Worksheet 12

1 Inner product

(1.a) Let's define a new inner product in $\mathbb{R}^3$ with a symmetric positive-definite matrix $W \in \mathbb{R}^{3\times 3}$ as follows:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_W := \boldsymbol{u}^\top W \boldsymbol{v}$$

Show that this defines an inner product on $\mathbb{R}^3$ and write down the induced norm. If

$$W = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

give a vector that is orthogonal to $[1, 0, 0]^\top$ in the W-inner product.

(1.b) Now let's consider a symmetric positive semi-definite matrix. Does the above definition still define an inner product and why/why not?

2 Orthogonal polynomial with weight

We assume an inner product on $[-1, 1]$ with weight $\omega(x) := 1 - x^2$, i.e.,

$$\langle p, q \rangle = \int_{-1}^{1} \omega(x) p(x) q(x) \, \mathrm{d}x.$$

We define the following polynomials, which are orthogonal with respect to this inner product:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = 2x, \quad \varphi_2(x) = 5x^2 - 1$$

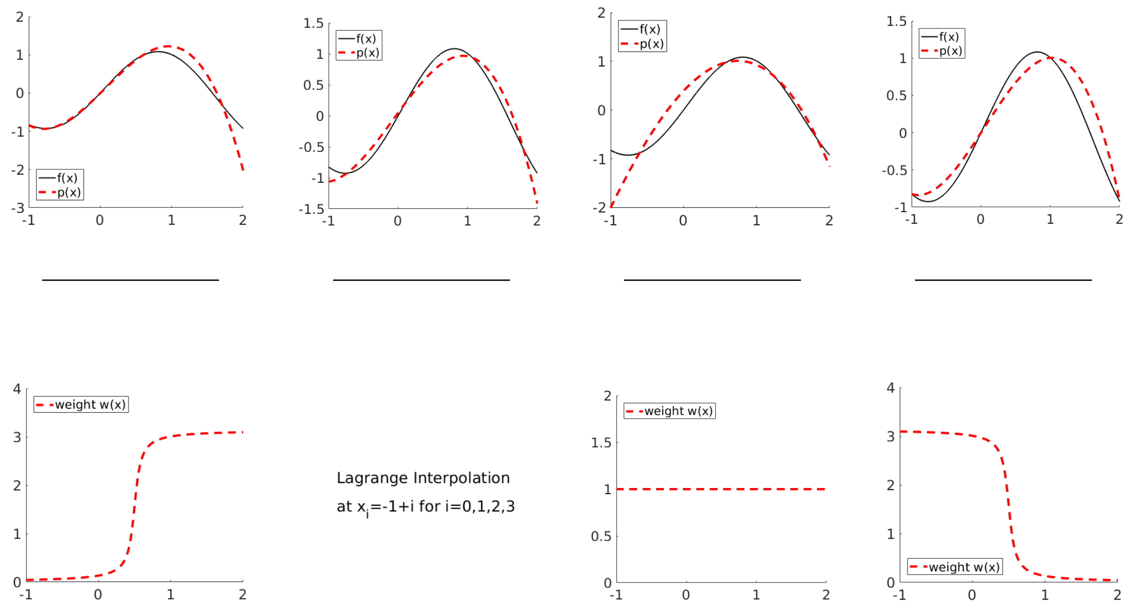
(2.a) Verify that φ_0 and φ_1 are orthogonal on $[-1, 1]$ with respect to the weight ω.

(2.b) Are the φ_j's orthonormal under this inner product?

(2.c) Find the polynomial $p \in \boldsymbol{P}_2$ for which $\int_{-1}^{1} (1 - x^2)(p(x) - x^3)^2 \, \mathrm{d}x$ is minimal.

3 Best 2-norm approximation

(3.a) The upper row in the below figure shows a function f together with a polynomial approximation. For three plots, the optimal best 2-norm fit for three different weights $w(x)$ is used, and one is the result of an Lagrange interpolation. Match the approximations in the upper row with the information (weight functions or interpolation points) in the lower row.



(A) (B) (C) (D)

(3.b) Let $\{\varphi_0, \varphi_1, \varphi_2\}$ be a system of ortho*normal* polynomials on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$ given by

$$\varphi_0(x) = \sqrt{\frac{2}{\pi}}, \quad \varphi_1(x) = 2x\sqrt{\frac{2}{\pi}}, \quad \varphi_2(x) = (4x^2 - 1)\sqrt{\frac{2}{\pi}}.$$

Given $f(x) = \frac{2}{\sqrt{1-x^2}}$, find the polynomial best fit of degree 2 in the weighted 2-norm.