1 Poisson kernel in the upper plane

[From §2.2 of Stein and Shakarchi 2003] Solve the Laplace equation on the upper plane with boundary data:

$$u(x,0) = f(x). \tag{1}$$

Specifically, find the Green's function such that

$$u(x,y) = f(x) * \mathcal{P}_y(x). \tag{2}$$

Hint: use Fourier transform. Remember that product of Fourier transforms is convolution of the original functions.

Remark: This Green's function we found is called the Poisson kernel. It is a conformal mapping (i.e.: Cayley transform) away from the Poisson kernel for the unit disk, which you will obtain in the HW. However, I do not recommend using this method to get the Poisson kernel for the unit disk. The derivation in the HW is better.

2 Green's identity and Green's function

(2.a) We take a 3D scalar field ψ and a 3D vector field $\mathbf{\Gamma}$ with sufficient smoothness defined on some region $U \subset \mathbb{R}^3$. Show the identity

$$\iiint_U \left(\psi\,\nabla\cdot\mathbf{\Gamma} + \mathbf{\Gamma}\cdot\nabla\psi\right)\,dV = \oiint_{\partial U} \psi\,(\mathbf{\Gamma}\cdot\boldsymbol{n})\,dS = \oiint_{\partial U} \psi\mathbf{\Gamma}\cdot d\boldsymbol{S}. \tag{3}$$

(2.b) Use (3) to show the Green's first identity. Take 3D scalar fields ψ and φ both with sufficient smoothness:

$$\iiint_U \left(\psi\,\nabla^2\varphi + \nabla\psi\cdot\nabla\varphi\right)\,dV = \oiint_{\partial U} \psi\,(\nabla\varphi\cdot\boldsymbol{n})\,dS = \oiint_{\partial U} \psi\,\nabla\varphi\cdot d\boldsymbol{S}. \tag{4}$$

(2.c) Show the Green's second identity:

$$\iiint_U \left(\psi\,\nabla^2\varphi - \varphi\,\nabla^2\psi\right)\,dV = \oiint_{\partial U} (\psi\nabla\varphi - \varphi\nabla\psi)\cdot d\boldsymbol{S}. \tag{5}$$

This shows that the Laplacian is a self-adjoint operator for functions vanishing on the boundary so that the right hand side of the above identity is zero.

(2.d) Suppose we have the Green's function of the Poisson equation on the bounded domain Ω. That is, we have $G(\boldsymbol{x};\boldsymbol{y})$ s.t.

$$\nabla^2_{\boldsymbol{x}} G = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{with BC} \quad G|_{\partial\Omega} = 0. \tag{6}$$

Use Green's second identity to obtain the solution to the Poisson equation

$$\nabla_{\boldsymbol{x}}^2 u = f \quad \text{with BC} \quad u|_{\partial\Omega} = g \tag{7}$$

from the Green's function.

Think about how would you get such a Green's function for general domains.

3 Energy minimization for Poisson and Laplace

(3.a) Suppose that we are interested in solving the Poisson's equation:

$$-\nabla^2 u = f(x) \tag{8}$$

for continuous $f(x)$, and with Dirichlet boundary conditions over $\partial\Omega$ for u. Show that the solution of this equation minimizes, among $u \in C^2(\Omega) \cap C(\overline{\Omega})$, with $u|_{\partial\Omega} = g(x)$, the *complementary energy*:

$$V[u] = \int_\Omega \left(\frac{1}{2} \|\nabla u\|^2 - f(x)u \right) \mathrm{d}V. \tag{9}$$

Inversely, show that the minimizer of V solves the Poisson's equation.

(3.b) Now we study Laplace with *Neumann boundary condition*. Suppose that the boundary can be divided into $\partial\Omega_D$ on which $u = g(x)$ is known, and $\partial\Omega_N$ on which $\partial u/\partial n = N(x)$ is known. Show that u minimizes:

$$W[u] = \int_\Omega \frac{1}{2} \|\nabla u\|^2 \,\mathrm{d}V - \int_{\partial\Omega_N} Nu \,\mathrm{d}S. \tag{10}$$

among $u \in C^2(\Omega) \cap C(\overline{\Omega})$, with $u|_{\partial\Omega_D} = g(x)$. Additionally, show the inverse.

References

Stein, Elias M. and Rami Shakarchi (Apr. 6, 2003). *Fourier Analysis: An Introduction*. Princeton University Press. 328 pp. ISBN: 978-0-691-11384-5.