1 Solving Ax = b and LU factorization

We will study the LU-factorization of the matrix

$$A := \begin{bmatrix} 3 & 3 & 0 \\ 6 & 4 & 7 \\ -6 & -8 & 9 \end{bmatrix}$$

into the product

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- (1.a) In practical Gaussian elimination, the matrices L_k , are never formed and multiplied explicitly. The multipliers ℓ_{jk} are computed and stored directly into L, and the transformations L_k are then applied implicitly (Trefethen and Bau 1997, p.151).
 - 1. Verify that Gaussian elimination could be written as the following loop:

Algorithm 20.1. Gaussian Elimination without Pivoting
$$U=A,\ L=I$$
 for $k=1$ to $m-1$ for $j=k+1$ to m
$$\ell_{jk}=u_{jk}/u_{kk}$$

$$u_{j,k:m}=u_{j,k:m}-\ell_{jk}u_{k,k:m}$$

- 2. Code this loop in MATLAB. Apply it to the matrix A and obtain the L and U matrices.
- (1.b) Use the LU factorization to solve the linear system Ax = b with $b = [1, 0, 0]^{\top}$ using one forward and one backward substitution.
- (1.c) Use the LU factorization to compute the determinant of A. Recall that for two matrices of appropriate sizes, $\det(AB) = \det(A) \det(B)$.
- (1.d) In the matrix A defined above, replace the (2,2)-entry by 6. What is the rank of A after this modification? Attempt to compute the LU factorization of A. What do you observe? How might you "fix" the problem?

2 Diagonally dominant matrix and pivoting

A matrix is called strictly (column) diagonal-dominant if the the absolute value of the diagonal entry in each column is larger than the sum of the absolute values of the other entries in that

column; i.e., for all i:

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ji}|$$

(2.a) Which of the following matrices is diagonally dominant?

$$B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

(2.b) When computing the LU factorization of a strictly diagonally dominant matrix, why is pivoting never necessary?

- 1. First argue why the first column does not require pivoting. Then use Gaussian elimination to generate the required zeros in the first column
- 2. Show that, the submatrix you obtain when removing the first column and row is again strictly diagonally dominant.

(2.c) Let's show that an LU decomposition without pivoting exists in a different way:

- 1. Why are the leading principal submatrices of a strictly diagonally dominant matrix also strictly diagonally dominant?
- 2. Show that a diagonally dominant matrix is always invertible using the following argument: If \mathbf{A} is not invertible, then there must exists a vector $\mathbf{v} \neq 0$ such that $\mathbf{A}\mathbf{v} = \mathbf{0}$. Call r the largest (in absolute value) entry of \mathbf{v} and consider multiplication of the r-th row.
- 3. Combine the previous two statements with a result from class to argue that the LU factorization of a strictly diagonally dominant matrix exists.

3 Schur complement

Assume $M \in \mathbb{R}^{(m+n)\times(m+n)}$ and we split them into blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times n}$. We also assume that M and all its leading submatrices are non-singular.

(3.a) Verify the formula

$$\begin{bmatrix} I \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ & D - CA^{-1}B \end{bmatrix}$$

for "elimination" of the block C. The matrix $D-CA^{-1}B$ is known as the *Schur complement* of A in M.

Worksheet 3-2

(3.b) Explain the above decomposition as a form of "block LU".

Extra: Write down the block LDU decomposition.

References

Trefethen, Lloyd N. and David III Bau (June 1997). Numerical Linear Algebra. SIAM. ISBN: 978-0-89871-361-9.