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February 23rd, 2024

MATH-UA.0325
Analysis

Worksheet 4

1

Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

2

Let $\{a_n\}_{n\geq 1}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1. \tag{1}$$

(2.a) Show that the sequence is bounded below.

(2.b) Show that this is a sequence of rational numbers.

(2.c) Prove that the sequence is monotonically decreasing.

(2.d) Deduce that $\{a_n\}_{n\geq 1}$ converges and find its limit.

Remark: This is an example of Cauchy sequence of rational numbers converging to an irrational number.

3

Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences. Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \tag{2}$$

4

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above. Prove that $L = \limsup a_n n$ has the following properties:

(4.a) For every $\epsilon > 0$ there are only finitely many n for which $a_n > L + \epsilon$.

(4.b) For every $\epsilon > 0$ there are infinitely many n for which $a_n > L - \epsilon$.

Remark: It is also true that there can be at most one real number with both of the above two properties.

5

Show that a sequence $\{a_n\}_{n\geq 1}$ is bounded if and only if $\limsup_{n\to\infty} |a_n| < \infty$.