

1 Blow-up of solution

[From Shearer and Levy 2015, Exercise 3.8]

(1.a) Use the method of characteristics to solve the initial value problem:

$$u_t + tu_x = u^2, \quad -\infty < x < \infty, \quad 0 < t < 1 \quad (1)$$

with initial condition

$$u(x, 0) = \frac{1}{1 + x^2}. \quad (2)$$

Solution: We have the ODE equations

$$\frac{dt}{d\tau} = 1 \quad (3)$$

$$\frac{dx}{d\tau} = t \quad (4)$$

$$\frac{du}{d\tau} = u^2. \quad (5)$$

We have the solution

$$t = \tau + t_0 \quad (6)$$

$$x = \frac{\tau^2}{2} + C \quad (7)$$

$$u = \frac{-1}{\tau + D}. \quad (8)$$

Since the boundary data is at $t = 0$, we pick $t_0 = 0$ for convenience. This give us $\tau = t$. We match the boundary condition

$$u(x_0, 0) = \frac{1}{1 + x_0^2} \quad (9)$$

$$\Rightarrow u = \frac{-1}{t - 1 - x_0^2} \quad (10)$$

To eliminate x_0 we find $x_0 = x(\tau = 0)$ as a function of (x, t) . Using (7) we have

$$x_0 = C = x - \frac{t^2}{2}. \quad (11)$$

Together we have the solution

$$u(x, t) = \frac{1}{1 - t + (x - \frac{t^2}{2})^2} \quad (12)$$

(1.b) Show that the solution blows up as $t \rightarrow 1^-$:

$$\lim_{t \rightarrow 1^-} \max_x u(x, t) = \infty. \quad (13)$$

Remark: for a similar problem, see Olver 2014, Exercise 2.2.11.

Solution: Take the limit we have

$$\lim_{t \rightarrow 1^-} \max_x u(x, t) = \frac{1}{(x - 1/2)^2} \quad (14)$$

and it will blow up at $x = 1/2$.

References

- Olver, Peter J. 2014. *Introduction to Partial Differential Equations*. Undergraduate Texts in Mathematics. Cham: Springer International Publishing. ISBN: 978-3-319-02098-3. <https://doi.org/10.1007/978-3-319-02099-0>.
- Shearer, Michael, and Rachel Levy. 2015. *Partial Differential Equations: An Introduction to Theory and Applications*. Princeton University Press, March 1, 2015. ISBN: 978-1-4008-6660-1.