Worksheet 5 Partial Solutions

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Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of non-negative numbers such that $\sum_{n\geq 1} a_n < \infty$. Show that

$$\lim_{n\to\infty} na_n = 0. \tag{1}$$

Proof. Pick any $\epsilon > 0$. Because $\sum_{n\geq 1}\{a_n\}$ converges, by the Cauchy criterion, there exists $N_1 \in \mathbb{N}$ such that for all $p > 0$,

$$\sum_{N_1+1}^{N_1+p} a_n < \frac{\epsilon}{2}. \tag{2}$$

Because $\{a_n\}_{n\geq 1}$ is decreasing,

$$pa_{N_1+p} \leq \sum_{N_1+1}^{N_1+p} a_n < \frac{\epsilon}{2}. \tag{3}$$

Moreover, the sequence $\{a_n\}_{n\geq 1}$ converges to 0. There exists $N_2 \in \mathbb{N}$ s.t. for all $p \geq N_2$,

$$a_{N_1+p} < \frac{\epsilon}{2N_1} \tag{4}$$

$$\Rightarrow \quad N_1 a_{N_1+p} < \frac{\epsilon}{2}. \tag{5}$$

Adding two together we have, for all $p \geq N_2$,

$$(N_1+p)a_{N_1+p} < \epsilon \tag{6}$$

Change of variable, we have for all $m \geq N_1 + N_2$, $ma_m < \epsilon$. This proves $\lim_{n\to\infty} na_n = 0$.

$\square$