

1 Convergence order of quadrature

(1.a) We would like to integrate a function on $[0, 1]$ using the composite trapezoid rule with sub-interval size h . We name the result T_h . How does the error ($e_h = |T_n - I|$) scale with h ?

(1.b) In real application, we do not know the true value of the integral. To verify the order of convergence of our code, we can calculate this quantity:

$$\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}.$$

How does this quantity scale with h ?

(1.c) We will show this using the code from last session.

2 Inner product

(2.a) Let's define a new inner product in \mathbb{R}^3 with a symmetric positive-definite matrix $W \in \mathbb{R}^{3 \times 3}$ as follows:

$$\langle \mathbf{u}, \mathbf{v} \rangle_W := \mathbf{u}^\top W \mathbf{v}$$

Show that this defines an inner product on \mathbb{R}^3 and write down the induced norm. If

$$W = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

give a vector that is orthogonal to $[1, 0, 0]^\top$ in the W -inner product.

(2.b) Now let's consider a symmetric positive semi-definite matrix. Does the above definition still define an inner product and why/why not?

3 Orthogonal Polynomial

We assume an inner product on $[-1, 1]$ with weight $\omega(x) := 1 - x^2$, i.e.,

$$\langle p, q \rangle = \int_{-1}^1 \omega(x) p(x) q(x) dx.$$

We define the following polynomials, which are orthogonal with respect to this inner product:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = 2x, \quad \varphi_2(x) = 5x^2 - 1$$

(3.a) Verify that φ_0 and φ_1 are orthogonal on $[-1, 1]$ with respect to the weight ω .

(3.b) Are the φ_j 's orthonormal under this inner product?

(3.c) Find the polynomial $p \in \mathbf{P}_2$ for which $\int_{-1}^1 (1 - x^2)(p(x) - x^3)^2 dx$ is minimal.