CSE 477: Introduction to Computer Security

Lecture – 8

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Outline

- Number theory review
- RSA Cryptosystem
- RSA Implementation

Prime numbers

- Prime number p :
  - p is an integer (Integers are like whole numbers, but they also include negative numbers, but no fractions allowed)
  - $p \geq 2$
  - The only divisors of p is 1 and p
- Examples
  - 2, 7, 19 are primes
  - -3, 0, 1, 6 are not primes
- Prime decomposition (aka factorization) of a positive integer n:
  - $n = p_1^{e_1} \times p_2^{e_2} \times \cdots \times \ p_k^{e_k}$
- Example:
  - $200 = 2^3 \times 5^2$
- Fundamental Theorem of Arithmetic:
  - The prime decomposition of a positive integer is unique

Greatest Common Divisor (GCD)

- The greatest common divisor (GCD) of two integers a and b, denoted gcd(a, b), is the largest positive integer that divides both a and b
- Examples:
  - gcd(18, 30) = 6
  - gcd(0, 20) = 20
  - gcd(-21, 49) = 7
- Two integers a and b are said to be relatively prime if gcd(a, b) = 1
- Example:
  - 15 and 28 are relatively prime, as gcd(15,28) = 1

Modular arithmetic

- Modulo operator for a positive integer n:
  - $r = a \; mod \; n$, here, r and a are integers and r is the reminder
- It is equivalent to: $a = r + kn$
  - Here, k is the quotient, also denoted with q
- Example:
  - 29 mod 13 = 3 13 mod 13 = 0 -1 mod 13 = 12
  - 29 = 3 + 2 x 13 13 = 0 + 1 x 13 -1 = 12 + (-1) x 13
- Modulo and GCD
  - gcd(a, b) = gcd(b, a mod b)
- Example:
  - gcd(21, 12) = 3 gcd(12, 21 mod 12) = gcd(12, 9) = 3

Euclid's GCD algorithm

- Euclid's algorithm for computing the GCD repeatedly applies the formula
  - gcd(a, b) = gcd(b, a mod b)
- Example: gcd(412, 260) = 4

Algorithm ***EuclidGCD(a,b)***
  Input integers *a* and *b*
  Output gcd(*a*,*b*)
  if *b* = *0*
    return *a*
  else
    return *EuclidGCD(b, a mod b)*

a	412	260	152	108	44	20	4
b	260	152	108	44	20	4	0

Multiplicative Inverse

- The **residues** modulo a positive integer n are the set
  - $Z_n = \{0, 1, 2, \ldots, (n-1)\}$
- Let x and y be two elements of Z_n such that: $xy \; mod \; n = 1$
  - Then we say that y is the multiplicative inverse of x in Z_n
  - and we write $y = x^{-1} \; mod \; n$
- Example:
  - Multiplicative inverses of the residues modulo 11

x	0	1	2	3	4	5	6	7	8	9	10
x^{-1}		1	6	4	3	9	2	8	7	5	10

Multiplicative Inverse

- Theorem:
  - An element x of Z_n has a multiplicative inverse if and only if x and n are relatively prime
- Example:
  - The elements of Z_{10} with a multiplicative inverse are 1, 3, 7, 9

x	0	1	2	3	4	5	6	7	8	9
x^{-1}		1		7				3		9

- Corollary:
  - If p is prime, every nonzero residue in Z_p has a multiplicative inverse

x	0	1	2	3	4	5	6	7	8	9	10
x^{-1}		1	6	4	3	9	2	8	7	5	10

Powers

- Let p be a prime
- The sequences of successive powers of some elements of Z_p exhibit repeating sub-sequences
- The sizes of the repeating sub-sequences and the number of their repetitions are the divisors of $p - 1$
- Example ($p = 7$), all operations mod p

	x	x^2	x^3	x^4	x^5	x^6
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1

Modular exponentiation

- The form $x^y \ mod \ n$ is called the modular exponentiation
- It has several properties
- If n is not prime, e.g. $n = 10$, there are modular powers equal to 1 only for the elements of Z_n that are relatively prime with n
  - That is, those elements whose gcd with n is 1
  - For $n = 10$, these elements are 1, 3, 7, 9
- If n is prime, e.g. $n = 13$, every nonzero element of Z_n has a power equal to 1

	y								
	1	2	3	4	5	6	7	8	9
1^y	1	1	1	1	1	1	1	1	1
2^y	2	4	8	6	2	4	8	6	2
3^y	3	9	7	1	3	9	7	1	3
4^y	4	6	4	6	4	6	4	6	4
5^y	5	5	5	5	5	5	5	5	5
6^y	6	6	6	6	6	6	6	6	6
7^y	7	9	3	1	7	9	3	1	7
8^y	8	4	2	6	8	4	2	6	8
9^y	9	1	9	1	9	1	9	1	9

	y											
	1	2	3	4	5	6	7	8	9	10	11	12
1^y	1	1	1	1	1	1	1	1	1	1	1	1
2^y	2	4	8	3	6	12	11	9	5	10	7	1
3^y	3	9	1	3	9	1	3	9	1	3	9	1
4^y	4	3	12	9	10	1	4	3	12	9	10	1
5^y	5	12	8	1	5	12	8	1	5	12	8	1
6^y	6	10	8	9	2	12	7	3	5	4	11	1
7^y	7	10	5	9	11	12	6	3	8	4	2	1
8^y	8	12	5	1	8	12	5	1	8	12	5	1
9^y	9	3	1	9	3	1	9	3	1	9	3	1
10^y	10	9	12	3	4	1	10	9	12	3	4	1
11^y	11	4	5	3	7	12	2	9	8	10	6	1
12^y	12	1	12	1	12	1	12	1	12	1	12	1

Fermat's Little Theorem

- Theorem
  - Let p be a prime. For each nonzero residue x of Z_p, we have
    - $x^{p-1} \bmod p = 1$
- Example ($p = 5$):
- $1^4 \bmod 5 = 1$ $\qquad$ $2^4 \bmod 5 = 16 \bmod 5 = 1$
- $3^4 \bmod 5 = 81 \bmod 5 = 1$ $\qquad$ $4^4 \bmod 5 = 256 \bmod 5 = 1$
- Corollary
  - Let p be a prime. For each nonzero residue x of Z_p, the multiplicative inverse of x is $x^{p-2} \bmod p$
- Proof
  - $x(x^{p-2}) \bmod p = x^{p-1} \bmod p = 1$

Euler's Theorem

- The multiplicative group of Z_n, denoted with Z_n^*, is the subset of elements of Z_n relatively prime with n
- The totient function of n, denoted with $\Phi(n)$ is the size of Z_n^*, $\Phi(n) = |Z_n^*|$
- Example :
  - $Z_{10}^* = \{1,3,7,9\}$ $\quad\quad \Phi(10) = 4$
- If p is prime, we have:
  - $Z_p^* = \{1,2,3, \dots, (p-1)\}$ $\quad\quad \Phi(p) = p - 1$
- Theorem:
  - For each element x of Z_n^* we have: $x^{\Phi(n)} \, mod \, n = 1$
- Example $(n = 10)$
  - $3^{\Phi(10)} \, mod \, 10 = 3^4 \, mod \, 10 = 81 \, mod \, 10 = 1$
  - $7^{\Phi(10)} \, mod \, 10 = 7^4 \, mod \, 10 = 2401 \, mod \, 10 = 1$
  - $9^{\Phi(10)} \, mod \, 10 = 9^4 \, mod \, 10 = 6561 \, mod \, 10 = 1$

RSA Cryptosystem

- ***RSA*** is named after its inventors, Ronal Rivest, Adi Shamir, and Leonard Adleman
- First published in 1977
- It is based on the practical difficulty of the factorization of the product of two large prime numbers
- One of the most widely used cryptosystems
- Because of its implications, the inventors have received Turing prize in 2002, the so-called Noble prize of CS

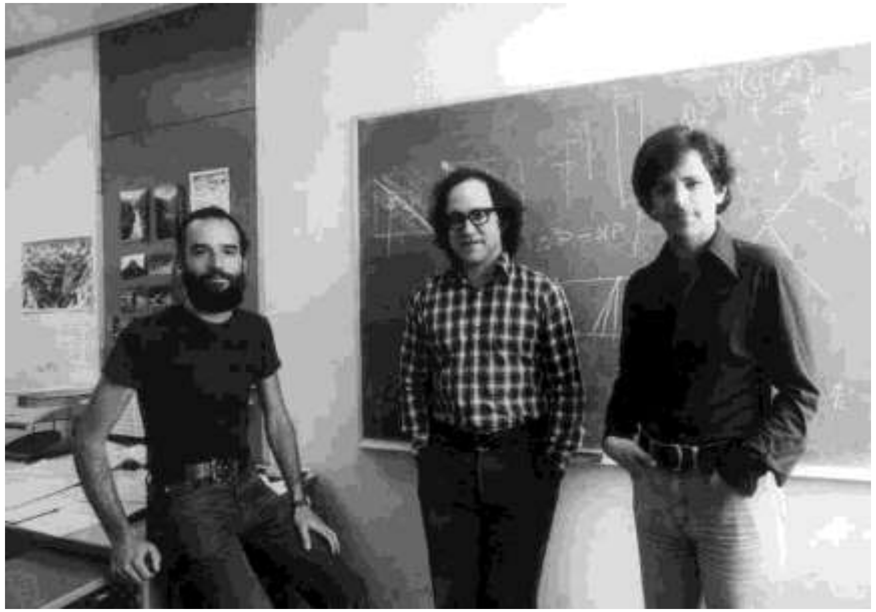


Figure 10: The inventors of the RSA cryptosystem, from left to right, Adi Shamir, Ron Rivest, and Len Adleman, who received the Turing Award in 2002 for this achievement. (Image used with permission from Ron Rivest and Len Adleman.)

RSA Cryptosystem

- Setup
  - $n = pq$, here p and q should be large prime numbers (e.g. 1024 digits)
  - e is chosen such that it is relatively prime to $\Phi(n)$
    - That is $\gcd(e, \Phi(n)) = 1$
    - $\Phi(n) = \Phi(p)\Phi(q) = (p - 1)(q - 1)$
  - d is inverse of e in $Z_{\Phi(n)}$
    - That is $de \bmod \Phi(n) = 1$
- Keys
  - Public key, $K_e = (n, e)$
  - Private key, $K_d = d$
- Encryption
  - Plaintext , M
  - Ciphertext, $C = M^e \bmod n$
- Decryption
  - $M = C^d \bmod n$

- Setup
  - $p = 7, q = 17$
  - $n = 7 \times 17 = 119$
  - $\Phi(n) = (p - 1)(q - 1) = 6 \times 16 = 96$
  - $e = 5$
  - $d = 77$
- Keys
  - Public key: $(119{,}5)$
  - Private key: 77
- Encryption
  - $M = 19$
  - $C = 19^5 \bmod 119 = 66$
- Decryption
  - $M = 66^{77} \bmod 119 = 19$

RSA Cryptosystem

- Setup
  - $p = 5, q = 11$
  - $n = 5 \times 11 = 55$
  - $\Phi(n) = 4 \times 10 = 40$
  - $e = 3$
  - $d = 27\ (3.27 = 81 = 2.40 + 1)$
- Encryption
  - $C = M^3\ mod\ 55$
- Decryption
  - $M = C^{27}\ mod\ 55$

M	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
C	1	8	27	9	15	51	13	17	14	10	11	23	52	49	20	26	18	2
M	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
C	39	25	21	33	12	19	5	31	48	7	24	50	36	43	22	34	30	16
M	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
C	53	37	29	35	6	3	32	44	45	41	38	42	4	40	46	28	47	54

RSA Correctness

- Ciphertext, $C = M^e \ mod \ n$ with the assumption that $\gcd(M, n) = 1$
  - This is probably the case, since p and q are very large, they are unlikely to be a factor of M
- We need to show that $C^d \ mod \ n = M$
- Since $ed \ mod \ \Phi(n) = 1$, there is an integer k such that:
  - $ed = k\Phi(n) + 1$
- Since M is relatively prime with n, we get the following from Euler's theorem
  - $M^{\Phi(n)} \ mod \ n = 1$
- Now,
  - $C^d mod \ n = M^{ed} \ mod \ n = M^{k\Phi(n)+1} \ mod \ n$
  - $= MM^{k\Phi(n)} mod \ n = M(M^{\Phi(n)})^k \ mod \ n = M.1^k = M$

RSA Security

- The security of the RSA cryptosystem is based on the difficulty of finding d, given e and n
- If we knew $\varphi(n) = (p-1)(q-1)$, it would be easy to compute d from e
- Thus, Bob needs to keep p and q secret (or even destroy all knowledge of them), since anyone who knows the values of p and q immediately knows the value of $\varphi(n)$
- Anyone who knows the value of $\varphi(n)$ can compute $d = e^{-1} \bmod \varphi(n)$, using the extended Euclidian algorithm
- Thus, the security of the RSA cryptosystem is closely tied to factoring n, which would reveal the values of p and q
- Fortunately, since this problem has shown itself to be hard to solve, we can continue to rely on the security of the RSA cryptosystem, provided we use a large enough modulus

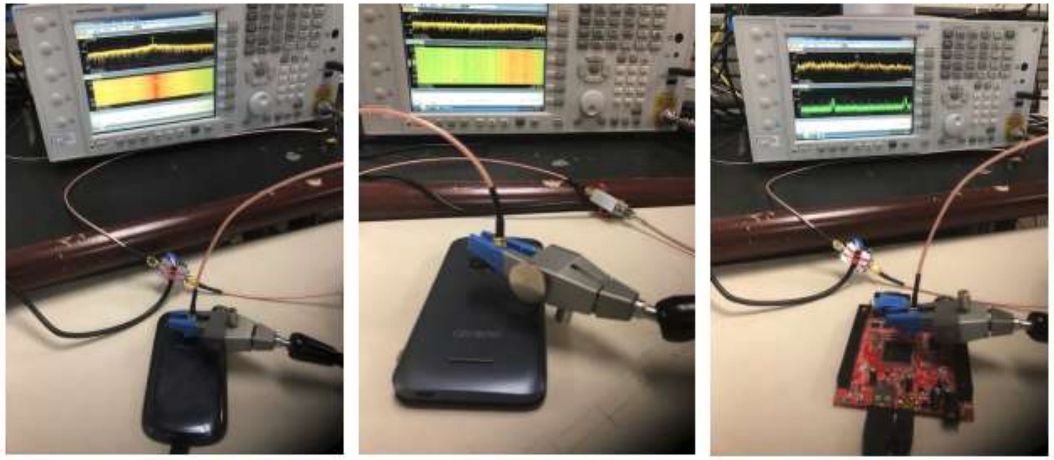
RSA Security

- In 1999, 512-bit challenge factored in 4 months using 35.7 CPU-years
  - 160 175-400 MHz SGI and Sun
  - 8 250 MHz SGI Origin
  - 120 300-450 MHz Pentium II
  - 4 500 MHz Digital/Compaq
- In 2005, a team of researchers factored the RSA-640 challenge number using 30 2.2GHz CPU years
- In 2004, the prize for factoring RSA-2048 was $200,000
- Current practice is 2,048-bit keys
- Estimated resources needed to factor a number within one year

Length (bits)	PCs	Memory
430	1	128MB
760	215,000	4GB
1020	342×10^6	170GB
1620	1.6×10^{15}	120TB

RSA Security

- Side channel attacks have also been demonstrated on RSA, based on measuring the time taken by decryption and/or the power consumption of the CPU performing the operation
- A very recent (30 August, 2018) research presented at USENIX'18, authors were able to retrieve the encryption keys from mobile device within seconds and without physical access to the devices
- The attack recovers the exponent's bits during modular exponentiation from analog signals that are unintentionally produced by the processor
    - https://www.usenix.org/system/files/conference/usenixsecurity18/sec18-alam.pdf



RSA Determinism

- We must take some care in how we use the RSA cryptosystem, however, because of its deterministic nature
- For example, suppose we use the RSA algorithm to encrypt two plaintext messages, M_1 and M_2, into the respective ciphertexts, C_1 and C_2, using the same public key
- Because RSA is deterministic, we know that, in this case, if $C_1 = C_2$, then $M_1 = M_2$
- Unfortunately, this fact could allow a cryptanalyst to infer information from ciphertexts encrypted from supposedly different plaintexts
- There is an alternative cryptosystem which can handle this issue

RSA Implementation

- The implementation of the RSA cryptosystem requires various algorithms
- Overall
  - Representation of integers of arbitrarily large size and arithmetic operations on them
- Encryption
  - Modular power
- Decryption
  - Modular power
- Setup
  - Generation of random numbers with a given number of bits (to generate candidates p and q)
  - Primality testing (to check that candidates p and q are prime)
  - Computation of the GCD (to verify that e and $\Phi(n)$ are relatively prime)
  - Computation of the multiplicative inverse (to compute d from e)

Repeated squaring

- RSA requires modular exponentiation in the form of $x^y \ mod \ n$ for its encryption and decryption functions
- One simple approach to calculate x^y then perform the modular operation
  - This is fully impractical in case y is large, as in the case of RSA
- Let's assume that we would like to compute $2^{1234} \ mod \ 789$
- If we compute 2^{1234} at first and then reduce it to 789, we will need to deal with very large numbers even though the final will contain only 3 digits
- Another approach could be
  - Perform each multiplication and then calculate the remainder
- But it would require to perform the multiplication 1234 times which will be too slow to be practical
- In such cases, another method is utilised called repeated squaring

Repeated squaring

- $2^2 \; mod \; 789 = 4$
- $2^4 \; mod \; 789 = 4^2 = 16$
- $2^8 \; mod \; 789 = 16^2 = 256$
- $2^{16} \; mod \; 789 = 256^2 = 49$
- $2^{32} \; mod \; 789 = 49^2 = 34$
- $2^{64} \; mod \; 789 = 34^2 = 367$
- $2^{128} \; mod \; 789 = 367^2 = 559$
- $2^{256} \; mod \; 789 = 559^2 = 37$
- $2^{512} \; mod \; 789 = 37^2 = 580$
- $2^{1024} \; mod \; 789 = 580^2 = 286$

- $1234 = 1024 + 128 + 64 + 16 + 2$
- $2^{1234} = 2^{1024+128+64+16+2}$

  $= 2^{1024} . 2^{128} . 2^{64} . 2^{16} . 2^2$

  $= 286.559.367.49.4 \; mod \; 789$

  $= 481$
- If we want compute $a^b \; mod \; n$
  - We can do it with at most $2 \log_2(b)$ multiplications $mod \; n$
  - We never have to deal with numbers larger than n^2

Multiplicative inverse calculation

- Given integers a and b, there are integers i and j such that
  - $ia + jb = \gcd(a, b) = d$
- Example: a=21, b=15, d=3, i=3, j=-4
  - 3=3x21 + (-4)15 = 63 - 60 = 3
- Given positive integers a and b, the extended Euclid's algorithm computes a triplet (d,i,j) such that
  - d = gcd(a,b)
  - d = ia + jb
- To test the existence of and compute the inverse of $x \in Z_n$,
  - we execute the extended Euclid's algorithm on the input pair (x,n)
- Let (d,i,j) be the triplet returned where d = ix + jn
- If d = 1, i is the multiplicative inverse of x in Z_n
- If d > 1, x has no inverse in Z_n

Primality testing

- Yet another important computation that is often used in modern cryptography is ***primality testing***
- In this instance, we are given a positive integer, n, and we want to determine if n is prime or not
- That is, we want to determine if the only factors of n are 1 and n itself
- Fortunately, there are efficient methods for performing such tests
  - None of these methods actually factor n
- They just indicate whether n is prime or not

Primality testing

- Fermat Primality Test:
  - Let $n > 1$ be an integer. Choose a random integer a with $1 < a < n - 1$.
  - If $a^{\mathrm{n-1}} \bmod \mathrm{n} \neq 1$, then n is composite
  - If $a^{\mathrm{n-1}} \bmod \mathrm{n} = 1$, then n is probably prime
- Given an efficient way of performing primality testing, actually generating a random prime number is relatively easy
- This simplicity is due to an important fact about numbers
  - the number of prime numbers between 1 and any number *n* is at least *n*/ln *n*, for *n* ≥ 4
- Selecting a random number between the range, check for its primality
  - if we repeat this process a logarithmic number of times, testing each number generated for primality, then one of our generated numbers is expected to be prime

Typical RSA use-case

- Even with an efficient implementation, the RSA cryptosystem is orders-of-magnitude slower than the AES symmetric cryptosystem
- Thus, a standard approach to encryption is as follows:
  - Encrypt a secret key, K, with the RSA cryptosystem for the AES symmetric cryptosystem
  - Encrypt with AES using key K
  - Transmit the RSA-encrypted key together with the AES-encrypted document
- The above method illustrates a common use of public-key cryptography in conjunction with a symmetric cryptosystem

The lecture slides can be found in the following location!