1 Polynomial interpolation basics

True or False?

- For the nodes $x_0 = 0, x_1 = 1, x_2 = 2$, the Lagrange interpolation polynomial $L_0(x)$ is $-x^2 + 1$.
- We compute the Hermite interpolant with 3 distinct nodes of a function f that is a polynomial of degree 4. Then this Hermite interpolant is identical to f. (In short: Hermite interpolation with 3 nodes is exact for polynomials of degree 4.)
- Hermite interpolation with 4 distinct nodes is exact for polynomials of degree 6.

2 Lagrange interpolation polynomial example

Let $x_0, \dots, x_n$ be distinct interpolation nodes, and let

$$p_n(x) = \sum_{k=0}^{n} L_k(x)(x_k)^j,$$

where j is an integer and $n \geq j > 0$. What are the values of $p_n(0)$ and $p_n(1)$?

3 Hermite interpolation polynomial example

Recall that the Hermite interpolation of a function f at the points x_0, x_1, x_2 has the form

$$p(x) = \sum_{j=0}^{2} H_j(x) f(x_j) + \sum_{j=0}^{2} K_j(x) f'(x_j).$$

(3.a) Show that the polynomial

$$-\frac{1}{\pi}x^2 + x$$

is the Hermite interpolation polynomial of $f(x) := \sin(x)$ based on the nodes $x_0 = 0$, $x_1 = \pi$.

(3.b) Show that the polynomial $K_2(x)$ in this representation for $x_0 = 0, x_1 = 1, x_2 = 2$ is given by

$$\frac{1}{4}x^5 - x^4 + \frac{5}{4}x^3 - \frac{1}{2}x^2.$$

4 Deriving a new quadrature rule

Given $f : [0,1] \to \mathbb{R}$, you want to derive a new quadrature rule that does uses not only function values, but also gradient values:

$$\int_0^1 f(x)\,\mathrm{d}x \approx \alpha_0 f(0) + \alpha_1 f'(0) + \alpha_2 f(1). \tag{1}$$

(4.a) First, find polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$, with the following properties:

$$\begin{aligned} J_0(0) &= 1, \quad J_0'(0) = 0, \quad J_0(1) = 0 \\ J_1(0) &= 0, \quad J_1'(0) = 1, \quad J_1(1) = 0 \\ J_2(0) &= 0, \quad J_2'(0) = 0, \quad J_2(1) = 1. \end{aligned}$$

(*Hint:* For each J_i, make an ansatz for a quadratic polynomial using the monomial basis.)

Given f, you can now define a polynomial approximation $p \in \mathcal{P}_2$ via

$$p(x) = f(0)J_0(x) + f'(0)J_1(x) + f(1)J_2(x). \tag{2}$$

The polynomial p is an approximation to f in the sense that $p(0) = f(0)$, $p'(0) = f'(0)$ and $p(1) = f(1)$.

(4.b) Use the polynomial p derived in (2) and the same method used to derive the Newton-Cotes quadrature rules, to find the coefficients α_0, α_1 and α_2 in (1).

(4.c) Use your new quadrature rule to approximate $\int_0^1 \exp(2x)\sin^2(x)\,\mathrm{d}x$, and also compare with Simpson's rule. The exact value of this integral is $1.2668\ldots$

5 Trapezoidal rule for smooth periodic functions

We investigate how the (composite) trapezoidal rule performs for smooth, periodic functions. Consider integrating the smooth, periodic function $f(x) = e^{\sin x}$ over a single period. The exact value of the integral is

$$I(f) = \int_0^{2\pi} e^{\sin x}\,\mathrm{d}x = 7.95492652101284527\ldots.$$

(5.a) Write down the composite trapezoidal rule $T_N(f)$ on equispaced nodes $0 = x_0 \leq \cdots \leq x_N = 2\pi$ for estimating the value of this integral.

(5.b) Simplify your expression for $T_N(f)$ using the periodicity of f.

(5.c) Show that $T_N(f)$ is equivalent to both a left-endpoint Riemann sum and a right-endpoint Riemann sum approximation to $I(f)$.

(5.d) Compute $T_N(f)$ for various progressively larger N. Plot the quadrature errors against N on (i) a log-log plot, and (ii) a `semilogy` plot. What is the order of accuracy of the trapezoidal rule for smooth, periodic functions?