses waves, i.e. how the various field u',v',w',b' and p' are related to each other and to the wave number and propagation speed.
d. Determine the group velocity of the waves. How is it related to the propagation speed?

2. (Inertial Oscillations) Rotation adds a term $f\hat{\mathbf{z}} \times \mathbf{v}$ to the left hand side of the momentum equation (1) in the first problem:

$$\frac{d\mathbf{V}}{dt} = f\hat{\mathbf{z}} \times \mathbf{v} - \nabla p + b\mathbf{k} \tag{4}$$

In the limit of strong rotation and uniform density ($b = 0$), one can neglect variations in z, and obtain the shallow water equations (a form of which are given in the next problem).
a. Assuming constant f and a flat bottom, derive the shallow water equations. Discuss the pressure term and its relationship to the fluid thickness.
b. Linearize the equations about a state of rest and derive the dispersion relation.
c. Show that in the long-wave limit, the dispersion relation is approximately $\omega = f$. Explain the mathematical meaning of 'long-wave limit'.
d. Describe the motion in this limit.

3. (Equatorial waves) Consider the shallow water equations on an equatorial β-plane:

$$\frac{du}{dt} = \beta y v - g\frac{\partial h}{\partial x} \tag{5}$$

$$\frac{dv}{dt} = -\beta y u - g\frac{\partial h}{\partial y} \tag{6}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0 \tag{7}$$

Here, (u, v) is the horizontal velocity, h is the fluid thickness and g is the gravitational acceleration. The Coriolis parameter $f = \beta y$ is proportional to the distance from the equator, which is located at $y = 0$.
a. Linearize the equations of motion for an atmosphere of thickness H_0 at rest.
b. We will seek a special solution of the linearized equations. For this, assume a solution of the form

$$u'(x, y, t) = \Re(U(y)exp(ikx - \omega t)) \tag{8}$$

$$v' = 0 \tag{9}$$

$$h'(x, y, t) = \Re(H(y)exp(ikx - \omega t)) \tag{10}$$

in the equations. First, combine the equations for u' and h' to obtain a wave equation. What is the corresponding propagation speed? Use it to determine a relationship between $U(y)$ and $H(y)$.
c. Consider now the linearization of equation (6) and recall our assumption about the perturbation meridional velocity v'. By injecting the solution from (b), you should obtain an ODE for $U(y)$. Solve this ODE.
d. Discuss the nature of the solutions. What is the structure of the solutions? Are both eastward and westward propagating solutions equally valid?

4. Air over the Sahara subsides continuously from above and is thus very dry and warm (this is the descending branch of the Hadley cell). The environmental lapse rate $\Gamma_e = dT_e/dz$ is less than the adiabatic lapse rate Γ_d, implying that energy is lost as parcels descend. Given a subsidence velocity w, derive an expression for the energy flux to space over the Sahara (in W/m^2). [Hint: use the thermodynamic equation

$$\frac{dH}{dt} - \frac{dp}{dt} = Q, \tag{11}$$

with Q the heating rate per unit *volume*, and take advantage of the hydrostatic balance to relate the pressure change dp/dt to the vertical velocity w.] If the observed lapse rate is $\Gamma \approx 7$ K/km, air density is 0.5 kg/m^3, heat capacity is 1000J/kg and the observed flux of heat to space $J \approx 30$ W/m^2 is spread uniformly over the entire depth of the atmosphere (10km) , what is the subsidence rate w?

GFD Written Examination

January 2014

Please attempt all four problems.

1. (Topographic waves) The f-plane shallow water quasigeostrophic (QG) model in the presence of bottom topography is

$$q_t + J(\psi, q) = 0, \qquad q = \nabla^2\psi - k_d^2\psi + \frac{f}{H_0}\eta_b(x, y)$$

where ψ is the streamfunciton, k_d^{-1} is the deformation radius, f is the (constant) Coriolis parameter, η_b is the bottom topography, H_0 is the mean thickness and $J(\psi, q) = \psi_x q_y - \psi_y q_x$. We take the topography to be a simple slope

$$\eta_b = \gamma x.$$

(a) What conditions must γ satisfy to be consistent with the assumptions of the QG approximation?

(b) Linearize the governing equation for a small perturbation, and obtain the dispersion relationship for a perturbation of the form $\psi' = \hat{\psi}\exp[i(kx + ly - \omega t)]$. What is the direction of propagation for these waves?

(c) Determine the group velocity of these waves. Hoe does it compare to the phase velocity?

(d) Draw a schematic representation of these waves that explains their phase velocity, and relate them to other waves that can arise in QG.

2. (Energy conservation) We start from the Boussinesq equations

$$\frac{D\mathbf{v}}{Dt} = -\nabla\phi + b\mathbf{k}$$
$$\frac{Db}{Dt} = 0$$
$$\nabla \cdot \mathbf{v} = 0$$

where $\mathbf{v} = (u, v, w)$ is the three-dimensional velocity, $D/Dt = \partial_t + \mathbf{v} \cdot \nabla$ is the advective derivative, ϕ/ρ_0 is the pressure potential, $b = -g\rho/\rho_0$ is the buoyancy and ρ_0 is a reference density.

(a) Obtain an expression for the rate of change of the kinetic energy $|\mathbf{v}|^2/2$ and show that its generation is associated with a vertical transport of buoyancy.

(b) Derive the conservation equation for the total energy $E = |\mathbf{v}|^2/2 + bz$.

(c) Consider now a small perturbation (u', v', w', b', ϕ') about a rest state with a background buoyancy profile $B(z) = N^2 z$. Write the linearized equations. Now consider an oscillatory solution of the form

$$(u', v', w', b', \phi') = (\hat{u}, \hat{v}, \hat{w}, \hat{b}, \hat{\phi})e^{i(kx+ly+mz-\omega t)}$$

and show that the domain-averaged buoyancy flux is always zero.

(d) Show that for a growing mode, with imaginary frequency $\sigma = \Im(\omega) > 0$, one can only have a positive contribution from the buoyancy flux if the background buoyancy decreases with height ($N^2 < 0$).

3. (Residual circulation) The QG equations for a two-layer fluid with equal layer thicknesses H are

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \quad i = 1, 2,$$

where the potential vorticity in each layer is

$$q_1 = \nabla^2 \psi_1 + \frac{k_d^2}{2}(\psi_2 - \psi_1) + \beta y$$
$$q_2 = \nabla^2 \psi_2 + \frac{k_d^2}{2}(\psi_1 - \psi_2) + \beta y,$$

with ψ_1 and ψ_2 the stream functions in each layer. We assume the system is in a zonal channel (periodic in x) with rigid walls at $y = +L$ and $y = -L$.

(a) Consider first a steady flow given by $u_1 = U = -u_2$. Determine the streamfunction, thickness and potential vorticity distribution of this flow.

(b) Linearize the equations of motion for a small perturbation about this steady state.

(c) The eddy potential enstrophy in each layer is defined as

$$\mathcal{Z}_i = \frac{\overline{q_i'^2}}{2},$$

where the overline $\overline{(\cdot)}$ indicates an average in x. Derive the governing equations for $\mathcal{Z}_1$ and $\mathcal{Z}_2$.

(d) Show that for a growing mode, the potential vorticity flux in each layer $\overline{v_i' q_i'}$ must be down the potential vorticity gradient.

(e) The thickness perturbation in each layer is is given by

$$h_1 = \frac{k_d^2 H}{2 f_0}(\psi_2 - \psi_1) = -h_2.$$

Why? Show that the potential vorticity flux is proportional to the mass transport, i.e.

$$\overline{v_i' q_i'} = \frac{2 f_0}{H} \overline{v_i' h_i'}.$$

(f) Using this result, explain how an unstable mode would affect the zonal mean mass distribution. [Bonus: How does this affect the zonal wind?]

4. (Forced shallow water motion) The rotating shallow water equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f\mathbf{k} \times \mathbf{u} = -g' \nabla \eta$$
$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(H + \eta)\mathbf{u}] = 0$$

where $\mathbf{u} = (u, v)$ is the horizontal velocity, H is the (constant) resting depth, η is the deviation height (so that $H+\eta$ is the thickness of the fluid), f is the (constant) Coriolis parameter, and g' is the reduced gravity.

(a) Assume there is a mass source (or sink) $S(x, y, t)$ with units mass/(area x time), injecting water of the same density ρ_0 as the shallow water fluid. What is the form of the forcing to the right hand side of the height equation?

(b) Linearize the forced equations about a state of rest. Assume the forcing $S = S(y)$, independent of x and t, so that derivatives with respect to x vanish. In addition, assume $\partial v / \partial t = 0$. What do you notice about the two velocity components?

(c) Derive the equation for the perturbation height

$$\eta - L_d^2 \frac{\partial^2 \eta}{\partial y^2} = \frac{S(y)}{\rho_0} t.$$

where $L_d = \sqrt{g'H}/f$ is the deformation radius.

(d) Assuming $\eta(y, t = 0) = 0$, find a solution for $\eta(y, t)$ when there S is a mass sink distributed over a strip of width $2L$, i.e. assume

$$S(y) = \begin{cases} -S_0 & |y| < L \\ 0 & |y| > L \end{cases}$$

with $S_0 > 0$. Comment on the structure of the solution.

(e) Find a solution in the extreme case of $L \ll L_d$, i.e. take $S(y) = 2LS_0\delta(y)$. Comment on the structure of the solution.

GFD Written Examination

August 2014

Please attempt all four problems.

1. Shallow water

(a) Starting from the rotating hydrostatic primitive equations for a constant-density fluid of mean depth H_0, derive the rotating shallow water equations

$$\begin{aligned}\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f_0 \hat{z} \times \boldsymbol{u} &= -g \nabla \eta \\ \partial_t \eta + \nabla \cdot (h \boldsymbol{u}) &= 0,\end{aligned}$$

where $h(x, y, t) = H_0 + \eta(x, y, t)$.

(b) Show that these conserve forms of potential vorticity, mass and energy.

(c) Linearize the equations about a state of rest, and find the dispersion relation for shallow water gravity waves. Find their phase and group velocities. Also, what is the meaning of the 0 frequency solution?

(d) Now find the dispersion relation for linear shallow water Kelvin waves along a straight north-south boundary at $x = 0$ by looking for solutions with $u = 0$. Describe the wave physically. What is the typical width of the wave?

2. **Jets**

Consider the shallow water quasigeostrophic (SWQG) equations on the β-plane, unbounded in y and periodic in x. The SWQG potential vorticity (PV) is $q = \beta y + \nabla^2\psi - L_D^{-2}\psi$, where ψ is the streamfunction. Now suppose an initial PV distribution $q|_{t=0} = \beta y$ is acted on by a localized stirring around $y = 0$, producing a perfectly homogenized layer of PV in the region $-b < y < b$, and leaving the PV equal to its initial state outside this band of width $2b$. The integral of the stirring over all space is zero, so no PV is injected into the flow.

Find the zonally-averaged velocity $\overline{u}(y) = -\partial_y\overline{\psi}$ associated with this final-state PV distribution. Remember that the streamfunction and velocity must both be continuous at all points, even though the PV is not. Write your answer in terms of the velocity scale $u_r = \beta L_D^2$. Sketch your solution. Comment on the asymmetry between eastward and westward jets in your solution.

Show that in the limit $b/L_D \to 0$, the velocity is approximately $\overline{u} \approx (\beta/2)(y^2 - b^2)$ for $|y| < b$.

3. Starting from the barotropic vorticity equations, and assuming periodicity in x, derive the "Taylor identity," a relation between the vorticity flux and the momentum flux given by

$$\overline{v'\zeta'} = -\partial_y \overline{u'v'}$$

where an overbar denotes a zonal average, primes are deviations from the zonal average and $\zeta' = \partial_x v' - \partial_y u'$ is the vorticity in the $\hat{z}$ direction. From this, show that the mean flow $\overline{u}$ is driven by the vorticity flux, i.e. $\partial_t \overline{u} = \overline{v'\zeta'}$.

4. **Two-D Boussinesq equations**

The Boussinesq equations are

$$\frac{D\boldsymbol{v}}{Dt} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla\phi + b\hat{\boldsymbol{z}}$$
$$\frac{Db}{Dt} = 0$$
$$\nabla \cdot \mathrm{v} = 0$$

where $\boldsymbol{v} = (u, v, w)$ and $\boldsymbol{u} = (u, v)$ are three- and two-dimensional velocities, $D/Dt = \partial_t + \mathrm{v} \cdot \nabla$ is the advective derivative, ϕ/ρ_0 is the pressure potential, $b = -g\rho/\rho_0$ is the buoyancy, $\hat{\boldsymbol{z}}$ is a unit vector in the vertical direction and ρ_0 is a reference density.

Assuming no variations in the x direction, write the equations in terms of buoyancy and vorticity in the $\hat{x}$ direction. Show that the vorticity is driven by lateral gradients in buoyancy.

Show that if the equation for u is retained, the vorticity is driven by ageostrophic deviations from hydrostatic balance $f\partial_z u + \partial_y b$.

GFD Written Examination

January 2015

Please attempt all four problems.

1. **Hurricane wind**

The shallow water equations in polar coordinates (r, ϕ) can be written as

$$\frac{Du}{Dt} + \frac{uv}{r} + fv = -\frac{g}{r}\frac{\partial h}{\partial \phi} \tag{1}$$

$$\frac{Dv}{Dt} - \frac{u^2}{r} - fu = -g\frac{\partial h}{\partial r} \tag{2}$$

where

$$u = r\,\frac{D\phi}{Dt} \quad \text{and} \quad v = \frac{Dr}{Dt}$$

are the azimuthal and radial velocities, respectively, h is the layer thickness, a is the radius of the sphere, g is the gravitational acceleration, ϕ is the azimuthal angle, r is the distance from the origin, $f = 2\Omega$ is the Coriolis parameter, and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{1}{r}u\frac{\partial}{\partial \phi} + v\frac{\partial}{\partial r}$$

is the material derivative.

(a) Show that for an axisymmetric flow (for which there is no dependence on the azimuthal angle ϕ), the quantity $M = ur + fr^2/2$ is conserved along the flow trajectories (i.e. $DM/Dt = 0$).

(b) Provide a physical interpretation for M and its two components ur and $fr^2/2$.

(c) Consider now a steady flow with

$$M = 0 \quad \text{for } r < r_0$$
$$u = 0 \quad \text{for } r > r_0.$$

Find the thickness profile $h(r)$ associated with this flow and sketch the solution for u and h. [This would occur for example if you have a point mass source at $r = 0$, which can be thought of as air being injected into the upper troposphere within a narrow hurricane eyewall.]

(d) Could you use the quasi-geostrophic approximation to study this flow? Justify your answer.

(e) Repeat the computation as in (c) but assume

$$M = \frac{1}{2}fr_0^2 \quad \text{for } r < r_0$$
$$u = 0 \quad \text{for } r > r_0.$$

Discuss what happens near the origin. [This would occur for example if you have a point mass sink at $r = 0$, which can be thought of as air leaving the lower troposphere within a narrow hurricane eyewall.]

2. **Rayleigh-Kuo criterion for QGSW**

The QG Shallow Water equations are:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad q = \nabla^2\psi - k_d^2\psi + \beta y$$

where $J(f, g) = f_x g_y - f_y g_x$. The velocity field is related to the streamfunction by $u = -\psi_y$ and $v = \psi_x$. Assume the lateral boundary conditions to be periodic in both x and y.

(a) Linearize the equations about a background state given by $u = \overline{U}(y)$, $v = 0$. What is the background PV gradient $\overline{q}_y$?

(b) Consider a perturbation of the form

$$\psi'(x, y, t) = \tilde{\psi}(y) \exp[ik(x - ct)].$$

Derive the Rayleigh equation for this system. You should get a a second order ODE of the form:

$$\frac{d^2\tilde{\psi}}{\partial y^2} = F(\overline{U}, \overline{q}_y, c, k)\tilde{\psi}.$$

(c) Obtain a necessary conditions for instability based on this ODE .

3. Rossby edge waves

Consider the quasigeostrophic equations in a horizontally periodic, vertically semi-infinite domain, with a flat boundary at $z = 0$ and constant stratification N,

$$\begin{aligned} &q_t + J(\psi, q) = 0, \quad z > 0, \\ &b_t + J(\psi, b) = 0, \quad z = 0, \\ &\psi \to 0 \quad \text{as} \quad z \to \infty, \end{aligned}$$

where $q = \beta y + \nabla^2\psi + s^2\psi_{zz}$ is the QG potential vorticity, $b = f\psi_z|_{z=0}$ is the surface buyoancy, and $s = f/N$. Horizontal velocities are given by geostrophic balance, $(u, v) = (-\psi_y, \psi_x)$.

(a) Given a background zonal mean velocity

$$U(z) = \Lambda\left(z + \frac{z^2}{2H}\right)$$

(where H is an arbitrary depth scale) compute the background meridional surface buoyancy and PV gradients, B_y and Q_y, respectively (include the β term in Q_y). Find the surface shear Λ_0 such that $Q_y = 0$.

(b) Setting $\Lambda = \Lambda_0$ to ensure $Q_y = 0$, and assuming no initial PV, $q|_{t=0} = 0$, the PV will remain 0 for all time. Use this assumption to derive the vertical structure and dispersion relation for plane waves on this background flow; i.e. linearize about the mean state, and assume a solution of the form $\psi(x, y, z, t) = \Re\left[\hat{\psi}(z)\exp(kx + \ell y - \omega t)\right]$, and find $\hat{\psi}(z)$ and an expression for ω. Describe in words the vertical structure of these waves, including their dependence on horizontal scale, and discuss any length scales that arise.

(c) Find the phase and group velocities for these waves, and show that they can only propagate energy westward, i.e. in the negative x direction (by contrast, sufficiently short standard QG Rossby waves can propagate energy eastward).

(d) Assuming arbitrary Λ, under what conditions on might the flow be baroclinically unstable? Comment on the role of β here — how does it contrast with its role in the two-layer baroclinic instability problem? [Recall that in the two-layer problem, a sufficiently large β will shut off the instability entirely].

4. TEM

Consider the hydrostatic primitive equations with constant stratification,

$$\begin{aligned} D_t u - fv &= -\phi_x \\ D_t v + fu &= -\phi_y \\ 0 &= -\phi_z + b \\ u_x + v_y + w_z &= 0 \\ D_t b + N^2 w &= 0 \end{aligned}$$

where $D_t = \partial_t + u\partial_x + v\partial_y + w\partial_z$ is the advective derivative, f is the (constant) Coriolis parameter and N is the (constant) buoyancy frequency.

(a) Derive the governing equatiosn for the zonally averaged wind and buoyancy fields, $\overline{u}$, $\overline{v}$, $\overline{w}, \overline{b}$, and separate the contribution from the mean flow and the eddies (u',v',w' and b'). (An overbar denotes the zonal average, a prime denotes a deviation from this average.)

(b) Assuming small Rossby number, apply quasigeostrophic scaling to arrive at

$$\overline{u}_t = f\overline{v} - (\overline{u'v'})_y \tag{3}$$
$$\overline{b}_t = -N^2\overline{w} - (\overline{v'b'})_y, \tag{4}$$

where the zonally averaged zonal velocity and buoyancy fields are linked by thermal wind balance, $f\overline{u}_z = -\overline{b}_y$.

(c) Use the quasigeostrophic equations to relate the eddy momentum flux $\overline{v'u'}$ and the eddy buoyancy flux $\overline{v'b'}$ to the eddy PV flux $\overline{v'q'}$ (this is the Eliassen-Palm flux relation). Recall that, in terms of the QG eddy streamfunction, $q' = \nabla^2\psi' + (f/N)^2\psi'_{zz}$, $b' = f\psi'_z$, $u' = -\psi'_y$, and $v' = \psi'_x$.

(d) Since $\overline{v}_y + \overline{w}_z = 0$, one may define a streamfunction $\overline{\psi}$ for the zonally averaged wind, such that $(\overline{v}, \overline{w}) = (-\overline{\psi}_z, \overline{\psi}_y)$. Now define a new *residual* streamfunction

$$\psi^* = \overline{\psi} + \frac{1}{N^2}\overline{v'b'}$$

and associated residual velocities v^* and w^*. Write the zonally averaged equations (3) in terms of the residual velocities and find that the equations are now forced by only one eddy flux term, $\overline{v'q'}$.

(e) Assuming an unstable baroclinic wave has the property that $\overline{v'u'} = 0$ and $\overline{v'b'} = \exp -(y^2/L^2)\exp(-z/H)$, show that the eddies act to decelerate the mean flow. Discuss the structure of the ageostrophic velocity field $\overline{v}, \overline{w}$.

GFD Written Examination

August 2015

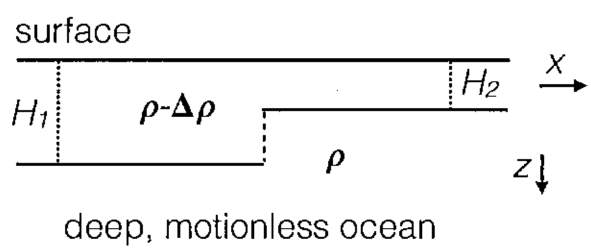
Please attempt all four problems.

1. A storm passes over the ocean, thickening the mixed layer to a depth H_1, while it remains at a depth of H_2 in the region missed by the storm, as shown in the figure below. The perturbation is independent of y (into the board in the figure) and you may assume that the ocean exists in an infinite f-plane with constant Coriolis parameter f_0. The density of the deep ocean is ρ and of the mixed layer, $\rho - \Delta\rho$, where $\Delta\rho \ll \rho$. The ocean below is effectively infinitely deep and motionless.

   (a) Find the solution for the state of the ocean after it adjusts geostrophically to this purturbation. Note any assumptions that you must make along the way.

   (b) Calculate the fraction of the potential energy released in the adjustment process that has been converted to kinetic energy in the final steady state.

   (c) If not all the potential energy released in the adjustment process was converted to kinetic energy, what happened to it?



2. Consider two different atmospheres, the first on planet "Iso," where the temperature is uniform, and a second, planet "Adia," where the potential temperature is uniform. Both atmospheres are in hydrostatic balance, and the temperature at the surface is given by T_s.

   (a) Define hydrostatic balance. What does it mean, physically?

   (b) What is the potential temperature? How does it change, as a function of height, for the atmosphere of constant temperature?

   (c) How does the temperature change, as a function of height, for the atmosphere of constant potential temperature?

   (d) What is the Brunt-Väisälä frequency, N? (It's often defined in terms of its square, N^2.) What does it represent, physically?

   (e) What is N^2 for the two different atmospheres?

   (f) How will the density change with height in the two atmospheres? Will either have a finite vertical extent (that is, the density falls to zero at some height)? If so, what is that height, expressed as a function of T_s?

   (g) Comment on what you might expect in terms of the weather in the two planets? Where do you think the atmosphere will be more prone to instability and turbulent motions?

3. **Linear Boussinesq equations**

The three-dimensional hydrostatic Boussinesq equations are

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + f\hat{\boldsymbol{z}}\times\boldsymbol{u} &= -\nabla\phi \\ 0 &= -\phi_z + b \\ b_t + \boldsymbol{v}\cdot\nabla b + N^2 w &= 0 \\ \nabla\cdot\boldsymbol{u} + w_z &= 0 \end{aligned}$$

where $\boldsymbol{u} = (u, v)$, $\boldsymbol{v} = (u, v, w)$, $\nabla = (\partial_x, \partial_y)$ and $\phi = p/\rho_0$ is the pressure potential.

(a) For constant f and N derive the dispersion relation for linear plane inertia–gravity waves, in which all fields are proportional to $\exp[i(kx + ly + mz - \omega t)]$. Using the special case $l = 0$ demonstrate that inertia–gravity waves (the non-stationary solutions) have zero potential vorticity q, where q is defined by

$$q = v_x - u_y + \frac{f}{N^2} b_z.$$

(b) Allowing for $N = N(z)$, show that the domain-integrated energy with density

$$E = \frac{1}{2}\left(u^2 + v^2 + w^2 + \frac{b^2}{N^2}\right)$$

is conserved. (This is not true in the fully nonlinear equations.) Show that the quasi-geostrophic potential vorticity

$$q = v_x - u_y + f\left(\frac{b}{N^2}\right)_z$$

is conserved locally, i.e., $q_t = 0$.

(c) Returning to the original Boussinesq equations (not the wave solution), assuming rigid-lid boundary conditions at $z = 0$ and $z = H$, project the linear Boussinesq equations onto vertical modes $F_j(z)$, satisfying

$$F_j'' + \frac{N^2}{f^2}\lambda_j^2 F_j = 0, \quad F_j(0) = F_j(H) = 0,$$

where λ_j is an eigenvalue, and the solutions $F_j(z)$ are orthogonal in the sense that $H^{-1}\int_0^H N^2 F_i F_j dz = \delta_{ij}$ (which you should be able to prove, but you don't need to do it here). Be careful about the boundary conditions: some variables will be expanded in F_j (e.g. w, which satisfies the same boundary conditions as F_j), others in F_j' (e.g. pressure and velocity). You should find shallow water equations for each mode.

4. **PV inversion**

Find the quasi-geostrophic stream function corresponding to a point charge of potential vorticity in an unbounded three-dimensional domain, i.e.

$$\psi_{xx} + \psi_{yy} + \frac{f^2}{N^2}\psi_{zz} = q = \delta(x)\delta(y)\delta(z)$$

with constant f and N. This is Green's function for the PV inversion problem. Show that the balanced flow around the point charge is cyclonic and that the stratification surfaces $b + N^2 z$ are bending towards the point charge from above and below.

GFD Written Examination

January 2016

Please attempt all four problems.

1. **Kelvin circulation theorem**

The momentum equation for an inviscid fluid is

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{\nabla p}{\rho},$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative, $\boldsymbol{v}$ is the velocity field, p is the pressure and ρ is the density. The circulation around a material loop is defined as

$$\Gamma = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}.$$

(a) Relate the circulation to the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$.

(b) Show that if $p = p(\rho)$, then the circulation is constant (this is Kelvin's Circulation Theorem):

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0.$$

(c) Discuss how Kelvin's circulation theorem must be adjusted in a rotating reference frame.

2. **Rossby waves in SWQG**

The QG Shallow Water equations are:

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad q = \nabla^2\psi - k_d^2\psi + \beta y$$

where $J(f,g) = f_x g_y - f_y g_x$. The velocity field is related to the streamfunction by $u = -\psi_y$ and $v = \psi_x$. Assume the lateral boundary conditions to be periodic in both x and y.

(a) Linearize the equations about a background state given by $u = v = 0$. What is the background PV gradient?

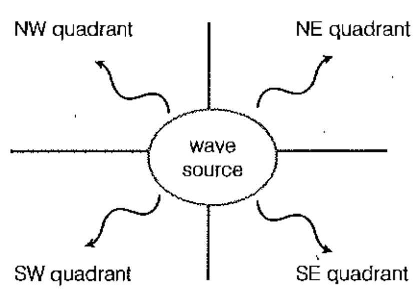
(b) Consider a perturbation of the form

$$\psi'(x,y,t) = \mathrm{Re}\left(\tilde{\Psi}\exp[i(kx + ly - \omega t)]\right),$$

where Re denotes the real part of a complex number, k and l are the zonal and meridional wave number and ω is the angular speed. Derive the dispersion relationship of the form $\omega = f(k,l)$.

(c) Compute the phase and group velocity in the zonal and meridional directions. Show that, while the zonal phase velocity is always negative, the zonal group velocity can be either positive or negative.

(d) Consider a localized wave source, as sketched on the figure. Discuss the types of Rossby waves you would expect to find in the four different quadrants (NW, SW, NE and SE): what are the directions of their phase lines, and where do you expect to find long and short waves?



3. **Lee waves**

The non-rotating Boussinesq equations in the x-z plane, linearized about a state of rest are

$$u_t + \phi_x = 0, \quad w_t + \phi_z = b, \quad b_t + N^2 w = 0, \quad u_x + w_z = 0.$$

Consider a mean zonal current U flowing over topography $z_{\mathrm{bot}}(x) = h\cos(kx)$, where U, h and k are all fixed, positive constants. Instead of re-linearizing about this mean flow, transform to a coordinate system moving with the flow, which yields time-dependent topography $z_{\mathrm{bot}}(x+Ut) = h\cos[k(x+Ut)]$. Thus internal waves of zonal wavenumber k and frequency $\omega = -Uk$ are continuously forced by the boundary.

(a) Show that in the limit of small topographic height h, the bottom boundary condition becomes $w|_{z=0} = Ukh\sin(kx - \omega t)$.

(b) Remembering that ω and k are fixed by the forcing, show that the dispersion relation then becomes

$$m = \pm\sqrt{\frac{N^2}{U^2} - k^2}.$$

What sign must m have to be a physical solution? [Hint: consider the vertical group velocity.]

(c) Describe the structure of the wave in the two cases $k > N/U$ and $k < N/U$, and explain these results in physical terms.

* In the latter case, at what rate is momentum transferred upward by the waves? Supposing this momentum is dissipated somewhere, what will this do to the mean flow? [skip this part if out of time]

4. **Stokes Drift**

Consider the position $x(t)$ of a particle starting at $x(0) = 0$ in a one-dimensional compressional wave

$$u = U\sin(kx(t) - \omega t) = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

(a) Defining a phase average

$$\overline{u} = \frac{1}{T}\int_t^{t+T} u(t')\,\mathrm{d}t', \qquad T = \frac{\omega}{2\pi},$$

Taylor expand u about $\overline{x}$ at fixed time and average to show that the mean particle trajectory is approximately

$$\overline{x} \approx \frac{kU^2}{2\omega}t.$$

What must be true of ω, k and U for this to be a reasonable approximation?

(b) For a general three-dimensional wave flow,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t))$$

show that the same kind of argument leads to the Stokes drift estimate

$$\frac{\mathrm{d}\overline{\boldsymbol{x}}}{\mathrm{d}t} \approx \overline{\left(\left(\int^t \boldsymbol{u}\,\mathrm{d}t\right)\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}} \equiv \boldsymbol{u}^{\mathrm{Stokes}}.$$

(c) For what sorts of waves do you expect a non-zero Stokes drift? Consider, for example, waves in the Shallow Water vs. Boussinesq equations. [Hint: what is the relevance of the word "compressional" in the first part?]

GFD Written Examination

August 2016

Please attempt all four problems.

1. **Inertia-gravity waves**

   The linear rotating Boussinesq equations in the xz-plane are

   $$u_t - fv + P_x = 0, \quad v_t + fu = 0, \quad w_t + P_z = 0, \quad u_x + w_z = 0,$$

   where $P = p/\rho$. In a plane wave all fields are given by

   $$w = \operatorname{Re} \hat{w} \exp[i(kx + mz - \omega t)]$$

   etc... where Re denotes the real part.

   (a) Show that a *single* plane wave is an exact solution of the fully nonlinear equations.

   (b) Derive the dispersion relation

   $$\omega^2 = f^2 \frac{m^2}{k^2 + m^2}.$$

   Explain what is meant by *Taylor columns* in rapidly rotating flows and identify the regime ofthe dispersion relationship that corresponds to this phenomenon.

   (c) Define and compute the group velocity components $\boldsymbol{u}_g = (u_g, w_g)$ and show that the horizontal group and phase velocities have opposite signs. At fixed frequency, how does the group velocity magnitude scale as a function of wavelength?

   (d) What is the physical significance of the group velocity vector? Explain what is meant by a "radiation condition" in the context of waves generated at the boundary at $z = 0$ and propagating into the upper half plane. What does this imply for $\operatorname{sgn}(m)$ and $\operatorname{sgn}(\omega)$?

2. **Balanced vortex**

For an axisymmetric flow, the shallow water momentum equations in polar coordinates are:

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{uv}{r} = -f_0 v$$
$$\frac{\mathrm{D}v}{\mathrm{D}t} - \frac{u^2}{r} = +f_0 u - g\frac{\partial h}{\partial r}$$

Here, u and v are the radial and azimuthal velocity, h is the water thickness, f_0 is the Coriolis parameter, g is the gravitational acceleration, and $\mathrm{D}/\mathrm{D}t = \partial_t + u\partial_x + v\partial_y$ is the material derivative.

(a) Show that the angular momentum $M = ur + \frac{1}{2}f_0 r^2$ is materially conserved, i.e. $\mathrm{D}M/\mathrm{D}t = 0$. Discuss how this conservation law is related to Kelvin's circulation theorem.

(b) When one neglects the material derivative of the radial wind, gradient wind balance is

$$g\frac{\partial h}{\partial r} = \frac{u^2}{r} + f_0 u.$$

Express the gradient wind balance in terms of the angular momentum and thickness gradient.

(c) Consider a vortex in which the angular momentum is given by

$$M(r) = \frac{f_0}{2}\begin{cases} 0 & r < r_0 \\ r^2 & \text{otherwise} \end{cases}$$

Determine both the thickness and velocity distributions assuming gradient wind balance.

(d) Consider a vortex in which the angular momentum is given by

$$M(r) = \frac{f_0}{2}\begin{cases} r_0^2 & r < r_0 \\ r^2 & \text{otherwise} \end{cases}$$

Determine both the thickness and velocity distribution in this vortex. Discuss what happens near the center of the domain.

3. **Lee waves**

Using the 2D non-rotating Boussinesq equations with a background stratification in the x-z plane,

$$u_t + uu_x + wu_z + P_x = 0 \tag{1}$$
$$w_t + uw_x + ww_z + P_y = b \tag{2}$$
$$u_x + w_z = 0 \tag{3}$$
$$b_t + ub_x + wb_z + wN^2 = 0, \tag{4}$$

where $b = -g\rho/\rho_0$ and $P \doteq p/\rho_0$, and $\bar{b}_z = N^2$ is the mean stratification. Consider a mean zonal current U flowing over topography $z_b(x) = h\cos(kx)$, where U, h and k are all fixed, positive constants. Transform to a coordinate system moving with the flow, which yields time-dependent topography $z_b(x + Ut) = h\cos[k(x + Ut)]$, and then linearize the flow. Introduce a streamfunction such that $u = \psi_z$ and $w = -\psi_x$, seek a plane wave solution $\propto \exp[ik(x + Ut)]$. This would yield internal waves of zonal wavenumber k and frequency $\omega = -Uk$ that are continuously forced by the boundary.

(a) Show that in the limit of small topographic height h, the bottom boundary condition becomes $w|_{z=0} = Ukh\sin(kx - \omega t)$.

(b) Remembering that ω and k are fixed by the forcing, show that the dispersion relation then becomes

$$m = \pm\sqrt{\frac{N^2}{U^2} - k^2}.$$

What sign must m have to be a physical solution?

(c) Describe the structure of the wave in the two cases $k > N/U$ and $k < N/U$, and explain these results in physical terms.

4. **Barotropic instability**

Consider the linearized barotropic vorticity equation, where ψ is the peturbation streamfunction, β the gradient in planetary vorticity, and U is a zonally uniform background flow.

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + (\beta - U_{yy})\frac{\partial \psi}{\partial x} = 0$$

(a) Assume a wave-like solution $\psi = \mathrm{Re}\,\tilde{\psi}(y)e^{ik(x-ct)}$ and derive *Rayleigh's equation*,

$$(U-c)(\tilde{\psi}_{yy} - k^2\tilde{\psi}) + (\beta - U_{yy})\tilde{\psi} = 0.$$

(b) Next, use Rayleigh's equation to derive a necessary (albeit not necessarily sufficient) condition for instability: $\beta - U_{yy}$ must change sign within the domain. This is known as the *Rayleigh-Kuo inflection point criterion.*

As a hint to get you started, the key is whether c can be part imaginary, i.e. $c = c_r + ic_i$ with $c_i \neq 0$. You'll want to divide Rayleigh's equation by $(U-c)$, multiply by the complex conjugate of $\tilde{\psi}$, and integrate over the domain (assuming that $\tilde{\psi}(y)$ vanishes at the boundaries) to establish an equation for the peturbation amplitude $|\tilde{\psi}|^2$.

(c) Lastly, we specify the zonal flow U, supposing it to be a Gaussian shaped jet give by

$$U(y) = U_0 \exp(-y^2/L^2),$$

on a β-plane. There are thus three parameters in the problem: U_0, L, and β. How would increasing each of these parameters tend to enhance/decrease the likelihood of instability? In each case, please provide a sentence or two physical explanation of why (or why not) the parameter impacts the likelihood of instability.

GFD Written Examination

January 2017

Please attempt all four problems. You must show and justify your work for credit.

1. **Thermodynamics**

   Consider dry air over the Sahara desert in the descending branch of the Hadley cell. The environmental lapse rate $\Gamma_e = dT_e/dz$ (where T_e is the temperature of the environment) in this region is -7 K km^{-1}. The ultimate goal of this question will be to use satellite observations over this region to estimate the actual descent rate.

   (a) To get started, we begin with a more basic question. Suppose dry air descends adiabatically. How will its temperature change with height? (This quantity is known as the dry adiabatic lapse rate, Γ_d.) For simplicity, you can assume that gravitational accelleration is 10 ms^{-2} and the heat capacity of dry air at constant pressure $c_p = 1000$ J kg^{-1}.

   (b) Suppose that air parcels leave from an initial height z. Assuming the displacement to be adiabatic, show that after a time δt, the parcel would be warmer than its environment by an amount $w\Lambda_e\delta t$, where w is the descent rate and

   $$\Lambda_e = \Gamma_e + \frac{g}{c_p}.$$

   (c) Now suppose that the displacement is not adiabatic, but rather that the parcel cools radiatively such that its temperature is *always the same as* its environment. (In this case, the circulation would be in equilibrium!) Show that the radiative rate of energy loss per unit mass must be $c_p w \Lambda_e$.

   (d) Assume that air is descending through the depth of the atmosphere. What then is the net radiative energy loss to space per unit horizontal area?

   (e) Finally, radiative measurements show that energy is lost to space at a net, annually averaged rate of 20 W m^{-2} over the Sahara. Estimate the vertically, annually averaged subsidence velocity.

2. **Inertial Gravity Waves**

If both rotation and stratification are present, the two-dimensional Boussinesq equations are

$$u_t - fv + P_x = 0, \quad v_t + fu = 0, \quad w_t + P_z = b, \quad b_t + N^2 w = 0, \quad u_x + w_z = 0. \tag{1}$$

(a) Show that the corresponding dispersion relation has the roots

$$\omega = 0, \quad \omega^2 = N^2 \frac{k^2}{k^2 + m^2} + f^2 \frac{m^2}{k^2 + m^2}. \tag{2}$$

(b) The unsteady waves are called *inertia–gravity waves*. Show that their frequency is bounded between f and N.

(c) Show that the root $\omega = 0$ corresponds to a flow in hydrostatic and geostrophic balance.

(d) How would the number of roots and their functional form change if w_t is neglected in the vertical momentum equation?

3. **Vorticity Dynamics**

Consider the vorticity equation for an incompressible ideal fluid in 3D:

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \vec{\nabla}\vec{u}. \tag{3}$$

(a) The vorticity is a material vector field. Explain *briefly* what this statement means.

(b) Show that for any scalar field $\theta(\vec{x}, t)$,

$$\frac{D}{Dt}(\vec{\omega} \cdot \vec{\nabla}\theta) = (\vec{\omega} \cdot \vec{\nabla})\frac{D\theta}{Dt}.$$

You may want to use the following identity from vector calculus:

$$(\vec{A} \cdot \vec{\nabla}\vec{B} - \vec{B} \cdot \vec{\nabla}\vec{A}) \cdot \vec{\nabla}f = \vec{A} \cdot \vec{\nabla}(\vec{B} \cdot \vec{\nabla}f) - \vec{B} \cdot \vec{\nabla}(\vec{A} \cdot \vec{\nabla}f).$$

(c) The vorticity equation (3) is only valid in a Galilean (or inertial) reference frame. Show that when the reference frame is rotating at a rate Ω in the vertical direction, the vorticity equation can be rewritten as

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \vec{\nabla}\vec{u} + 2\Omega\vec{k} \cdot \vec{\nabla}\vec{u}, \tag{4}$$

where $\vec{k}$ is the unit vector in the vertical direction. Provide a physical interpretation of the equation and explain the factor 2 on the right-hand side.

(d) Discuss how the additional term in (4) affects the evolution of the vorticity in the case of a purely vertical velocity field and a purely horizontal velocity field.

4. **Instability**

In this problem you will consider baroclinic instability from an energetic point of view, using parcel theory. As a warm-up, we'll consider gravitational instability (convection).

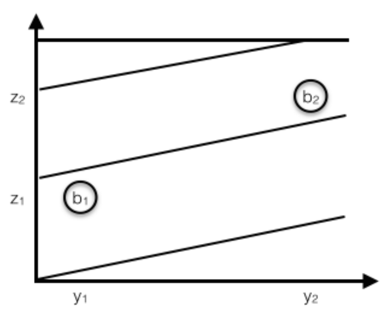
(a) Consider two parcels of fluid at rest in a vertically stratified fluid, with $N^2 = db/dz$, where $b = -g\rho/\rho_0$. Parcels 1 and 2 have densities ρ_1 and ρ_2 and are located initially at z_1 and z_2, respectively. If the two parcels are swapped instantaneously, **show** that the change in potential energy of the system is

$$\Delta P = P_{\text{final}} - P_{\text{initial}} = \rho_0 \Delta b \Delta z$$

where $\Delta b = b_2 - b_1$ and $\Delta z = z_2 - z_1 > 0$. Thus $\Delta P < 0$ if $\Delta b < 0$, i.e. there is a loss of potential energy if the buoyancy decreases with height (denser fluid over lighter).

(b) The loss of potential energy is a gain in kinetic energy, which is $\Delta K = K_{\text{final}} = 2 \times (1/2)\rho_0 w^2$, where w is the vertical velocity of each parcel. If there's an instability, the z positions of each parcel grow exponentially, i.e. $z \propto e^{\omega t}$. Thus $w = dz/dt = \omega z$. **Show**, therefore, by setting $\Delta P = \Delta K$, that $\omega \sim |N|$ (which is the result one finds in a full instability analysis).

(c) Now consider two parcels, at (y_1, z_1) and (y_2, z_2), in a background state with both lateral and vertical buoyancy gradients, as in the figure. Take $N^2 = \partial b/\partial z$ and $M^2 = \partial b/\partial y$ both to be constant. Thus $\delta b = M^2 \delta y + N^2 \delta z$, and so the slope of the isopycnals is $s_b = -M^2/N^2$.



If the parcels are swapped, the slope along which they are exchanged is $s = \Delta z/\Delta y$, where $\Delta y = y_2 - y_1$ and $\Delta z = z_2 - z_1$. The change in potential energy will be just as before, $\Delta P = \rho_0 \Delta b \Delta z$. By rewriting Δb in terms of the background stratification, **show** that this is

$$\Delta P = \rho_0 N^2 \Delta y^2 s(s - s_b).$$

(d) Now **find** s such that ΔP is maximally negative. Setting $\Delta P|_{\max} = \Delta K = \rho_0 v^2$, assuming an instability means the positions of the parcels $y \propto e^{\omega t}$, and using thermal wind balance, **derive** an estimate for the Eady growth rate,

$$\omega \sim \frac{f}{\sqrt{Ri}}, \quad \text{where} \quad Ri = \frac{N^2}{u_z^2}.$$

GFD Written Examination

August 2017

Please attempt all four problems.

1. **Shallow water**

   (a) Starting from the rotating (constant f) hydrostatic primitive equations for a constant-density fluid of mean depth H_0, derive the rotating shallow water equations

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f_0 \hat{\boldsymbol{z}} \times \boldsymbol{u} = -g \boldsymbol{\nabla} \eta$$
$$\partial_t \eta + \boldsymbol{\nabla} \cdot (h \boldsymbol{u}) = 0,$$

   where $h(x, y, t) = H_0 + \eta(x, y, t)$ and $\boldsymbol{u} = (u, v)$

   (b) Now write the equations in vorticity-divergence form: using a Helmholtz decomposition $\boldsymbol{u} = \boldsymbol{\nabla}\phi + \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi$, where $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla}^2 \phi \equiv \delta$ is the divergence of the flow and $\hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\nabla}^2 \psi \equiv \zeta$ is the vorticity, write the equations in terms of δ and ζ instead of u and v. The following vector identity may be useful: $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u} + (1/2) \boldsymbol{\nabla} |\boldsymbol{u}|^2$.

   (c) Derive from these a conservation law for potential vorticity.

   (d) Linearize the equations about a state of rest, and find the dispersion relation for shallow water gravity waves. Find their phase and group velocities. What is the significance of the 0-frequency solution?

2. **Geostrophic adjustment**

The strong winds associated with a tropical storm can thicken the mixed-layer in the ocean. Consider that, right after the passing of a storm, the mixed layer depth is given by

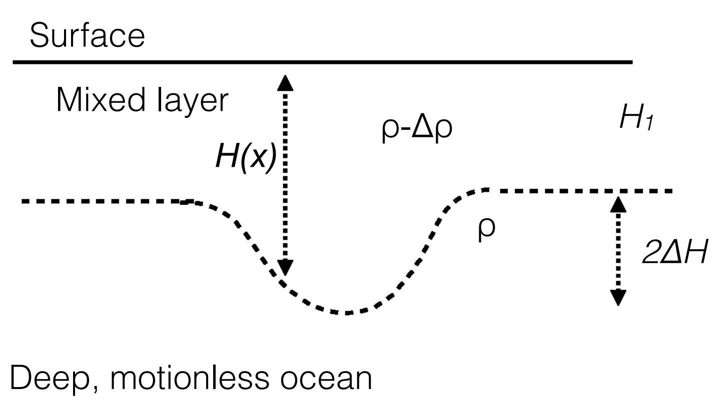
$$H(x) = \begin{cases} H_1 + \Delta H[1 + \cos(\frac{\pi x}{L})], & -L < x < L \\ H_1, & \text{otherwise} \end{cases}$$

as shown in the figure below. The perturbation is independent of y (into the board in the figure) and you may assume that the ocean is an infinite f-plane with constant Coriolis parameter f_0. The mixed layer has a uniform density $\rho - \Delta\rho$, while the rest of the ocean below it as a density ρ. The ocean below is effectively infinitely deep and motionless.

(a) Find the solution for the state of the ocean after it adjusts to establish geostrophic balance following this perturbation. Note any assumptions that you must make along the way.

(b) Calculate the fraction of potential energy converted in the adjustment process to kinetic energy in the final steady state. How does this fraction depend on the width of the track?

(c) If not all the potential energy was converted to kinetic energy, what happened to it?



3. **Instability of stratified shear flow**

Show that a sufficiently stratified fluid can suppress Kelvin-Helmholtz shear instability. Consider the Boussinesq equations in the x-z plane,

$$u_t + uu_x + wu_z = -\phi_x \tag{1}$$
$$w_t + uw_x + ww_z = -\phi_z + b \tag{2}$$
$$u_x + w_z = 0 \tag{3}$$
$$b_t + ub_x + wb_z + wN^2 = 0 \tag{4}$$

where $b = -g\rho/\rho_0$ and $\phi = p/\rho_0$, and $\bar{b}_z = N^2$ is the mean stratification.

(a) Linearize these about a linear mean shear $\bar{u}(z) = \Lambda z$.

(b) Introduce a streamfunction such that $u = \psi_z$ and $w = -\psi_x$, seek a plane wave solution $\propto \exp[ik(x - ct)]$.

(c) Derive the Taylor-Goldstein equation,

$$(\bar{u} - c)(\psi_{zz} - k^2\psi) + \left(\frac{N^2}{\bar{u} - c} - \bar{u}_{zz}\right)\psi = 0.$$

(d) Take the boundary conditions to be $\psi(0) = \psi(H) = 0$. Derive an amplitude equation.

[*Hint:* You may find it useful to introduce $\Psi = \sqrt{\bar{u} - c}\,\psi$].

(e) (bonus) Show that a necessary condition for instability is that the Richardson number

$$\mathrm{Ri} < \frac{1}{4}, \quad \text{where} \quad \mathrm{Ri} = \frac{N^2}{\Lambda^2}.$$

4. **Stokes problem**

Consider an incompressible fluid of density ρ and dynamic viscosity μ lying over an infinite horizontal plate located at $z = 0$ and occupying the semi-infinite domain $z > 0$. The plate oscillates along the x direction with a velocity

$$U(t) = a\cos(\omega t), \quad a > 0, \quad \omega > 0, \tag{5}$$

and as a result the fluid is set in motion.

(a) Assuming that the velocity field has the form $\boldsymbol{u} = (u, 0, 0)$, where u depends on z and t only, and that the pressure is constant, show that the momentum equation for u reduces to the diffusion equation

$$u_t = \nu u_{zz}, \quad \nu = \mu/\rho. \tag{6}$$

(b) Using the ansatz $u(z,t) = \mathrm{Re}\{e^{i\omega t} f(z)\}$, and assuming no-slip boundary conditions at the plate, compute the solution to (6) remaining finite as $z \to \infty$.

(c) Derive an expression for the shear stress $\tau(t)$ at the plate. In a few words, comment on the phase relationship between $U(t)$ and $\tau(t)$.

GFD Written Examination

January 2018

Please attempt all four problems.

1. **Two-layer equations**

   Consider the hydrostatic rotating primitive equations with a flat bottom, a free upper surface and a total mean fluid depth H. The fluid is stably stratified, with two immiscible layers of densities ρ_1 and $\rho_2 > \rho_1$, with resting thicknesses $H/2$ (the layers have equal thicknesses).

   (a) Derive the two-layer equations for this system. You should find a coupled set of shallow water (SW) equations.

   (b) Starting from the conservation of SW potential vorticity for each layer, informally derive the two-layer quasigeostrophic (QG) equations. Write these in terms of streamfunctions ψ_1 and ψ_2 for each layer. Discuss the relative magnitudes and meanings of the two length-scales that arise.

   (c) The QG PVs for each layer, q_1 and q_2 are linearly related to ψ_1 and ψ_2. Write the relations as a 2×2 matrix equation. Diagonalize this matrix. What do these two eigenvectors represent?

   (d) Now project the two-layer equations onto the eigenvectors to get the two-mode equations. Linearize these modal equations, and find the plane-wave solutions. What is the meaning of the two wave modes you find? How do their phase speeds differ?

2. **Waves on the equatorial β-plane**

   Start with the rotating shallow water equations on the equatorial β-plane (i.e. with $f_0 = 0$), and linearize about a state of rest.

   (a) First look for equatorial Kelvin waves with vanishing meridional velocity $v = 0$. Show that the solutions for u and h have the form $F(x - ct)G(y)$, where $c = \sqrt{g'H_1}$ with $g' = g(\rho_2 - \rho_1)/\rho_0$. These waves are translating in the positive x-direction at speed c. Now use the meridional momentum equation to find an expression for $G(y)$. What is the width in y of the wave?

   (b) Now look for plane wave solutions of the linear equatorial β-plane equations, with the form $u(x, y, t) = U(y) \exp[i(kx - \omega t)]$ and so forth for $V(y)$ and $H(y)$. Eliminate U and H to find a single equation ODE for $V(y)$ of the form $V'' + r(y)V = 0$, where $r(y)$ is a function you should derive. Do not try to solve the equation, but do analyze it as best you can (e.g. consider the sign of $r(y)$...).

3. **Motion of a particle on a rotating bowl**

Consider a circular, concave surface, with radius R, rotating about its center at a constant rate Ω. What must the shape of the surface be to exactly balance the centrifugal acceleration at any point? (i.e. so that, without friction, a ball placed at any position on the surface with no initial velocity in the rotating frame will stay put).

Now assume the bowl is shaped exactly as derived above, and consider the motion of a frictionless ball launched from the edge of the bowl with an initial velocity $-v_0\hat{r}_R$, where $\hat{r}_R$ is the unit vector pointing away from the center of the disk *in the rotating frame.* Describe the path of the ball in both the rotating and inertial frames of reference.

Recall velocities and accelerations in the inertial (I) and rotating (R) frames are related as

$$\boldsymbol{v}_I = \boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}, \quad \boldsymbol{a}_I = \boldsymbol{a}_R + 2\boldsymbol{\Omega} \times \boldsymbol{v}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

where $\boldsymbol{r}(t) = (x(t), y(t))$ is the position of the ball relative to the center of the bowl (we ignore the vertical direction, or height on the surface of the bowl).

4. Consider the barotropic vorcitiy equation

$$\zeta_t + J(\psi, \zeta) + \beta\psi_x = 0, \quad \zeta = \boldsymbol{\nabla}^2\psi$$

in a periodic-in-x channel of length L_x, with walls at $y = 0, L_y$. In the statements to follow, an overline denotes a zonal average $\overline{(\cdot)} = L - x^{-1} \int_0^{L_x} (\cdot)\ dx$.

(a) First, assuming perturbations (which we'll refer to as "eddies" or "waves" and will denote with primes) are small, show that

$$\frac{\partial}{\partial t}\left(\frac{\overline{\zeta'^2}}{2}\right) = -\overline{v'\zeta'}\,\overline{\zeta}_y$$

(include the β term in $\overline{\zeta}_y$). Under what conditions will the eddy enstrophy grow?

(b) Now show that $\overline{v'\zeta'} = \partial_y(\overline{u'v'})$, and hence

$$\frac{1}{\overline{\zeta}_y}\frac{\partial}{\partial t}\left(\frac{\overline{\zeta'^2}}{2}\right) = \partial_y(\overline{u'v'}).$$

Integrate in y and derive a necessary condition for instability.

GFD Written Examination

August 2018

Please attempt all four problems.

1. **Adiabatic Atmospheres**

   Consider two atmospheres, A and B, each composed of the same ideal gas, with gas constant R and heat capacity c_p, in the presence of a constant gravitational acceleration g. Atmosphere A is isotropic (constant temperature), and atmosphere B is isentropic (constant entropy, or constant potential temperature). Suppose both have the same temperature at the surface, T_s.

   (a) How does density vary as a function of height in the two atmospheres? Please express your answer in terms of T_s, R, c_p, g, etc... Does either have a finite extent? If so, what is?

   (b) What would happen if a parcel of air was adiabatically displaced by a small vertical distance in atmosphere A? Would it stay still, continue moving, or oscillate? If it would oscillate, what would be the frequency?

   (c) What would happen if a parcel of air was displaced in atmosphere B?

   (d) What region (layer) of our atmosphere that would most closely correspond to atmosphere A? To atmosphere B?

2. **Rotating Shallow Water**

The two-dimensional shallow water (SW) equations in a rotating frame are

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) = 0$$

$$\frac{D\boldsymbol{u}}{Dt} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} + g\boldsymbol{\nabla} h = 0,$$

where the velocity $\boldsymbol{u} = (u, v)$ and the layer depth h are functions of (x, y, t).

(a) Briefly describe the physical meaning of each term in these equations.

(b) Define 'potential vorticity' and derive its conservation law.

(c) Assuming that $\boldsymbol{u} = \epsilon \boldsymbol{u}'$ and $h = H + \epsilon h'$ with constant H and $\epsilon \ll 1$, derive the linearized SW equations at $O(\epsilon)$ and show that

$$q' \equiv \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \frac{f}{H} h' \quad \text{satisfies} \quad \frac{\partial q'}{\partial t} = 0.$$

Now consider the y-independent linear adjustment problem defined by the following initial conditions at $t = 0$:

$$u'_0 = v'_0 = 0, \quad h'_0 = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}.$$

(d) Show that as $t \to +\infty$ the final steady state satisfies

$$u' = 0, \quad g\frac{\mathrm{d}h'}{\mathrm{d}x} = fv', \quad \frac{\mathrm{d}v'}{\mathrm{d}x} - \frac{f}{H}h' = -\frac{f}{H}h'_0.$$

Find h' and v' (the deformation scale defined as $L_D = \sqrt{gH}/f$ is helpful here) and sketch h'_0, h' and v'.

(e) Compute the disturbance energy contained in $x \in [-L, L]$

$$\mathcal{E}_L \equiv \frac{1}{2} \int_{-L}^{+L} \left(u'^2 + v'^2 + gh'^2/H\right) \mathrm{d}x$$

for both the initial and final state. Show that the difference between these energies has a finite limit as $L \to \infty$. Where is the missing energy?

3. **Dispersive waves**

GFD is characterized by dispersive waves, which arise in the quasigeostrophic, shallow water and primitive equations. Here we consider the dynamics of simpler linear dispersive PDE

$$u_t = u_{xxx}.$$

(a) Find the dispersion relation for plane waves $u(x,t) = A \exp i(kx - \omega t)$ in this system. Find the phase velocity and group velocity. Explain what these are.

(b) Consider now a wave *packet* with an envelope (amplitude) that varies slowly in space and time, relative to the wavelength and period of the wave, and find an evolution equation for this amplitude.

One way to proceed is to use multiscale asymptotics. First, define the parameter $\epsilon \ll 1$ as the ratio of the wavelength to the envelope's spatial scale, as well as the ratio of the wave period to the envelope's temporal scale. Then define fast and slow variables $\tilde{x} = x$, $X = \epsilon x$, $\tilde{t} = t$ and $T = \epsilon t$, and assume

$$u(x,t) \to u(\tilde{x}, X, \tilde{t}, T).$$

The partial derivatives become $\partial_t \to \partial_{\tilde{t}} + \epsilon \partial_T$ and $\partial_x \to \partial_{\tilde{x}} + \epsilon \partial_X$. [Using this, for example, the second derivative $\partial_{xx} \to (\partial_{\tilde{x}} + \epsilon \partial_X)(\partial_{\tilde{x}} + \epsilon \partial_X)$, etc...]

Expand the dependent variable $u = u^0 + \epsilon\, u^1 + \epsilon^2\, u^2 + \cdots$, insert into the PDE, along with the multiscale derivatives, then group the terms of the equations in like powers of ϵ. At $O(1)$, you should have the original linear PDE in the fast variables for u^0, and so a wave solution like in part (a) should work, with the exception that the amplitude $A = A(X,T)$ (for this problem, ignore the dependence of the phase on the slow variables).

At $O(\epsilon)$, you should find a wave equation for u^1, forced by a function of the $O(1)$ solution. To prevent resonance, demand that the forcing term be 0; this results in a PDE for $A(X,T)$.

Show that this is a transport equation for A, with transport velocity equal to the group velocity found in part (a).

4. **Shallow Water QG**

(a) Referring to the rotating shallow water equations at the start of problem 2, write the total height as a mean height H plus a deviation, i.e. $h(x, y, t) = H + \eta(x, y, t)$. State clearly and succinctly the asymptotic assumptions necessary to derive the f-plane quasigeostrophic (QG) equations. Define any relevant nondimensional numbers.

(b) Under these assumptions, derive the f-plane vorticity and height equations in the quasigeostrophic limit:

$$\partial_t \zeta + J(\psi, \zeta) + f\delta = 0$$
$$\partial_t \eta + J(\psi, \eta) + H\delta = 0,$$

where $\psi = (g/f)\eta$ is the streamfunction, $\zeta = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u} = v_x - u_y = \boldsymbol{\nabla}^2\psi$ is the vorticity and $\delta = \boldsymbol{\nabla} \cdot \boldsymbol{u} = u_x + v_y$ is the divergence. The Jacobians are $J(a, b) = a_x b_y - a_y b_x$.

(c) Eliminate the divergence δ to find the equation for the conservation of QG SW potential vorticity,

$$\partial_t q + J(\psi, q) = 0, \quad q = \boldsymbol{\nabla}^2\psi - L_D^{-2}\psi$$

where $L_D = \sqrt{gH}/f$.

(d) Notice that because ψ is proportional to η, the nonlinear term in the height equation $J(\psi, \eta) = 0$. Eliminate time-derivative terms in the vorticity and height equations to find a single eliptic equation for the divergence. Explain its relevance.

GFD Written Examination

January 2019

Please attempt all four problems. Good luck!

1. **Baroclinic instability**

   The QG equations for a two-layer fluid are:

$$
\begin{aligned}
\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0, \qquad q_1 = \nabla^2\psi_1 + \frac{\kappa_d^2}{2}(\psi_2 - \psi_1) \\
\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0, \qquad q_2 = \nabla^2\psi_2 + \frac{\kappa_d^2}{2}(\psi_1 - \psi_2)
\end{aligned}
$$

   where $J(f, g) = f_x g_y - f_y g_x$. Assume the lateral boundary conditions to be periodic in both x and y.

   (a) Derive the conserved energy for the system, define the kinetic energy (KE) and available potential energy (APE), and write down the horizontally-integrated energy budget equations for each. Give a physical interpretation of the energy exchange term between KE and APE.

   (b) Linearize the governing equations about the background state with $-\partial\psi_1/\partial y = u_1 = U$ and $-\partial\psi_2/\partial y = u_2 = -U$. What are the mean PV gradients in each layer?

   (c) Write the linearized equations in terms of the barotropic and baroclinic stream functions, $\psi \equiv (\psi_1 + \psi_2)/2$ and $\tau \equiv (\psi_1 - \psi_2)/2$, respectively. Then assume a plane-wave solution of the form $\psi' = \mathrm{Re}\, \hat{\psi} \exp[i(kx + ly - \omega t)]$ (and likewise for τ), and compute the dispersion relationship.

   (d) For what wavenumbers is the flow stable and unstable? Relate the stability criterion to the PV gradients. In the unstable regime, what is the energy source for the growing disturbance?

2. **Topographic waves**

The f-plane shallow water quasigeostrophic (QG) model in the presence of perturbed bottom topography is

$$q_t + J(\psi, q) = 0, \qquad q = \nabla^2\psi - \kappa_d^2\psi + \frac{f}{H}\eta_b(x, y)$$

where ψ is the stream function, q is the potential vorticity, κ_d^{-1} is the deformation radius, f is the (constant) Coriolis parameter, η_b is the bottom topography perturbation, H is the mean thickness and $J(\psi, q) = \psi_x q_y - \psi_y q_x$. We take the topography to be a simple slope

$$\eta_b = \gamma x.$$

(a) Linearize the governing equation for a small perturbation, and obtain the dispersion relationship for a perturbation of the form $\psi' = \hat{\psi}\exp[i(kx + ly - \omega t)]$. If $k = 0$ what is the direction of phase propagation for these waves?

(b) Determine the group velocity of these waves. How does it compare to the phase velocity?

(c) Draw a schematic representation of the PV dynamics of these waves that explains their phase velocity, and relate them to other waves that can arise in QG theory.

3. **Inertial waves**

The linear rotating Boussinesq equations in the xz-plane are

$$u_t - fv + P_x = 0, \quad v_t + fu = 0, \quad w_t + P_z = 0, \quad u_x + w_z = 0, \tag{1}$$

where $P = p/\rho$. In a plane wave all fields are given by $w = \hat{w}\exp(i[kx + mz - \omega t])$ etc., with real parts understood.

(a) Show that a *single* plane wave is an exact solution of the fully nonlinear equations.

(b) Derive the dispersion relation

$$\omega^2 = f^2 \frac{m^2}{k^2 + m^2}. \tag{2}$$

Explain what is meant by *Taylor columns* in rapidly rotating flows and identify the regime of (2) that corresponds to this phenomenon.

(c) Define and compute the group velocity components $\boldsymbol{u}_g = (u_g, w_g)$ and show that the horizontal group and phase velocities have opposite signs. At fixed frequency, how does the group velocity magnitude scale as a function of wavelength?

(d) What is the physical significance of the group velocity vector? Explain what is meant by a "radiation condition" in the context of waves generated at the boundary at $z = 0$ and propagating into the upper half plane. What does this imply for sgn(m) and sgn(ω)?

4. **Surface QG**

Consider the quasigeostrophic equations in a horizontally periodic, vertically semi-infinite domain, with a flat boundary at $z = 0$ and constant Coriolis parameter f and stratification N:

$$
\begin{aligned}
&q_t + J(\psi, q) = 0, \quad z > 0,\\
&b_t + J(\psi, b) = 0, \quad z = 0,\\
&\psi \to 0 \ \text{ as } \ z \to \infty,
\end{aligned}
$$

where

$$q = \psi_{xx} + \psi_{yy} + \frac{f^2}{N^2}\psi_{zz} \tag{3}$$

is the QG potential vorticity and

$$b(x, z, t) = f\psi_z(x, y, 0, t) \tag{4}$$

is the surface buoyancy.

(a) Show that if $q = 0$ everywhere then the flow is determined solely by b. Do this by deriving a well-posed diagnostic inversion problem for finding ψ from b.

(b) Show that the vertical flow structure decays exponentially away from the surface with a decay scale that decreases with horizontal scale, so that small horizontal scales are trapped near the surface. What is the decay scale? Hint: use a horizontal Fourier expansion

$$\psi(x, y, z, t) = \sum_{k,l} \hat{\psi}_{k,l}(z, t)\exp(i[kx + ly]) \tag{5}$$

and work out the vertical structure of each mode.

(c) Show that the surface temperature variance

$$\frac{1}{2}\int b^2\, dxdy \tag{6}$$

is conserved in time. What does this imply for the time dependence of

$$\frac{1}{2}\int (u_0^2 + v_0^2)\, dxdy \tag{7}$$

where (u_0, v_0) are the horizontal velocities at $z = 0$? Hint: use the Fourier expansion.

GFD Written Examination

January 2020

Please attempt all four problems.

1. **Geostrophic adjustment**

Consider a layer of fluid with density ρ, bounded above by a rigid lid, and overlying a deep abyss of motionless water with density $\rho + \Delta\rho$, where $\Delta\rho \ll \rho$. The equations describing this system are the reduced-gravity shallow water equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y}$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

where (u, v) are the horizontal velocities, h is the depth of the layer, and $g' = g\Delta\rho/\rho$ is the reduced gravity. Note that h may be 0 if there is an outcrop.

(a) Show that this system conserves the following form of potential vorticity (PV),

$$Q = \frac{f + \partial v/\partial x - \partial u/\partial y}{h}.$$

(b) Consider now an initial state with $u = v = 0$ and

$$h|_{t=0} = \begin{cases} H & x < 0 \\ 0 & x > 0 \end{cases}$$

with no variation in y, and assume f is constant. The front will slump, and after a long time will evolve to a new steady state. Use the equations of motion, and conservation of PV to derive steady, long-time solutions for v and h, denoting the new outcrop position at $x = a$. For boundary conditions, assume $h \to H$ as $x \to -\infty$ and $h = 0$ at $x = a$. Sketch the solution. Use volume conservation $\int_{-\infty}^{0}(H - h)\, dx = \int_0^a h\, dx$ to find a. What is the length scale that emerges?

(c) Compute the initial and final state available potential energies and final state kinetic energy. Did the adjustment conserve energy? If not, where did the extra energy go?

2. **Reduced gravity SWQG model**

(a) Starting from the shallow water equations in problem 1, but now assuming $f = f_0 + \beta y$, derive the β-plane reduced gravity shallow water quasigeostrophic (SWQG) system,

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad \text{where} \quad q = \nabla^2\psi - R^{-2}\psi + \beta y$$

and $R = \sqrt{g'H}/f$ is the internal deformation scale. What three scaling assumptions are needed to derive these equations?

(b) Show that $\psi = -Uy + A\cos(kx + \ell y - \omega t)$ is an exact solution to the *nonlinear* equations, so long as the appropriate dispersion relationship is satisfied. What is this dispersion relationship for these Rossby waves?

(c) Compute the westward phase speed of the Rossby waves. Note that the phase speed is not a simple Doppler shift of Rossby waves in a resting mean state — explain what is going on (consider the mean PV gradient and the mechanism of Rossby waves).

3. Hurricane Inflow

Consider an axisymmetric atmosphere with constant f. The equations of motion in cylindrical coordinates are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + f v_\theta$$
$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} - f v_r$$

where v_θ is the tangential velocity component and v_r is the radial velocity component.

(a) Show that the angular momentum

$$M = v_\theta r + \frac{f r^2}{2}$$

is conserved following a parcel trajectory.

(b) Determine the tangential velocity of an air parcel initially at rest at $r = r_0$ as it moves toward the center, conserving its angular momentum.

(c) If the angular momentum of the flow is uniform, density is uniform and the flow is in gradient wind balance (i.e. the balance that arises when you neglect the advection and time-derivative terms in the equation for v_r), determine the pressure field as function of the radius.

(d) Take $f = 10^{-4}$ s^{-1}, $\rho = 1$ kg m^{-3}, $r_0 = 10^6$ m and $p(r_0) = 10^5$ Pa. Determine the distances at which the pressure reaches 90000 Pa and 0 Pa. Comment on the possibility of $p = 0$.

4. **Barotropic instability conditions**

Consider the barotropic vorcitiy equation

$$\zeta_t + J(\psi, \zeta) + \beta\psi_x = 0, \quad \zeta = \nabla^2\psi$$

in a periodic-in-x channel of length L_x, with walls at $y = 0, L_y$. In the statements to follow, an overline denotes a zonal average $\overline{(\cdot)} = L_x^{-1}\int_0^{L_x}(\cdot)\,dx$, and a prime indicates a deviation from this mean ("eddies").

(a) First, assuming small-amplitude eddies, show that the evolution of the mean square eddy vorticity (the eddy enstrophy) is given by

$$\frac{\partial}{\partial t}\left(\frac{\overline{\zeta'^2}}{2}\right) = -\overline{v'\zeta'}\,\overline{\zeta}_y$$

(include the β term in $\overline{\zeta}_y$). Under what conditions will the eddy enstrophy grow?

(b) Now show that $\overline{v'\zeta'} = \partial_y(\overline{u'v'})$, and hence

$$\frac{1}{\overline{\zeta}_y}\frac{\partial}{\partial t}\left(\frac{\overline{\zeta'^2}}{2}\right) = \partial_y(\overline{u'v'}).$$

Integrate in y and derive a necessary condition for instability.