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Worksheet 8

1

[Exercise 3.2.1 of Lebl 2023] Using the definition of continuity directly prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is continuous.

2

[Exercise 3.2.4 and 3.2.4 of ibid.] Is $f : \mathbb{R} \to \mathbb{R}$ continuous? Prove your assertion.

(2.a)

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1}$$

(2.b)

$$f(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2}$$

3

[Exercise 3.2.14 of ibid.] Suppose $f : [-1, 0] \to \mathbb{R}$ and $g : [-1, 0] \to \mathbb{R}$ are continuous and $f(0) = g(0)$. Define $h : [-1, 0] \to \mathbb{R}$ by $h(x) := f(x)$ if $x \leq 0$ and $h(x) := g(x)$ if $x > 0$. Show that h is continuous.

4

[Exercise 3.2.10 of ibid.] Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

5

[Exercise 3.4.7 of ibid.] Let $f : (0, 1) \to \mathbb{R}$ be a bounded continuous function. Show that the function $g(x) := x(1 - x)f(x)$ is uniformly continuous.

6

[Exercise 3.2.16 of ibid.] Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and such that $f(x+y) = f(x)+f(y)$ for every x and y. Show that $f(x) = ax$ for some $a \in \mathbb{R}$. Hint: Show that $f(nx) = nf(x)$, then show f is continuous on $\mathbb{R}$. Then show that $f(x)/x = f(1)$ for all rational x.

References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I.* version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.