Ryan Shìjié Dù
November 17th, 2023

Worksheet 10

1 Polynomial interpolation and linear algebra

In lecture you have learned the theorem which states:

For $n \geq 1$ and distinct $n+1$ data pairs $(x_0, y_0), \ldots, (x_n, y_n)$ there exists a unique $p_n(x) \in P_n$, an n-th order polynomial such that $p_n(x_i) = y_i$ for $i = 0, \ldots, n$.

We will try to prove this using linear algebra.

(1.a) We can write an n-th order polynomial as

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

This gives us $n+1$ free variables to solve. Frame the problem of finding $p_n(x)$ s.t. $p_n(x_i) = y_i$ for $i = 0, \ldots, n$ as a matrix problem $X\boldsymbol{a} = \boldsymbol{y}$.

(1.b) Show that since $x_i \neq x_j$ for all i, j, we have the matrix X we constructed has full rank.

(1.c) Think about the uniqueness and existence claim in the theorem in linear algebra language.

2 Interpolation basics

- True or False? For the nodes $x_0 = 0, x_1 = 1, x_2 = 2$, the Lagrange interpolation polynomial $L_0(x)$ is $-x^2 + 1$.
- True or False? We compute the Hermite interpolant with 3 distinct nodes of a function f that is a polynomial of degree 4. Then this Hermite interpolant is identical to f. (In short: Hermite interpolation with 3 nodes is exact for polynomials of degree 4.)
- True or False? Hermite interpolation with 4 distinct nodes is exact for polynomials of degree 6.
- True or false: Let p_n be the Lagrange interpolant to a function f with $n+1$ interpolation points, and $e_n(x) = |p_n(x) - f(x)|$. The interpolation error $\|e_n\|_\infty$ *always* gets arbitrarily small for large n, i.e., $\|e_n\|_\infty \to 0$ as $n \to \infty$.

3 Lagrange interpolation polynomial example

Let $x_0, \ldots, x_n$ be distinct interpolation nodes, and let

$$p_n(x) = \sum_{k=0}^{n} L_k(x)(x_k)^j,$$

where j is an integer and $n \geq j > 0$. What is the $p_n(x)$ function? What are the values of $p_n(0)$ and $p_n(1)$?

4 Hermite interpolation polynomial example

Recall that the Hermite interpolation of a function f at the points x_0, x_1, x_2 has the form

$$p(x) = \sum_{j=0}^{2} H_j(x) f(x_j) + \sum_{j=0}^{2} K_j(x) f'(x_j).$$

(4.a) Show that the polynomial

$$-\frac{1}{\pi}x^2 + x$$

is the Hermite interpolation polynomial of $f(x) := \sin(x)$ based on the nodes $x_0 = 0$, $x_1 = \pi$.

(4.b) Show that the polynomial $K_2(x)$ in this representation for $x_0 = 0, x_1 = 1, x_2 = 2$ is given by

$$\frac{1}{4}x^5 - x^4 + \frac{5}{4}x^3 - \frac{1}{2}x^2.$$

5 Error bound for interpolation

This requires MATLAB or Python: we interpolate the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \exp(3x)$ using the nodes $x_i = i/2$, $i = 0, 1, 2$ by a quadratic polynomial $p_2 \in \boldsymbol{P}_2$. Compare the exact interpolation error $E_f(x) := f(x) - p_2(x)$ at $x = 3/4$ with the estimate

$$|E_f(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|,$$

where $M_{n+1} = \max_{z \in [0,1]} |f^{(n+1)}(z)|$, $f^{(n+1)}$ is the $(n+1)$st derivative of f, and $\pi_{n+1}(x) = (x - x_0)(x - x_1)(x - x_2)$.