Worksheet 3

1

Let A be a non-empty bounded subset of $\mathbb{R}$ and suppose $\sup A \notin A$. Show that there exists an increasing sequence of points $\{a_n\}_{n\geq 1}$ in A such that $\lim_{n\to\infty} a_n = A$.

2

Let S be a non-empty bounded subset of $\mathbb{R}$.

(2.a) Prove that $\inf S \leq \sup S$.

(2.b) What can you say about S if $\inf S = \sup S$.

3

Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and let

$$-A = \{-a : a \in A\}. \tag{1}$$

Prove that $\inf A = -\sup(-A)$.

4

Let S and T be two non-empty bounded subsets of $\mathbb{R}$ and let

(4.a) [Exercise 1.1.4 of Lebl 2023] Prove that if $S \subset T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(4.b) Prove that $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$.

5

(5.a) [Proposition 2.2.7 of ibid.] Show that if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$.

(5.b) Show (via an example) that the converse is not true.

6

Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers. Show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty \iff \lim_{n\to\infty} a_n = 0. \tag{2}$$

7

Show that the sequence

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \frac{1}{2}. \tag{3}$$

8

[Exercise 2.2.12 of Lebl 2023] Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers such that $\{a_n\}_{n\geq 1}$ is bounded and $\{b_n\}_{n\geq 1}$ converges to 0. Show that the sequence $\{a_n b_n\}_{n\geq 1}$ converges to 0.

9

Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers and let $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n > a$. Show that there exists $n_0 \in \mathbb{N}$ such that $a_n > a$ for all $n > n_0$.

References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I.* version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.