Worksheet 7

1

(1.a) [Exercise 3.1.9 of Lebl 2023] Let c_1 be a cluster point of $A \subset \mathbb{R}$ and c_2 be a cluster point of $B \subset \mathbb{R}$. Suppose $f : A \to B$ and $g : B \to \mathbb{R}$ are functions such that $f(x) \to c_2$ as $x \to c_1$ and $g(y) \to L$ as $y \to c_2$.

Let $h(x) := g(f(x))$, we have "chain rule" for limits: $h(x) \to L$ as $x \to c_1$, if we also suppose that $g(c_2) = L$ (that is, $g(x)$ is continuous at c_2).

(1.b) [Exercise 3.1.14 of Lebl 2023] Show via a counterexample that the assumption of $g(c_2) = L$ is neccessry.

2

[Exercise 3.2.1 of Lebl 2023] Using the definition of continuity directly prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is continuous.

3

[Example 3.2.6 of Lebl 2023] Show that the functions $\sin(x)$ and $\cos(x)$ are continuous.
Hint: use the sum-to-product trigonometric identities.

4

[Example 3.2.12 of Lebl 2023] Show that the popcorn function (or the Thomae function) is continuous. The popcorn function is defined as $f : (0,1) \to \mathbb{R}$

$$f(x) := \begin{cases} 1/k & \text{if } x = m/k, \text{ where } m, k \in \mathbb{N} \text{ and have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \tag{1}$$

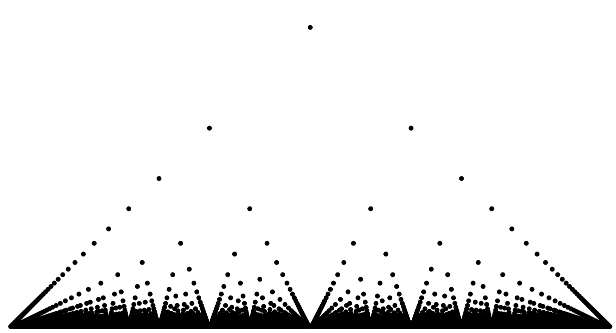

Figure 1: Graph of the popcorn function.

5

[Exercise 3.2.4 and 3.2.4 of Lebl 2023] Is $f : \mathbb{R} \to \mathbb{R}$ continuous? Prove your assertion.

(5.a)

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2}$$

(5.b)

$$f(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{3}$$

6

[Exercise 3.2.10 of Lebl 2023] Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I.* version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.