1 Diagonally dominant matrix and pivoting

A matrix is called strictly (column) diagonal-dominant if the absolute value of the diagonal entry in each column is larger than the sum of the absolute values of the other entries in that column; i.e., for all i:

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ji}|$$

(1.a) Which of the following matrices is diagonally dominant?

$$B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

(1.b) When computing the LU factorization of a strictly diagonally dominant matrix, why is pivoting never necessary?

- 1. First argue why the first column does not require pivoting. Then use Gaussian elimination to generate the required zeros in the first column
- 2. Show that, the submatrix you obtain when removing the first column and row is again strictly diagonally dominant.

(1.c) Let's show that an LU decomposition without pivoting exists in a different way:

- 1. Why are the leading principal submatrices of a strictly diagonally dominant matrix also strictly diagonally dominant?
- 2. Show that a diagonally dominant matrix is always invertible using the following argument: If A is not invertible, then there must exists a vector $\vec{v} \neq 0$ such that $A\vec{v} = \vec{0}$. Call r the largest (in absolute value) entry of \vec{v} and consider multiplication of the r-th row.
- 3. Combine the previous two statements with a result from class to argue that the LU factorization of a strictly diagonally dominant matrix exists.



2 Schur complement

Assume $M \in \mathbb{R}^{(m+n)\times(m+n)}$ and we split them into blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times n}$. We also assume that M and all its leading submatrices are non-singular.

(2.a) Verify the formula

$$\begin{bmatrix} I \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ & D - CA^{-1}B \end{bmatrix}$$

for "elimination" of the block C. The matrix $D-CA^{-1}B$ is known as the Schur complement of A in M.

(2.b) Explain the above decomposition as a form of "block LU".

Extra: Write down the block LDU decomposition.