3 Blow-up of solution

[From Shearer and Levy 2015, Exercise 3.8]

(3.a) Use the method of characteristics to solve the initial value problem:

$$u_t + t u_x = u^2, \quad -\infty < x < \infty,\ 0 < t < 1 \tag{1}$$

with initial condition

$$u(x,0) = \frac{1}{1+x^2}. \tag{2}$$

Solution: We have the ODE equations

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1 \tag{3}$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = t \tag{4}$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = u^2. \tag{5}$$

We have the solution

$$t = \tau + t_0 \tag{6}$$
$$x = \frac{\tau^2}{2} + C \tag{7}$$
$$u = \frac{-1}{\tau + D}. \tag{8}$$

Since the boundary data is at $t = 0$, we pick $t_0 = 0$ for convenience. This give us $\tau = t$. We match the boundary condition

$$u(x_0, 0) = \frac{1}{1+x_0^2} \tag{9}$$
$$\Rightarrow \quad u = \frac{-1}{t - 1 - x_0^2} \tag{10}$$

To eliminate x_0 we find $x_0 = x(\tau = 0)$ as a function of (x, t). Using (7) we have

$$x_0 = C = x - \frac{t^2}{2}. \tag{11}$$

Together we have the solution

$$u(x,t) = \frac{1}{1 - t + (x - \frac{t^2}{2})^2} \tag{12}$$

(3.b) Show that the solution blows up as $t \to 1^-$:

$$\lim_{t \to 1^-} \max_x u(x,t) = \infty. \tag{13}$$

Remark: for a similar problem, see Olver 2014, Exercise 2.2.11.

Solution: Take the limit we have

$$\lim_{t \to 1^-} \max_x u(x,t) = \frac{1}{(x-1/2)^2} \tag{14}$$

and it will blow up at $x = 1/2$.

References

Olver, Peter J. 2014. *Introduction to Partial Differential Equations.* Undergraduate Texts in Mathematics. Cham: Springer International Publishing. ISBN: 978-3-319-02098-3. https://doi.org/10.1007/978-3-319-02099-0.

Shearer, Michael, and Rachel Levy. 2015. *Partial Differential Equations: An Introduction to Theory and Applications.* Princeton University Press, March 1, 2015. ISBN: 978-1-4008-6660-1.