1 Domain of dependence

[From Olver 2014, Exercise 2.2.12] A sensor situated at position $x = 1$ monitors the concentration of a pollutant $u(t, 1)$ as a function of t for $t \geq 0$. Assuming that the pollutant is transported with wave speed $c = 3$, at what locations x can you determine the initial concentration $u(0, x)$?

Remark: this is a first example of an inverse problem. To explain a sub-class of inverse problems: "forward" problem is the evolution of the PDE from the initial condition, and inverse problem tries to infer information about the initial condition from observations of the solution at a later time (and a specific location). Things get significantly more difficult when diffusion, modeled by the heat equation, is in the dynamics. Inverse problem is a big field with active research. We will come back to explore more of it later on.

2 Initial and boundary conditions

[From Olver 2014, Exercise 2.2.14] Let $c > 0$. Consider the uniform transport equation

$$u_t + cu_x = 0 \tag{1}$$

restricted to the quarter-place $Q = \{x > 0, t > 0\}$ and subject to initial conditions

$$u(0, x) = f(x) \quad \text{for} \quad x \geq 0 \tag{2}$$

along with the boundary condition

$$u(t, 0) = g(t) \quad \text{for} \quad t \geq 0. \tag{3}$$

(2.a) For which initial and boundary conditions does a classical solution to this initial-boundary value problem exists? Write down a formula for the solution.

(2.b) On which regions are the effects of the initial conditions felt? What about the boundary conditions? Is there any interaction between the two?

3 Blow-up of solution

[From Shearer and Levy 2015, Exercise 3.8]

(3.a) Use the method of characteristics to solve the initial value problem:

$$u_t + tu_x = u^2, \quad -\infty < x < \infty,\ 0 < t < 1 \tag{4}$$

with initial condition

$$u(x, 0) = \frac{1}{1 + x^2}. \tag{5}$$

(3.b) Show that the solution blows up as $t \to 1^-$:

$$\lim_{t\to 1^-} \max_x u(x,t) = \infty. \tag{6}$$

Remark: for a similar problem, see Olver 2014, Exercise 2.2.11.

4 Transport in higher dimensions

[Adapted from Olver 2014, Exercise 2.2.31] We will consider transport equation in 2D. In vector form, the transport equation is

$$\frac{D\rho}{Dt} := \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0 \tag{7}$$

where we define the material derivative D/Dt. In 2D, the velocity vector is

$$\boldsymbol{u}(x,y,t) = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix}. \tag{8}$$

We also write out the dot product in (7):

$$\boldsymbol{u} \cdot \nabla \rho = u\rho_x + v\rho_y. \tag{9}$$

All together we have the transport equation

$$\frac{D\rho}{Dt} = 0 \tag{10}$$

$$\iff \quad \rho_t + u(x,y,t)\rho_x + v(x,y,t)\rho_y = 0. \tag{11}$$

(4.a) Define a characteristic curve, and prove that the solution is constant along it.

(4.b) Apply the method of characteristics to solve the initial value problem

$$\rho_t + y\rho_x - x\rho_y = 0, \quad u(0,x,y) = e^{-(x-1)^2-(y-1)^2}. \tag{12}$$

(4.c) Describe the behavior of your solution.

5 Symmetries of the wave equation

[From Shearer and Levy 2015, Exercise 4.3] Show that if $u(x,t) \in C^3$ is a solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{13}$$

then so are the following functions:

(5.a) For any $y \in \mathbb{R}$, the function $u(x-y,t)$

(5.b) Both u_x and u_t.

(5.c) For any $a \in \mathbb{R}$, the function $u(ax,at)$.

References

Olver, Peter J. 2014. *Introduction to Partial Differential Equations.* Undergraduate Texts in Mathematics. Cham: Springer International Publishing. ISBN: 978-3-319-02098-3. https://doi.org/10.1007/978-3-319-02099-0.

Shearer, Michael, and Rachel Levy. 2015. *Partial Differential Equations: An Introduction to Theory and Applications.* Princeton University Press, March 1, 2015. ISBN: 978-1-4008-6660-1.