# 1 Norms Equivalency

Two norms in a finite-dimensional linear space $X$ (e.g.: $\mathbb{R}^n$), $\|\cdot\|_a$ and $\|\cdot\|_b$, are called equivalent if there is a constant $c$ such that for all $x$ in $X$,

$$\|\boldsymbol{x}\|_a \leq c\,\|\boldsymbol{x}\|_b\,, \qquad \|\boldsymbol{x}\|_b \leq c\,\|\boldsymbol{x}\|_a\,. \tag{1}$$

**(1.a)** Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, and we know that an algorithm produces a sequence of vectors $\{\boldsymbol{e}_n\}_{n\geq 1}$, $\|\boldsymbol{e}_n\|_a \to 0$ as $n \to \infty$. What could we conclude about $\|\boldsymbol{e}_n\|_b$'s behavior for $n \to \infty$?

**(1.b)** We first show that the vector norms on $\mathbb{R}^n$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, are equivalent. To do this prove the inequality:

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \sqrt{n}\,\|\boldsymbol{x}\|_\infty\,.$$

**(1.c)** The induced matrix norm on $\mathbb{R}^{n\times n}$: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent as well. Prove the inequality

$$\begin{aligned}\|\boldsymbol{A}\|_\infty &\leq \sqrt{n}\,\|\boldsymbol{A}\|_2\,,\\ \|\boldsymbol{A}\|_2 &\leq \sqrt{n}\,\|\boldsymbol{A}\|_\infty\,.\end{aligned}$$

**(1.d)** (Challenge) Prove that: in a finite-dimensional linear space, all norms are equivalent; that is, any two satisfy (1) with some $c$, depending on the pair of norms (Lax 2007, p.217).

One inequality is relatively simple, the other one requires some big theorems from analysis. Read about the proof in Lax's book if you are interested.

# 2 Condition numbers based on different norms

**(2.a)** Let $A \in \mathbb{R}^{n\times n}$ be defined by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Calculate $\kappa_1(A)$ and $\kappa_\infty(A)$. We see that a matrix can be well or ill-conditioned depending on the choice of norms.

**(2.b)** Indeed, we solve $A\boldsymbol{x} = \boldsymbol{b}$ and $A(\boldsymbol{x} + \Delta\boldsymbol{x}) = (\boldsymbol{b} + \Delta\boldsymbol{b})$ where

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad \Delta\boldsymbol{b} = \begin{bmatrix} \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Check that we have for both norms:

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \kappa(A)\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}.$$

## 3 Conditional number for the Hilbert matrix

The Hilbert matrix $H \in \mathbb{R}^{n\times n}$ is a matrix with entries

$$h_{ij} = \frac{1}{i+j-1}.$$

**(3.a)** Using MATLAB or Python, compute the 2-norm-based condition numbers for $n = 3, 5, 10, 20, 25$.

**(3.b)** Let's consider a relative right hand side perturbation $\delta\boldsymbol{b}$ of a linear system with $\|\delta\boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2 \approx 10^{-15}$. Write down the corresponding bounds $\|\delta\boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$ from the theory we discussed in class.

**(3.c)** Now, let's compute the actual error. Use the right-hand side vector with entries $b_i = \sum_{j=1}^{n}(j/(i+j-1))$ chosen such that the solution vector has entries $x_i = i$. Now, Compute the numerical solutions[1] $\boldsymbol{x}$, then re-compute $\boldsymbol{b} = H\boldsymbol{x}$ and compare the relative right-hand side error and the relative error in the solutions. How much are these better than the estimates you got from the condition number?

## References

Lax, Peter D. (Sept. 2007). *Linear Algebra and Its Applications.* John Wiley & Sons. ISBN: 978-0-471-75156-4.

[1]Note that all these computations contain tiny errors due to the final precision of computer computations.