Worksheet 2

1 Defining the rationals

[5.4.2 of Hajłasz 2018] Take the set $\mathcal{R} = \{(a,b), a, b \in \mathbb{Z}, b \neq 0\}$. For (a,b) and $(c,d) \in \mathcal{R}$, we write $(a,b) \sim (c,d)$ if $ad = bc$. Check if this is an equivalence relation.

2

Let $(F, +, \cdot, <)$ be an ordered field, show

(2.a) Take $a \in F$ such that $a > 0$, then

$$a^{-1} > 0. \tag{1}$$

(2.b) Take $a, b \in F$ such that $0 < a < b$, then

$$0 < b^{-1} < a^{-1}. \tag{2}$$

(2.c) Take $a, b, c \in F$, show that

$$2ab \leq a^2 + b^2 \tag{3}$$

and

$$ab + bc + ac \leq a^2 + b^2 + c^2. \tag{4}$$

3

Let S be a non-empty bounded subset of $\mathbb{R}$.

(3.a) Prove that $\inf S \leq \sup S$.

(3.b) What can you say about S if $\inf S = \sup S$.

4

Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and let

$$-A = \{-a : a \in A\}. \tag{5}$$

Prove that $\inf A = -\sup(-A)$.

5

Let S and T be two non-empty bounded subsets of $\mathbb{R}$ and let

(5.a) [1.1.4 of Lebl 2023] Prove that if $S \subset T$, then $\inf T \le \inf S \le \sup S \le \sup T$.

(5.b) Prove that $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$.

6

[1.2.9 of ibid.] Let A and B be two non-empty bounded subsets of $\mathbb{R}$ and let

$$S = \{a + b : a \in A \text{ and } b \in B\}. \tag{6}$$

(6.a) Prove that $\sup S = \sup A + \sup B$.

(6.b) Prove that $\inf S = \inf A + \inf B$.

7

(7.a) Show that if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$.

(7.b) Show (via an example) that the converse is not true.

References

Hajłasz, Piotr (2018). *Introduction to Analysis.* URL: https://sites.pitt.edu/~hajlasz/Teaching/Math1530Fall2018/selection.pdf.

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I.* version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.