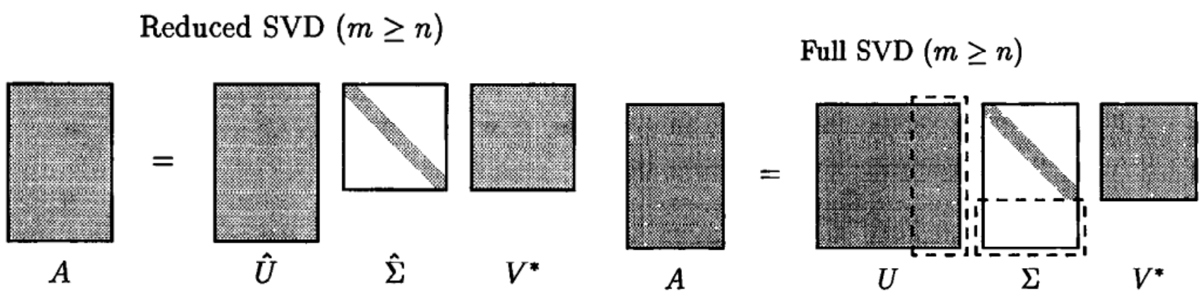
Worksheet 10

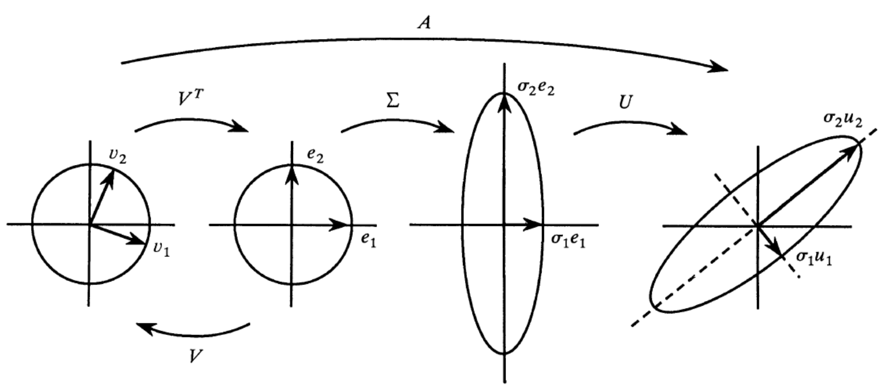
1 Singular Value Decomposition (SVD) basics

(1.a) Like QR, SVD has the full and reduced form [TB97]:



Note that the full form has U which spans the whole of $\mathbb{R}^n$.

(1.b) We can interpret the full form of SVD as a change of basis, then a scaling, and then another change of basis (Figure from [Str93]).



2 Some properties of SVD

We list (and attempt to prove) some properties of SVD:

(2.a) $\text{range}(A) = \text{span}(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r)$ and $\text{null}(A) = \text{span}(\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n)$.

(2.b) The rank of A is r, the number of nonzero singular values.

(2.c) We have $\|A\|_2 = \sigma_1$, the largest singular value. And $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$.

You will show the second property in your homework.

3 Low-rang approximation using SVD

(3.a) Show that A is the sum of r *rank-one* matrices:

$$A = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^\top.$$

(3.b) For any ν with $0 \leq \nu \leq r$, define

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^\top.$$

Then we have

$$\|A - A_\nu\|_2 = \inf_{\substack{B \in \mathbb{R}^{m \times n} \\ \operatorname{rank}(B) \leq \nu}} \|A - B\|_2 = \sigma_{\nu+1}.$$

Note that a similar theorem is also true for the Frobenius norm.

4 Polynomial interpolation and linear algebra

In lecture you have learned the theorem which states:

For $n \geq 1$ and distinct $n+1$ data pairs $(x_0, y_0), \ldots, (x_n, y_n)$ there exists a unique $p_n(x) \in P_n$, an n-th order polynomial such that $p_n(x_i) = y_i$ for $i = 0, \ldots, n$.

We will try to prove this using linear algebra.

(4.a) We can write an n-th order polynomial as

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

This gives us $n+1$ free variables to solve. Frame the problem of finding $p_n(x)$ s.t. $p_n(x_i) = y_i$ for $i = 0, \ldots, n$ as a matrix problem $X\boldsymbol{a} = \boldsymbol{y}$.

(4.b) Show that since $x_i \neq x_j$ for all i, j, we have the matrix X we constructed has full rank.

(4.c) Think about the uniqueness and existence claim in the theorem in linear algebra language.

References

[Str93] Gilbert Strang. The Fundamental Theorem of Linear Algebra. *The American Mathematical Monthly*, 100(9):848–855, 1993.

[TB97] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra.* SIAM, June 1997.