Worksheet 10

1 Poisson's equation with Robin boundary conditions

[From Shearer and Levy 2015, Problem 8.3] Consider Poisson's equation on a bounded open set $U \in \mathbb{R}^n$ with Robin boundary conditions

$$\nabla^2 u = f(\boldsymbol{x}), \quad \boldsymbol{x} \in U, \tag{1}$$

$$\text{with} \quad \frac{\partial u}{\partial \boldsymbol{n}} + \alpha u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial U. \tag{2}$$

(1.a) Prove that if $\alpha > 0$, then the energy method can be used to show uniqueness of solutions $u \in C^2(U) \cap C(\overline{U})$

(1.b) For $\alpha = 0$, show that solutions are unique up to a constant.

(1.c) Design an example to show that uniqueness can fail if $\alpha < 0$. (Hint: Choose $n = 1$.)

2 Conservation laws of KdV

[From ibid., Problem 12.2] Given the KdV equation

$$u_t + uu_x + \gamma u_{xxx} = 0 \tag{3}$$

with the condition that the solution decays sufficiently rapidly as $|x| \to \infty$.

(2.a) Show that the momentum and kinetic energy (when u interpreted as a velocity) are conserved:

$$\int_{\mathbb{R}} u \, \mathrm{d}x; \qquad \int_{\mathbb{R}} u^2 \, \mathrm{d}x. \tag{4}$$

(2.b) Find η (depending on γ) so that the integral

$$\int_{\mathbb{R}} \frac{1}{2}(u_x^2 - \eta u^3) \, \mathrm{d}x \tag{5}$$

is conversed.

3 Separation of variables for the Helmholtz equation

[From Olver 2014, Problem 4.3.18] Use separation of variables to solve the Helmholtz boundary value problem on the unit square:

$$\nabla^2 u = k^2 u \tag{6}$$

$$\text{with} \quad u(x,0) = 0, u(x,1) = f(x), u(0,y) = 0, u(1,y) = 0 \tag{7}$$

4 Energy minimization

[From Fall 2015 of Applied Differential Equations qualifying exam at UCLA, Problem 7[1]] Consider K be the set of functions $u : [0, 2] \to \mathbb{R}$ of the form

$$u(x) = \begin{cases} v(x) & \text{for} \quad x \in [0, 1) \\ w(x) & \text{for} \quad x \in (1, 2] \end{cases} \tag{8}$$

and $v \in C^2[0, 1)$, $w \in C^2(1, 2]$, with the property that $v(0) = w(2) = 0$ and $[u] := w(1) - v(1) = a$. You should assume that the value of the constant a is known. Define

$$E(u) = \frac{1}{2} \int_0^1 (u_x)^2 \,\mathrm{d}x + \int_1^2 (u_x)^2 \,\mathrm{d}x + \frac{w(1) + v(1)}{2} b \tag{9}$$

where b is another constant, whose value you can assume is known. Show that there exists $h(x)$ that minimizes E over all functions $u \in K$, and solve for $h(x)$.

5 Green's identity and Green's function

(5.a) We take a 3D scalar field ψ and a 3D vector field $\mathbf{\Gamma}$ with sufficient smoothness defined on some region $U \subset \mathbb{R}^3$. Show the identity

$$\iiint_U (\psi \, \nabla \cdot \mathbf{\Gamma} + \mathbf{\Gamma} \cdot \nabla \psi) \; dV = \oiint_{\partial U} \psi \, (\mathbf{\Gamma} \cdot \boldsymbol{n}) \; dS = \oiint_{\partial U} \psi \mathbf{\Gamma} \cdot d\boldsymbol{S}. \tag{10}$$

(5.b) Use (10) to show the Green's first identity. Take 3D scalar fields ψ and φ both with sufficient smoothness:

$$\iiint_U \left(\psi \, \nabla^2 \varphi + \nabla \psi \cdot \nabla \varphi\right) \; dV = \oiint_{\partial U} \psi \, (\nabla \varphi \cdot \boldsymbol{n}) \; dS = \oiint_{\partial U} \psi \, \nabla \varphi \cdot d\boldsymbol{S}. \tag{11}$$

(5.c) Show the Green's second identity:

$$\iiint_U \left(\psi \, \nabla^2 \varphi - \varphi \, \nabla^2 \psi\right) \; dV = \oiint_{\partial U} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot d\boldsymbol{S}. \tag{12}$$

This shows that the Laplacian is a self-adjoint operator for functions vanishing on the boundary so that the right hand side of the above identity is zero.

(5.d) Suppose we have the Green's function of the Poisson equation on the bounded domain Ω. That is, we have $G(\boldsymbol{x}; \boldsymbol{y})$ s.t.

$$\nabla^2_{\boldsymbol{x}} G = \delta(\boldsymbol{x} - \boldsymbol{y}) \quad \text{with BC} \quad G|_{\partial\Omega} = 0. \tag{13}$$

Use Green's second identity to obtain the solution to the Poisson equation

$$\nabla^2_{\boldsymbol{x}} u = f \quad \text{with BC} \quad u|_{\partial\Omega} = g \tag{14}$$

from the Green's function.

Think about how would you get such a Green's function for general domains.

[1] https://ww3.math.ucla.edu/wp-content/uploads/2021/09/ade-15F.pdf

References

Olver, Peter J. (2014). *Introduction to Partial Differential Equations*. Undergraduate Texts in Mathematics. Cham: Springer International Publishing. ISBN: 978-3-319-02098-3. DOI: 10.1007/978-3-319-02099-0.

Shearer, Michael and Rachel Levy (Mar. 1, 2015). *Partial Differential Equations: An Introduction to Theory and Applications*. Princeton University Press. 287 pp. ISBN: 978-1-4008-6660-1.