Material taken from [TB97, Lecture 6 and 10].

1 Projectors

A projector is a square matrix P that satisfies

$$P^2 = P.$$

(1.a) Assume P is a projector, show that $I - P$ is also a projector.

(1.b) We can show that

$$\text{range}(I - P) = \text{null}(P);$$
$$\text{null}(I - P) = \text{range}(P);$$
$$\text{range}(P) \cap \text{null}(P) = 0.$$

An orthogonal projector is a projector whose has the subspaces $\text{range}(P)$ and $\text{null}(P)$ orthogonal.

n.b.: An orthogonal projector P is not an orthogonal matrix! Why?

(1.c) Show that if $P = P^\top$ symmetric, the projector P is orthogonal (Hint: take one vector in $\text{range}(P)$ and one in $\text{null}(P)$, show that they must be orthogonal to each other).

The reverse direction holds as well. Therefore the two definition are equivalent.

(1.d) A special case of orthogonal projection is projection onto a vector:

$$P_v = \frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

Show that it is indeed an orthogonal projector with range $\text{span}(\boldsymbol{v})$.

(1.e) Another orthogonal projection is

$$P_{\perp v} = I - \frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

What is its null space? What is its range?

2   Geometric interpretation of Householder reflectors

(2.a) Name $H(\boldsymbol{v})$ the linear subspace orthogonal to the vector $\boldsymbol{v}$. A reflector across $H(\boldsymbol{v})$ is

$$F_{\boldsymbol{v}} = I - 2\frac{\boldsymbol{v}\boldsymbol{v}^{\top}}{\boldsymbol{v}^{\top}\boldsymbol{v}}.$$

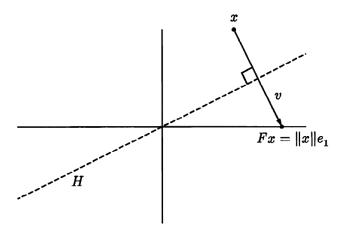Compare this with P_v and $P_{\perp v}$, interpret the formula geometrically.

Figure 10.1. *A Householder reflection.*

(2.b) To use Householder for QR decomposition, we want $F\boldsymbol{x} = c\boldsymbol{e}_1$. We know that $c = \pm \|\boldsymbol{x}\|_2$. Explain this geometrically.

(2.c) From this we have that

$$\boldsymbol{v} = \boldsymbol{x} - F\boldsymbol{x} = \boldsymbol{x} \pm \|\boldsymbol{x}\|\,\boldsymbol{e}_1.$$

From a geometric point of view, why is this the correct formula.

References

[TB97] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra*. SIAM, June 1997.