Chapter-1

Deductive proof: A deductive proof consists of a sequence of statements whose truth leads us from some initial statement, called the hypothesis or the given statement (s), to a conclusion statement.

If $\underbrace{H}_{\text{Hypothesis}}$ then $\underbrace{C}_{\text{conclusion}}$ / H implies C / H only if C / C if H / Whenever H holds, C follows

Theorem 1.3: If $x \geq 4$, then $2^x \geq x^2$.

Solution: First, notice that ~~to~~ the hypothesis H is "$x \geq 4$". This hypothesis has a parameter, x, and thus is neither true nor false. Rather, it's truth depends on the value of the parameter x; e.g., H is true for $x = 6$ and false for $x = 2$.

Likewise, the conclusion C is "$2^x \geq x^2$". This statement also uses parameter x and is true for certain values of x and not others. For example, C is false for $x = 3$, since $2^3 = 8$, which is not as large as $3^2 = 9$. On the other hand, C is true for $x = 4$, since $2^4 = 4^2 = 16$. For $x = 5$, the statement is also true, since $2^5 = 32$ is at least as larger as $5^2 = 25$. Thus, we can say that $2^x \geq x^2$ will be true whenever $x \geq 4$.

Here, as $2^x > x^2$

$$\text{So, } 2^{x+1} \geq (x+1)^2$$
$$\Rightarrow 2^x . 2 \geq (x+1)^2$$
$$\Rightarrow x^2 . 2 \geq (x+1)^2$$
$$\Rightarrow 2 \geq \left(\frac{x+1}{x}\right)^2$$
$$\therefore 2 \geq \left(1+\frac{1}{x}\right)^2$$

Theorem 1.4: If x is the sum of the squares of four positive integers, then $2^x \geq x^2$.

Solution: Let a, b, c and d be the four positive integers.

$$\text{So, } x = a^2+b^2+c^2+d^2 \quad \cdots\cdots (1)$$

$$\text{and } a \geq 1,\ b \geq 1,\ c \geq 1,\ d \geq 1 \quad \cdots\cdots (2)$$

from (2) and properties of arithmetic we get,

$$a^2 \geq 1,\ b^2 \geq 1,\ c^2 \geq 1,\ d^2 \geq 1 \quad \cdots\cdots (3)$$

from (1), (3) and properties of arithmetic, we get, $x \geq 4$ ..... (4)

x is at least, $1+1+1+1=4$

from (4) and theorem 1.3, we get,

$$2^x \geq x^2 \ldots\ldots (5)$$

Additional forms of proof:

Proofs about sets: $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$

* # The contrapositive: Every if-then statement has an equivalent form that in some circumstances is easier to prove. The contrapositive of the statement "if H then C" is "if not C then not H". A statement and its contrapositive are either both true or both false, so we can prove either to prove the other.

Statement	Justification
(1) x is in $R \cup (S \cap T)$	Given
(2) x is in R or x is in $S \cap T$	(1) and definition of union
(3) x is in R or x is in both S and T	(2) and definition of intersection

④ x is in $R \cup S$	② and definition of union
⑤ x is in $R \cup T$	③ and definition of union
⑥ x is in $(R \cup S) \cap (R \cup T)$	④, ⑤ and definition of intersection

Again,

Statement	Justification
① x is in $(R \cup S) \cap (R \cup T)$	Given
② x is in $R \cup S$	(1) and definition of intersection
③ x is in $R \cup T$	(1) and " " "
④ x is R or x is in ~~Both~~ S and T	②, ③ and reasoning about unions
⑤ x is in R or x is in $S \cap T$	④ and definition of intersection
⑥ x is in $R \cup (S \cap T)$	⑤ and definition of union

Converse & Contrapositive: The converse of an if-then statement is the "other direction", that is, the converse of "if H then C" is "If C then H". Unlike, the contrapositive, which is logically equivalent to the original, the converse is not equivalent to the original statement.

Example 1.11: "If $x \geq 4$, then $2^x \geq x^2$" $\to$ main statement

"If not $2^x \geq x^2$ then not $x \geq 4$ $\to$ Contrapositive

$\Rightarrow$ "If $2^x < x^2$ then $x < 4$ $\to$ Another form of "

Proof by contradiction:

Completing the proof by showing that something known to be false, starting by assuming the hypothesis is true and the conclusion false.

Example: Prove that $\sqrt{5}$ is an irrational number.

$\Rightarrow$ Let $\sqrt{5}$ be a rational number.

So, $\sqrt{5} = \frac{p}{q}$ [$p, q \in \mathbb{Z}$, $q \neq 0$ and p, p are co-prime and $q > 1$]

$\Rightarrow 5 = \frac{p^2}{q^2}$ [squaring both sides]

Multiplying both sides by q, we get,

$\Rightarrow 5q = \frac{p^2}{q}$

Here, $5q$ are clearly is an integer but $\frac{p^2}{q}$ is not as p and q are coprime numbers and $q > 1$.

So, $5q \neq \frac{p^2}{q}$

$\therefore \sqrt{5} \neq \frac{p}{q}$

$\therefore$ Therefore, $\sqrt{5}$ is an irrational number.

Proof by counterexamples: It shows that a given statement can't possibly be correct by showing an instance that contradicts a universal statement.

Theorem 1.13: All primes are odd.

→ The integer 2 is a prime, but is even.

Theorem 1.14: There is no pair of integers a and b such that a mod b = b mod a

Soln: There is three cases. (1) $a > b$, (2) $a < b$, (3) $a == b$

$a \bmod b = pb + q$, $q = [0, b-1]$

If $a > b$ then

$a \bmod b = c$ is a unique integer between 0 and $b-1$

and $b \bmod a = b$

So, $b \bmod a > a \bmod b$.

If $a < b$, then

$b \bmod a = a'$ is a unique integer between 0 and $a-1$

and $a \bmod b = a$

$\therefore a \bmod b > b \bmod a$

If $a = b$ then,

$$a \bmod b = a \bmod a = 0$$
$$b \bmod a = b \bmod b = 0$$

So, if $a \neq b$, then there is no pair of integers a and b such that $a \bmod b = b \bmod a$.

Theorem 1.15: $a \bmod b = b \bmod a$ if and only if $a = b$.

→ same as 1.14

Inductive proof: It is essential when dealing with recursively defined objects. It is used to prove a statement is true for all values of n.

Basis step: $S(i)$ is true for $i = 0$ or 1

$i \rightarrow$ Particular integer

Inductive step: If $S(n)$ is true, $S(n+1)$ is also true.

Theorem 1.16: For all $n \geq 0$: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Soln:

Base step: For $n=0$,

$\sum_{i=1}^{0} i^2$ has no terms, so the sum is 0.

$$\sum_{i=1}^{0} i^2 = 0$$

Inductive step: Let, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Again,
$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)\{2(n+1)+1\}}{6}$$
$$= \frac{(n+1)(n+2)(2n+3)}{6}$$
$$= \frac{1}{6}(2n^3+3n^2+6n^2+9n+4n+6)$$
$$= \frac{1}{6}(2n^3+9n^2+13n+6) \quad \cdots\cdots (1)$$

We can also write,
$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$
$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$
$$= \frac{1}{6}\{(n^2+n)(2n+1)+6n^2+12n+6\}$$
$$= \frac{1}{6}\{2n^3+n^2+2n^2+n+6n^2+12n+6\}$$
$$= \frac{1}{6}(2n^3+9n^2+13n+6) \quad \cdots (2)$$

[Proved]

Example: If $x \geq 4$, then $2^x \geq x^2$

Soln:

Base step: for $x=4$, $2^x = 2^4 = 16$

$x^2 = 4^2 = 16$

$\therefore 2^x = x^2 \ldots\ldots (1)$

Inductive step: For $x \geq 4$

Let, $2^{x+1} \geq (x+1)^2 \longrightarrow x = x+1$

$\Rightarrow 2^x \cdot 2 \geq (x+1)^2$

$\Rightarrow x^2 \cdot 2 \geq x^2 + 2x + 1$ [from (1)]

$\Rightarrow 2x^2 \geq x^2 + 2x + 1$

$\Rightarrow x^2 \geq 2x + 1$

$\Rightarrow x \geq 2 + \frac{1}{x}$ [dividing both sides by x]

For $x \geq 4$, the maximum value of $(2 + \frac{1}{x})$ is 2.25, so, L.H.S. $\geq$ R.H.S.

$\therefore 2^x \geq x^2$