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Worksheet 1

MATH-UA.0325
Analysis

1 De Morgan's laws

Let X be a universe set, and $A, B \subset X$ subsets of it. Recall that in this case we simply denote $X \setminus A$ by A^c.

(1.a) Show the De Morgan's laws which state that

- The complement of the union is the intersection of complements. That is, $(A \cup B)^c = A^c \cap B^c$.
- The complement of the intersection is the union of complements. That is, $(A \cap B)^c = A^c \cap B^c$.

(1.b) Show De Morgan's laws for infinite unions and intersections. That is, if $A_n \subset X$ is a collection of sets for $n \in \mathbb{N}$

- $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$.
- $\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c$.

2 Some Logic

Here we will do some logic that will benefit proof construction.

(2.a) De Morgan's laws for statements If P is some sentence or formula, then $\neg P$ is called the denial of P. Convince yourself that the following are tautologies:

$$\neg(P \text{ or } Q) \Leftrightarrow (\neg P \text{ and } \neg Q); \tag{1}$$

$$\neg(P \text{ and } Q) \Leftrightarrow (\neg P \text{ or } \neg Q). \tag{2}$$

(2.b) We often deal with statements like P implies Q. Convince yourself that

$$P \Rightarrow Q \Leftrightarrow \neg P \text{ or } Q. \tag{3}$$

Use this to show

$$P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P. \tag{4}$$

The second statement is the contrapositive of the first. It is sometimes easier to prove the second statement, which we also call prove by contradiction.

Use the above De Morgan's laws for statements to show that

$$\neg(P \Rightarrow Q) \Leftrightarrow P \text{ and } \neg Q. \tag{5}$$

(2.c) There are versions of De Morgan's laws for quantifiers:

$$\neg(\forall x\ P(x)) \Leftrightarrow \exists x\ \neg P(x); \tag{6}$$

$$\neg(\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x). \tag{7}$$

(2.d) Try to negate this sentence!

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, |x| < \delta \Rightarrow |f(x)| < \epsilon. \tag{8}$$

This is essentially the infamous $\epsilon - \delta$ definition of continuity we will learn later in this class.

3

Let $f : X \to Y$, $A, B \subset X$, and $P, Q \subset Y$.

(3.a) Show

$$f(A \cup B) = f(A) \cup f(B); \tag{9}$$
$$f(A \cap B) \subset f(A) \cap f(B); \tag{10}$$
$$f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q); \tag{11}$$
$$f^{-1}(P \cap Q) = f^{-1}(P) \cap f^{-1}(Q). \tag{12}$$

(3.b) Give an example of $f : X \to Y$, A, and B where $f(A \cap B) \neq f(A) \cap f(B)$. Hint: design a case where A and B are disjoint but $f(A)$ and $f(B)$ are not disjoint.

(3.c) Show that if f is injective, then $f(A \cap B) = f(A) \cap f(B)$.

4

Let $f : A \to B$ and $g : B \to C$ be two bijective functions. Show $g \circ f$ is bijective.

5

Let $f : X \to Y$ be a function between sets and $U \subset A$ and $V \subset B$.

(5.a) Show that

$$f(f^{-1}(V)) \subset V \quad \text{and} \quad U \subset f^{-1}(f(U)). \tag{13}$$

(5.b) Give an example that shows the above relation cannot be equality.