1 Convergence order of quadrature

(1.a) We would like to integrate a function on $[0, 1]$ using the composite trapezoid rule with sub-interval size h. We name the result T_h. How does the error ($e_h = |T_n - I|$) scale with h?

(1.b) In real application, we do not know the true value of the integral. To verify the order of convergence of our code, we can calculate this quantity:

$$\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}.$$

How does this quantity scale with h?

(1.c) We will show this using the code from last session.

2 Inner product

(2.a) Let's define a new inner product in $\mathbb{R}^3$ with a symmetric positive-definite matrix $W \in \mathbb{R}^{3\times 3}$ as follows:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_W := \boldsymbol{u}^T W \boldsymbol{v}$$

Show that this defines an inner product on $\mathbb{R}^3$ and write down the induced norm. If

$$W = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

give a vector that is orthogonal to $[1, 0, 0]^T$ in the W-inner product.

(2.b) Now let's consider a symmetric positive semi-definite matrix. Does the above definition still define an inner product and why/why not?

3 Orthogonal Polynomial

We assume an inner product on $[-1, 1]$ with weight $\omega(x) := 1 - x^2$, i.e.,

$$\langle p, q \rangle = \int_{-1}^{1} \omega(x) p(x) q(x) \, \mathrm{d}x.$$

We define the following polynomials, which are orthogonal with respect to this inner product:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = 2x, \quad \varphi_2(x) = \frac{15}{4}x^2 - \frac{3}{4}$$

(3.a) Verify that φ_0 and φ_1 are orthogonal on $[-1, 1]$ with respect to the weight ω.

(3.b) Are the φ_j's orthonormal under this inner product?

(3.c) Find the polynomial $p \in \boldsymbol{P}_2$ for which $\displaystyle\int_{-1}^{1} (1 - x^2)(p(x) - x^3)^2 \, \mathrm{d}x$ is minimal.