# Worksheet 11



## 1

[Exercise 5.1.3 of Lebl 2023] Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}_{k=1}^{\infty}$ of $[a, b]$ such that

$$\lim_{k\to\infty} (U(P_k, f) - L(P_k, f)) = 0. \tag{1}$$

Show that $f$ is Riemann integrable and that

$$\int_a^b f = \lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f). \tag{2}$$

## 2

[Example 5.1.14 of Lebl 2023] Let us show $\frac{1}{1+x}$ is integrable on $[0, b]$ for all $b > 0$.

## 3

[Exercise 5.2.14(a) of Lebl 2023] Let $f : [a, b] \to \mathbb{R}$ be a function. Show that if $f$ is increasing, then it is Riemann integrable.

## 4

[Exercise 5.1.14 of Lebl 2023] Construct functions $f$ and $g$, where $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, $g : [0, 1] \to [0, 1]$ is one-to-one and onto, and such that the composition $f \circ g$ is not Riemann integrable.

## 5

[Exercise 5.2.5 of Lebl 2023] Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x$.

## 6

[Exercise 5.2.10 of Lebl 2023] Suppose $f : [a, b] \to \mathbb{R}$ is bounded and has finitely many discontinuities. Show that as a function of $x$ the expression $|f(x)|$ is bounded with finitely many discontinuities and is thus Riemann integrable. Then show

$$\left| \int_a^b f(x)\, \mathrm{d}x \right| \leq \int_a^b |f(x)|\, \mathrm{d}x. \tag{3}$$

## 7

[Exercise 5.2.4 of Lebl 2023] Prove the mean value theorem for integrals: If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a, b]$ such that

$$\int_a^b f = f(c)(b - a). \tag{4}$$

## 8

[Exercise 5.3.10 of Lebl 2023] A function $f$ is an odd function if $f(x) = -f(-x)$, and $f$ is an even function if $f(x) = f(-x)$. Let $a > 0$. Assume $f$ is continuous. Prove:

**(8.a)** If $f$ is odd, then $\int_{-a}^a f = 0$.

**(8.b)** If $f$ is even, then $\int_{-a}^a f = 2\int_0^a f$.

## References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I*. version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.