Worksheet 7

1 Geometric interpretation of Householder reflectors

(1.a) Name $H(\boldsymbol{v})$ the linear subspace orthogonal to the vector $\boldsymbol{v}$. A reflector across $H(\boldsymbol{v})$ is

$$F_{\boldsymbol{v}} = I - 2\frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

Compare this with P_v and $P_{\perp v}$, and interpret the formula geometrically.

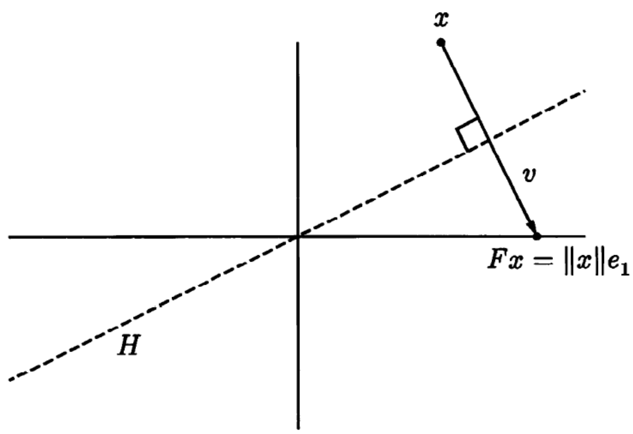

Figure 10.1. *A Householder reflection.*

(1.b) To use Householder for QR decomposition, we want $F\boldsymbol{x} = c\boldsymbol{e}_1$. We know that $c = \pm\|\boldsymbol{x}\|_2$. Explain this geometrically.

(1.c) From this, we have that

$$\boldsymbol{v} = \boldsymbol{x} - F\boldsymbol{x} = \boldsymbol{x} \pm \|\boldsymbol{x}\|\,\boldsymbol{e}_1.$$

From a geometric point of view, why is this the correct formula?

2 Gershgorin disks and the power method

Consider the matrix

$$A = \begin{bmatrix} -6 & 2 & 0.3 & 0 & -0.7 \\ 2 & -4 & 0.1 & 0.05 & 0 \\ 0.3 & 0.1 & 2 & 0.1 & 0.1 \\ 0 & 0.05 & 0.1 & 4 & 0 \\ -0.7 & 0 & 0.1 & 0 & 6 \end{bmatrix}$$

and recall the definition of the Gershgorin disks:

$$D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}|\}.$$

(2.a) Argue that all eigenvalues of A are real.

(2.b) What are the Gershgorin disks for A? Use them to give a set, $D \subset \mathbb{R}$, that contains all eigenvalues of A.

(2.c) Can you conclude that the eigenvalue with the largest absolute value is simple?

(2.d) Argue that A is invertible. Conclude that all diagonally dominant matrix is invertible.

(2.e) True or False? Let $A \in \mathbb{R}^{n\times n}$ and D_i, $i = 1, 2, \ldots, n$, be the Gerschgorin disks of A. If $0 \in \bigcup_{i=1}^n D_i$ then A is singular.

(2.f) Write down the first iteration of the power method starting from $\boldsymbol{x}_0 = (0, 0, 0, 0, 1)^T$. You don't need to normalize. Explain why $\boldsymbol{x}_0 = \mathbf{0}$ is not a suitable starting point.

(2.g) The eigenvalues of A, after rounding, are $\{-7, -3, 2, 4, 6\}$. Which eigenvalue direction will the sequence of the previous question converge to?

3 Eigenvectors as stationary points of Rayleigh quotient

For H a real symmetric matrix, we define Rayleigh quotient as a function $\mathbb{R}^n \to \mathbb{R}$:

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top H \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} = \frac{q(\boldsymbol{x})}{p(\boldsymbol{x})}.$$

We will show that $\boldsymbol{v}$ is a stationary point (i.e.: $\nabla R(\boldsymbol{v}) = 0$) of the Rayleigh quotient if and only if it is an eigenvector of H (cf. Lax 2007, p.114-116 and Trefethen and Bau 1997, p.203-204).

(3.a) To characterize a point such that $\nabla R(\boldsymbol{v}) = 0$, we need to know the gradient of $R(\boldsymbol{x})$ at $\boldsymbol{v}$. We could do this, but an alternative approach is to take $t \in \mathbb{R}$ and calculate

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} R(\boldsymbol{v} + t\boldsymbol{y})\right|_{t=0}$$

for all $\boldsymbol{y} \in \mathbb{R}^n$. In particular, we can get the gradient by picking $\boldsymbol{y} = \boldsymbol{e}_i$.

(3.b) Using the above calculation, show the iff claim in the main text of the problem.

References

Lax, Peter D. (Sept. 2007). *Linear Algebra and Its Applications*. John Wiley & Sons. ISBN: 978-0-471-75156-4.

Trefethen, Lloyd N. and David III Bau (June 1997). *Numerical Linear Algebra*. SIAM. ISBN: 978-0-89871-361-9.