1 Conditional numbers based on different norms

(1.a) Let $A \in \mathbb{R}^{n\times n}$ be defined by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Calculate $\kappa_1(A)$ and $\kappa_\infty(A)$. We see that a matrix can be well or ill-conditioned depending on the choice of norms.

(1.b) Indeed, we solve $A\boldsymbol{x} = \boldsymbol{b}$ and $A(\boldsymbol{x} + \Delta\boldsymbol{x}) = (\boldsymbol{b} + \Delta\boldsymbol{b})$ where

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad \Delta\boldsymbol{b} = \begin{bmatrix} \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Check that we have for both norms:

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \kappa(A)\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}.$$

2 Conditional numbers and pivoted LU

(2.a) Solve the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A := \begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

What is $\kappa_\infty(A)$?

Consider a small perturbation $\Delta\boldsymbol{b} = [10^{-3}, 0]^\top$ being added to the right hand side, and solve again. Repeat with $\Delta\boldsymbol{b} = [0, 10^{-3}]^\top$. You should see that small perturbations can, but do not have to have a large effect even for badly conditioned systems.

(2.b) Verify the following LU decomposition of a matrix A without pivoting:

$$A := \begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix} \begin{bmatrix} 10^{-4} & 1 \\ 0 & 1 - 10^4 \end{bmatrix}$$

We have seen in the previous problem that solving a system with the matrix L is sensitive to errors, i.e., it is poorly conditioned. However, the original A matrix is well-conditioned.

Now the LU factorization of A with pivoting is

$$PA = \begin{bmatrix} 1 & 1 \\ 10^{-4} & 1 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ 10^{-4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 - 10^{-4} \end{bmatrix}$$

We see that the LU factors with pivoting are better conditioned.

3 Conditional Number for Solving Linear System

(3.a) Find $\|A\|_2$ for the matrix

$$A = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix},$$

where $\varepsilon \in (0, 1)$ (Hint: for symmetric matrices A, the eigenvalues of $A^T A$ are simply the squares of the eigenvalues of A).

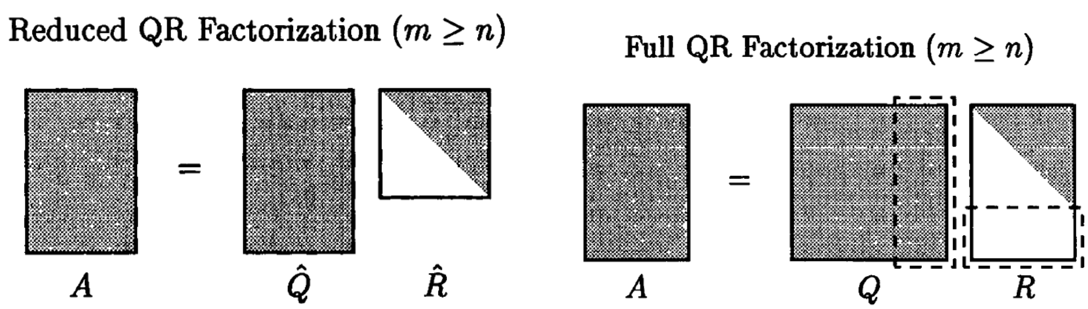
(3.b) Continued from previous item: Suppose that you have two systems

$$\begin{aligned} x_1 + \varepsilon x_2 &= b_1 \\ \varepsilon x_1 + x_2 &= b_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{x}_1 + \varepsilon \tilde{x}_2 &= \tilde{b}_1 \\ \varepsilon \tilde{x}_1 + \tilde{x}_2 &= \tilde{b}_2 \end{aligned}$$

where $\tilde{\boldsymbol{b}} = (\tilde{b}_1, \tilde{b}_2)^T$ is approximately equal to $\boldsymbol{b} = (b_1, b_2)^T$, with a 5% relative error, that is $\frac{\|\tilde{\boldsymbol{b}} - \boldsymbol{b}\|_2}{\|\boldsymbol{b}\|_2} \leq 0.05$. Find an upper bound for the relative error $\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}$ where $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2)^T$ and $\boldsymbol{x} = (x_1, x_2)^T$. This upper bound will depend on ε.

4 Two forms of QR

(4.a) We have two forms of QR:



(4.b) We can interpret the formula for the solution of the least-squared problem

$$\hat{R}\boldsymbol{x} = \hat{Q}^\top \boldsymbol{b}$$

by using the full form of QR.

5 Least squares and infections disease

Let us assume an infectious disease with the following reported new infections I_i on each day t_i, for $i = 1, \dots, 10$. Using least squares fitting, we would like to understand the nature of this growth.

Table 1: Number of new infections I_i on days t_i.

t_i:	1	2	3	4	5	6	7	8	9	10
I_i:	14	20	21	24	15	45	67	150	422	987

We consider two models to describe the connection between time (i.e., days) t and the number of new infections, both with 3 unknown parameters (a, b, c):

$$I(t) = a + bt + ct^2 \qquad \text{(polynomial model)}$$

$$I(t) = a + bt + c\exp(t) \qquad \text{(exponential model)}$$

Our goal is to figure out which model describes the progression of the infections better, and we use least squares fitting to figure that out. Note that if a model would fit the data perfectly, $I(t_i) = I_i$ for all i. In general, you will not be able to find parameters that satisfy this, and thus have to use least squares fitting (sometimes this is also called *regression*).

(5.a) Formulate, assuming the polynomial model, the least squares problem for the parameters $\boldsymbol{x} = [a, b, c]^T$ by specifying the matrices A and the vector $\boldsymbol{b}$:

$$\min_{\boldsymbol{x} \in \mathbb{R}^3} \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2$$

(5.b) Same as above, but for the exponential model.

(5.c) Use a QR-factorization in MATLAB or Python to solve these problems and plot the data as points, as well as the model as a line. Repeat using the normal equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$.

(5.d) To decide which model describes the data better, we need to compute the distance between the model and the data points. Take a look at the proof from class for how the QR factorization can be used to solve least squares problems. In particular, we found that:

$$\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 \geq \|\boldsymbol{b}_2\|_2^2,$$

where $\boldsymbol{b}_2 = \hat{\hat{Q}}^\top \boldsymbol{b}$. We also found that this inequality is an equality if $\boldsymbol{x}$ solves the least squares problem. Thus, the norm or $\boldsymbol{b}_2$ is a measure of how well the model fits the data. Use this to decide which of the two models above describes the data better.