Worksheet 7

Material taken from [TB97, Lecture 6 and 10].

1 Projectors

A projector is a square matrix P that satisfies

$$P^2 = P.$$

(1.a) Assume P is a projector, show that $I - P$ is also a projector.

(1.b) We can show that

$$\begin{aligned} &\mathrm{range}(I - P) = \mathrm{null}(P); \\ &\mathrm{null}(I - P) = \mathrm{range}(P); \\ &\mathrm{range}(P) \cap \mathrm{null}(P) = 0. \end{aligned}$$

An orthogonal projector is a projector whose has the subspaces range(P) and null(P) orthogonal.

(1.c) Show that if $P = P^\top$ symmetric, the projector P is orthogonal (Hint: take one vector in range(P) and one in null(P), show that they must be orthogonal to each other)

The reverse direction holds as well. Therefore the two definition are equivalent.

(1.d) A special case of orthogonal projection is projection onto a vector:

$$P_v = \frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

Show that it is indeed an orthogonal projector with range span($\boldsymbol{v}$).

(1.e) Another orthogonal projection is

$$P_{\perp v} = I - \frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

What is its null space? What is its range?

2 Geometric interpretation of Householder reflectors

(2.a) Name $H(\boldsymbol{v})$ the linear subspace orthogonal to the vector $\boldsymbol{v}$. A reflector across $H(\boldsymbol{v})$ is

$$F_{\boldsymbol{v}} = I - 2\frac{\boldsymbol{v}\boldsymbol{v}^\top}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

Compare this with P_v and $P_{\perp v}$, interpret the formula geometrically.

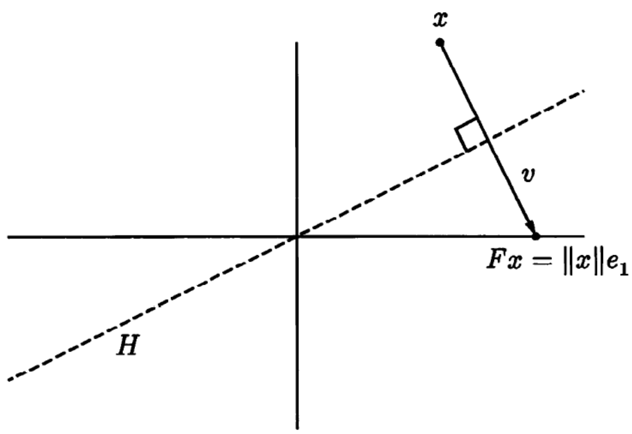

Figure 10.1. ***A Householder reflection.***

(2.b) To use Householder for QR decomposition, we want $F\boldsymbol{x} = c\boldsymbol{e}_1$. We know that $c = \pm\|\boldsymbol{x}\|_2$. Explain this geometrically.

(2.c) From this we have that

$$\boldsymbol{v} = \boldsymbol{x} - F\boldsymbol{x} = \boldsymbol{x} \pm \|\boldsymbol{x}\|\,\boldsymbol{e}_1.$$

From a geometric point of view, why is this the correct formula.

References

[TB97] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra.* SIAM, June 1997.