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Worksheet 2

Numerical Analysis

1 Solving $Ax = b$ and LU factorization

We will study the LU-factorization of the matrix

$$A := \begin{bmatrix} 3 & 3 & 0 \\ 6 & 4 & 7 \\ -6 & -8 & 9 \end{bmatrix}$$

into the product

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

(1.a) In practical Gaussian elimination, the matrices L_k, are never formed and multiplied explicitly. The multipliers ℓ_{jk} are computed and stored directly into L, and the transformations L_k are then applied implicitly [TB97, p.151].

1. Verify that Gaussian elimination could be written as the following loop:

Algorithm 20.1. Gaussian Elimination without Pivoting

$U = A,\ L = I$
for $k = 1$ **to** $m - 1$
  for $j = k + 1$ **to** m
    $\ell_{jk} = u_{jk}/u_{kk}$
    $u_{j,k:m} = u_{j,k:m} - \ell_{jk} u_{k,k:m}$

2. Apply this loop at the matrix A and obtain the L and U matrices.

(1.b) Use the LU factorization to solve the linear system $Ax = b$ with $b = [1, 0, 0]^\top$ using one forward and one backward substitution.

(1.c) Use the LU factorization to compute the determinant of A. Recall that for two matrices of appropriate sizes, $\det(AB) = \det(A)\det(B)$.

2 Block matrices and MATLAB matrix operations

(2.a) Let's practice creating matrices in the computer:

1. Create a matrix of all ones in your favorite coding language.
2. Create a matrix where its entries are independent standard Gaussian random variables (i.e.: with density $\mathcal{N}(0,1)$).

(2.b) Now let's operate on these matrices:

1. Sum two matrices.
2. Multiply a matrix with an appropriately sized vector.
3. Multiply two matrices (Try some non-square matrices).

(2.c) Suppose we split up a matrix of size $\mathbb{R}^{(m+n)\times(m+n)}$ into blocks:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where $A_{11} \in \mathbb{R}^{n\times n}$, $A_{22} \in \mathbb{R}^{m\times m}$, $A_{12} \in \mathbb{R}^{n\times m}$, and $A_{21} \in \mathbb{R}^{m\times n}$.

We have the block matrix multiplication formula:

$$AB = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{22}B_{12} + A_{22}B_{22} \end{array}\right]$$

You could try to prove this at home. In session we will verify this formula numerically by creating examples of A and B.

1. Make A and B to be Gaussian random matrices.
2. How do you compare two matrices numerically?

(2.d) (Challenge for you) Could you calculate the determinant of $A \in \mathbb{R}^{(m+n)\times(m+n)}$ by calculating the determinants of only m-by-m and n-by-n matrices?

Hint: try Gaussian elimination on block matrices. Then follow the procedure in (1.c).

References

[TB97] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra.* SIAM, June 1997.