1 Heat flux and the Robin boundary condition

(1.a) Write the heat equation as a conservation law. What is the heat flux function?

Relating the flux to its gradient is called the Fourier's law or the Fick's law.

(1.b) Imagine we have a metal rod where one end of it is submerged in cold water. The water has temperature T_L and is moving. The heat flux due to convection is

$$q = h(T - T_L) \tag{1}$$

where h is called the convection coefficient.

This heat flux must be equal to the heat flux at the end of rod which follows the Fourier's law. From this obtain that $\theta = T - T_L$ must satisfy the homogeneous robin boundary condition.

2 Alternative statements of the method of images

[From Shearer and Levy 2015, Problem 5.3] We have Φ the fundamental solution of the heat equation.

(2.a) Let $g : [0, \infty) \to \mathbb{R}$ be a bounded integrable function. Prove directly that

$$u(x,t) = \int_0^\infty (\Phi(x-y,t) - \Phi(x+y,t))g(y)\,\mathrm{d}y \tag{2}$$

is an odd function of $x \in \mathbb{R}$ for each $t > 0$.

(2.b) Let $h : \mathbb{R} \to \mathbb{R}$ be an odd bounded integrable function. Prove that

$$u(x,t) = \int_{-\infty}^\infty \Phi(x-y,t)h(y)\,\mathrm{d}y \tag{3}$$

is an odd function of $x \in \mathbb{R}$ for each $t > 0$. That is, the symmetry in the initial data is carried through to the same symmetry in the solution.

3 Maximum and variance of the fundamental solution

[Adapted from Olver 2014, Problem 8.1.6]

(3.a) What is the maximum value of the fundamental solution at time t?

(3.b) Calculate the "variance" of the fundamental solution:

$$\operatorname{var}(u(t)) = \int_{\mathbb{R}} x^2 u(t,x)\,\mathrm{d}x. \tag{4}$$

One could do this directly. Alternatively, calculate how the variance change in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{var}(u(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} x^2 u(t,x)\,\mathrm{d}x. \tag{5}$$

(3.c) Can you justify the claim that its width is proportional to $\sqrt{t}$?

4 Advection-diffusion equation

[From Shearer and Levy 2015, Problem 5.10] Devise a change of variable corresponding to a moving frame of reference to solve the initial value problem for the advection-diffusion equation with constant speed c

$$u_t + cu_x = ku_{xx}, \quad -\infty < x < \infty, t > 0, \tag{6}$$
$$u(x,0) = g(x), \quad -\infty < x < \infty \tag{7}$$

5 Heat equation with damping

[From Shearer and Levy 2015, Problem 5.9] Consider the initial value problem

$$u_t + du = ku_{xx}, \quad -\infty < x < \infty, t > 0, \tag{8}$$
$$u(x,0) = g(x), \quad -\infty < x < \infty \tag{9}$$

with constant d, and given integrable function g.

(5.a) Use the change of variable $u(x,t) = e^{-dt}v(x,t)$ to find u using the fundamental solution.

(5.b) What is the effect of the constant d?

(5.c) Suppose $d = d(t)$ is a given continuous function. What would be a suitable change of variable to solve the problem?

References

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