# 1 Diagonally dominant matrix and pivoting

A matrix is called strictly (column) diagonal-dominant if the absolute value of the diagonal entry in each column is larger than the sum of the absolute values of the other entries in that column; i.e., for all $i$:

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ji}|$$

**(1.a)** Which of the following matrices is diagonally dominant?

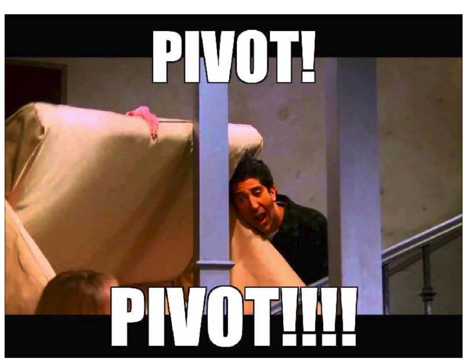
$$B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

**(1.b)** When computing the LU factorization of a strictly diagonally dominant matrix, why is pivoting never necessary?

1. First argue why the first column does not require pivoting. Then use Gaussian elimination to generate the required zeros in the first column
2. Show that, the submatrix you obtain when removing the first column and row is again strictly diagonally dominant.

**(1.c)** Let's show that an LU decomposition without pivoting exists in a different way:

1. Why are the leading principal submatrices of a strictly diagonally dominant matrix also strictly diagonally dominant?
2. Show that a diagonally dominant matrix is always invertible using the following argument: If $A$ is not invertible, then there must exists a vector $\vec{v} \neq 0$ such that $A\vec{v} = \vec{0}$. Call $r$ the largest (in absolute value) entry of $\vec{v}$ and consider multiplication of the $r$-th row.
3. Combine the previous two statements with a result from class to argue that the LU factorization of a strictly diagonally dominant matrix exists.



## 2 Schur complement

Assume $M \in \mathbb{R}^{(m+n)\times(m+n)}$ and we split them into blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$. We also assume that $M$ and all its leading submatrices are non-singular.

**(2.a)** Verify the formula

$$\begin{bmatrix} I & \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ & D - CA^{-1}B \end{bmatrix}$$

for "elimination" of the block $C$. The matrix $D - CA^{-1}B$ is known as the *Schur complement* of $A$ in $M$.

**(2.b)** Explain the above decomposition as a form of "block LU".

Extra: Write down the block LDU decomposition.