1 Domain of dependence

[From Shearer and Levy 2015, Exercise 4.6] Suppose u satisfies the wave equation with $c = 1$ for $x > 0$. We want to find a solution of the PDE for $x > 0, t > 0$, that satisfy the initial conditions

$$u(x, 0) = \phi(x) \tag{1}$$
$$u_t(x, 0) = \psi(x), \tag{2}$$

and the boundary condition

$$u_x(0, t) = 0. \tag{3}$$

(1.a) Solve for $u(x, t)$

(1.b) Assume $\mathrm{supp}\phi = \mathrm{supp}\psi = [1, 2]$, where can you guarantee that $u = 0$ for $x > 0, t > 0$.

2 Fourier series and the expansions of π

(2.a) Show the Fourier series of periodic extension of x on the interval $(-\pi, \pi)$ is

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx). \tag{4}$$

(2.b) Show this expansion of π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \tag{5}$$

(2.c) Show the solution to the Basel problem is

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}. \tag{6}$$

3 Challenging wave equation problem

[From Spring 2019 of Applied Differential Equations qualifying exam at UCLA, Problem 1[1]] Let $u(x, t)$ solve the initial value problem

$$\begin{cases} u_{tt} + u_{xt} - 2u_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = g(x), \\ u_t(x, 0) = h(x). \end{cases} \tag{7}$$

1. https://ww3.math.ucla.edu/wp-content/uploads/2021/09/ade-19S.pdf

(3.a) Derive a formula for u in terms of g and h, when g and h are C^2.

Hint: Consider how to simplify the equation into something more obviously like the wave equation by making a change of coordinate system: $(x, t) \to (\zeta, t)$ where $\zeta = x - vt$ for v appropriately determined.

(3.b) Next consider the boundary value problem

$$\begin{cases} u_{tt} + u_{xt} - 2u_{xx} = 0, & x \in [0, 1], t > 0, \\ u(0, t) = u(1, t) = 0. \end{cases} \tag{8}$$

Show that a smooth solution u to above problem must be zero if $u(x, 0) = u_t(x, 0) = 0$.

Hint: use an energy argument. Try the energy for the wave equation.

4 Equipartition of energy

[From Evans 2010, Exercise 2.24] Let u solve the initial-value problem for the wave equation in one dimension:

$$u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \tag{9}$$

$$u = 0, u_t = 0 \quad \text{on} \quad \mathbb{R} \times \{t = 0\}. \tag{10}$$

Suppose g, h have compact support. We define the kinetic energy

$$k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t)\, \mathrm{d}x \tag{11}$$

and the potential energy

$$p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t)\, \mathrm{d}x. \tag{12}$$

We know that the total energy $k(t) + p(t)$ is constant in t. Show that $k(t) = p(t)$ for all large enough times t.

References

Evans, Lawrence C. 2010. *Partial Differential Equations.* American Mathematical Soc. ISBN: 978-0-8218-4974-3.

Shearer, Michael, and Rachel Levy. 2015. *Partial Differential Equations: An Introduction to Theory and Applications.* Princeton University Press, March 1, 2015. ISBN: 978-1-4008-6660-1.