Ryan Shìjié Dù
November 3rd, 2023

Worksheet 8

1 Computing eigenvalues via the Power Iteration

Given is the following matrix:

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix},$$

It has eigenvalues and eigenvectors:

$$\lambda_1 = 0,\ \boldsymbol{v}_1 = \begin{bmatrix} 0.41 \\ -0.82 \\ 0.41 \end{bmatrix}, \quad \lambda_2 = -6,\ \boldsymbol{v}_2 = \begin{bmatrix} 0.71 \\ 0.0 \\ -0.71 \end{bmatrix}, \quad \lambda_3 = 3,\ \boldsymbol{v}_3 = \begin{bmatrix} -0.58 \\ -0.58 \\ -0.58 \end{bmatrix}.$$

(1.a) Calculate the first iterate of the power method when $\boldsymbol{x}_0 = (0, 1, 1)^T$.

(1.b) Which eigenvalue direction will the sequence defined in (1.a) converge to?

(1.c) Give an initialization vector such that the power method does *not* converge to the direction of the largest (in absolute value) eigenvalue.

(1.d) Write a simple program implementing the power method for the matrix A.

1. Use the Rayleigh quotient to calculate estimates of the eigenvalues for each iteration.
2. What is the order of convergence of the eigenvector estimates and the eigenvalue estimates? What is the speed of convergence?
3. Could you explain the relationship between the two convergence speeds? (Hint: last week we showed that eigenvectors $\boldsymbol{v}$ are stationary points of the Rayleigh quotient)

2 The Inverse Iteration

Take A to be the matrix above, and let $\theta \in \mathbb{R}$ and let $\boldsymbol{x}_0 \in \mathbb{R}^3$.

(2.a) Define the *Inverse Iteration* (also called *Inverse Power Method*) to calculate eigenvectors of A near θ.

(2.b) If $\theta = 2$, where will the sequence defined in (i) converge to and why?

(2.c) If $\theta = -2$, where will the sequence defined in (i) converge to and why?

(2.d) Write a simple program implementing the inverse power method. Do the same sub-tasks as the ones in (1.d).

3 The Rayleigh Quotient Iteration

It is irresistible to use the eigenvalues estimates from the Rayleigh quotient to update θ for each step in the inverse iteration. The resulting algorithm is the Rayleigh Quotient Iteration (Trefethen and Bau 1997):

Algorithm 27.3. Rayleigh Quotient Iteration

$v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$
$\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$ = corresponding Rayleigh quotient
for $k = 1, 2, \dots$
  Solve $(A - \lambda^{(k-1)} I) w = v^{(k-1)}$ for w — apply $(A - \lambda^{(k-1)} I)^{-1}$
  $v^{(k)} = w / \|w\|$ — normalize
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ — Rayleigh quotient

(3.a) Implement the Rayleigh Quotient Iteration.

1. What is the order of convergence of the eigenvector estimates and the eigenvalue estimates?
2. Could you explain this (high) order of convergence?

References

Trefethen, Lloyd N. and David III Bau (June 1997). *Numerical Linear Algebra*. SIAM. ISBN: 978-0-89871-361-9.