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April 5th, 2024

Worksheet 9

1

[Exercise 4.1.9 of Lebl 2023] Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable Lipschitz continuous function. Prove that f' is a bounded function.

2

[Exercise 4.1.11 of Lebl 2023] Suppose $f : I \to \mathbb{R}$ is bounded, $g : I \to \mathbb{R}$ is differentiable at $c \in I$, and $g(c) = g'(c) = 0$. Show that $h(x) := f(x)g(x)$ is differentiable at c. Hint: You cannot apply the product rule.

3

[Exercise 4.1.13 of Lebl 2023] Suppose $f : (-1, 1) \to \mathbb{R}$ is a function such that $f(x) = xh(x)$ for a bounded function h.

(3.a) Show that $g(x) := (f(x))^2$ is differentiable at the origin and $g'(0) = 0$.

(3.b) Find an example of a continuous function $f : (-1, 1) \to \mathbb{R}$ with $f(0) = 0$, but such that $g(x) := (f(x))^2$ is not differentiable at the origin.

4

[Exercise 4.1.16 of Lebl 2023] Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, $f(c) = 0$, and $f'(c) > 0$. Prove that there is a $\delta > 0$ such that $f(x) < 0$ whenever $c - \delta < x < c$ and $f(x) > 0$ whenever $c < x < c + \delta$.

5

[Exercise 4.4.6 of Lebl 2023] Let $f(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$. Show that f is differentiable at all x, that $f'(0) > 0$, but that f is not invertible on any open interval containing the origin.

References

Lebl, Jiri (July 11, 2023). *Basic Analysis I: Introduction to Real Analysis, Volume I*. version 6.0. URL: https://www.jirka.org/ra/realanal.pdf.