1 Green's identity and Green's function

(1.a) We take a 3D scalar field ψ and a 3D vector field $\mathbf{\Gamma}$ with sufficient smoothness defined on some region $U \subset \mathbb{R}^3$. Show the identity

$$\iiint_U (\psi \nabla \cdot \mathbf{\Gamma} + \mathbf{\Gamma} \cdot \nabla \psi)\, dV = \oiint_{\partial U} \psi\, (\mathbf{\Gamma} \cdot \boldsymbol{n})\, dS = \oiint_{\partial U} \psi \mathbf{\Gamma} \cdot d\boldsymbol{S}. \tag{1}$$

(1.b) Use (1) to show the Green's first identity. Take 3D scalar fields ψ and φ both with sufficient smoothness:

$$\iiint_U \left(\psi\, \nabla^2 \varphi + \nabla \psi \cdot \nabla \varphi\right) dV = \oiint_{\partial U} \psi\, (\nabla \varphi \cdot \boldsymbol{n})\, dS = \oiint_{\partial U} \psi\, \nabla \varphi \cdot d\boldsymbol{S}. \tag{2}$$

(1.c) Show the Green's second identity:

$$\iiint_U \left(\psi\, \nabla^2 \varphi - \varphi\, \nabla^2 \psi\right) dV = \oiint_{\partial U} (\psi \nabla \varphi - \varphi \nabla \psi) \cdot d\boldsymbol{S}. \tag{3}$$

This shows that the Laplacian is a self-adjoint operator for functions vanishing on the boundary so that the right hand side of the above identity is zero.

(1.d) Suppose we have the Green's function of the Poisson equation on the bounded domain Ω. That is, we have $G(\boldsymbol{x}; \boldsymbol{y})$ s.t.

$$\nabla^2_{\boldsymbol{x}} G = \delta(\boldsymbol{x} - \boldsymbol{y}) \quad \text{with BC} \quad G|_{\partial\Omega} = 0. \tag{4}$$

Use Green's second identity to obtain the solution to the Poisson equation

$$\nabla^2_{\boldsymbol{x}} u = f \quad \text{with BC} \quad u|_{\partial\Omega} = g \tag{5}$$

from the Green's function.

Think about how would you get such a Green's function for general domains.

2 Energy minimization for Poisson and Laplace

(2.a) Suppose that we are interested in solving the Poisson's equation:

$$-\nabla^2 u = f(x) \tag{6}$$

for continuous $f(x)$, and with Dirichlet boundary conditions over $\partial\Omega$ for u. Show that the solution of this equation minimizes, among $u \in C^2(\Omega) \cap C(\overline{\Omega})$, with $u|_{\partial\Omega} = g(x)$, the *complementary energy*:

$$V[u] = \int_\Omega \left(\frac{1}{2} \|\nabla u\|^2 - f(x) u\right) \mathrm{d}V. \tag{7}$$

Inversely, show that the minimizer of V solves the Poisson's equation.

(2.b) Now we study Laplace with *Neumann boundary condition*. Suppose that the boundary can be divided into $\partial\Omega_D$ on which $u = g(x)$ is known, and $\partial\Omega_N$ on which $\partial u/\partial n = N(x)$ is known. Show that u minimizes:

$$W[u] = \int_{\Omega} \frac{1}{2} \|\nabla u\|^2 \, \mathrm{d}V - \int_{\partial\Omega_N} Nu \, \mathrm{d}S. \tag{8}$$

among $u \in C^2(\Omega) \cap C(\overline{\Omega})$, with $u|_{\partial\Omega_D} = g(x)$. Additionally, show the inverse.

3 Euler–Tricomi equation

The Euler–Tricomi equation is a PDE useful in the study of transonic flow:

$$u_{xx} + xu_{yy} = 0. \tag{9}$$

This equation changed its type over x. Categorize this equation.