Worksheet 4

1 Calculating Pivoted-LU

Compute by hand an LU factorization with pivoting ($PA = LU$) of the matrix:

$$A := \begin{bmatrix} -2 & 0 & 6 \\ -3 & 6 & 9 \\ -1 & 4 & 5 \end{bmatrix}.$$

Double check your result using MATLAB's or Python's LU-function!

2 Matrix Norms Basics

(2.a) Compute $\|A\|_\infty$ and $\|A\|_1$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 7 & 2 & 3 & 5 \\ 2 & -4 & 3 & 8 \\ -3 & 5 & 3 & 1 \end{bmatrix}.$$

(2.b) Show that for symmetric positive definite (i.e., all eigenvalues are positive) matrices $A \in \mathbb{R}^{n\times n}$, the 2-norm condition number can also be computed as the ratio between the largest and the smallest eigenvalue of A, i.e.: $\kappa_2(A) = \lambda_{\max}/\lambda_{\min}$. Hint: Think about what the largest eigenvalue of A^{-1} is.

3 Norms Equivalency

Two norms in a finite-dimensional linear space X (e.g.: $\mathbb{R}^n$), $\|\cdot\|_a$ and $\|\cdot\|_b$, are called equivalent if there is a constant c such that for all x in X,

$$\|x\|_a \leq c\,\|x\|_b\,, \qquad \|x\|_b \leq c\,\|x\|_a\,. \tag{1}$$

(3.a) Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, and we know that an algorithm produce a sequence of vectors $\{e_n\}_{n\geq 1}$, $\|e_n\|_a \to 0$ as $n \to \infty$. What could we conclude about $\|e_n\|_b$'s behavior for $n \to \infty$.

(3.b) We first show that the vector norms on $\mathbb{R}^n$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, are equivalent. To do this prove the inequality:

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty\,.$$

(3.c) The induced matrix norm on $\mathbb{R}^{n\times n}$: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent as well. Prove the inequality

$$\|A\|_\infty \leq \sqrt{n}\,\|A\|_2\,,$$
$$\|A\|_2 \leq \sqrt{n}\,\|A\|_\infty\,.$$

(3.d) (Challenge) Prove that: in a finite-dimensional linear space, all norms are equivalent; that is, any two satisfy (1) with some c, depending on the pair of norms [Lax07, p.217].

One inequality is relative simple, the other one requires some big theorems from analysis. Read about the proof in Lax's book if you are interested.

References

[Lax07] Peter D. Lax. *Linear Algebra and Its Applications.* John Wiley & Sons, September 2007.