1 Alternative formula for Fourier transform

In the HW you have one definition of the Fourier transform and its associated inversion formula:

$$\hat{f}(\xi) = \int f(x) e^{-i\xi x} \, \mathrm{d}x, \tag{1}$$

$$f(x) = \int \hat{f}(\xi) e^{+i\xi x} \, \mathrm{d}\xi \cdot \frac{1}{2\pi}. \tag{2}$$

An alternative formula that absorbs the 2π factor into the Fourier kernal is:

$$\hat{f}(\xi) = \int f(x) e^{-2\pi i\xi x} \, \mathrm{d}x, \tag{3}$$

$$f(x) = \int \hat{f}(\xi) e^{+2\pi i\xi x} \, \mathrm{d}\xi. \tag{4}$$

Show that the second version is also valid. I would use the second version in this note.

2 Some example dispersion relations

(2.a) The Korteweg–De Vries equation is a nonlinear, dispersive PDE that models waves on shallow water surfaces

$$u_t + u_{xxx} - 6uu_x = 0. \tag{5}$$

It is clear that zero is a solution. Linearize the equation around the zero solution. That is, assume the amplitude of the solution is small. We get

$$w_t + w_{xxx} = 0. \tag{6}$$

What is the dispersion relation? What is the phase velocity and group velocity?

(2.b) Kuramoto-Sivashinsky equation or the flame equation model a flame front.

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2} u_x^2 = 0. \tag{7}$$

It is known for its chaotic behavior. Again linearize around the zero solution to get the linear PDE

$$w_t + w_{xxxx} + w_{xx} = 0. \tag{8}$$

Calculate the same three quantities.

(2.c) We study the sine-Gordon equation

$$u_{tt} = c^2 u_{xx} - \sin(u). \tag{9}$$

Linearize around the zero solution. Calculate the same three quantities.

3 Numerical diffusion and dispersion

[From §10.9 of LeVeque 2007] We will study finite difference scheme for the advection equation

$$u_t + au_x = 0. \tag{10}$$

We assume $a > 0$ and $a = O(1)$.

(3.a) We could use the upwinding scheme

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{\Delta x}\left(U_j^n - U_{j-1}^n\right). \tag{11}$$

Show that the upwinding scheme is a consistent approximation of the PDE using Taylor expansion.

(3.b) Take a function $v(x,t)$ that satisfy the upwinding scheme exactly:

$$v(x, t+\Delta t) = v(x,t) - \frac{a\Delta t}{\Delta x}\left[v(x,t) - v(x-\Delta x, t)\right]. \tag{12}$$

Show that up to an $O(\Delta t^2)$ approximation the PDE that v satisfies is

$$v_t + av_x = \frac{1}{2}a\Delta x\left(1 - \frac{a\Delta t}{\Delta x}\right)v_{xx}. \tag{13}$$

This is called the modified equation of the upwinding scheme.

Therefore the numerical error when we use the upwinding scheme is diffusive in nature.

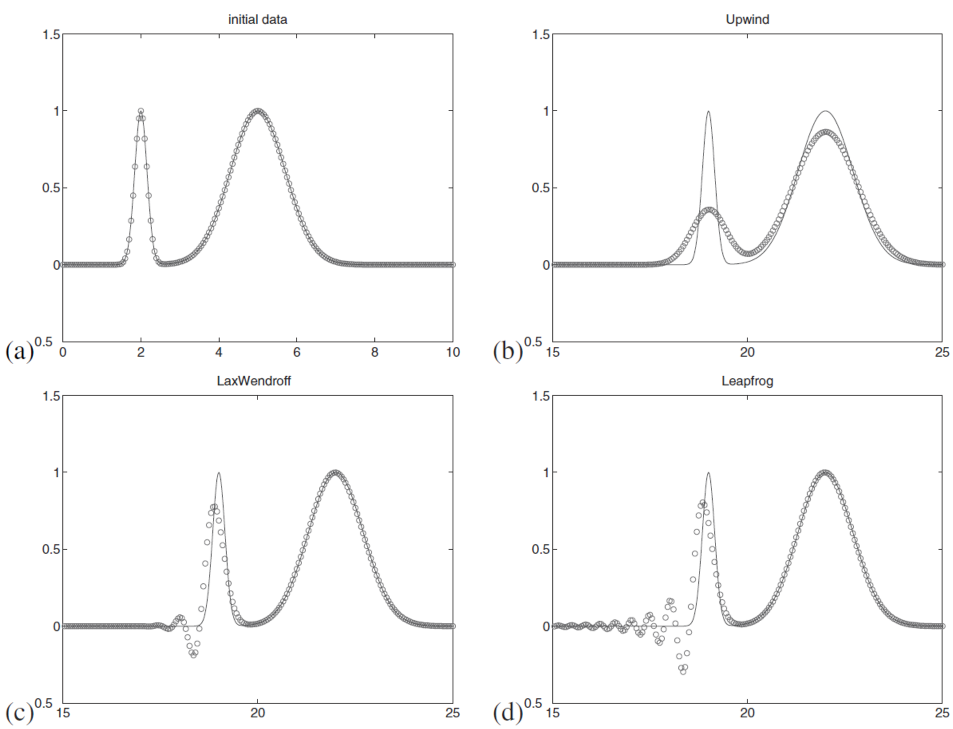


Figure 10.4. *The numerical experiments on the advection equation described in Section* 10.8.

(3.c) A better approximation to the advection equation gives the Lax-Wendroff scheme:

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right). \tag{14}$$

A rather long calculation[1] gives the modified equation of the Lax-Wendroff scheme:

$$v_t + av_x + \frac{1}{6}a\Delta x^2\left(1 - \left(\frac{a\Delta t}{\Delta x}\right)^2\right)v_{xxx} = 0. \tag{15}$$

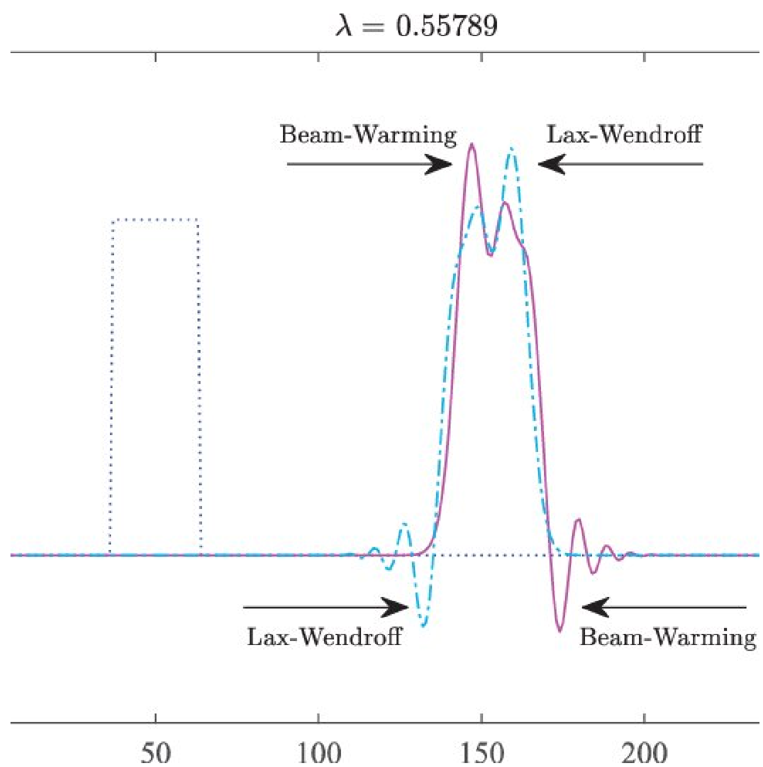
Calculate the dispersion relationship for waves in this PDE. Are the waves dispersive? What is the group velocity? We say that the error for the Lax-Wendroff is dispersive in nature.

For stability, we require the ratio

$$\frac{a\Delta t}{\Delta x} < 1. \tag{16}$$

Compare the group velocity to the advection velocity. Use this to explain the trailing wavy error in the above figure for Lax-Wendroff.

(3.d) The above plot also show the Leapfrog scheme, which also has dispersive error. Another scheme is the Beam-Warming scheme. We see from the figure below that its error is also dispersive, but the error lead the solution. You could do the same calculation and see that $c_g > a$.



[1] see https://guillod.org/teaching/m2-b004/TD2-solution.pdf

4 Riemann-Lebesgue Lemma

The mathematical theorem that is behind the method of stationary phase is the Riemann-Lebesgue Lemma.

Let $f : \mathbb{R} \to \mathbb{R}$ be a (Lebesgue) integrable function. Then we have

$$\lim_{\xi \to \infty} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, \mathrm{d}x = 0 \tag{17}$$

Intuitively, as ξ increases the oscillations in the integral increase and become much faster than any variation in $f(x)$; successive oscillations thus cancel and the integral becomes very small.

(4.a) Show this result assuming $f(x)$ is the indicator function for some compact set. Extend the result to step functions which is piecewise constant.

(4.b) Compact (Lebesgue) integrable function can be approximated arbitrary well by step functions. That is, for all $\epsilon > 0$ there is a step function φ such that

$$\left| \int_M f \, \mathrm{d}x - \int_M \varphi \, \mathrm{d}x \right| < \epsilon. \tag{18}$$

Use this fact to show that the Riemann-Lebesgue Lemma holds for compact (Lebesgue) integrable function.

(4.c) Finally, show that Riemann-Lebesgue Lemma holds for any (Lebesgue) integrable function. That is, function such that

$$\int_{\mathbb{R}} |f| \, \mathrm{d}x < \infty. \tag{19}$$

References

LeVeque, Randall J. (Jan. 1, 2007). *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*. SIAM. 343 pp. ISBN: 978-0-89871-783-9.