1 QR decomposition via Householder

(1.a) Construct the QR factorization of the following matrix using Householder reflectors:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}.$$

(1.b) Use the factorization to determine $|\det(A)|$

2 Gershgorin disks and the power method

Consider the matrix

$$A = \begin{bmatrix} -6 & 2 & 0.3 & 0 & -0.7 \\ 2 & -4 & 0.1 & 0.05 & 0 \\ 0.3 & 0.1 & 2 & 0.1 & 0.1 \\ 0 & 0.05 & 0.1 & 4 & 0 \\ -0.7 & 0 & 0.1 & 0 & 6 \end{bmatrix}$$

and recall the definition of the Gershgorin disks:

$$D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}.$$

(2.a) Argue that all eigenvalues of A are real.

(2.b) What are the Gershgorin disks for A? Use them to give a set, $D \subset \mathbb{R}$, that contains all eigenvalues of A.

(2.c) Can you conclude that the eigenvalue with the largest absolute value is simple?

(2.d) Argue that A is invertible. Conclude that all diagonally dominant matrix is invertible.

(2.e) True or False? Let $A \in \mathbb{R}^{n \times n}$ and D_i, $i = 1, 2, \ldots, n$, be the Gerschgorin disks of A. If $0 \in \bigcup_{i=1}^{n} D_i$ then A is singular.

(2.f) Write down the first iteration of the power method starting from $\boldsymbol{x}_0 = (0, 0, 0, 0, 1)^T$. You don't need to normalize. Explain why $\boldsymbol{x}_0 = \mathbf{0}$ is not a suitable starting point.

(2.g) The eigenvalues of A, after rounding, are $\{-7, -3, 2, 4, 6\}$. Which eigenvalue direction will the sequence of the previous question converge to?

3 Eigenvectors as stationary points of Rayleigh quotient

For H a real symmetric matrix, we define Rayleigh quotient as a function $\mathbb{R}^n \to \mathbb{R}$:

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top H \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} = \frac{q(\boldsymbol{x})}{p(\boldsymbol{x})}.$$

We will show that $\boldsymbol{v}$ is a stationary point (i.e.: $\nabla R(\boldsymbol{v}) = 0$) of the Rayleigh quotient if and only if it is an eigenvector of H (cf. [Lax07, p.114-116] and [TB97, p.203-204]).

(3.a) To characterize a point such that $\nabla R(\boldsymbol{v}) = 0$, we need to know the gradient of $R(\boldsymbol{x})$ at $\boldsymbol{v}$. We could do this, but an alternative approach is to take $t \in \mathbb{R}$ and calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} R(\boldsymbol{v} + t\boldsymbol{y})\bigg|_{t=0}$$

for all $\boldsymbol{y} \in \mathbb{R}^n$. In particular, we can get the gradient by picking $\boldsymbol{y} = \boldsymbol{e}_i$.

(3.b) Using the above calculation, shows the iff claim in the main text of the problem.

References

[Lax07] Peter D. Lax. *Linear Algebra and Its Applications.* John Wiley & Sons, September 2007.

[TB97] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra.* SIAM, June 1997.