

Material taken from [?, Lecture 6 and 10].

1 Projectors

A projector is a square matrix P that satisfies

$$P^2 = P.$$

(1.a) Assume P is a projector, show that $I - P$ is also a projector.

(1.b) We can show that

$$\begin{aligned}\text{range}(I - P) &= \text{null}(P); \\ \text{null}(I - P) &= \text{range}(P); \\ \text{range}(P) \cap \text{null}(P) &= 0.\end{aligned}$$

An orthogonal projector is a projector whose has the subspaces $\text{range}(P)$ and $\text{null}(P)$ orthogonal.

n.b.: An orthogonal projector P is not an orthogonal matrix! Why?

(1.c) Show that if $P = P^\top$ symmetric, the projector P is orthogonal (Hint: take one vector in $\text{range}(P)$ and one in $\text{null}(P)$, show that they must be orthogonal to each other).

The reverse direction holds as well. Therefore the two definition are equivalent.

(1.d) A special case of orthogonal projection is projection onto a vector:

$$P_v = \frac{\mathbf{v}\mathbf{v}^\top}{\mathbf{v}^\top \mathbf{v}}.$$

Show that it is indeed an orthogonal projector with range $\text{span}(\mathbf{v})$.

(1.e) Another orthogonal projection is

$$P_{\perp v} = I - \frac{\mathbf{v}\mathbf{v}^\top}{\mathbf{v}^\top \mathbf{v}}.$$

What is its null space? What is its range?

2 Geometric interpretation of Householder reflectors

(2.a) Name $H(v)$ the linear subspace orthogonal to the vector v . A reflector across $H(v)$ is

$$F_v = I - 2 \frac{vv^\top}{v^\top v}.$$

Compare this with P_v and $P_{\perp v}$, interpret the formula geometrically.

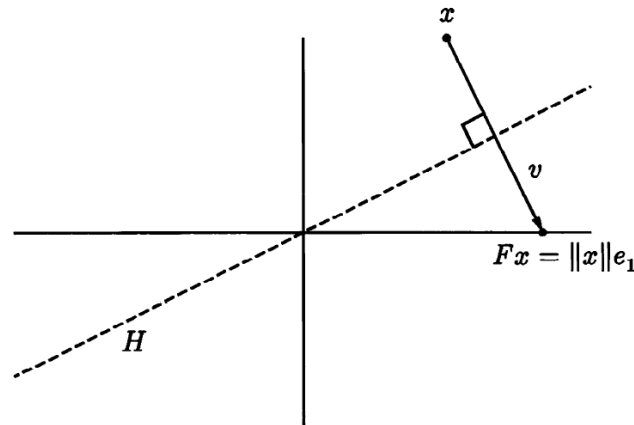


Figure 10.1. A Householder reflection.

(2.b) To use Householder for QR decomposition, we want $Fx = ce_1$. We know that $c = \pm \|x\|_2$. Explain this geometrically.

(2.c) From this we have that

$$v = x - Fx = x \pm \|x\| e_1.$$

From a geometric point of view, why is this the correct formula.

References

[TB97] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra*. SIAM, June 1997.