When to stop the iteration ?

① residual $|f(x_k)|$

$$\varepsilon > |f(x_k)| = |f(x_k) - f(x^*)| = |f'(\xi)|\,|x_k - x^*| \geq p\, e_k .$$

$\uparrow$ $\xi \in [x_k, x^*]$

$$\Rightarrow \quad e_k < \frac{\varepsilon}{p}$$

Here we assumed in an interval I containing x^*,

$|f'(x)| \geq p$ ($|f'|$ is bounded from below)

What if $p = 0$?

Keep expanding the Taylor series and look at the higher derivatives.

or try $f(x) + (x - x^*)$?

② Step size $|x_k - x_{k-1}|$

$$\varepsilon > |x_k - x_{k-1}| \geq |x_{k-1} - x^*| - |x_k - x^*| \qquad \text{(triangular inequality)}$$

we also have $\displaystyle \frac{|x_k - x_{k-1}|}{|x_k - x^*|} \geq \underbrace{\frac{|x_{k-1} - x^*|}{|x_k - x^*|}}_{\lim_{k\to\infty} \downarrow\ = \frac{1}{L}} - 1$

$\Rightarrow$ $\exists\, k$ large enough s.t. $\displaystyle \frac{|x_k - x_{k-1}|}{|x_k - x^*|} \geq \frac{1}{2L}$

$\Rightarrow$ $\varepsilon > |x_k - x_{k-1}| \geq \left(\frac{1}{2L} - 1\right) |x_k - x^*|$ ↖ play with the constant to make sure RHS is positive.

$\Rightarrow$ $e_k < \varepsilon \Big/ \left(\frac{1}{2L} - 1\right)$ for k large enough.