1 Deriving a new quadrature rule

Given $f : [0,1] \to \mathbb{R}$, you want to derive a new quadrature rule that does uses not only function values, but also gradient values:

$$\int_0^1 f(x)\,\mathrm{d}x \approx \alpha_0 f(0) + \alpha_1 f'(0) + \alpha_2 f(1). \tag{1}$$

(1.a) First, find polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$, with the following properties:

$$\begin{aligned} J_0(0) &= 1, \quad J_0'(0) = 0, \quad J_0(1) = 0 \\ J_1(0) &= 0, \quad J_1'(0) = 1, \quad J_1(1) = 0 \\ J_2(0) &= 0, \quad J_2'(0) = 0, \quad J_2(1) = 1. \end{aligned}$$

(*Hint:* For each J_i, make an ansatz for a quadratic polynomial using the monomial basis.)

Given f, you can now define a polynomial approximation $p \in \mathcal{P}_2$ via

$$p(x) = f(0)J_0(x) + f'(0)J_1(x) + f(1)J_2(x). \tag{2}$$

The polynomial p is an approximation to f in the sense that $p(0) = f(0)$, $p'(0) = f'(0)$ and $p(1) = f(1)$.

(1.b) Use the polynomial p derived in (2) and the same method used to derive the Newton-Cotes quadrature rules, to find the coefficients α_0, α_1 and α_2 in (1).

(1.c) Use your new quadrature rule to approximate $\int_0^1 \exp(2x)\sin^2(x)\,\mathrm{d}x$, and also compare with Simpson's rule. The exact value of this integral is $1.2668\ldots$.

2 Trapezoidal rule for smooth periodic functions

We investigate how the (composite) trapezoidal rule performs for smooth, periodic functions. Consider integrating the smooth, periodic function $f(x) = e^{\sin x}$ over a single period. The exact value of the integral is

$$I(f) = \int_0^{2\pi} e^{\sin x}\,\mathrm{d}x = 7.95492652101284527\ldots.$$

(2.a) Write down the composite trapezoidal rule $T_N(f)$ on equispaced nodes $0 = x_0 \le \cdots \le x_N = 2\pi$ for estimating the value of this integral.

(2.b) Simplify your expression for $T_N(f)$ using the periodicity of f.

(2.c) Show that $T_N(f)$ is equivalent to both a left-endpoint Riemann sum and a right-endpoint Riemann sum approximation to $I(f)$.

(2.d) Compute $T_N(f)$ for various progressively larger N. Plot the quadrature errors against N on (i) a log-log plot, and (ii) a `semilogy` plot. What is the order of accuracy of the trapezoidal rule for smooth, periodic functions?

3 Convergence order of quadrature

(3.a) We would like to integrate a function on $[0, 1]$ using the composite trapezoid rule with sub-interval size h. We name the result T_h. How does the error ($e_h = |T_h - I|$) scale with h?

(3.b) In real application, we do not know the true value of the integral. To verify the order of convergence of our code, we can calculate this quantity:

$$\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}.$$

How does this quantity scale with h?

(3.c) We will show this using the code from the last problem.