Worksheet 4 Solutions

1

Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

Proof. Since $\{a_n\}_{n\geq 1}$ is Cauchy, we know that it is bounded by $A \in \mathbb{R}$. Additionally, for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ s.t.

$$|a_n - a_m| < \frac{\epsilon}{2A} \quad \text{for all} \quad n, m \geq N_\epsilon.$$

Then we have for all $n, m \geq N_\epsilon$

$$|a_n^2 - a_m^2| = |a_n + a_m||a_n - a_m| < 2A \cdot \frac{\epsilon}{2A} = \epsilon$$

This work for all $\epsilon > 0$ so the sequence $\{a_n^2\}_{n\geq 1}$ is also Cauchy. □

2

Let $\{a_n\}_{n\geq 1}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1. \tag{1}$$

(2.a) Show that the sequence is bounded below by $\sqrt{2}$.

Proof. It is clear that the sequence is positive. Then we see

$$a_{n+1} - \sqrt{2} = \frac{a_n}{2} + \frac{1}{a_n} - \sqrt{2} = \frac{a_n^2 + 2 - 2\sqrt{2}a_n}{2a_n} = \frac{(a_n - \sqrt{2})^2}{2a_n} > 0.$$

Therefore $a_n > \sqrt{2}$ for all n. □

(2.b) Show that this is a sequence of rational numbers.

Proof. The set $\mathbb{Q}$ is a field and has the closure property. Therefore $a_n \in \mathbb{Q}$ for all n. □

(2.c) Prove that the sequence is monotonically decreasing.

Proof. For all $n \in \mathbb{N}$

$$a_{n+1} - a_n = \frac{-a_n}{2} + \frac{1}{a_n} = \frac{-a_n^2 + 2}{2a_n} < 0$$

where we used the fact that $a_n > \sqrt{2}$. $\{a_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. □

(2.d) Deduce that $\{a_n\}_{n\geq 1}$ converges and find its limit.

Proof. $\{a_n\}_{n\in\mathbb{N}}$ is bounded and monotonically decreasing therefore it converges.

We name $\lim_{n\to\infty} a_n = a$. By the recurrence relation a needs to satisfy

$$a = \frac{a}{2} + \frac{1}{a} \tag{2}$$

which has positive solution $\sqrt{2}$. Thus, $\lim_{n\to\infty} a_n = \sqrt{2}$. □

Remark: This is an example of Cauchy sequence of rational numbers converging to an irrational number.

3

Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences. Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \tag{3}$$

Proof. By definition,

$$\begin{aligned}
\limsup_{n\to\infty}(a_n + b_n) &= \lim_{N\to\infty} \sup\{(a_n + b_n), n \geq N\} \\
\limsup_{n\to\infty} a_n &= \lim_{N\to\infty} \sup\{a_n, n \geq N\} \\
\limsup_{n\to\infty} b_n &= \lim_{N\to\infty} \sup\{b_n, n \geq N\}.
\end{aligned}$$

We name $A_N = \sup\{a_n, n \geq N\}$ and $B_N = \sup\{b_n, n \geq N\}$. Then for all $n \geq N$, $A_N \geq a_n$, $B_N \geq b_n$, and thus $A_N + B_N \geq a_n + b_n$. $A_N + B_N$ is an upper bound of $\{a_n + b_n, n \geq N\}$.
Since $\sup\{a_n + b_n, n \geq N\}$ is the least upper bound

$$\begin{aligned}
& A_N + B_N \geq \sup\{a_n + b_n, n \geq N\} \\
\Rightarrow \quad & \sup\{a_n, n \geq N\} + \sup\{b_n, n \geq N\} \geq \sup\{a_n + b_n, n \geq N\}.
\end{aligned}$$

We can take the limit to infinity of both sides and we have

$$\lim_{N\to\infty} \sup\{(a_n + b_n), n \geq N\} \leq \lim_{N\to\infty} \left(\sup\{a_n, n \geq N\} + \sup\{b_n, n \geq N\}\right).$$

Because both sup are bounded

$$\begin{aligned}
& \lim_{N\to\infty} \left(\sup\{a_n, n \geq N\} + \sup\{b_n, n \geq N\}\right) = \lim_{N\to\infty} \sup\{a_n, n \geq N\} + \lim_{N\to\infty} \sup\{b_n, n \geq N\} \\
\Rightarrow \quad & \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.
\end{aligned}$$

□

4

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above. Prove that $L = \limsup a_n$ has the following properties:

(4.a) For every $\epsilon > 0$ there are only finitely many n for which $a_n > L + \epsilon$.

Proof. Name $v_N = \sup\{a_n, n \geq N\}$. Then by definition $L = \limsup a_n = \lim_{N\to\infty} v_N$. We also know that if $\{v_N\}_{N\geq 1}$ is a decreasing sequence and $v_n \geq L$.

For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for all $n \geq N$,

$$\begin{aligned} & |v_n - L| < \epsilon \\ \Rightarrow \quad & v_n < L + \epsilon \\ \Rightarrow \quad & \sup\{a_m, m \geq n\} < L + \epsilon \\ \Rightarrow \quad & a_n < L + \epsilon. \end{aligned}$$

This leaves only some or all $a_n, n < N$ (only finitely amount) that could have

$$a_n > L + \epsilon.$$

□

(4.b) For every $\epsilon > 0$ there are infinitely many n for which $a_n > L - \epsilon$.

Proof. We know there exists a subsequence $\{a_{k_n}\}_{n\geq 1}$ that converges to L. Then for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$

$$|a_{k_n} - L| < \epsilon \Rightarrow L - \epsilon < a_{k_n}$$

This means there are infinitely many a_{k_n} thus a_n larger than $L - \epsilon$. □

Remark: It is also true that there can be at most one real number with both of the above two properties.

5

Show that a sequence $\{a_n\}_{n\geq 1}$ is bounded if and only if $\limsup_{n\to\infty} |a_n| < \infty$.

Proof. "$\Rightarrow$": We know

$$\limsup_{n\to\infty} |a_n| \leq \sup\{|a_n|, n \geq 1\}.$$

However the RHS is bounded because $\{a_n\}_{n\geq 1}$ is bounded.

"$\Leftarrow$": We know from Problem (4.a) that for $\epsilon = 1$ there are only finitely many n for which $|a_n| > \limsup_{n\to\infty} |a_n| + 1$. Then for some finite N

$$\sup_n\{|a_n|\} \leq \max\{a_1, \dots, a_N, \limsup_{n\to\infty} |a_n| + 1\}. \tag{4}$$

This is finite because it is a maximum over finite elements. Finally, $\sup_n\{|a_n|\} < \infty$ implies the sequence is bounded.

$\square$