THE TROJAN ASTEROID BELT:
PROPER ELEMENTS, STABILITY, CHAOS AND FAMILIES

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Abstract. I have computed proper elements for 174 asteroids in the 1 : 1 resonance with Jupiter, that is for all the reliable orbits available (numbered and multi–opposition). The procedure requires numerical integration, under the perturbations by the four major planets, for 1,000,000 years; the output is digitally filtered and compressed into a "synthetic theory" (as defined within the *LONGSTOP* project). The proper modes of oscillation of the variables related to eccentricity, perihelion, inclination and node define proper elements. A third proper element is defined as the amplitude of the oscillation of the semimajor axis associated with the libration period; because of the strong nonlinearity of the problem, this component cannot be determined by a simple Fourier transform to the frequency domain. I therefore give another definition, which results in very good stability with time. For 87% of the computed orbits, the stability of the proper elements –at least over 1 Myr– is within the following bounds: 0.001 AU in semimajor axis, 0.0025 in eccentricity and sine of inclination. Half of the cases with degraded stability of the proper elements are found to be chaotic, with e–folding times between $16,000$ and $660,000\,yr$; in some other cases, chaotic behaviour does not result in a significantly decreased stability of the proper elements (stable chaos). The accuracy and stability of these proper elements is good enough to allow a search for asteroid families; however, the dynamical structure of the Trojan belt is very different from the one of the main belt, and collisional events among Trojans can result in a distribution of fragments difficult to identify. The occurrence of couples of Trojans with very close proper elements is proven not to be statistically significant in almost all cases. As the only exception, the couple 1583 Antilochus – 3801 Thrasimedes is significant; however, it is not easy to account for it by a conventional collisional theory. The Menelaus group is confirmed as a strong candidate collisional family; Teucer and Sarpedon could be considered as significant clusters. A number of other clumps are detected (by the same automated clustering method used for the main belt by Zappalà et al., 1990,1992), but the total number of Trojans with reliable orbits is not large enough to detect many significant candidate families.

Key words: asteroids–libration–proper elements–stability–chaos–families

1. The Other Asteroid Belt

The motion of the asteroids of the so–called *main belt* has been for a very long time an important source of dynamical problems to be handled by Celestial Mechanics. Out there there is another asteroid belt, the *Trojan belt*, which was believed, by some, to be less relevant to the history and dynamical structure of the Solar System only because the larger distance and the lower albedo biased the rate of asteroid discovery; now the number of Trojans is believed to be of the same order as that of the main belt asteroids (Schoemaker et al., 1992).

The problem of the 1:1 libration orbits has always fascinated the specialists of Celestial Mechanics, to the point that to collect a complete list of references is an impossible task. The theory I found most useful is the one by Érdi (1988; and references therein), which is based on a 3–D model with Jupiter in a fixed

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elliptic orbit, and which I shall often quote later. However, very few people have tried to work on the orbits of the real objects according to a realistic model of the perturbations by all the major planets; the most important work on this has been done by Schubart and Bien (Bien and Schubart, 1987; Schubart and Bien, 1987; and references therein). This might be due to the intrinsic difficulty of the problem: the very high inclination of many Trojans makes it very difficult to use any theory truncated with respect to the powers of inclination and eccentricity; the singularity of collision occurring for values of the critical argument just outside the libration region makes both analytical and semianalytical theories much more difficult.

However, the problem is there for us to solve, and the fact that the already available tools are not suitable to tackle the problem is not a good excuse not to try. Therefore I decided to collect as much information as possible on the behaviour of Trojan orbits and on the dynamical structure of the Trojan belt, by means of a large set of numerical integrations of orbits, in the framework of a realistic model of the gravitational perturbations by the major planets.

Now that CPU time is not any more a scarce resource (how many personal workstations are really used during the night?) the main problem with numerical integrations is to understand what the results mean. Unless spectacular instabilities occur, very little information is contained in the final state, even after a very long integration. The main challenge is then to compress the information –contained in hundreds of megabytes of orbital elements time series– in a theory comparable in form and readability to a simple analytical theory. Such a problem has been solved for non–resonant planetary orbits with the *synthetic theory* method (Carpino et al., 1987), and some techniques (such as digital filtering) can be copied from that case.

However, it turns out that there are some features of the orbits of the real Trojans which cannot be accounted for by a synthetic theory of the same form of the one successfully used to represent the orbits of the major outer planets. This arises from one of the same reasons discussed above, namely from the fact that the three *small parameters* occurring in a theory of the Trojans, the sine of the libration amplitude, the proper eccentricity, and the sine of the proper inclination, can have values as high as $0.58, 0.22, 0.61$ (see later, e.g. Tables 1 and 2, and Figure 10). Thus a theory which essentially forces a linearisation of the results through linear spectral analysis cannot represent well the orbit over a long period; this even for perfectly regular, quasiperiodic orbits. Here I propose a modified synthetic theory technique, which can be shown to represent the dynamical behaviour of Trojan orbits over a long time span (of the order of millions of years) and with satisfactory accuracy (typically within a thousandth of AU in semimajor axis and a few thousandths in $e, \sin I$), and this for most orbits of real Trojans.

Two pieces of information can be extracted by this technique from the output of a number of numerical integrations of Trojan orbits. *Proper elements*, that is three characteristic orbital parameters which, unlike the osculating elements, are almost constant (within a narrow range of variations) for a very long time, are computed for all the Trojans observed for long enough to provide reliable osculating orbital elements. *Resonances* and *chaotic behaviour* can be studied from the few difficult cases in which the stability of the proper elements is poor.

The proper elements, being almost constant, allow to search for *families* of objects which have been on very close orbits a long time ago, and which could be interpreted as the remains of a catastrophic collision. Information on resonances and chaos, especially when combined with the data on the time invariance of the proper elements, allows to discuss the stability of the Trojan belt. In this paper these consequences of my computations are only tentatively explored, and a long list of open problems is given. However, I hope to have provided –to myself as well as to others– a large enough set of constraints which should allow to found a future dynamical model of the Trojan belt on a solid basis.

This paper is organized as follows. Section 2 contains the definitions of the proper elements, and information on how they have been computed for 174 Trojan asteroids with well known orbits; the proper elements themselves are given in two Tables at the end of the paper. Section 3 discusses the accuracy of the proper elements, by means of a stability test using the running box method; also the results on the estimation of the maximum Lyapounov characteristic exponent are given, and the –more or less understood– relationship between stability and chaos is briefly discussed. Section 4 presents a preliminary search for asteroid families in the Trojan belt; although the method used is consistent with the most rigorous ones used for the main belt (Zappalà et al., 1990, 1992), the results are limited, simply because the number of Trojans observed so far is not large enough. Section 5 gives a list of problems, which I believe are relevant to understand the dynamics and the history of the Trojan belt. For these, either I have not been able to find a solution, or (worse) the problem looks more difficult now than when I started studying the dynamics of Trojans; a significant amount of future work, including possibly my own, needs to be done.

2. Computation of Proper Elements

The critical argument of the 1:1 resonance is $\lambda - \lambda'$, where λ is the mean longitude of the Trojan and λ' the mean longitude of Jupiter; since this argument cannot circulate, but librates around either $\pi/3$ or $-\pi/3$, it can be expressed as:

$$\lambda - \lambda' = \chi + D\cos(\theta) + \ldots \tag{2.1}$$

where θ is the argument of libration, $\chi = \pm\pi/3 + \ldots$, D is the libration amplitude, and the little dots stand for anything else, which is supposed to be "higher order". As a result of the libration, the semimajor axis a of the Trojan also oscillates $\pi/2$ out of phase:

$$a = a' + d\sin(\theta) + \ldots \tag{2.2}$$

around a mean value given by the semimajor axis a' of Jupiter. In the simplest possible approximation, D and d are constant and θ is a linear function of time, with a fixed frequency f: $\theta = f(t - t_o)$. This approximation is not too bad, if the dynamics of the Trojan is studied within a simplified model, in which Jupiter is on a fixed elliptic orbit (restricted, 3–D, 3–body model): as shown by Érdi (1988), the little dots are indeed of order 2 with respect to a small parameter, which is of the same order of D and d; moreover, the ratio d/D is given by:

$$\frac{d}{D} = \sqrt{3\mu}a' + \ldots \simeq 0.278 \tag{2.3}$$

where μ is the ratio (mass of Jupiter)/(mass of the Sun + mass of Jupiter), d is measured in AU, D in radians. However, the theory by Érdi does not account for the perturbation from all the other planets. When the perturbation of the outer planets (mostly the one by Saturn) is added in, two effects arise. First, Saturn (and the other planets) perturb directly the orbit of the asteroid, inducing both short periodic and long periodic changes in the orbital elements. Second, the orbital elements of Jupiter are not constant any more; this indirectly affects the orbit of the Trojans, and makes it hard to use a 3–body type theory as an intermediary orbit, because the secular changes in the eccentricities and inclinations are slow but by no means small.

For the orbits which are not locked in a resonance, an approximate representation is available for the solution of the complete N–body problem: the classical linear theory of secular perturbations. The nonsingular orbital elements $h = e\sin(\varpi), k = e\cos(\varpi)$ (ϖ is the longitude of the perihelion) and $p = \sin(I)\sin(\Omega), \sin(I)\cos(\Omega)$ are represented as a Fourier series:

$$\begin{aligned} h =&\ e_P \sin(gt+\phi) + \textstyle\sum_k \xi_k \sin(g_k t + \phi_k) + \ldots \\ k =&\ e_P \cos(gt+\phi) + \textstyle\sum_k \xi_k \sin(g_k t + \phi_k) + \ldots \\ p =&\ \sin(I_P)\sin(st+\psi) + \textstyle\sum_k \eta_k \sin(s_k t + \psi_k) + \ldots \\ q =&\ \sin(I_P)\cos(st+\psi) + \textstyle\sum_k \eta_k \sin(s_k t + \psi_k) + \ldots \end{aligned} \tag{2.4}$$

with twice as many *secular fundamental frequencies* g_k, s_k as there are perturbing planets, and two *proper frequencies* g, s; the forced amplitudes ξ_k, η_k are proportional to the proper eccentricities e_k and to the sine of the proper inclinations I_k of the perturbing planets, while the *linear proper eccentricity and inclination* e_P, I_P are function of the initial conditions of the asteroid. In (2.4) the little dots stand for the terms which are of higher order, with respect to the small parameters e_P, e_k, I_P, I_k; some higher order terms can also be computed, to improve the accuracy in the representation of the solution, and to compute nonlinear *proper eccentricity and inclination* more accurately constant (Milani and Knežević, 1990, 1992). There are two main difficulties in applying a similar procedure to the Trojan case. First, the coefficients of an analytic theory (e.g., the ratios $\xi_k/e_k, \eta_k/\sin(I_k)$) are not simple combinations of Laplace coefficients, but more complicated functions of the orbital elements for which no explicit expression is known. Second, the little dots standing for the higher order terms are not negligible at all when the proper inclinations are as high as it is typical of the Trojan belt; even an higher

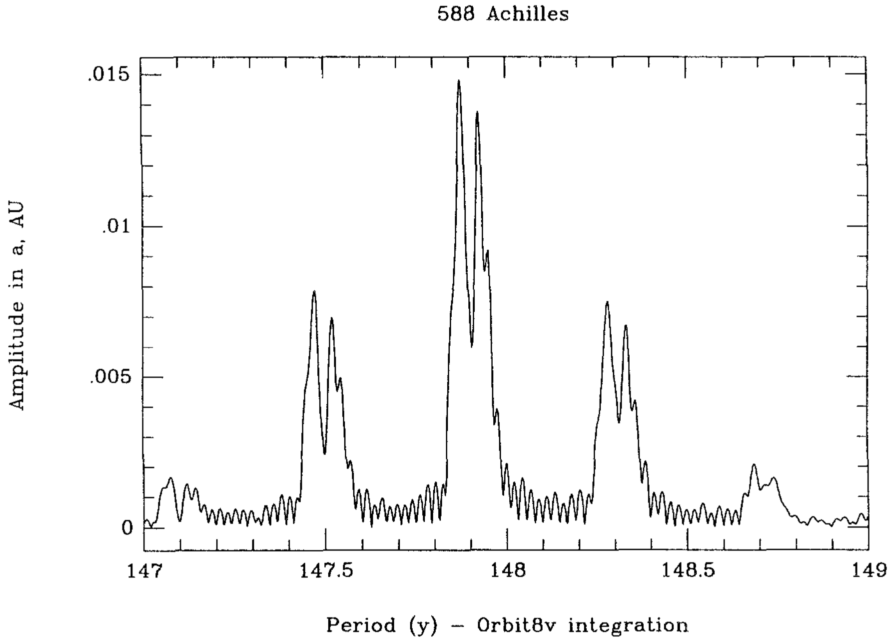


Fig. 1. Spectrum of the changes in the semimajor axis of *588 Achilles*: the region around the frequency f of the libration of the critical argument is shown. The two main features on the sides of the main line correspond to the frequencies $f \pm (g_5 - g_6)$. The finer structure inside each line cannot be interpreted because the data time span of 1 Myr is not long enough.

order analytical theory (analogous to Milani and Knežević), for the very fact of using an expansion in powers of the inclination, could not give accurate results.

All these difficulties notwithstanding, some theory will one day be manufactured to describe the full complexity of the dynamical behaviour of the Trojans; the semianalytical theories (see Williams, 1969; Morbidelli, 1991) are, in principle, the most promising. However, I always found useful, before constructing a theory, to know what the results are for a representative set of examples; numerical integration can of course provide us with such a set. The problem is how to represent the solution given by the numerical integration.

The output of any numerical integration is a time series for the osculating orbital elements. A well established technique (Carpino et al., 1987; Schubart and Bien, 1984) allows to generate a mean elements time series by digital filtering; the problem of the short periodic perturbations by Saturn et al. can therefore be handled in a comparatively easy way. As for the long periodic perturbations, the technique of the *synthetic theory*, used to transform the output time series in a Fourier series, and developed for *Project LONGSTOP*, accounts for higher and higher degree terms by assuming a linear superposition of harmonics, formed by integer combinations of the secular fundamental frequencies (Carpino et al., 1987;

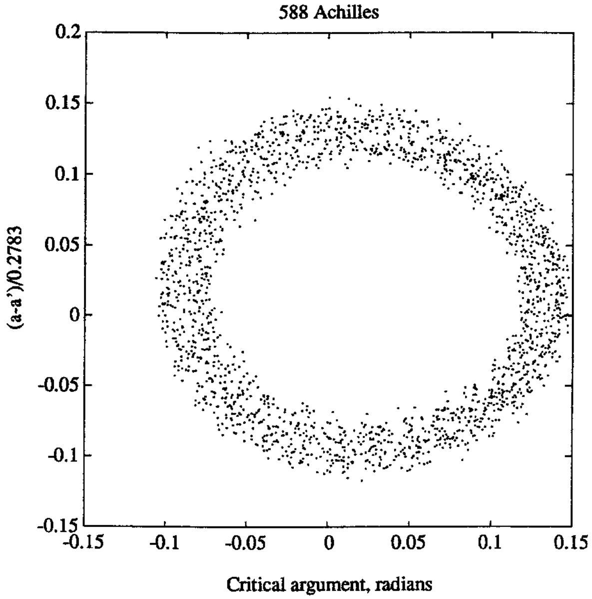


Fig. 2. Critical argument $\lambda - \lambda' - \pi/3$, in radians, and renormalized difference in semimajor axis between the Trojan and Jupiter, that is $(a - a')/(0.2783\, AU)$, for *588 Achilles*; the librations are not exactly circular, nor the center is at the origin of the coordinate axis, but the average amplitude of libration is well defined.

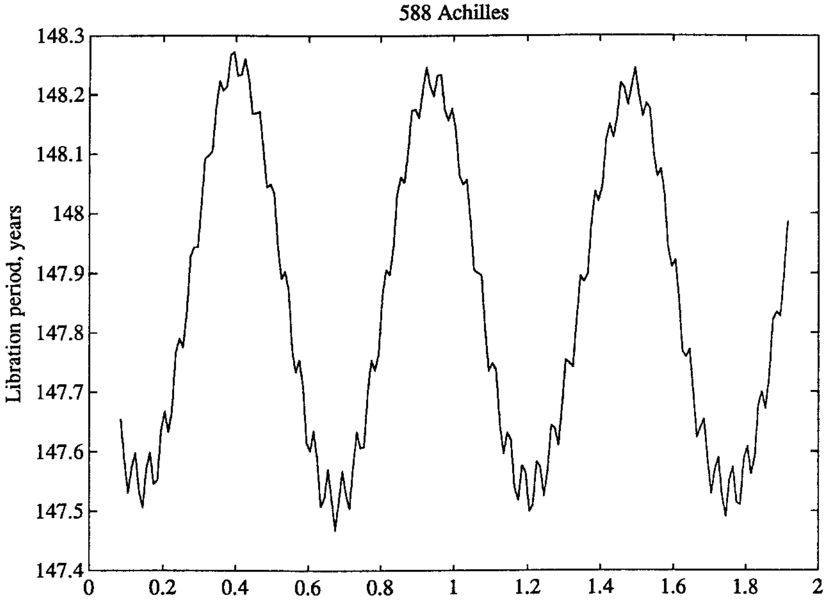
Nobili et al., 1989). The analogous of the *LONGSTOP* technique, applied to the Trojan case, would be to represent the little dots of equations (2.1), (2.2), (2.4) as trigonometric terms with frequencies obtained from integer combinations of f, g, s and g_k, s_k. The reason why this would not work is shown in Figure 1, where I have plotted a portion of the Fourier transform of the semimajor axis of *588 Achilles*, corresponding to frequencies near the libration frequency f. All the spectral lines in this figure can be interpreted as combination frequencies, e.g. the separation between the bunch of lines around 147.9 y and the one around 147.5 y corresponds to the frequency $g_6 - g_5$. However, the synthetic theory technique works only if the series of higher and higher degree combination frequencies converges, and quickly enough; as it is seen in Figure 1, the combination lines decrease too slowly to allow an accurate representation of the solution with a Fourier series of reasonable length.

To understand what is happening to the semimajor axis of a Trojan asteroid, I have computed the argument of libration θ from (2.1) and (2.2), neglecting all the little dots and taking also into account (2.3); that is, θ is the polar angle of the point with cartesian coordinates $(0.2783 \cdot (\lambda - \lambda' - \chi), a - a')$. Since the libration is not a circle –centered at the origin– in the plane of the above defined variables (see Figure 2), an oscillation with period equal to the libration period is superimposed to the long term trend of θ; moreover, there are short periodic perturbations induced by Saturn et al.. The time series for the angle θ is then digitally filtered, in such a way that all the oscillations with periods between 2.5 y and 250 y are removed; the main short periodic perturbation has a period of $\simeq$ 20 y, and the libration period is $\simeq$ 150 y, thus both the short periodic perturbations and the oscillation with period equal to the libration period are removed.

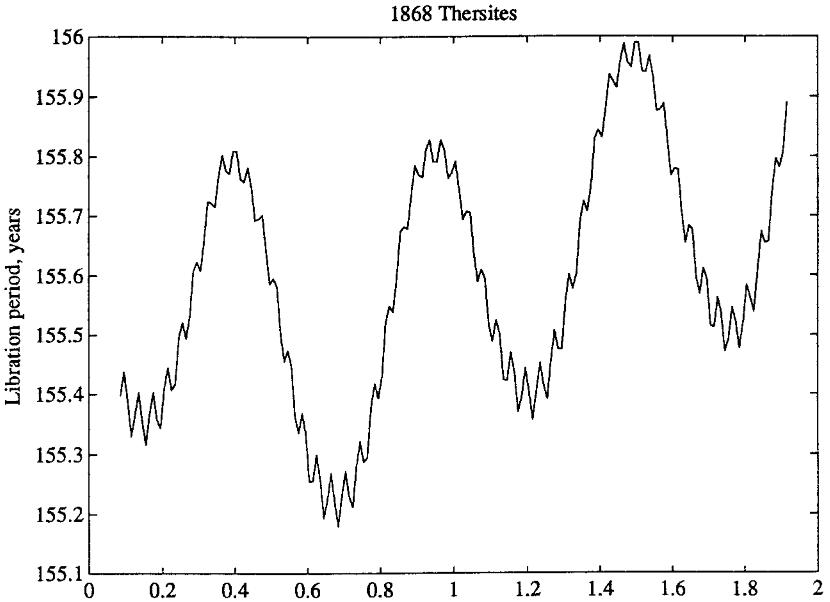
Since the filter has coefficients with sum 1 and even symmetry, linear trends are not distorted. This applies provided the angular variables are handled as real numbers (that is, the angular variables are continuous functions of time, unlike the principal values which jump by 2π at each revolution). Then the filtered time series for θ can be used to estimate a libration period (e.g. by linear fit) with short periodic perturbations removed. The result of this computation for *588 Achilles* and *1868 Thersites* are shown in Figure 3; the libration period changes over a long period (with arguments such as $(g_5 - g_6)t$) and the extremes of this oscillation correspond to the main side lines of the spectrum of Figure 1.

The standard representation for an oscillation with variable period is of the form $A\cos(ft + \tau)$ where τ oscillates slowly; the attempt to represent the changes in a of a Trojan as a Fourier series with arguments linear functions of time fails because the amplitude of the oscillation of τ is large, as shown (again for *588 Achilles*, and for *2594*) in Figure 4. However, by some kind of dumb luck the amplitude A is not changing in a long periodic way as τ does, and this both for the semimajor axis and for the critical argument; that is both D and d are approximately constant for a long time. This can be understood in an intuitive way, by using again the formula (2.3); the ratio of the amplitudes in the two conjugate variables depends only weakly upon the amplitudes themselves: the little dots in (2.3) are actually second order with respect to D (Érdi, 1988, eq. (8) and (4)). Therefore the area of the libration curve is approximately proportional to the square of any one of the two amplitudes, and the amplitudes d and D are, to a good approximation, functions of the adiabatic invariant obtained by representing the problem as a slowly varying restricted elliptic problem.

The proper element related to the libration amplitude can therefore be defined as the amplitude of the oscillation of either a or $\lambda - \lambda'$ with argument θ; for the latter variable, the filtered time series is used, while the variables a, a', λ, λ' are represented by a time series passed through a different filter, which removes the short periodic perturbations (periods between 2.5 y and 50 y) but not the periods close to 150 y. The only tricky point is that the Fourier component with argument θ of either a or $\lambda - \lambda'$ needs to be computed in line with the numerical integration, because the frequency f is comparatively fast and a time series for 1 My would be too large to be stored, if sampled well enough to compute a component with a period $\simeq$ 150 y. For this purpose I have modified my own numerical integration program by adding two parallel filters (with dark bands for periods up to 50 and 250 y respectively, designed according to Carpino et al., 1987), and the computation of θ as defined above; the sums to be used to compute the Fourier transforms with



(a)



(b)

Fig. 3. Changes in the period of libration, in years, as a result of longer period perturbations: (a) for *588 Achilles* the main changes are correlated with the changes in the eccentricity of Jupiter, that is with the frequency $g_5 - g_6$; (b) for *1868 Thersites*, with larger amplitude of libration, a more complex behaviour appears.

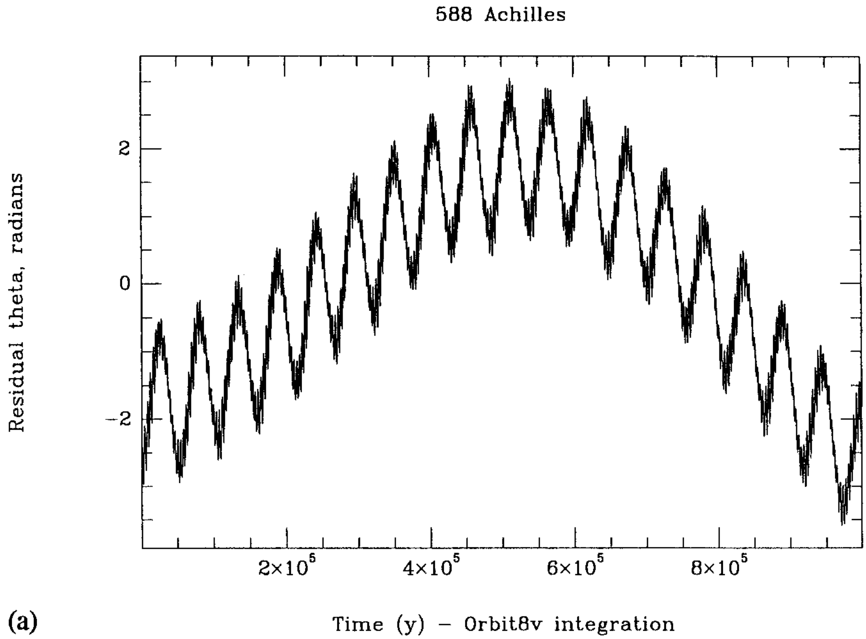
respect to θ are accumulated and written to disk at long intervals. This procedure I have used to compute the proper elements d and D, and the frequency f, for all the Trojan asteroids for which a good enough orbit is available from the observations.

The computation started from a list of osculating elements for Trojan asteroids, including 85 numbered asteroids, 86 observed at two oppositions, and 3 single opposition with a very long arc ($> 90\, days$); the orbital elements were for the same reference epoch (December 1991). The catalogue of orbits was provided to me by E. Bowell of the Lowell Observatory, as a byproduct of his continuing effort to maintain and improve a large catalogue of asteroid orbital data. For the unnumbered asteroids I am using in this paper an arbitrary numbering (with numbers above *6000*); to actually identify the indicated asteroid, the readers must refer to Table 2 where international designators and survey catalogue numbers are given.

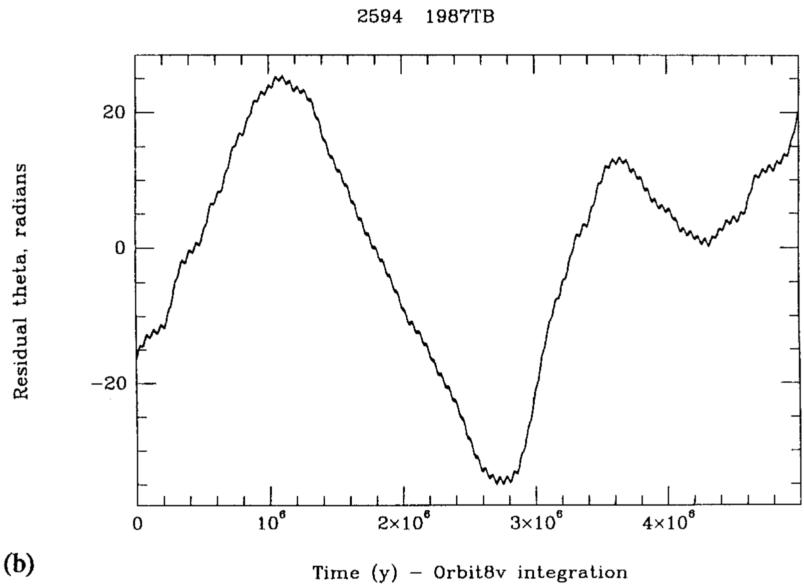
In the same catalogue there were 97 other single opposition orbits, but I have not used them because the osculating orbital elements are too inaccurate. Table II of Van Houten et al. (1991) shows that the orbital elements of Trojans, derived from single opposition observations during the main surveys specifically targeted at the Trojan belt, are too poor to be useful, having an uncertainity an order of magnitude larger than the one introduced by my computation of proper elements (see Section 3 and Table 3). It is somewhat surprising that the surveys, which have been conducted with the specific purpose of discovering Trojans, have used observations spread over an arc of only 16 days, shorter by a factor of 2 than the arc length used in the Palomar–Leiden survey for main belt asteroids; this when it is well known that a longer arc is needed to determine with the same accuracy a Trojan orbit.

For each one of the 174 Trojans I have used, an orbit was numerically computed for a time span of 1 $Million\ years$, within a rather complete model, fully accounting for the perturbations by the four major outer planets; the accuracy of the integration was more than enough for our purposes (Milani and Nobili, 1988). The amount of computational resources used was not very significant; my portable workstation was running background jobs for a few weeks. To reduce the effect of the unmodelled perturbations by the inner planets, the initial conditions of both the outer planets and the asteroids have been referred to the center of mass of the inner solar system; this technique (essentially equivalent to the use of jacobian coordinates) sharply reduces the difference in the fundamental secular frequencies g_k, s_k between computations including and not including the inner planets; e.g. the value of the frequency g_6 of the perihelion of Saturn, as found by numerical integrations for $> 5\ My$, is $28.164\ arcsec/y$ without barycentric correction (Milani and Knežević, 1992), and 28.236 with barycentric correction; the best value, obtained by the *LONGSTOP* project with a complicated procedure to remove the effect of the inner planets on the initial conditions, is 28.246 (Nobili et al., 1989).

The values of the amplitudes d and D, and of the libration frequency f, are listed in the first 3 columns in Table 1 (for numbered asteroids) and Table 2 (for



(a)



(b)

Fig. 4. The libration argument θ has been fitted to a straight line (thus computing the frequency f) and the residuals, in radians, plotted over the entire integration time span: (a) for *588 Achilles*, over 1 Myr, a regular pattern appears; (b) for *2594 1977 RR*, over 5 Myr, the irregular behaviour of θ suggests chaos.

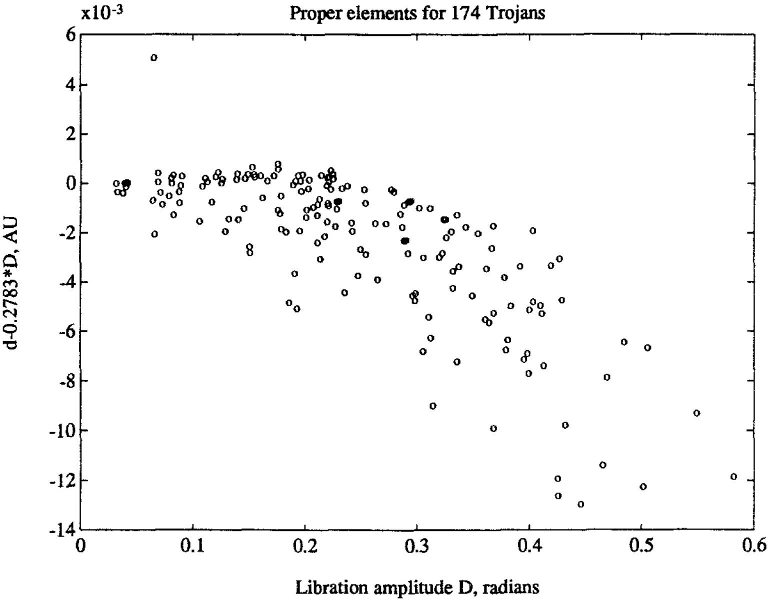


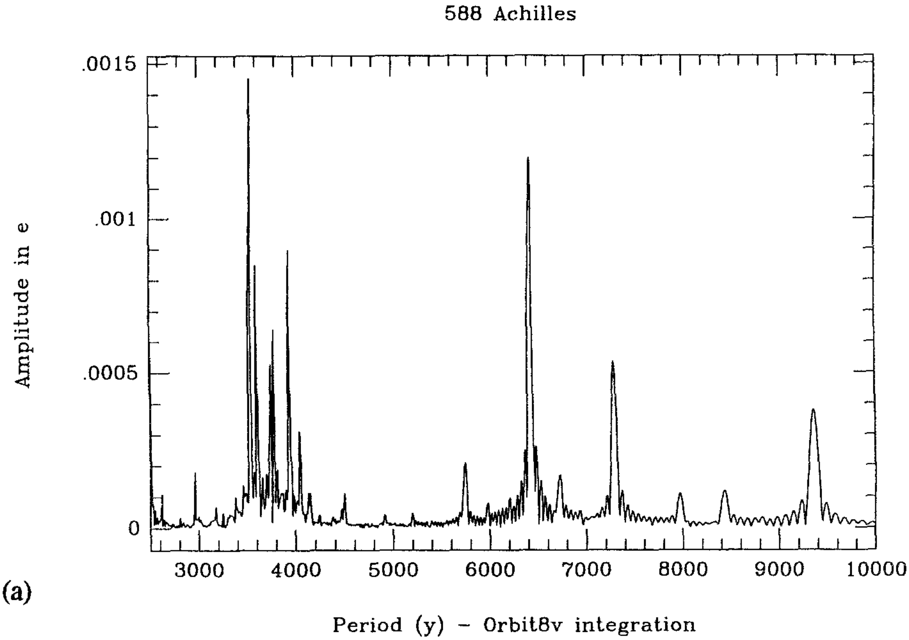
Fig. 5. The amplitude of libration d of the critical argument, in radians, and the amplitude of oscillation D of the semimajor axis, after the simple linear trend of eq. (2.3) has been removed. The trend corresponds to the higher order terms in Érdi (1988) theory, but there is also a wide scatter.

unnumbered two opposition orbits). These data can also be used to check a posteriori the accuracy of the approximation (2.3) used to compute θ; Figure 5 shows the residuals of d with respect to the simple law $d = 0.2783 \cdot D$; there is a visible trend corresponding to the higher order terms (see Érdi, 1988, eq. (8)), but also some scatter, indicating the limitations of a theory based upon a very simplified model.

For the computation of the other two proper elements, the orbital elements h, k, p, q of the asteroid are taken from the digitally filtered time series; the perturbations with periods between 2.5 y and 250 y are removed almost perfectly (the attenuation factor is $< 10^{-3}$) by the filter, and the filtered time series is essentially equivalent to a set of mean elements. For the eccentricity related variables h, k the proper eccentricity is essentially defined by (2.4): the components with the known frequencies g_5, g_6, g_7 are determined by Fourier analysis and removed from the data; the largest nonlinear forced term, the one with frequency $2g_6 - g_5$, is also removed (this term is important, see Milani and Knežević, 1992). The residuals after this removal of the forced terms are analysed to find the fundamental frequency g; to this purpose I compute the angle $gt + \phi$ as a time series, again accounting for the number of revolutions (this can be done provided the time series for h, k is sampled at time intervals shorter than π/g); a linear fit to the angle gives the best estimate of the frequency g and the amplitude of the Fourier component with frequency g is used as proper eccentricity e_P. The values of e_P and g computed in this way are given in columns 4 and 5 of the Tables 1 and 2.

The above procedure is somewhat simplistic, in that the highly nonlinear character of the Trojan dynamics results in the presence of important nonlinear coupled modes; however, the combination frequencies containing f should correspond to periods shorter than 250 y, which are removed by the digital filter. The main problem arises from the coupled modes containing both g and s, and also possibly some of the g_k, s_k; they result in the splitting of the spectral line with frequency g into a complicated multiplet. However, for most Trojans the combination lines are not very strong and the determination of the proper mode with frequency g is not significantly impaired, at the level of accuracy of this work. Figure 6 shows the spectrum of the variable h for frequencies close to g, after removal of both the forced terms g_5, g_6, g_7 and $2g_6 - g_5$ and of the proper mode g. For *588 Achilles*, the multiplet surrounding g (made visible by the removal of the dominant line) has the highest amplitude line at the frequency $g - 2s$. The frequency $g - 2s$ corresponds to the behaviour described by Bien and Schubart (1987) and by Érdi (1988) with the geometric construction of an ellipse rotating with frequency s. It would be possible to compute proper elements in a more refined way by adding the $g - 2s$ line to the theory; the improvement would not be very significant, since there are many more spectral lines not accounted for by Érdi's theory. Even in a case with large inclination, such as *2363 Cebriones*, the $g - 2s$ frequency corresponds to the largest, but by no means dominant, spectral line in the residuals; although the residuals after fitting the forced and proper modes are larger, the proper eccentricity is still stable enough (see Table 3).

The computation of the proper inclination was somewhat more problematic. As a matter of principle, (2.4) could be used again, and the forced terms could be identified and removed before proceeding to the determination of s and I_P. However, in this case the forced modes with frequencies s_6 and s_7 are much smaller than the proper s mode; if they are determined before removing the proper mode, the relative accuracy is often $> 10\%$. The only case in which the forced modes are large is when they are near resonant, and this means that the separation of the forced mode, from the proper mode with nearby frequency, is difficult. As for the s_8 mode, it is larger than the s_7 mode, but its period is $1,871,000\,y$ and there is no way to compute it accurately with only $1\,My$ of data. As a result, I have taken the simplistic approach of computing the proper angle $st + \psi$ as the polar angle of (q, p) (accounting for the number of revolutions) and then fitting s; of course p, q have been computed with inclination and node with respect to the invariable plane of the outer solar system (as determined by *LONGSTOP*). The sine of the proper inclination is then taken as the amplitude of the Fourier component with frequency s. The accuracy of this procedure is not really limited by the "noise" left by the forced terms, but by the time span. I have used the method by Ferraz-Mello (1981) to correctly separate the target frequency from the constant (or longer period) terms, but this is not always enough. The period of the node



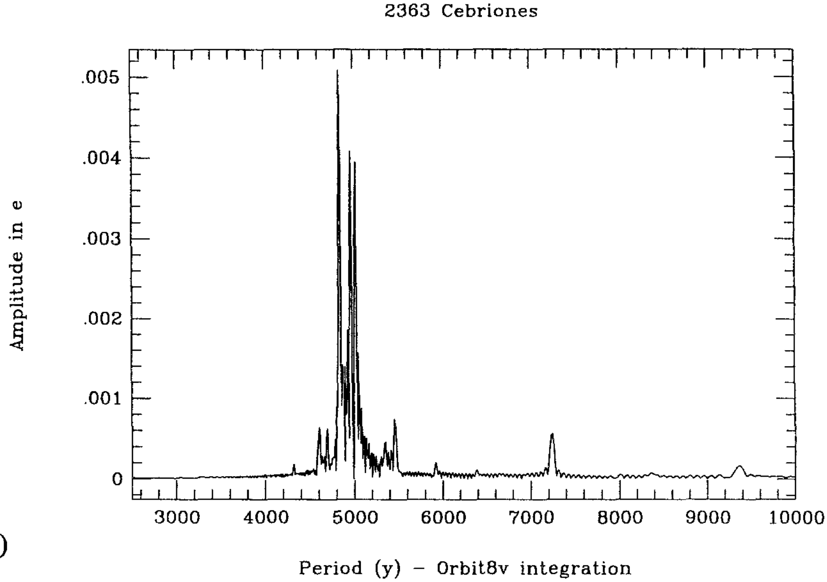


Fig. 6. The spectrum of the mean element $h = e \sin \varpi$ after removal of the main lines, namely the ones with frequencies $g, g_5, g_6, g_7, 2g_6 - g_5$: (a) for *588 Achilles*, the residuals contain a multiplet around the removed proper mode g (corresponding to a period of $3,763\, yr$), but also other lines not accounted for by any existing theory; (b) for *2363 Cebriones*, with a proper inclination of $32.7°$, the $g - 2s$ line predicted by existing theories is the largest, but there are many more forming a complex multiplet around the removed g mode (at $4,990\, yr$).

of a Trojan can range between $\simeq 2{,}700{,}000\, y$ (for *1208 Troilus*) and $\simeq 38{,}000\, y$ (for *2594* and *6031*). When it is comparable to the integration time span, neither the frequency nor the amplitude are well determined. In 4 such cases, I decided to extend the numerical integration to 5 My; this was done for *1208, 2146, 4833* and *6057* because of their long nodal periods, and also for 3 more asteroids for another reason which will be apparent from Section 3. For all the 174 Trojans $\sin I_P$ and s are given in columns 6 and 7 of Tables 1 and 2; the last two columns specify whether the libration is around L_4 (*Greek*, or leading Trojan) or L_5 (*Trojan*, or trailing Trojan), and the integration time span.

The Tables of this paper, as well as some auxiliary information, are available in computer readable form from the `ftp anonymous server` of my department (*INTERNET* address `gauss.dm.unipi.it`, that is 131.114.6.55; look in the `pub/propel/trojans` directory).

3. Accuracy, Stability, Chaos

The level of accuracy in the proper elements, which is actually required, is limited to the physically meaningful range, and the latter is limited both by the observational accuracy of the astrometry and by the impossibility to study collisional processes between too small asteroids that far. A level of accuracy corresponding to about 0.001 (rms) in both proper e and $\sin I$, to about 0.01 AU in a (that is, in d), and to about 20 m/s in relative velocity after escape of the fragments of a catastrophic disruption, can be considered fully satisfactory, at the present state of the art. The question is whether I have achieved this level of accuracy.

I have always been very diffident toward internal consistencies and formal covariances as a way to assess the accuracy of a result; what is needed is a test which directly verifies the property of the proper elements which makes them useful, namely their stability over a long time span. This is comparatively easy to do for main belt asteroids, if the proper elements are computed by an analytical formula: the time series of osculating elements obtained as output of an accurate numerical integration is processed to give the corresponding time series for the proper elements; the changes of the supposedly constant proper elements directly measure their accuracy (Milani and Knežević, 1990; 1992). For the Trojans, such an analytic theory does not exist, and the proper elements as defined in this paper are themselves derived from the output of a long numerical integration. The only external check I could think of, consists of a change in the time span over which the output of the numerical integration is analysed. I have taken a running box of length $500{,}000\, y$, and moved it by steps of $50{,}000\, y$, thus obtaining 11 separate computations of my proper elements (for the 7 orbits which have been computed for 5 My, the box was 2.5 My long and the shifts were by $250{,}000\, y$). The stability of the proper elements and of the proper frequencies is defined as the difference between the maximum and the minimum of these 11 values. To compare both with the physically meaningful levels discussed above, and with the values given by

Milani and Knežević (1990; 1992), you should take into account that the value (maximum–minimum) generally corresponds to $\simeq 2\sqrt{2}$ times the rms value of the changes with respect to the mean value.

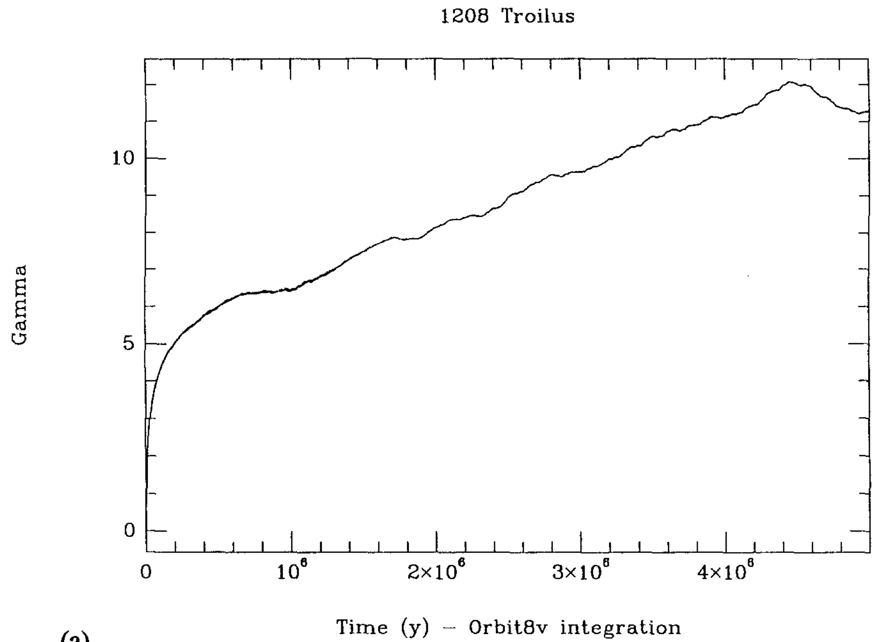
The results of this test have been encouraging; as an example, the values for *588 Achilles* are given in the first line of Table 3; they are so small, that the statement –given in Section 2– that a more refined theory is not needed appears fully justified in this case. Of course there are also some asteroids for which the results are much less satisfying. Table 3 lists all the cases in which either $\Delta d > 0.001\ AU$ or $\Delta e_P > 0.0025$ or $\Delta(\sin I_P) > 0.0025$: there are 7, 10 and 14 such cases respectively, a total of 22 asteroids with "degraded" proper elements. This proportion of difficult cases is not very different from the one occurring in the main belt, and does not inhibit the use of a proper elements catalogue to search for asteroid families (see Milani and Knežević, 1992; Zappalà et al., 1992). However, we would like to understand why some proper elements are less proper than others.

Beside the example of *588*, Table 3 contains also some other interesting cases, including all the 7 orbits computed for 5 My. The main indicators available to detect irregular dynamical behaviour are the *Lyapounov characteristic exponents (LCE)*. An orbit is by definition *chaotic* when some Lyapounov exponents are positive; all the Lyapounov exponents are zero when the solution is a conditionally periodic orbit of the kind described by the Kolmogorov–Arnold–Moser theorem; it is not known whether other cases can occur with a non vanishing probability.

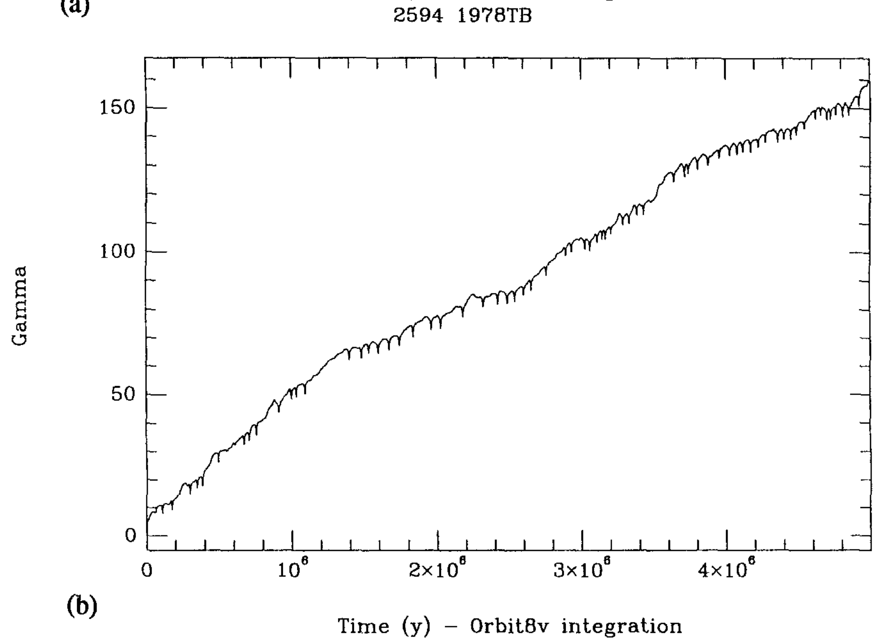
For each one of the 174 Trojan orbits computed numerically, I have simultaneously solved the *equation of variations*, that is the linearized differential equation for relative motion of two nearby orbits. The initial conditions for the variational equations were randomly chosen displacement vectors $v(0) = v_0$. It is possible to obtain an indication on the value of the maximum Lyapounov exponent by monitoring the function

$$\gamma(t) = \log \frac{|v(t)|}{|v_0|} \tag{3.1}$$

where $v(t)$ is the solution of the variational equations. With probability 1 (with respect to a random choice of the initial conditions v_0) the maximum Lyapounov exponent is: $\chi = \lim_{t\to+\infty} \gamma(t)/t$. Any calculation being finite, only an estimate can be obtained by means of the function $\gamma(t)$. I have used a linear fit to $\gamma(t)$ over the available time span to estimate the maximum LCE; the constant absorbs most of the initial increase of the function $\gamma(t)$ due to the linear divergence of nearby orbits occurring even in a regular orbit, and this allows the detection of a slower linear trend (see Figure 7). In this way a positive LCE χ can generally be detected with an integration time span of between 6 and 7 times $1/\chi$ ($1/\chi$ is the *Lyapounov time*). With a time span of 1 My all the cases in which the fit to $\gamma(t)$ gives a slope $\simeq 6 \times 10^{-6}\ y^{-1}$ or larger can be interpreted as positive detections of a positive LCE; of course there are always marginal cases, in which visual inspection of the plot of $\gamma(t)$ is recommended before drawing any conclusion; thus a number of



(a)



(b)

Fig. 7. The divergence ratio γ (see eq. (3.1)) as a function of time. The presence of a linear trend indicates a positive Lyapounov exponent. (a) for *1208 Troilus*: although the time span is too short for strong chaotic effects, the best fit value of $1.5 \times 10^{-6}\, yr^{-1}$ indicates the orbit is indeed chaotic. The use of the classical formula $\gamma(t)/t$ at the end of the integration time span would give a wrong value and would not allow a reliable detection of chaos. (b) for *2594* the presence of chaos is quite obvious; the Lyapounov time is only $\simeq 35,000\, yr$.

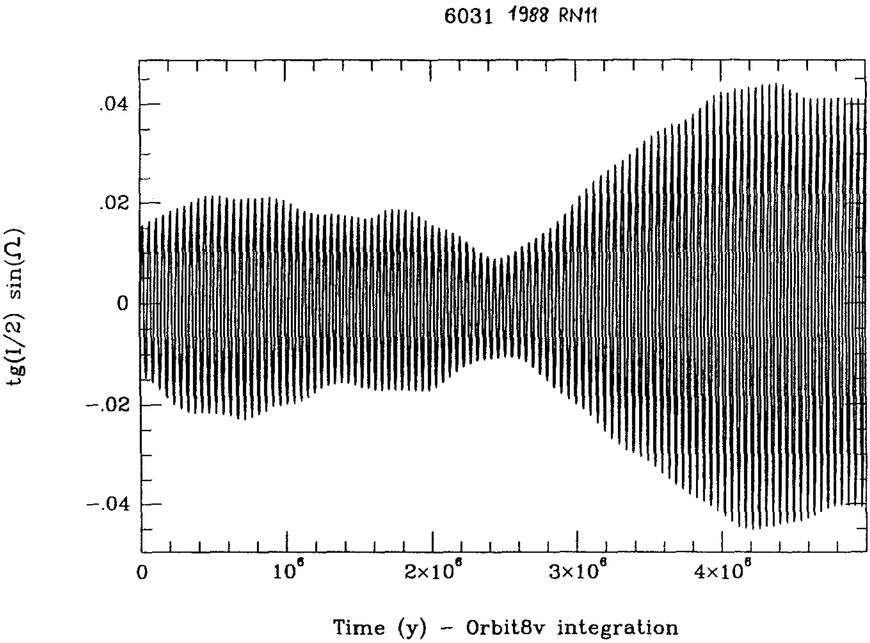


Fig. 8. The worst case of unstable chaos: *6031 88RN11* undergoes a shift in the orbital behaviour after $\simeq 2\,Myr$ which results in a change in the proper elements by a factor $\simeq 2$. As an example the element $tg(I/2)\sin\Omega$ shows a wild change in the amplitude of oscillation.

cases with a result of the fit in the range 4 to $5 \times 10^{-6}\,y^{-1}$ are likely to be chaotic, but a longer integration is needed for a positive detection.

The values of the maximum LCE, computed with the method discussed above, are given in the last column of Table 3; all the orbits for which the value was $\geq 6 \times 10^{-6}$ have been included in the Table. 24 such cases have been found among the 174 orbits computed. For the orbits which have been computed for $5\,My$, even a smaller value of the maximum LCE can be considered as a positive detection of chaos: the value 1.5×10^{-6} for *1208 Troilus* is indeed a positive detection, and indicates a chaotic behaviour with a Lyapounov time longer than any known example in the asteroid belt (Figure 7a). Some other Trojans have remarkably large LCE, such a *2594* (Figure 7b); 3 of these strongly chaotic cases (*2594, 6031, 6067*) have also been computed for $5\,My$ to look for large scale instability. Figure 8 shows the most striking case of large scale instability, occurring for asteroid *6031*, with I_P changing by a factor $\simeq 2$.

For each entry, the instabilities of the proper elements and frequencies as derived from the running box test are also given, in the same order as in the Tables 1 and 2. For all the other orbits I have computed, neither chaotic behaviour nor instability of the proper elements (above the levels discussed above) was detected. For the asteroids which are listed in Table 3, the main problem is to understand the origin and long term dynamical effect of both instability and chaoticity. I cannot claim I have solved all the problems posed by Table 3; on the contrary, I understand only a few of these dynamical and computational problems, and a list of unsolved problems is given in Section 5. My experience with the study of the dynamics of a large population of objects indicates that it is very important to find a right balance between quantity and quality; in other words, extending the study to a larger sample and investigating in depth a few peculiar cases are both useful, and it would be wrong to give up any of the two. Therefore I will try to give at least a first approximation classification of all the "exceptional" cases of Table 3, to be able to point out the cases worthy of further study, without giving up my approach based upon large scale computations and the study of general dynamical properties of large populations.

All the confirmed chaotic orbits and orbits with degraded proper elements, a total of 37 cases, belong to one of the following classes:

(A) chaotic orbits, with an instability of the proper elements d and D of significant size, although not very large in absolute terms: *1437, 3451, 4709, 4835, 4946.* Asteroid *4835* is also remarkable for its very high eccentricity (however it is not unique, since the recently discovered *1991XX* has a comparable eccentricity). In some of these cases there is some instability in the proper element I_P, but not in e_P; this might be a clue to the critical arguments involved in the chaotic behaviour, but I do not have an explanation for the occurrence of chaos.

(B) stable chaotic orbits, for which a positive LCE does not correspond to a significantly increased instability of the proper elements and frequencies: *1404, 1868, 1869, 2207, 2797, 4489, 4543, 4708, 4754, 4827, 6001, 6049, 6056, 6079.* Stable chaos is a poorly understood phenomenon, but it is now clear that it does occur in many orbits of real solar system bodies, sometimes in very extreme forms (Milani and Nobili, 1992). The resonances involved are likely to be of high order; on the other hand, with a ratio f/g always above $\simeq$ 18, a ratio g/s above $\simeq$ 12, and a ratio n/f always above $\simeq$ 12, low order resonances are hard to find in the Trojan belt.

(C) unstable chaos, with both positive LCE and large scale instabilities: *1173, 2594, 6031, 6067, 6073.* The only resonances which are known to occur in the Trojan belt are secular resonances involving the node, mainly $s - s_6$, which is certainly responsible for the spectacular instability of *6031* shown in Figure 8, and is also implied in the behaviour of *1173* and *6073*. This resonance has already been studied (see Nakai, and Kinoshita, 1985; Bien and Schubart, 1986), but a wild behavior such as that of *6031* has never been suspected. Another puzzle arises from the fact that *1871* is also close to the $s - s_6$ resonance but it shows no indications of chaos. It is somewhat surprising to find a large scale instability, as in the case of *6031*, for an object for which there are no a priori reasons to suspect a dynamical age shorter than that of most asteroids; on the other hand, there are no indications of a transition to a *comet* class orbit, that is to a Jupiter crossing state. For *2594, 6067* the origin of the chaotic behaviour is unknown: these are large libration amplitude and low inclination orbits, and this of course implies stronger perturbation by Jupiter; however, the Trojan with the largest libration amplitude is *6075* with $D = 33.3°$, and it does not show any indication of chaotic behaviour. I have not been able to find any resonance of a "reasonable" order which could be responsible for their behavior. The only possible explanation is that some prejudices on what is a "reasonable" resonance need to be abandoned.

(D) orbits with large I_P, and very slow s: *1208, 2146, 2363, 3317, 4833, 4834, 4867, 6047, 6057.* The main problem is to understand whether the difficulty is computational (that is, the proper elements have not been computed accurately because of the need to solve for effects with a period close to the data time span) or dynamical (that is, there are secular resonances involved). In the cases of *4833* and *6057*, whose node precesses with a period of $1,026,000$ and $699,000\ y$ respectively, the "cure" consisting in an increased integration time span has worked reasonably well; not so for *1208 Troilus* and *2146 Stentor*, for which there are very complicated long periodic perturbations. *1208* has $s = -0.15\ arcsec/y$ and it is not clear whether the time span of 5 My is enough; on the other hand the influence of some secular resonance cannot be excluded. For *2146* $s = +3.28\ arcsec/y$ and there is the possibility that the secular small divisor $2s - 2g_5$ plays a significant rôle, as proposed by Bien and Schubart (1987); the finding of a (very slow) chaotic behaviour might be rated as a confirmation of this hypothesis. All the cases in this class have periods of the node above $280,000\ y$, and it could be argued that for the 5 cases in which the orbit has been computed for 1 My and the running box was only $500,000\ y$ long, the proper element I_P computed with half data is likely to be much less accurate of the one computed with all the data, thus the instability estimate of Table 3 should be pessimistic. However, not all the cases with slow s show the same degradation.

(E) cases in which the proper eccentricity has a degraded stability, without any other sign of either chaos or instability: *884, 4057, 6023, 6050.* None of these cases has a high inclination, and the occurrence of complicated combination lines containing both g and s is not likely to be responsible; some other mechanism, which I do not understand, must be involved. Of course it is possible to investigate these cases with degraded e_P one by one, but this is not a solution. An algorithm for the computation of proper elements needs to be automatic and objective, to allow the processing of large and ever increasing catalogues of orbits; to find a specific fix for each case would not help.

As a conclusion, I am rather satisfied that the proper elements I have computed are as accurate as they should be. Out of the 22 cases which are not stable according to the specifications, 11 are chaotic; in these cases, it is already a remarkable result that proper elements can, in some cases, be only moderately unstable, that is the finding of various degrees of stable chaos, with very small average changes to proper elements over one Lyapounov time. Of the other 11 cases, 6 (may be 7) can be explained by the presence of very long periods which should be handled with even longer integrations, and only 4 to 5 seem to point to some dynamical phenomenon not accounted for in the algorithm used to compute proper elements. It is certainly possible to improve upon this result, but these improvements would be relevant only for a small minority of Trojans, and therefore I believe the present version of my proper elements is good enough to be used, e.g. to search for Trojan families.

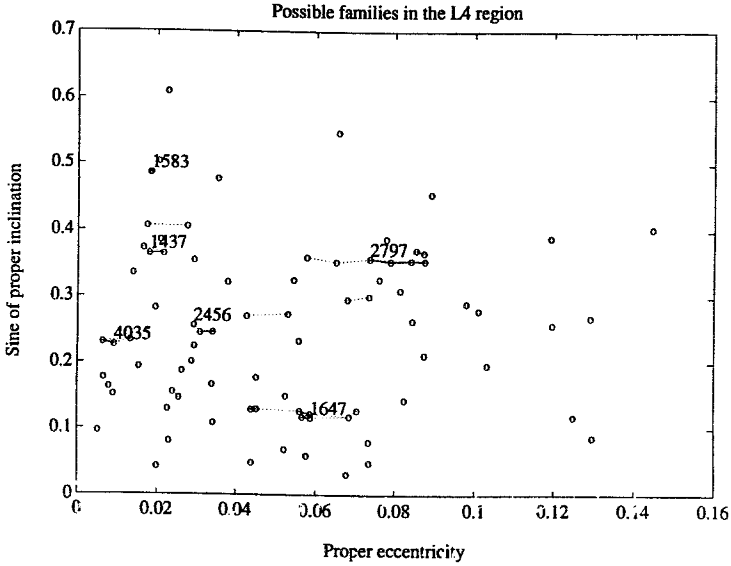
4. Search for Families

The outline of the procedure to identify asteroid families is discussed e.g. in Milani et al. (1992); once proper elements are available (together with some information on their stability), we need to define a distance between each couple of objects in the same region. There is unfortunately no unique optimal choice of a metric; Zappalà et al. (1990; 1992) have used two metrics which appear to be good compromise solution, and compared the results obtained with both, to increase the reliability of the detected families. Their metric d_1 (see Zappalà et al., 1992, eq. (2.1)) has the property of overweighing the differences in semimajor axis, metric d_2 overweighs the difference in inclination. For the Trojan belt the difficulty of the choice of a metric is compounded by the very peculiar geometry of the Trojan belt; it is indeed not a belt, but more like a couple of towers, with an height (in the direction perpendicular to the plane of the orbit of Jupiter) of more than 6 AU. When seen in the velocity space –were the arguments about dispersion of fragments from a catastrophic collision should apply– the shape is even more peculiar. A change of velocity, in the along track direction, of about 200 m/s, can result in the expulsion of a Trojan from the libration region, while a change of velocity of 7 Km/s in the direction perpendicular to the orbital plane can change a stable Trojan orbit in another equally stable (what happens with the radial direction, that is with the range in e_P, is less clear; see Section 5). As a result, the use of a metric which overweights the velocity difference in the along track direction by a factor 3, as the d_1 of Zappalà et al., changes the shape of the Trojan region in a very significant way; d_2 would on the contrary stretch the already very large separations in I_P. I am not claiming to have found an ideal solution to this problem; as a first attempt, I have used the "trivial" metric:

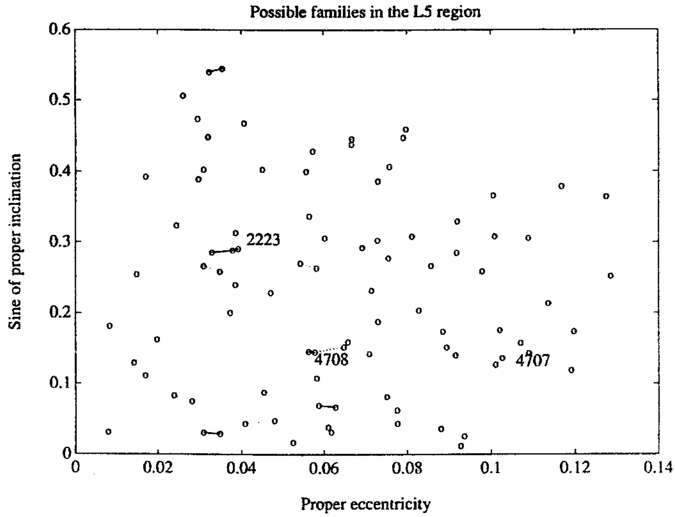
$$d_3 = \sqrt{\frac{1}{4}(\frac{\delta d}{a})^2 + 2(\delta e_P)^2 + 2(\delta \sin I_P)^2} \qquad (4.1)$$

which is obtained by ignoring the off diagonal terms in the equations linking the velocity difference components and the changes in the proper elements.

The next step is the classification, for which I have used the algorithm of *hierarchical clustering*, consistently with Zappalà et al. (1990; 1992). In short, the



(a)



(b)

Fig. 9. Proper elements e_P and $\sin I_P$: the third proper element d matters much less, because its contribution to the relative velocities is minor; however, the distances have been computed in 3 dimensions by eq. (4.1). The segments indicate a distance corresponding to less than 208 m/s (dotted line), and to less than 130 m/s (full line). (a) in the L4 swarm, the most remarkable features are the *1583–3801* couple and the clusters headed by *1647 Menelaus* and by *2797 Teucer*; (b) in L5 there is the *2223 Sarpedon* cluster. The others clumps may or may not be significant.

distances between all the couples of Trojans are sorted in ascending order; for each value of the cutoff Q, the objects with a distance $d_3 < Q$ are "clustered" together. This can be represented either as a graph, with the objects at distance less than Q joined by a line as in Figure 9, or as a "stalactite" table such as Table 4, were the rows correspond to a cutoff value (the unit is 10^{-2}) and list the objects joining the cluster at that cutoff, the columns correspond to the different clusters. The real difficulty is in deciding were to stop; the procedure has no natural termination, not until all the objects are clumped together at a very large cutoff.

I have not copied the procedure used by Zappalà et al. (1990) to decide were to place the cutoff (what they call the *Quasi Random Level*), because it would not work; being of a statistical nature, that procedure requires large numbers of asteroids to work reliably, and the number of Trojans is not large enough (the more complicated procedure used by Zappalà et al., 1992, could work). In preparing Table 4 and Figures 9a–9b I have assumed that a cutoff $Q = 0.016$ (which corresponds, multiplying by na, to $\simeq 208\ m/s$ in relative velocity; the dotted lines in the Figures and the top 3 lines in the Table) give results to be rated from marginal to unreliable; the results in which we can have some confidence should be obtained with a cutoff not above 0.01 ($130\ m/s$; the full lines in the Figures). I acknowledge that more work should be done to compute the correct value of the Quasi Random Level; this is just a first attempt, mostly with the purpose of deciding whether the presently available number of (good) Trojan orbits is enough to begin working more in depth on the family problem.

I am still left with the problem of deciding, for a given cutoff level, how many members a cluster must have to be rated significant. For this purpose, there is an argument I can apply to the Trojans which was not available for the main belt. There are two regions, around L4 and around L5, which are physically disjoint (no Greek ever gets as close as 3 AU from a Trojan); nevertheless, the proper elements, as defined here, are in the same range. It is enough to omit any check on the L4/L5 flag of Tables 1 and 2, when computing the distance between two Trojans, to find –in the list of distances sorted in increasing order– the spurious distances between an L4 object and an L5 one, together with the meaningful distances between objects in the same region. E.g., if I chose as cutoff $Q = 0.01$ ($\simeq 130\ m/s$), there are 13 couples at a lower distance in the L4 swarm, 6 in the L5 one, and 18 spurious couples formed with one in L4 and one in L5. Since the spurious couples have no possible physical interpretation, they are a measure of how many couples would be within such a distance by chance only: the number of chance occurrences in L4 and in L5 together should be statistically almost equal to the number of spurious ones, and indeed the numbers are close. I also would like to mention that a computation done with Poisson statistics, assuming a more or less uniform distribution, would predict much less than 18 spurious couples; this is a good example of the pitfalls of Poisson statistics when applied to a clumpy distribution. Therefore I do not use Poisson statistics at all, and I conclude from this test that the couples at distances $d_3 \leq 0.01$ are not statistically significant, and that at least 3 objects must belong to a cluster before it can be proposed as a family. I have not proven that 3 are enough, and I need to insist on the cautionary statement that I am listing "candidate", or "possible", families, to be confirmed with other independent information.

I need to stress, also for the purpose of comparing my work with that of Schoemaker et al. (1989; 1992), that the above statement does not imply that two Trojan asteroids with very close proper elements cannot be collisionally related; I am only saying that, with one exception (see below), there is no way to prove such a relationship on the basis of the statistical analysis of the dynamical information available. Two Trojans may well be fragments of the same parent body, but until we either find at least a third one near enough, or find some other evidence independent from the proper elements, we cannot prove it. Out of the many "Trojan couples" proposed by Schoemaker et al., some are here rejected (because my better computation of proper elements result in a larger distance), many are confirmed as couples but not as candidate families, and a few are confirmed as families because I have found at least one more neighbour; this is the case of Teucer in L4 and Sarpedon in L5.

The most striking result apparent from Table 4 is the couple *1583 Antilochus – 3801 Thrasimedes*; the actual distance is $d_3 = 0.0007$, which corresponds to less than $10\, m/s$. As a matter of fact, the differences in both e_P and $\sin I_P$ are of a few 10^{-4}; although the stability of the proper elements of *1583* and *3801* is very good, the difference is consistent with zero. The libration amplitude difference of 1.2° (corresponding to 0.006 AU in d) is significant and accounts for most of the distance. No other known Trojan is nearby (within $d_3 \leq 0.02$). Although I cannot use a Poisson statistics argument (not to contradict myself), it is clear that such a distance is unlikely to occur by chance (the nearest spurious L4/L5 couple is at a distance larger by a factor 3). The only conclusion I can draw so far is that this couple has some physical meaning and requires some dynamical explanation. The problem is, such a small relative velocity is hard to account even with a standard collisional model, taking into account that the escape velocity from *1583 Antilochus* is $\simeq 65\, m/s$. P. Farinella has proposed a non standard collisional model, involving binary asteroids; if we can confirm it, you will hear from us.

The other interesting results are the Menelaus family, which appears reliable, with between 4 and 8 members, depending from the cutoff (this family was also proposed in Schoemaker et al.), and the two less reliable but probably significant families of Teucer (4 to 6 members, but with visible chaining effect) and Sarpedon (3 members only, but very compact). The other six columns in Table 4 correspond to what I might call *clumps*, by using the Zappalà et al. (1992) terminology slightly out of context: some of them, but probably not all of them, could be real collisional families; I have not done enough work to be able to give a quantitative assessment of the probabilities.

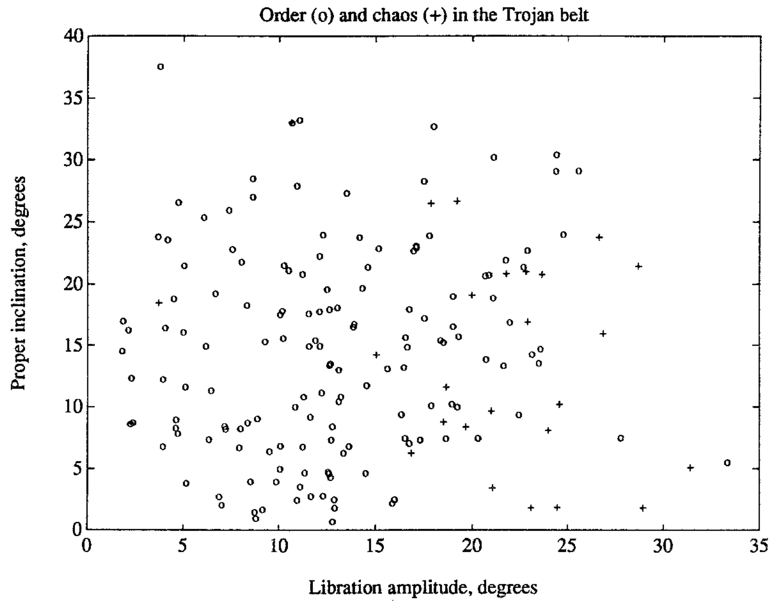
These results may appear limited, and indeed they are; however, I need to stress that the problem of identifying families among the Trojans is intrinsically more difficult than in the main belt. The size, in the velocity space, of the Trojan belt, is much smaller than the size of the main belt; on the other hand from the collisional processes in the Trojan belt, the relative velocities being comparable, the volume in the velocity space occupied by a family should be comparable. The most extreme case of this difficulty is when a fragment from a catastrophic collision ends up outside the stable libration region, and experiences a close approach to Jupiter; since ejection velocities of $\simeq 200\ m/s$ for fragments are possible (as it can be seen in the main belt families), a Trojan family can have as a member an object in a cometary orbit, in a meteorite type orbit, and (more likely) in an hyperbolic orbit of escape from the solar system. Further problems could result from the internal dynamical structure of the Trojan belt. E.g. by comparing with Table 3, it is found that 5 of the asteroids appearing in Table 4 have chaotic orbits; since 3 of them display stable chaos (case (B) of Section 3), the other 2 only moderately unstable chaos (case (A)), the results on possible families still stand, although it is hard to predict what could happen over a time span much longer than 1 Myr.

The main conclusion I can draw from this initial search for Trojan families is that the number of Trojans with good enough orbits is marginal for a reliable detection of families. Not only I have been able to use only 84 Trojans in L4 and 90 in L5, but the size distribution is quite limited, because most of the orbits I have used have been discovered by chance while looking for main belt asteroids. The data from the Trojan surveys are mostly useless for the reasons discussed in Section 2, and this has not allowed to use the data on enough faint asteroids, which have a considerable probability of being fragments from the larger ones. Thus the most urgent task, to be able to go forward with the families of Trojans, is either the discovery of a significant number of faint Trojans or at least the recovery of many of the 97 lost objects, observed at a single opposition.

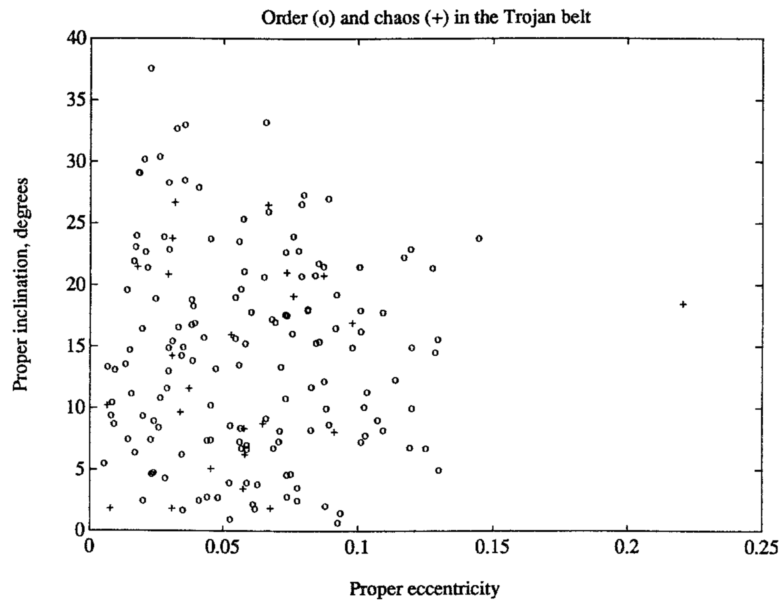
5. Open Problems

The availability of such a large set of data on the dynamics of Trojans solves some problems, and raises many more we were unable to see before. I would like to list the main ones:

(1) Figure 10 shows the distribution of "chaotic" orbits, defined here as positive identifications of Lyapounov exponents $\geq 6 \times 10^{-6}\ y^{-1}$, and "ordered" orbits, for which a positive LCE was not confirmed; they are marked with a + and an o respectively. Figure 10a, displaying libration amplitude D versus proper inclination I_P, definitely shows some pattern: chaos occurs more often for large libration amplitudes, and also more often (but not exclusively) at low inclination. The strength of the most intense perturbation by Jupiter increases with the libration amplitude; on the contrary the increase in inclination decreases the average perturbation, but not the maximum (even a very inclined orbit can have a close approach when the node is in the right position); thus the concentration of crosses and circles in different portions makes some sense, although one would like a quantitative



(a)



(b)

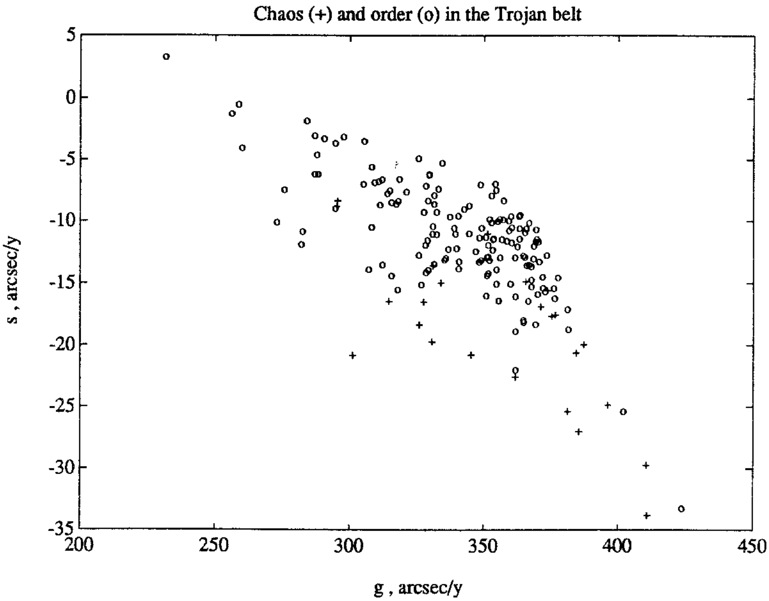
Fig. 10. Proper elements for all the 174 Trojans with well known orbits; the L4 and L5 swarms are superimposed. Circles indicate "regular" orbits, crosses are chaotic; the threshold for positive detection of chaos has been set at $6 \times 10^{-6}\, yr^{-1}$. (a) I_P and D (both in degrees). A large libration amplitude corresponds often, but by no means always, to chaotic orbits. (b) e_P and I_P (in degrees); apart the strange situation of *4835*, there is no obvious pattern.

model. However, there is no large region where all the orbits are chaotic. On the contrary I cannot see any special pattern in Figure 10b, which shows the proper eccentricity e_P versus proper inclination I_P. Then, why are some orbits chaotic, and others (with similar libration amplitude) either regular or at least with smaller LCE? Which are the resonances responsible for the chaos? As already pointed out in Section 3, it is not easy to find a resonance when the basic frequencies are so far apart; the distribution of the chaotic orbits in the frequency space (Figure 11) does not give any clue I can recognize. Of all the chaotic cases, I have an explanation in terms of resonances for only 3, namely *1173*, *6073* and *6031*, for which the $s - s_6$ resonance is involved.

(2) I have listed in Section 3 a total of 5 strongly unstable, 5 moderately unstable, and 14 stable chaotic orbits (again *1208* is excluded from this count because its chaos is much slower). Why are some chaotic orbits unstable, others stable? Of course when the resonance responsible is a major one, the corresponding chaotic layer is thicker, and, in a region of strong perturbations, resonance overlap can occur. However, excluding again the cases in the $s - s_6$ secular resonance, all the resonances involved should be of rather high order, given the ratios of the frequencies (Figure 11); why then is some of them more important than others? As an example, it is true that no Trojan has a ratio f/n smaller than $1/12$, but I cannot understand why $1/12$ is more important than $1/13$.

(3) Figure 10 also shows the overall shape of the Trojan region in the proper elements space (I have superimposed the L4 and the L5 swarms). Are the boundaries of that region stability boundaries? As an example, can we assume that no Trojan can have a larger libration amplitude than *6075*? And why there are no Trojans with $0.15 < e_P < 0.22$? Should we think that *4835* is a "comet", that is an object with a dynamical age significantly less than the age of the solar system? Although it is chaotic, it is only mildly so, much less than other moderate eccentricity Trojans. At least over 1 My, *4835* only shows a mild instability of the libration amplitude (which by the way could be the result of some long periodic effect); the hypothesis that it might be a temporary Trojan has no supporting evidence. Since another Trojan with comparable eccentricity has been found, it is difficult to invoke some exceptional protection mechanism. As a result of my computation of the orbit of *4835*, the problem of the high e boundary of the Trojan belt looks more difficult than before. An even more difficult problem is raised by the wild behaviour of *6031*; either there can be "comets" in the middle of the Trojan region, or a Trojan orbit can be unstable on a large scale such as in Figure 8 and still remain in the libration region for the age of the solar system: both hypothesis are somewhat paradoxical.

(4) The proper elements as defined in this paper are stable enough for many purposes, including the search for Trojan families. However, a proper element should be something more than an arbitrary function which changes little with



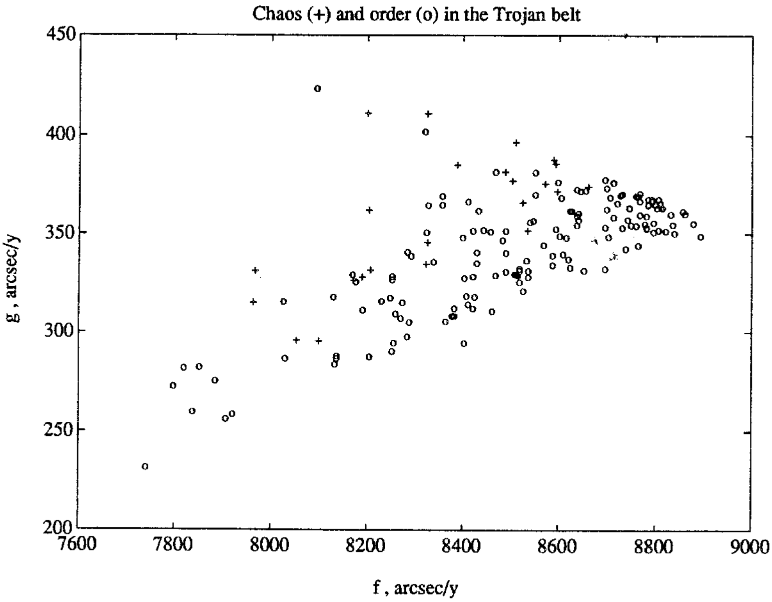


Fig. 11. The distribution of 174 Trojans in the frequency space; the units are $arcsec/yr$; crosses indicate positive detections of chaos. (a) perihelion frequency g versus node frequency s; more frequent chaos for shorter periods of the node, which occur for more strongly perturbed orbits. (b) libration frequency f versus g.

time over a particular solution. Even a large scale chaotic system can have a quasi–constant quantity, e.g. an average value of some quantity changing wildly with time but in a statistically regular way. The most chaotic systems, such as Anosov systems, are ergodic and have long term averages which tend to a limiting value, hence the averages over long time spans are quasi–integrals. A proper element should be the action variable of an approximating integrable system; that is, the set of proper elements should define a conditionally periodic orbit which approximates the exact solution for a very long time.

This is one reason why I did not follow the same procedure defined by Bien and Schubart (1987) to compute proper elements for Trojans. Bien and Schubart define the proper elements D and e_P in a way different, but not entirely so, from the one used here (warning: for Schubart and Bien, D is the difference between maximum and minimum of $\lambda - \lambda' - \chi$, that is twice the value reported in Tables 1 and 2). For them, the proper element is not the amplitude of a specific spectral line with a single frequency, but the average amplitude of the oscillation obtained by superimposing all the Fourier components with a frequency close to the proper one. E.g. the proper element d (in their notation, A) is the amplitude of the oscillation obtained by superimposing all the spectral lines shown in Figure 1. In my approach, the proper elements d and D are the amplitudes of a single Fourier component, only with an argument θ which is not linear in time. A similar difference occurs in the definition of e_P: I use the amplitude (in h, k) of the line with frequency g, Bien and Schubart use the superposition of all the spectral lines with frequencies close to g. The goal underlying my choice was to have a proper element which exactly defines an approximate solution, and is not only a statistical quantity. It is quite remarkable that the proper elements defined in a different way by Bien and Schubart and by me have almost the same values. The differences, for the 40 Trojans whose proper elements had been computed by Bien and Schubart, have a maximum of 0.92°, an rms of 0.33° in their D (printed in their Table 1 as an integer number of degrees). For e_P, a large discrepancy occurs for *1208* and *2594*, and this is understandable; for the others, the maximum difference is 0.0026 and the rms difference is 0.0007 (less than the value of the last digit printed by Bien and Schubart). For I_P, given by Bien and Schubart in degrees with only one digit after the decimal point, there are discrepancies up to 0.92° for *2594* and *1437*, for the others the maximum difference is 0.13° and the rms difference is 0.04°. Thus I have succeeded in giving a definition compatible with the one used previously, and which gives more information on the orbit of the Trojans, by referring explicitly to a conditionally periodic solution.

However, I am not very happy with the size of the residuals remaining after the removal of the forced and the proper modes, especially for h, k (see Figure 6). An unmodelled component with a typical rms of a few parts in 10^{-3} remains; this means that the theoretical goal of representing the solution as a sum of known harmonics has been achieved only roughly. For the inclination variables p, q a similar problem arises because I have not removed the forced terms (see Section

2); I have actually computed the forced terms, apart from s_8, but the accuracy is not very good. For the libration amplitude, the residuals are small but the use of the variable θ, whose time dependence is not known, also amounts to give up an explicit representation as a Fourier series in some known argument. I have to conclude that a real proper elements theory, which also implies an explicitly known approximate solution, has been achieved only with poor accuracy. The problem is how to obtain it, either with a synthetic method or otherwise.

(5) Secular resonances for Trojan orbits are not really understood, although quite a large numerical evidence has been accumulated (see e.g. Bien and Schubart 1986, Nakai and Kinoshita 1985). It would be helpful to be able to compute the width of resonances such as $s - s_6, s - s_7, s - s_8$ and $2s - 2g_5$, $2s - 2g_7$; it is somewhat of a paradox that the resonance widths appear (numerically) to be small, even for $s - s_6$, and then such a large scale instability as *6031* (Figure 8) can occur.

(6) To really understand the frequency of occurrence of Trojan families, we would need to develop a theory of the collisions between Trojans. Collision probabilities between main belt asteroids can be computed quite accurately (see e.g. Farinella and Davis, 1992), but the corresponding computations for Trojans are more difficult, because it is not possible to assume that the mean longitudes of two Trojans are not correlated. A new theory should be developed taking into account the model of the motion of the Trojans implied by the proper elements.

I am not listing the problems relative to the classification of Trojans into families, some of which have been mentioned in Section 4. I am also not sure I have really exhausted the list of open problems, but I believe there is enough work for us to do.

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TABLE 1: PROPER ELEMENTS AND FREQUENCIES

Numbered Trojan asteroids

No.	Name	d AU	D deg	f deg/y	e_P	g "/y	sin I_P	s "/y	L4 L5	M y
588	Achilles	.0314	6.45	2.434	.1032	344.44	.1967	-11.00	4	1
617	Patroclus	.0236	5.02	2.350	.1005	310.63	.3662	-6.76	5	1
624	Hektor	.0880	18.99	2.316	.0543	335.72	.3259	-12.95	4	1
659	Nestor	.0495	10.03	2.466	.1297	355.44	.0870	-16.40	4	1
884	Priamus	.0525	10.82	2.425	.0883	353.25	.1739	-12.31	5	1
911	Agamemnon	.0778	16.95	2.291	.0207	317.22	.3857	-8.59	4	1
1143	Odysseus	.0481	9.84	2.440	.0521	365.40	.0689	-10.90	4	1
1172	Aneas	.0481	10.15	2.366	.0602	331.45	.3056	-7.89	5	1
1173	Anchises	.1132	23.99	2.358	.0914	381.51	.1404	-25.40	5	1
1208	Troilus	.0468	10.63	2.200	.0354	258.55	.5446	-0.50	5	5
1404	Ajax	.0925	19.98	2.312	.0761	334.28	.3270	-15.00	4	1
1437	Diomedes	.1273	28.73	2.213	.0179	330.86	.3653	-19.74	4	1
1583	Antilochus	.1064	24.36	2.181	.0183	282.16	.4858	-10.84	4	1
1647	Menelaus	.0387	7.93	2.439	.0587	359.32	.1168	-9.96	4	1
1749	Telamon	.0660	13.61	2.422	.0686	365.95	.1185	-13.55	4	1
1867	Deiphobus	.0782	17.50	2.230	.0294	286.67	.4738	-6.22	5	1
1868	Thersites	.1060	22.88	2.313	.0979	345.35	.2906	-20.81	4	1
1869	Philoctetes	.1005	21.04	2.386	.0576	387.56	.0596	-19.97	4	1
1870	Glaukos	.0462	9.49	2.429	.0169	363.51	.1114	-9.48	5	1
1871	Astyanax	.1284	27.76	2.311	.0142	401.84	.1299	-25.37	5	1
1872	Helenos	.1091	23.55	2.313	.0148	364.69	.2538	-17.98	5	1
1873	Agenor	.0563	12.08	2.327	.1168	308.12	.3791	-10.48	5	1
2146	Stentor	.0162	3.76	2.150	.0226	231.51	.6093	3.28	4	5
2148	Epeios	.0224	4.61	2.430	.0239	354.56	.1553	-7.46	4	1
2207	Antenor	.0812	16.86	2.406	.0582	373.98	.1082	-15.61	5	1
2223	Sarpedon	.0655	13.88	2.358	.0379	340.41	.2882	-9.54	5	1
2241	79WM	.0596	12.59	2.365	.1008	329.03	.3081	-11.54	5	1
2260	Neoptolemus	.0192	4.03	2.385	.0195	334.39	.2828	-5.25	4	1
2357	Phereclos	.0337	6.88	2.448	.0480	363.27	.0478	-9.59	5	1
2363	Cebriones	.0784	17.99	2.177	.0323	259.73	.5397	-4.06	5	1
2456	Palamedes	.0785	16.63	2.358	.0293	351.53	.2566	-11.92	4	1
2594	78TB	.1433	31.42	2.278	.0454	410.95	.0879	-33.88	5	5
2674	Pandarus	.0345	7.01	2.461	.0879	360.41	.0361	-11.72	5	1
2759	Idomeneus	.0478	10.22	2.336	.0872	313.99	.3670	-7.76	4	1
2797	Teucer	.1039	22.81	2.275	.0737	327.72	.3579	-16.54	4	1
2893	Peiroos	.0781	16.44	2.372	.0471	356.11	.2287	-12.92	5	1
2895	Memnon	.0339	7.38	2.293	.0666	294.43	.4377	-3.64	5	1
2920	Automedon	.0986	21.69	2.271	.0164	325.69	.3733	-12.71	4	1
3063	Makhaon	.0605	12.64	2.389	.0557	349.14	.2339	-10.55	4	1
3240	Laocoon	.0541	11.07	2.444	.0775	365.30	.0622	-12.91	5	1
3317	Paris	.0216	4.71	2.292	.0790	290.29	.4475	-3.28	5	1

No.	Name	d AU	D deg	f deg/y	e_P	g "/y	sin I_P	s "/y	L4 L5	M y
3391	Sinon	.0903	19.28	2.339	.0426	351.56	.2711	-14.18	4	1
3451	84HA1	.1181	26.66	2.212	.0309	314.77	.4027	-16.52	5	1
3540	Protesilaos	.0786	17.10	2.297	.1192	306.90	.3898	-13.88	4	1
3548	Eurybates	.0890	18.62	2.388	.0438	376.54	.1290	-16.25	4	1
3564	Talthybius	.0444	9.26	2.394	.0844	337.31	.2645	-9.62	4	1
3596	Meriones	.0809	17.77	2.275	.0275	311.14	.4056	-8.68	4	1
3708	74FV1	.0087	1.79	1.802	.1283	330.78	.2511	-11.01	5	1
3709	Polypoites	.0584	12.46	2.339	.0138	328.30	.3353	-7.12	4	1
3793	Leonteus	.0489	10.47	2.335	.0576	318.40	.3603	-6.58	4	1
3794	Sthenelos	.0194	3.91	2.470	.1250	348.95	.1177	-13.12	4	1
3801	Thrasymedes	.1112	25.56	2.172	.0181	281.74	.4861	-11.89	4	1
4007	Euryalos	.0585	12.17	2.400	.0154	357.13	.1937	-9.85	4	1
4035	86WD	.0742	15.61	2.374	.0092	356.71	.2268	-11.46	4	1
4057	85TQ	.0599	12.26	2.440	.0736	367.80	.0487	-13.65	4	1
4060	Deipylos	.0490	10.19	2.403	.1292	331.51	.2690	-13.44	4	1
4063	Euforbo	.0477	10.04	2.371	.0735	331.42	.3011	-8.58	4	1
4068	Menestheus	.0821	17.52	2.341	.0680	340.79	.2963	-13.25	4	1
4086	85VK2	.0955	20.82	2.292	.0789	326.60	.3541	-15.10	4	1
4138	Kalchas	.0567	11.64	2.434	.0439	369.46	.0486	-11.66	4	1
4348	Poulydamas	.0408	8.36	2.439	.0893	353.43	.1516	-11.45	5	1
4489	88AK	.0994	21.77	2.280	.0293	331.34	.3551	-13.62	4	1
4501	Eurypylos	.0781	16.33	2.390	.0079	368.69	.1631	-13.04	4	1
4543	Phoinix	.1227	26.88	2.279	.0529	361.87	.2746	-22.61	4	1
4707	Khryses	.0392	7.99	2.450	.1091	351.63	.1429	-12.85	5	1
4708	Polydoros	.0937	19.65	2.381	.0578	375.54	.1450	-17.71	5	1
4709	Ennomos	.0862	19.23	2.237	.0319	295.53	.4484	-8.39	5	1
4715	89TS1	.0970	21.05	2.301	.0244	340.64	.3234	-13.82	5	1
4722	Agelaos	.0231	4.69	2.455	.1027	350.63	.1362	-11.29	5	1
4754	Panthoos	.0885	18.67	2.368	.0372	365.96	.2004	-14.92	5	1
4791	Iphidamas	.0610	13.47	2.260	.0797	287.89	.4591	-6.21	5	1
4792	Lykaon	.0617	12.75	2.416	.0563	362.61	.1461	-12.04	5	1
4805	Asteropaios	.0604	12.60	2.393	.0714	348.25	.2315	-11.34	5	1
4827	Dares	.0886	18.54	2.388	.0648	371.57	.1522	-16.95	5	1
4828	Misenus	.0566	11.85	2.385	.0856	339.28	.2660	-11.02	5	1
4829	Sergestius	.1054	22.39	2.352	.0198	381.70	.1625	-18.73	5	1
4832	Palinurus	.0573	12.06	2.371	.1090	328.13	.3054	-11.90	5	1
4833	Meges	.0485	11.03	2.196	.0656	256.04	.5477	-1.26	4	5
4834	Thoas	.0393	8.62	2.279	.0890	287.42	.4545	-4.60	4	1
4835	89BQ	.0231	3.71	1.500	.2207	301.21	.3158	-20.88	4	1
4836	Medon	.0614	13.00	2.358	.0813	331.02	.3103	-10.40	4	1
4867	89SZ	.0493	10.90	2.260	.0406	286.63	.4678	-3.04	5	1
4902	89AN2	.0115	2.36	2.433	.0090	354.27	.1515	-6.95	4	1
4946	88BW1	.1075	23.65	2.270	.0875	326.08	.3542	-18.39	4	1
5012	9507PL	.0613	12.57	2.435	.0735	366.91	.0803	-13.49	4	1

TABLE 2: PROPER ELEMENTS AND FREQUENCIES

Unnumbered, multi opposition Trojan asteroids

No.	Name	d AU	D deg	f deg/y	e_P	g "/y	sin I_P	s "/y	L4 L5	M y
6001	73SY	.0715	15.05	2.371	.0309	351.60	.2455	-11.06	4	1
6002	73SD1	.0830	17.29	2.399	.0560	372.99	.1270	-15.69	4	1
6003	73SH1	.0532	10.93	2.433	.0199	369.57	.0429	-10.68	4	1
6004	73SM1	.1090	23.46	2.321	.0133	369.37	.2345	-18.31	4	1
6005	73SO1	.0225	4.59	2.446	.0825	352.11	.1437	-9.83	4	1
6006	73SQ1	.0545	11.20	2.429	.0566	363.33	.1180	-11.44	4	1
6007	73SR1	.0349	7.18	2.428	.0256	357.37	.1464	-8.33	4	1
6008	73SW1	.0900	18.93	2.375	.0451	370.12	.1777	-15.90	4	1
6009	73SA2	.0701	14.48	2.416	.0230	373.55	.0811	-12.74	4	1
6010	85TG3	.0624	13.06	2.387	.0294	352.74	.2250	-10.10	4	1
6011	86TR6	.1010	21.58	2.336	.0064	366.61	.2308	-16.43	4	1
6012	86TS6	.0246	5.08	2.417	.0288	348.67	.2014	-7.04	4	1
6013	86VG1	.0353	7.56	2.328	.0778	308.01	.3879	-5.60	4	1
6014	87YU1	.0376	8.04	2.339	.0853	311.97	.3715	-6.61	4	1
6015	88BX1	.0924	21.06	2.190	.0203	275.33	.5030	-7.47	4	1
6016	88BY1	.0524	11.18	2.340	.0840	318.01	.3551	-8.37	4	1
6017	88QY	.1058	24.38	2.166	.0260	272.58	.5062	-10.10	5	1
6018	88RO	.0239	4.99	2.395	.0755	333.02	.2770	-7.38	5	1
6019	88RA1	.0086	1.84	2.363	.0692	329.55	.2922	-6.24	5	1
6020	88RF1	.1033	22.85	2.258	.0730	317.77	.3865	-15.52	5	1
6021	88RG1	.0782	17.08	2.286	.0170	315.49	.3924	-8.45	5	1
6022	88RG10	.0615	12.65	2.425	.0281	370.54	.0749	-11.68	5	1
6023	88RL10	.0628	12.85	2.442	.0776	367.80	.0433	-14.74	5	1
6024	88RN10	.0780	16.53	2.356	.0543	346.92	.2701	-12.41	5	1
6025	88RO10	.0697	15.15	2.298	.0296	314.86	.3890	-7.49	5	1
6026	88RR10	.0887	18.99	2.333	.0329	348.14	.2851	-13.26	5	1
6027	88RS10	.0627	12.87	2.435	.0617	370.75	.0315	-13.27	5	1
6028	88RY10	.0431	8.80	2.447	.0525	365.81	.0170	-10.62	5	1
6029	88RH11	.0768	15.86	2.420	.0610	376.24	.0385	-15.41	5	1
6030	88RM11	.0773	15.98	2.415	.0408	377.80	.0435	-14.58	5	1
6031	88RN11	.1102	23.08	2.387	.0309	385.58	.0308	-27.06	5	5
6032	88RY11	.0564	12.24	2.302	.0758	304.90	.4065	-6.99	5	1
6033	88RD12	.0608	12.53	2.424	.0238	369.93	.0830	-11.46	5	1
6034	88RE12	.0579	12.09	2.391	.0978	339.86	.2580	-12.19	5	1
6035	88RH12	.0433	8.86	2.443	.1071	351.09	.1575	-12.91	5	1
6036	88RP12	.0253	5.15	2.453	.0627	360.07	.0667	-9.58	5	1
6037	88RS12	.0553	11.31	2.440	.0751	364.80	.0815	-12.79	5	1
6038	88RT12	.0494	10.05	2.454	.1190	354.72	.1189	-15.05	5	1
6039	88RH13	.0870	18.49	2.349	.0581	351.11	.2629	-14.36	5	1
6040	88RL13	.0549	11.52	2.380	.0348	344.46	.2579	-8.73	5	1
6041	88SW1	.0656	13.83	2.370	.0917	336.70	.2845	-12.25	5	1
6042	88SK2	.0430	8.72	2.460	.0935	361.72	.0256	-12.92	5	1
6043	88SP2	.0109	2.27	2.303	.1137	338.91	.2132	-10.55	5	1

No.	Name	d AU	D deg	f deg/y	e_P	g "/y	sin I_P	s "/y	L4 L5	M y
6044	88SA3	.0785	16.74	2.341	.0812	335.25	.3079	-13.09	5	1
6045	88SG3	.0394	8.32	2.364	.0385	329.52	.3135	-6.19	5	1
6046	88SJ3	.0668	14.30	2.334	.0564	327.56	.3371	-9.26	5	1
6047	88SL3	.0193	4.15	2.323	.0557	305.29	.4000	-3.47	5	1
6048	88TH1	.0543	11.24	2.415	.0730	353.56	.1879	-11.40	5	1
6049	88TZ1	.0806	17.88	2.250	.0667	295.21	.4456	-8.85	5	1
6050	88TA3	.0857	17.85	2.396	.1021	361.73	.1755	-18.86	5	1
6051	89AU1	.0806	16.74	2.404	.0586	372.21	.1226	-15.38	4	1
6052	89AV2	.0947	20.63	2.292	.0650	328.35	.3531	-14.12	4	1
6053	89BL	.0310	6.34	2.443	.0706	355.93	.1278	-9.84	4	1
6054	89BW	.0099	2.12	2.352	.1010	329.05	.2798	-8.33	4	1
6055	89BB1	.0299	6.18	2.415	.1196	332.31	.2580	-11.06	4	1
6056	89CW1	.1146	24.57	2.330	.0066	384.78	.1767	-20.68	4	1
6057	89CH2	.0391	8.63	2.259	.0353	283.67	.4770	-1.85	4	5
6058	89CK2	.0213	4.49	2.366	.0379	325.59	.3225	-4.87	4	1
6059	89DJ	.1028	22.63	2.269	.0215	329.20	.3649	-13.92	4	1
6060	89EO11	.1105	24.76	2.229	.0173	315.46	.4068	-14.39	4	1
6061	89SC7	.1015	21.92	2.312	.0392	350.79	.2904	-15.99	5	1
6062	89TS2	.0627	13.06	2.400	.0082	360.45	.1812	-10.50	5	1
6063	89TU5	.0625	12.76	2.446	.0926	367.78	.0121	-15.29	5	1
6064	89TO11	.0970	20.68	2.342	.0384	361.73	.2396	-16.06	5	1
6065	89UC5	.0920	19.20	2.395	.1197	361.82	.1734	-21.99	5	1
6066	89UO5	.0861	18.34	2.345	.0308	351.99	.2658	-13.12	5	1
6067	89UX5	.1340	28.96	2.313	.0079	410.59	.0310	-29.76	5	5
6068	90DK	.0108	2.22	2.439	.0524	352.86	.1498	-7.90	4	1
6069	90TV12	.0279	6.06	2.301	.0573	297.58	.4285	-3.15	5	1
6070	90VU1	.0680	14.59	2.328	.1273	312.01	.3647	-13.50	5	1
6071	91EL	.0171	3.66	2.334	.1445	294.44	.4036	-8.97	4	1
6072	91GX1	.0633	13.18	2.399	.0263	359.28	.1876	-10.73	4	1
6073	4523PL	.1158	24.47	2.364	.0678	396.39	.0314	-24.91	4	1
6074	6541PL	.0964	20.26	2.375	.0227	381.36	.1295	-17.13	4	1
6075	6581PL	.1499	33.30	2.248	.0052	423.39	.0960	-33.26	4	1
6076	6591PL	.0794	16.52	2.401	.0452	371.97	.1296	-14.50	4	1
6077	9602PL	.0645	13.32	2.418	.0343	368.78	.1088	-12.00	4	1
6078	5030T2	.0191	3.92	2.427	.0874	342.67	.2118	-9.01	4	1
6079	5187T2	.0993	20.98	2.362	.0340	376.99	.1669	-17.60	4	1
6080	5493T2	.1073	23.08	2.321	.0341	364.87	.2463	-18.13	4	1
6081	2035T3	.0618	12.67	2.435	.1011	359.93	.1269	-15.05	5	1
6082	3104T3	.0317	6.68	2.368	.0919	321.07	.3295	-7.60	5	1
6083	3108T3	.0447	9.14	2.442	.0348	366.98	.0290	-10.14	5	1
6084	4035T3	.0545	11.50	2.366	.0729	332.32	.3023	-9.25	5	1
6085	4101T3	.0650	14.15	2.294	.0450	309.13	.4031	-6.87	5	1
6086	4179T3	.0417	8.51	2.445	.0587	363.29	.0693	-10.56	5	1
6087	4317T3	.0561	11.59	2.420	.0658	358.56	.1594	-11.61	5	1
6088	4369T3	.0352	7.21	2.438	.0709	355.19	.1425	-10.00	5	1
6089	5191T3	.0698	14.53	2.399	.0827	354.63	.2036	-13.87	5	1

TABLE 3: STABILITY OF PROPER ELEMENTS

No.	Name	Δd AU	ΔD deg	Δf deg/y	Δe_P	Δg "/y	$\Delta sin I_P$	Δs "/y	LCE $1/(10^5 y)$
588	Achilles	.00001	.002	.0010	.00025	.025	.00061	.025	0.3
884	Priamus	8	17	6	362	246	144	292	0.2
1173	Anchises	17	33	4	30	652	975	59	0.9
1208	Troilus	11	28	8	333	397	896	14	0.15
1404	Ajax	52	112	1	30	103	86	131	0.9
1437	Diomedes	204	519	57	168	264	237	593	1.7
1868	Thersites	52	125	17	46	92	33	144	1.1
1869	Philoctetes	24	52	7	15	123	160	63	1.5
2146	Stentor	49	115	4	507	198	136	4	0.06
2207	Antenor	12	29	8	44	45	49	40	0.6
2363	Cebriones	6	14	2	44	511	483	46	0.4
2594	78TB	50	133	25	1486	500	913	324	2.90
2797	Teucer	10	23	8	20	54	39	38	0.7
3317	Paris	3	7	3	176	406	252	38	0.2
3451	84HA1	131	326	27	73	129	257	420	1.9
3540	Protesilaos	70	165	17	31	149	63	116	0.4
4057	85TQ	1	11	35	574	843	66	532	0.0
4489	88AK	33	81	13	9	108	37	77	1.1
4543	Phoinix	45	125	19	35	119	55	128	1.1
4708	Polydoros	67	155	18	12	369	81	147	0.8
4709	Ennomos	95	221	14	36	31	410	105	0.7
4754	Panthoos	48	110	15	16	273	81	112	0.8
4827	Dares	46	108	15	13	264	110	95	1.0
4833	Meges	32	75	3	155	78	379	23	0.08
4834	Thoas	6	11	11	165	162	289	16	0.3
4835	89BQ	191	416	2317	43	177	48	45	0.6
4867	89SZ	15	38	14	100	174	305	5	0.4
4946	88BW1	166	397	32	52	304	150	432	0.7
6001	73SY	22	44	7	7	81	83	46	0.6
6023	88RL10	15	14	49	819	1.328	133	781	0.2
6031	88RN11	176	382	57	8658	2.882	3612	731	1.99
6044	88SA3	82	174	2	10	187	29	170	0.3
6047	88SL3	5	2	5	79	293	325	32	0.3
6049	88TZ1	13	33	8	41	16	51	27	0.7
6050	88TA3	18	29	16	297	431	91	321	0.3
6056	89CW1	12	36	6	5	121	70	55	0.9
6057	89CH2	9	21	3	134	165	99	16	0.08
6067	89UX5	140	352	39	3539	1.117	1159	500	6.38
6071	91EL	5	18	10	234	256	217	216	0.3
6073	4523PL	120	283	20	255	718	1169	640	2.7
6079	5187T2	34	71	5	11	232	90	113	1.2

TABLE 4: POSSIBLE TROJAN FAMILIES

L4 region (Greeks)

Family	Antilochus	Menelaus	Teucer	Diomedes	4035	Palamedes
1.6		1749 3548 6053 6076	3793 6052			2456
1.4						
1.2				2920	6004	
1.0			2797 4086		4035 6011	
0.8		6002 6051	4946 6016			
0.6				1437 6059		6001 6080
0.4		1647 6006				
0.2	1583 3801					

L5 region (Trojans)

Family	Sarpedon	88RA1	Polydoros	Kryses
1.6		6019 6044	4827	
1.4				4707 4722 6081
1.2				
1.0	6026	6084		
0.8				
0.6	2223 6061		4708 4792	