CSE 477: Introduction to Computer Security

Lecture – 9

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Outline

- Elgamal Crypto System
- Key Exchange
- Hash Function
- Digital Signature

Elgamal cryptosystem

- The ***Elgamal*** cryptosystem, named after its inventor, Taher Elgamal, is a public-key cryptosystem that uses randomization
  - Enabling independent encryptions of the same plaintext to produce different ciphertexts
- In the number system Z_p, all arithmetic is done modulo a prime number, p
- A number, g in Z_p, is said to be a ***generator*** or ***primitive root*** modulo p if, for each positive integer k in Z_p, there is a unique integer i such that $i = g^k \bmod p$
- For example,
  - 3 is a primitive root modulo 7
  - $3^0 \bmod 7 = 1$, $3^1 \bmod 7 = 3$, $3^2 \bmod 7 = 2$, $3^3 \bmod 7 = 6$, $3^4 \bmod 7 = 4$, $3^5 \bmod 7 = 5$, $3^6 \bmod 7 = 1$
  - Here, you get all the possible results: 1,3,2,6,4,5

  - 3 is not a primitive root modulo 11
  - $3^0 \bmod 11 = 1$, $3^1 \bmod 11 = 3$, $3^2 \bmod 11 = 9$, $3^3 \bmod 11 = 5$, $3^4 \bmod 11 = 4$, $3^5 \bmod 11 = 1$
  - You get 1, 3, 9, 5, 4 and this sequence repeats after 3^5

Elgamal cryptosystem

- It turns out that there are $\varphi(\varphi(p)) = \varphi(p - 1)$ generators for Z_p
  - So we can test different numbers until we find one that is a generator
- Once we have a generator g, we can efficiently compute $x = g^k \bmod p$, for any value k
- Conversely, given x, g, and p, the problem of determining k such that $x = g^k \bmod p$ is known as the ***discrete logarithm*** problem
- Like factoring, the discrete logarithm problem is widely believed to be computationally hard
- The security of the Elgamal cryptosystem depends on the difficulty of the discrete logarithm problem

Elgamal cryptosystem

- Setup:
  - Bob chooses a random large prime number, p, and finds a generator, g, for Z_p
  - He then picks a random number, x, between 1 and $p - 2$, and computes $y = g^x \bmod p$
  - The number, x, is Bob's secret key
  - His public key is the triple (p, g, y)
- When Alice wants to encrypt a plaintext message, M, for Bob, she begins by getting his public key, (p, g, y)
- She then generates a random number, k, between 1 and $p - 2$
- She then uses modular multiplication and exponentiation to compute two numbers:
  - $a = g^k \; mod \; p$
  - $b = My^k \; mod \; p$
  - The encryption of M is the pair (a,b)

Elgamal cryptosystem

- The decryption can be carried out by Bob in the following manner:
  - $M = b(a^x)^{-1}\, mod\, p$
- Correctness:
  - $b(a^x)^{-1} mod\, p = My^k \left(g^{kx}\right)^{-1} mod\, p$
    $= M(g^x)^k\, g^{-kx}\, mod\, p$
    $= Mg^{kx}g^{-kx}\, mod\, p$
    $= M\, mod\, p = M$

Elgamal cryptosystem

- Note that Bob doesn't need to know the random value, *k*, to decrypt a message that was encrypted using this value
- And Alice didn't need to know Bob's secret key to encrypt the message for him in the first place
- Instead, Alice got g^x, as *y*, from Bob's public key, and Bob got g^k, as *a*, from Alice's ciphertext
- Alice raised *y* to the power *k* and Bob raised *a* to the power *x*, and in so doing they implicitly computed a type of one- time shared key, g^{xk}
    - which Alice used for encryption and Bob used for decryption

Elgamal cryptosystem

- The security of this scheme is based on the fact that, without knowing *x*, it would be very difficult for an eavesdropper to decrypt the ciphertext, (*a*, *b*)
- Since everyone knows $y = g^x \bmod p$, from Bob's public key, the security of this scheme is therefore related to the difficulty of solving the discrete logarithm problem
- That is, Elgamal could be broken by an eavesdropper finding the secret key, *x*
  - This can be only done by solving the discrete logarithm problem which is believed to computationally hard
- Note that an Elgamal encryption is dependent on the choice of the random number, *k*
  - Different *k* should be chosen for different message encryption
- If *k* is reused she would be leaking information much like the one-time pad would leak information if we were to reuse a pad

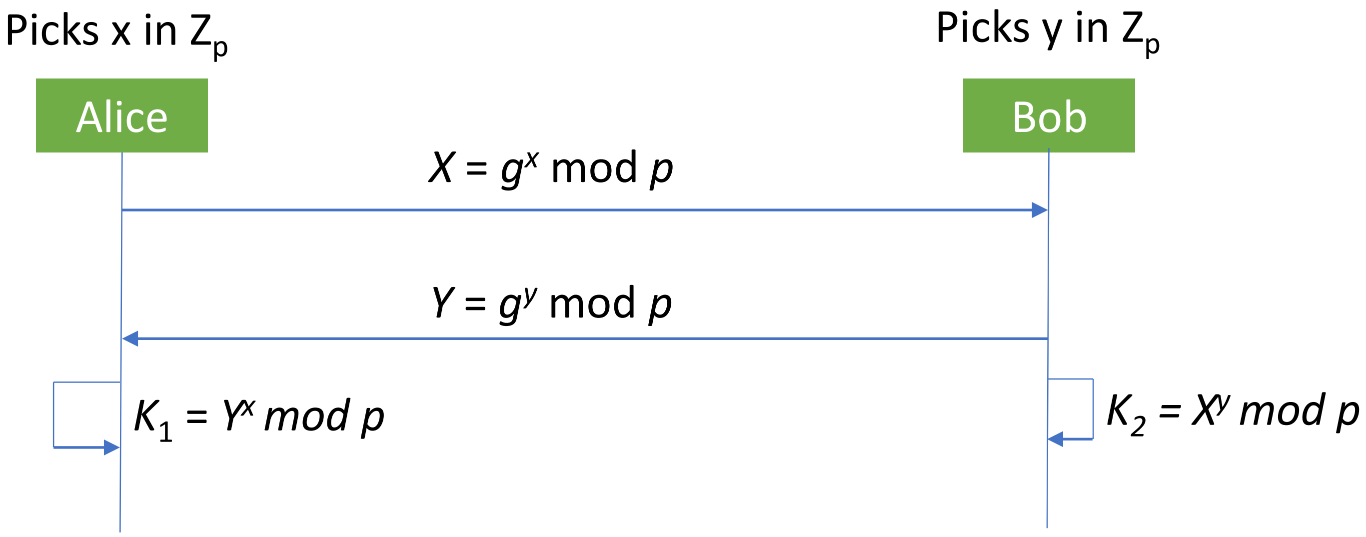
Key exchange

- The use of a symmetric cryptosystem requires that Alice and Bob agree on a secret key before they can send encrypted messages to each other
- Accomplish it by the one-time use of a private communication channel, such as an in-person meeting in a private room, or mailing in tamper-proof containers
- A ***key exchange protocol***, which is also called ***key agreement protocol***, is a cryptographic approach
    - to establish a shared secret key by communicating solely over an insecure channel, without any previous private communication
- Intuitively, the existence of a key exchange protocol appears unlikely
    - as the adversary can arbitrarily disrupt the communication between Alice and Bob
- Indeed, it can be shown that no key exchange protocol exists if the adversary can actively modify messages sent over the insecure channel
- Nevertheless, key exchange can be successfully accomplished if the adversary is limited to only passive eavesdropping on messages

Diffie-Hellman key exchange protocol (DH protocol)

- The classic ***Diffie-Hellman key exchange protocol*** (***DH protocol***), which is named after its inventors, Whitfield Diffie and Martin Hellman
  - based on modular exponentiation
- Assumed two public parameters have been established and are known to all participants (including the attacker):
  - a prime number, p, and a generator g, for Z_p
- Alice picks a random number $x \in Z_p$
- Bob picks a random number $y \in Z_p$

Diffie-Hellman key exchange protocol (DH protocol)



Diffie-Hellman key exchange protocol (DH protocol)

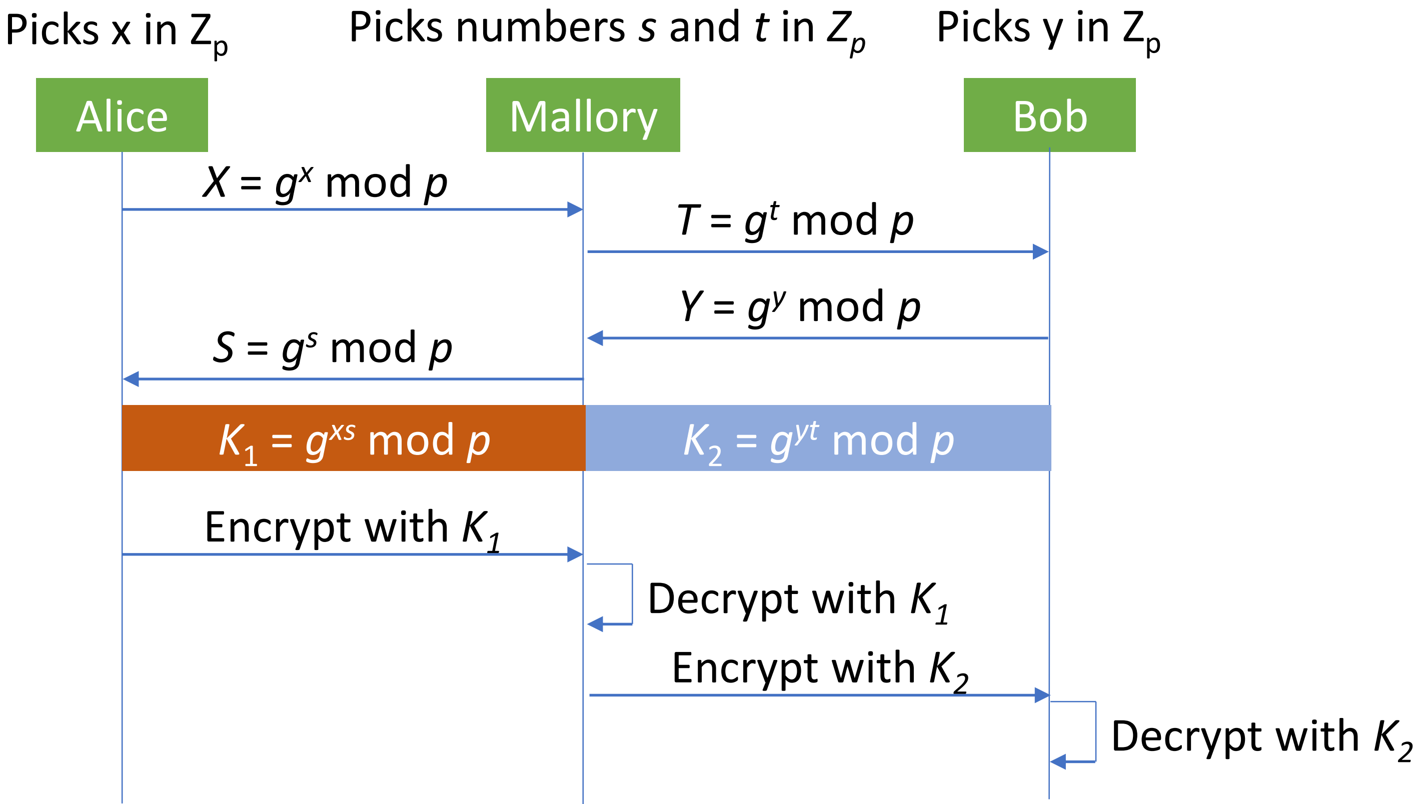
- The secret key essentially is $K = g^{xy} \; mod \; p$
- Proof:
  - $K_1 = Y^x \; mod \; p = (g^y)^x \; mod \; p = (g^x)^y \; mod \; p = X^y \; mod \; p = K_2$
- The security of the DH protocol is based on the assumption
  - It is difficult for the attacker to determine the key *K* from the public parameters and the eavesdropped values *X* and *Y*
- Indeed, recovering either *x* from *X* or *y* from *Y* is equivalent to solving the discrete logarithm problem
  - believed to be computationally hard

Diffie-Hellman key exchange protocol (DH protocol)

- Alice and Bob agree on $p = 23$ and $g = 5$
- Alice chooses $x = 6$ and sends $5^6 \bmod 23 = 8$ as X
- Bob chooses $y = 15$ and sends $5^{15} \bmod 23 = 19$ as Y
- Alice computes $K_1 = 19^6 \bmod 23 = 2$
- Bob computes $8^{15} \bmod 23 = 2$
- 2 is their shared secret!
- Clearly, much larger values of x, y, and p are required

Attack on DH protocol

- Even though it is secure against a passive attacker, the DH protocol is vulnerable to a man-in-the-middle attack if the attacker can intercept and modify the messages exchanged by Alice and Bob



Cryptographic Hash Functions

- A cryptographic ***hash function*** produces a compressed digest of a message with three properties: **deterministic**, ***one-way***, and ***collision-resistant***
- Deterministic property will ensure that a fixed input will produce a same output
- The hash value should be significantly smaller than a typical message
- For example, the commonly used standard hash function SHA-256 produces hash values with 256 bits
- The digest will be such that changing one bit of input should potentially impact every bit of output
- A hash function utilises several of the techniques employed in symmetric encryption, including substitution, permutation, exclusive-or, and iteration, in a way that provides the required diffusion

Hash Function: one way

- That is, given a message, M, it should be easy to compute a hash value, $H(M)$, from that message
- However, given only a value, x, it should be difficult to find a message, M, such that $x = H(M)$

Hash Function: collision resistance

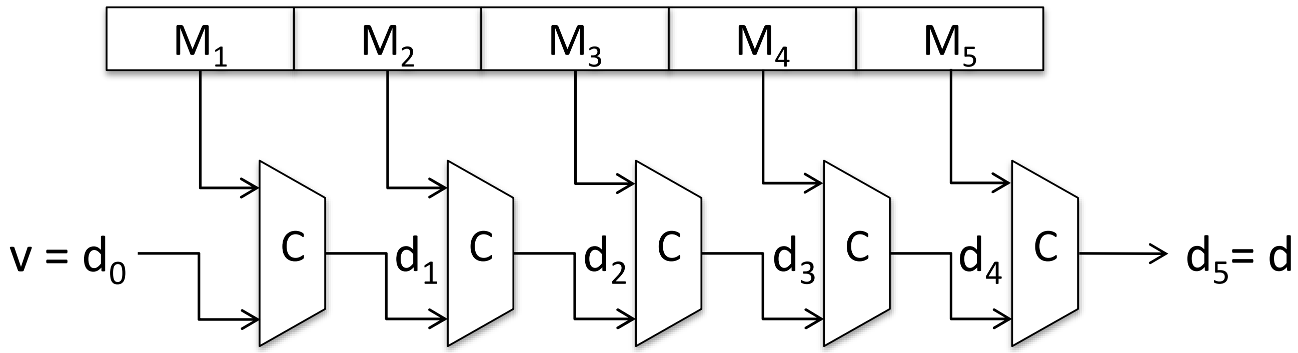
- A hash function, *H,* is a mapping of input strings to (smaller) output strings
- We say that *H* has ***weak collision resistance*** if, given any message, *M*, it is computationally difficult to find another message, $M' \neq M$, such that
  - $H(M') = H(M)$
- Hash function *H* has ***strong collision resistance*** if it is computationally difficult to compute two distinct messages, M_1 and M_2, such that
  - $H(M_1) = H(M_2)$
- That is, in weak collision resistance, we are trying to avoid a collision with a specific message, and in strong collision resistance we are trying to avoid collisions in general
- It is usually a challenge to prove that real-world cryptographic hash functions have strong collision resistance

Hash Function: construction

- A hash function utilises a building block called ***cryptographic compression function*** $C(X, Y)$
  - C takes as input two strings, X and Y, where X has fixed length m and Y has fixed length n
  - produces a hash value of length n
- Given a message M, we divide M into multiple blocks, $M_1, M_2, \ldots, M_k$, each of length m
  - The last block is padded in an unambiguous way with additional bits to make it of length m

Hash Function: construction

- For block $M_{1,}$ input M_1 with a fixed string *v* of length *n*, known as the ***initialization vector,*** to *C*
  - Denote, $d_1 = C(M_1,v)$
- Next, we apply the compression function to block M_2 and d_1
  - resulting in $d_2 = C(M_2, d_1)$
- These go on for all *k* blocks
- Final hash value, $H = d_k$
- Known as the Merkle-Damgård construction
  - Ralph Merkle & Ivan Damgård



Hash Function: practical implementation

- The currently recommended hash function for cryptographic applications are the ***SHA-256*** and ***SHA-512*** standardized by NIST
  - SHA stands for “Secure Hash Algorithm”
  - The numeric suffix refers to the length of the hash value
- Utilises the Merkle-Damgård construction
- SHA-256 employs a compression function with inputs of:
  - m = 512 bits and n = 256 bits and produces hash values of n = 256 bits
- For SHA-512: m = 1,024 and n = 512
- MD5 (Message Digest 5) hash function is still widely used in legacy applications
  - However, it is considered insecure as several attacks against it have been demonstrated
- For example, one can generate different PDF files or executable files with the same MD5 hash, a major vulnerability!

Hash Function: birthday attack

- The main way that cryptographic hash functions are attacked with is by compromising their collision resistance
- Sometimes this is done by careful cryptanalysis of the algorithms used to perform cryptographic hashing
- But it can also be done by using a brute-force technique known as a ***birthday attack***
- This attack is based on a nonintuitive statistical phenomenon that states that as soon as there are more than 23 people in a room, there is better than a 50-50 chance that two of the people have the same birthday
- If there are more than 60 people in a room, it is almost certain that two of them share a birthday
- Here, sharing birthday represents finding a collision

Hash Function: birthday attack

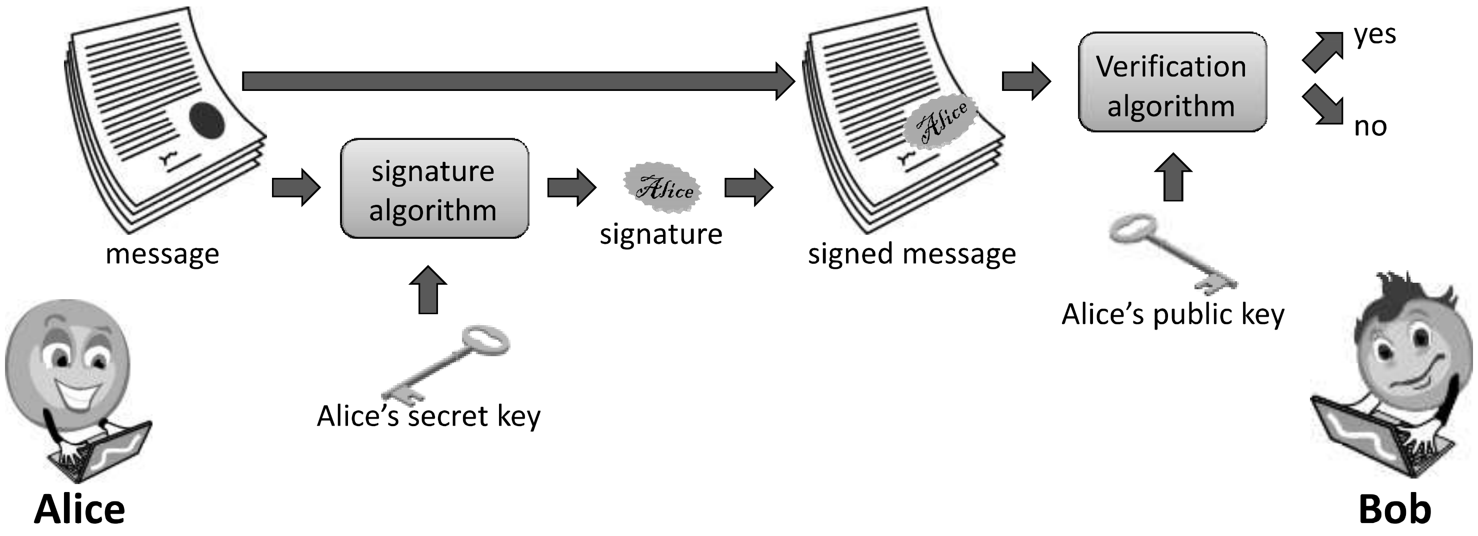
- The problem is to compute the approximate probability that in a group of n people, at least two have the same birthday
- The goal is to compute $P(A)$
  - the probability that at least two people in the room have the same birthday
- However, it is simpler to calculate $P(A')$
  - the probability that no two people in the room have the same birthday
- Because A and A' are the only two possibilities and are also mutually exclusive, $P(A) = 1 - P(A')$
- When events are independent of each other, the probability of all of the events occurring is equal to a product of the probabilities of each of the events occurring
- $P(A')$ can be described as 23 independent events
  - $P(A') = P(1) \times P(2) \times P(3) \times ... \times P(23)$

Hash Function: birthday attack

- For *Event 1*, there are no previously analysed people
  - The probability, *P*(1), that *Person 1* does not share his/her birthday with previously analysed people is 1, or 100% or 365/365 ignoring leap years
- For *Event 2*, the probability, *P*(2), that *Person 2* has a different birthday than *Person 1* is 364/365
- For *Event 3*, the probability, *P*(3), that *Person 3* has a different birthday than *Person 1* and *Person 2* is 363/365
- $P(A') = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{343}{365}$
  $= \left(\frac{1}{365}\right)^{23} \times (365 \times 364 \times \cdots \times 343)$
  $= 0.492703$
- $P(A) = 1 - P(A') = 1 - 0.492703 =$ 0.507297 or 50.7% > 50%

Digital signature

- A ***digital signature*** is a way for an entity to demonstrate the authenticity of a message by binding its identity with that message
- Alice uses her private key with a signature algorithm to produce a digital signature, $S_{\text{Alice}}(M)$, for a message, M
- Given Alice's public key, the message, M, and Alice's signature, $S_{\text{Alice}}(M)$
  - Bob verifies Alice's signature on M



Digital signature: properties

- Three important properties that we would like to have for a digital-signature scheme are the following:
- ***Nonforgeability:*** It should be difficult for an attacker, Eve, to forge a signature, $S_{\text{Alice}}(M)$, for a message, M, as if it is coming from Alice
- ***Nonmutability:*** It should be difficult for an attacker, Eve, to take a signature, $S_{\text{Alice}}(M)$, for a message, M, and convert $S_{\text{Alice}}(M)$ into a valid signature on a different message, N
- ***Non-repudiation:*** If a digital-signature scheme achieves these two properties, then it actually achieves ***nonrepudiation***
  - It should be difficult for Alice to claim she didn't sign a document, M, once she has produced a digital signature, $S_{\text{Alice}}(M)$, for that document

RSA signature

- In RSA encryption system, Bob creates a public key, (e, n), so that other parties can encrypt a message
  - $C = M^e \bmod n$
- In RSA signature, Bob instead signs a message, M, using his secret key, d
  - $S = M^d \bmod n$
- Verification:
  - Is it true that $M = S^e \bmod n$?
- Correctness, $de \ mod \ \Phi(n) = 1$:
  - $$S^e \ mod \ n = M^{de} \ mod \ n = M^{k\Phi(n)+1} \ mod \ n = MM^{k\Phi(n)} \ mod \ n$$
    $$= M \ mod \ n = M$$
- Verification utilises the same RSA encryption and decryption algorithms using the same public key (e, n)

RSA signature: security

- The nonforgeability of this scheme comes from the difficulty of breaking the RSA encryption algorithm
  - To forge a signature from Bob on a message, *M*, an attacker, Eve, would have to produce M^d mod *n*, without knowing *d*
- Unfortunately, the RSA signature scheme does not achieve non-mutability!
- For example, an attacker Eve, has two valid signatures from Bob for two messages M_1 and M_2
  - $S_1 = M_1^d mod\ n$ and $S_2 = M_2^d mod\ n$
- Now, Eve can produce a new signature
  - $S_1 . S_2\ mod\ n = (M_1 . M_2)^d\ mod\ n$
- This would validate as a verifiable signature from Bob on the message $M_1 . M_2$

Hash function + Digital signature

- For practical purposes, the above descriptions of the RSA digital-signature schemes are not what one would use in practice
- Firstly, the scheme is inefficient if the message, M, being signed is very long
  - For instance, RSA signature creation involves an encryption of the message, M, using a private key
- Secondly, one can construct valid RSA signatures on combined messages from existing RSA signatures as shown before
- For these reasons, real-world digital-signature schemes are usually applied to cryptographic hashes of messages, not to actual messages
- This approach significantly reduces the mutability risk for RSA signatures
  - For instance, since it is extremely unlikely that the product of two hash values, $H(M)$ and $H(N)$, would itself be equal to the hash of the product message, $M \cdot N$
- Moreover, signing a hash value is more efficient than signing a full message

The lecture slides can be found in the following location!