Worksheet 11

1

[Exercise 4.3.5 of Lebl 2023] If $f : [0,1] \to \mathbb{R}$ has $n+1$ continuous derivatives and $x_0 \in [a,b]$, prove

$$\lim_{x \to x_0} \frac{R_n^{x_0}(x)}{(x-x_0)^n} = 0. \tag{1}$$

2

[Exercise 5.4.11 of Lebl 2023] This is an example of infinitely differentiable function that is not analytic.

Since $(e^x)' = e^x$, it is easy to see that e^x is infinitely differentiable (has derivatives of all orders). Define the function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) := \begin{cases} e^{-1/x} & \text{if} \quad x > 0 \\ 0 \quad \text{if} \quad x \leq 0 \end{cases} \tag{2}$$

(2.a) Prove that for every $m \in \mathbb{N}$

$$\lim_{x \to 0^+} \frac{e^{-1/x}}{x^m} = 0. \tag{3}$$

(2.b) Prove that f is infinitely differentiable.

(2.c) Compute the Taylor series for f at the origin, that is,

$$\sum_{k=0}^{\infty} \frac{f^{(k)(0)}}{k!} x^k. \tag{4}$$

Show that it converges, but show that it does not converge to $f(x)$ for any given $x > 0$.

3

[Exercise 5.1.2 of Lebl 2023] Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) := x$. Show that $f \in \mathcal{R}([0,1])$ and compute $\int_0^1 f$ using the definition of the integral.

4

[Exercise 5.2.10 of Lebl 2023] Suppose $f : [a,b] \to \mathbb{R}$ is bounded and has finitely many discontinuities. Show that as a function of x the expression $|f(x)|$ is bounded with finitely many discontinuities and is thus Riemann integrable. Then show

$$\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x. \tag{5}$$

5

[Exercise 5.2.14(a) of Lebl 2023] Let $f : [a, b] \to \mathbb{R}$ be a function. Show that if f is increasing, then it is Riemann integrable.

6

[Exercise 5.3.10 of Lebl 2023] A function f is an odd function if $f(x) = -f(-x)$, and f is an even function if $f(x) = f(-x)$. Let $a > 0$. Assume f is continuous. Prove:

(6.a) If f is odd, then $\int_{-a}^{a} f = 0$.

(6.b) If f is even, then $\int_{-a}^{a} f = 2 \int_0^a f$.

References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I.* version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.