# Worksheet 10



## 1

**(1.a)** [Exercise 4.2.4 of Lebl 2023] Let $f : [a, b] \to \mathbb{R}$ be differentiable and let $c \in [a, b]$. Show there exists a sequence $\{x_n\}_{n=1}^{\infty}$ converging to $c$, $x_n \neq c$ such that

$$f'(c) = \lim_{n\to\infty} f'(x_n). \tag{1}$$

Do note this does not imply that $f'(x)$ is continuous (why?).

**(1.b)** Suppose that $\lim_{x\to c} f'(x)$ exists and is finite. Show this limit must be $f'(c)$.

## 2

[Exercise 4.2.10 of Lebl 2023] Let $f : (a, b) \to \mathbb{R}$ be an unbounded differentiable function. Show $f' : (a, b) \to \mathbb{R}$ is unbounded.

## 3

Let $f : [a, b] \to \mathbb{R}$ be differentiable. Assume that $f'$ is strictly increasing. Show that for any $c \in (a, b)$ such that $f'(c) = 0$ there exist $x_1, x_2 \in [a, b], x_1 < c < x_2$ such that

$$0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{2}$$

## 4

Assume $f : (1, \infty) \to \mathbb{R}$ is differentiable. If

$$\lim_{x\to\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f'(x) = c, \tag{3}$$

prove that $c = 0$.

## 5

Let $f : (0, 1) \to \mathbb{R}$ be differentiable function such that

$$|f'(x)| < 1 \quad \text{for all} \quad x \in (0, 1). \tag{4}$$

For $n \geq 2$ let $a_n = f(1/n)$. Show that

$$\lim_{n\to\infty} a_n \tag{5}$$

exists.

## 6 (Challenging)

Let $f : [0, 1] \to \mathbb{R}$ be continuous and be differentiable. Assume that $f(0) = 0$ and $f'$ is an increasing function on $(0, 1)$. Show that

$$g(x) = \frac{f(x)}{x} \tag{6}$$

is an increasing function on $(0, 1)$.

## 7 (Challenging)

Let $f : [a, b] \to \mathbb{R}$ be continuous and be differentiable on $(a, b)$, with $f(a) = f(b) = 0$. Prove that for every $\lambda \in \mathbb{R}$ there exists $x_0 \in (a, b)$ such that $f'(x_0) = \lambda f(x_0)$.

## References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I*. version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.