Solutions to Chapter 10, Susanna Epp Discrete Math 5th Edition

https://github.com/spamegg1

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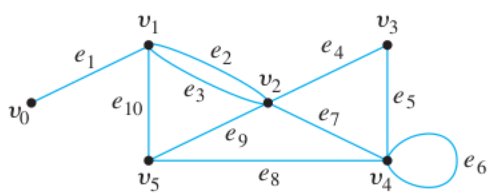
Contents

1 Exercise Set 10.1

1.1 Exercise 1

In the graph below, determine whether the following walks are trails, paths, closed walks, circuits, simple circuits, or just walks.



1.1.1 (a)

$v_0 e_1 v_1 e_{10} v_5 e_9 v_2 e_2 v_1$

Proof. trail (no repeated edge), not a path (has a repeated vertex, v_1), not a circuit □

1.1.2 (b)

$v_4 e_7 v_2 e_9 v_5 e_{10} v_1 e_3 v_2 e_9 v_5$

Proof. walk, not a trail (has a repeated edge, e_9), not a circuit □

1.1.3 (c)

v_2

Proof. closed walk (starts and ends at the same vertex), trail (no repeated edge since no edge), not a path or a circuit (since no edge) □

1.1.4 (d)

$v_5v_2v_3v_4v_4v_5$

Proof. circuit, not a simple circuit (repeated vertex, v_4) □

1.1.5 (e)

$v_2v_3v_4v_5v_2v_4v_3v_2$

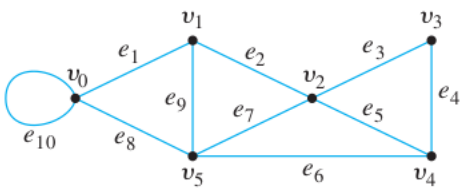
Proof. closed walk (starts and ends at the same vertex but has repeated edges, $\{v_2, v_3\}$ and $\{v_3, v_4\}$) □

1.1.6 (f)

$e_5e_8e_{10}e_3$

Proof. path □

1.2 Exercise 2



In the graph, determine whether the following walks are trails, paths, closed walks, circuits, simple circuits, or just walks.

1.2.1 (a)

$v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

Proof. walk (not closed), not a trail or circuit (has repeated edge e_2), not a path (has repeated vertex v_2), □

1.2.2 (b)

$v_2v_3v_4v_5v_2$

Proof. simple circuit (only has the first + last vertex repeated, no repeated edge) □

1.2.3 (c)

$v_4v_2v_3v_4v_5v_2v_4$

Proof. closed walk, not a trail, path, circuit (has repeated edge e_5 and vertex v_2) □

1.2.4 (d)

$v_2 v_1 v_5 v_2 v_3 v_4 v_2$

Proof. circuit, not simple circuit (has non-first, non-last vertex repeated: v_2) □

1.2.5 (e)

$v_0 v_5 v_2 v_3 v_4 v_2 v_1$

Proof. trail (no repeated edge), not a path (repeated vertex v_2), not a closed walk or circuit □

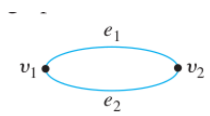
1.2.6 (f)

$v_5 v_4 v_2 v_1$

Proof. path (no repeated vertex), not a closed walk or circuit □

1.3 Exercise 3

Let G be the graph



and consider the walk $v_1 e_1 v_2 e_2 v_1$.

1.3.1 (a)

Can this walk be written unambiguously as $v_1 v_2 v_1$? Why?

Proof. No, because $v_1 v_2 v_1$ could also refer to $v_1 e_2 v_2 e_1 v_1$. □

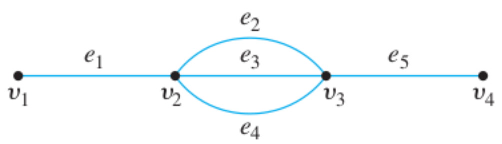
1.3.2 (b)

Can this walk be written unambiguously as $e_1 e_2$? Why?

Proof. Yes. □

1.4 Exercise 4

Consider the following graph:



1.4.1 (a)

How many paths are there from v_1 to v_4?

Proof. 3: $e_1e_2e_5, e_1e_3e_5, e_1e_4e_5$. □

1.4.2 (b)

How many trails are there from v_1 to v_4?

Proof. $3! + 3 = 9$ (The three paths from part (a) are also trails, and there are an additional 3! trails with vertices $v_1, v_2, v_3, v_2, v_3, v_4$. The reason is that from v_2 there are 3 choices of an edge to go to v_3, then 2 choices of a different edge to go back to v_2, and then 1 choice of a different edge to return to v_3. □

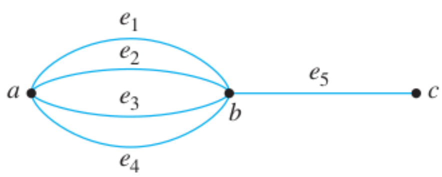
1.4.3 (c)

How many walks are there from v_1 to v_4?

Proof. Infinitely many. (Since a walk may have repeated edges, a walk from v_1 to v_4 may contain an arbitrarily large number of repetitions of edges joining a pair of vertices along the way.) □

1.5 Exercise 5

Consider the following graph:



1.5.1 (a)

How many paths are there from a to c?

Proof. 4: $e_1e_5, e_2e_5, e_3e_5, e_4e_5$. □

1.5.2 (b)

How many trails are there from a to c?

Proof. $4! + 4 = 28$ (The 4 paths from part (a) are also trails, and there are an additional 4! trails with vertices a, b, a, b, c. The reason is that from a there are 4 choices of an edge to go to b, then 3 choices of a different edge to go back to a, then 2 choices of a different edge to go back to b, and then 1 choice of a different edge to go to c. □

1.5.3 (c)

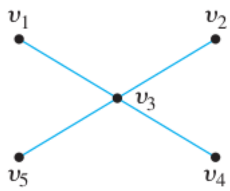
How many walks are there from a to c?

Proof. Infinitely many. (Since a walk may have repeated edges, a walk from a to c may contain an arbitrarily large number of repetitions of edges joining a pair of vertices along the way.) □

1.6 Exercise 6

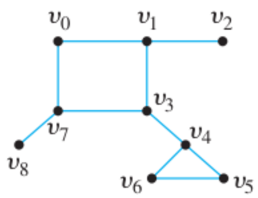
An edge whose removal disconnects the graph of which it is a part is called a bridge. Find all bridges for each of the graphs at the top of the next page.

1.6.1 (a)



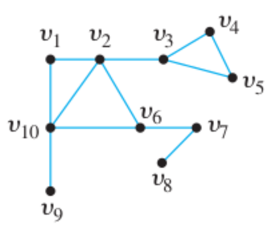
Proof. $\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_3\}$, and $\{v_5, v_3\}$ are all the bridges. □

1.6.2 (b)



Proof. $\{v_1, v_2\}, \{v_3, v_4\}, \{v_7, v_8\}$ are all the bridges. □

1.6.3 (c)



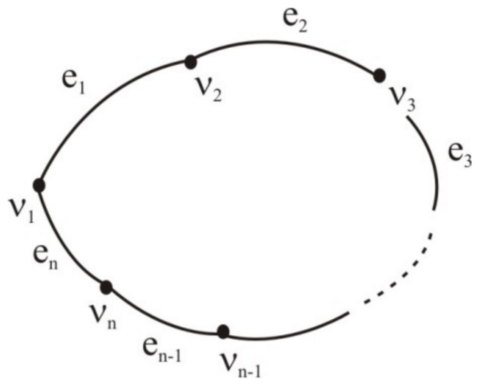
Proof. $\{v_2, v_3\}, \{v_6, v_7\}, \{v_7, v_8\}, \{v_9, v_{10}\}$ are all the bridges. □

1.7 Exercise 7

Given any positive integer n, (a) find a connected graph with n edges such that removal of just one edge disconnects the graph; (b) find a connected graph with n edges that cannot be disconnected by the removal of any single edge.

Proof. (a) Consider a line graph with n edges and $n+1$ vertices that looks like: $\cdot - \cdot - \ldots - \cdot - \cdot$: $v_1 e_1 v_2 e_2 \ldots v_n e_n v_{n+1}$. Then removing any one edge will break the line and disconnect the graph.

(b) Consider a closed loop graph with n edges and n vertices: $v_1 e_1 v_2 e_2 \ldots v_n e_n v_1$

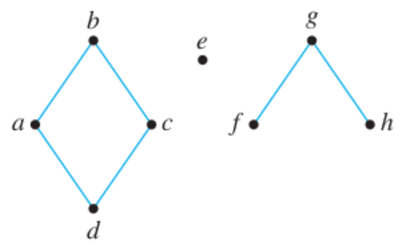


(so the last edge e_n connects back to the first vertex v_1, closing the loop). Then removing any one edge will break the loop, but it will remain connected. □

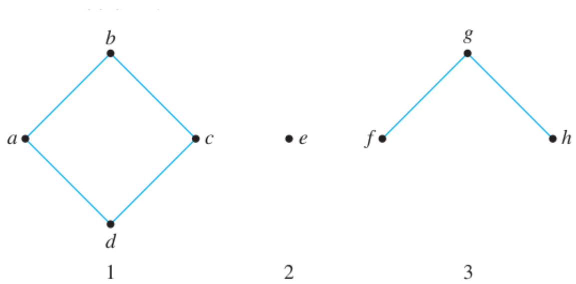
1.8 Exercise 8

Find the number of connected components for each of the following graphs.

1.8.1 (a)

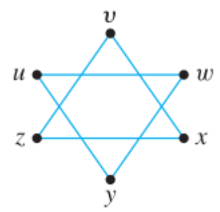


Proof. Three connected components, as shown in the next column.



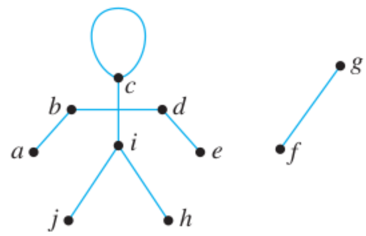
□

1.8.2 (b)



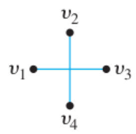
Proof. Two connected components: the two "triangles" $v - x - z$ and $u - w - y$. □

1.8.3 (c)



Proof. Three connected components: the "arms and shoulders" part $a - b - d - e$, the "head-torso-legs" part $c - i - j - h$ and the part $f - g$. □

1.8.4 (d)



Proof. Two connected components: $v_1 - v_3$ and $v_2 - v_4$. □

1.9 Exercise 9

Each of (a)–(c) describes a graph. In each case answer yes, no, or not necessarily to this question: Does the graph have an Euler circuit? Justify your answers.

1.9.1 (a)

G is a connected graph with five vertices of degrees 2, 2, 3, 3, and 4.

Proof. No. This graph has two vertices of odd degree, whereas all vertices of a graph with an Euler circuit have even degree. □

1.9.2 (b)

G is a connected graph with five vertices of degrees 2, 2, 4, 4, and 6.

Proof. Yes (connected, all vertices have even degree, so by Theorem 10.1.4). □

1.9.3 (c)

G is a graph with five vertices of degrees 2, 2, 4, 4, and 6.

Proof. Not necessarily (if G is connected then yes by Theorem 10.1.4). □

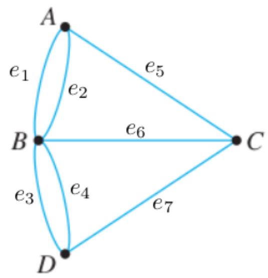
1.10 Exercise 10

The solution for Example 10.1.6 shows a graph for which every vertex has even degree but which does not have an Euler circuit. Give another example of a graph satisfying these conditions.

Proof. Just take two disconnected squares: graphs with 4 vertices each. Considered together as one graph, each vertex has degree 2 but there is no Euler circuit. □

1.11 Exercise 11

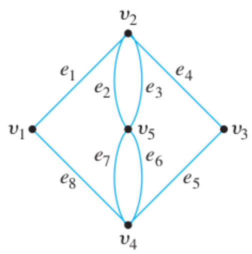
Is it possible for a citizen of Königsberg to make a tour of the city and cross each bridge exactly twice? (See figure below.) Explain.



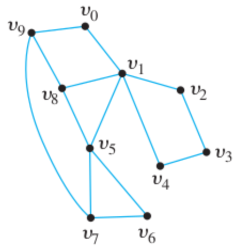
Proof. $Ae_1Be_2Ae_5Ce_6Be_3De_4Be_6Ce_7De_3Be_4De_7Ce_5Ae_1Be_2A$. (See below.) □

Determine which of the graphs in 12–17 have Euler circuits. If the graph does not have an Euler circuit, explain why not. If it does have an Euler circuit, describe one.

1.12 Exercise 12



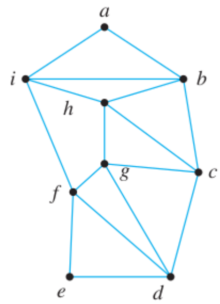
Proof. One Euler circuit is $e_4e_5e_6e_3e_2e_7e_8e_1$. □



1.13 Exercise 13

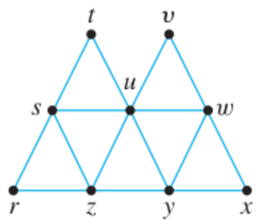
Proof. No Euler circuit, since v_1 has odd degree (5). □

1.14 Exercise 14



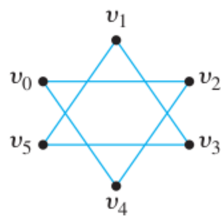
Proof. One Euler circuit is $iabihbchgcdgfdefi$. □

1.15 Exercise 15

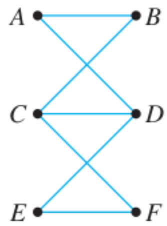


Proof. One Euler circuit is $rzyxwyuzsuwvutsr$. □

1.16 Exercise 16



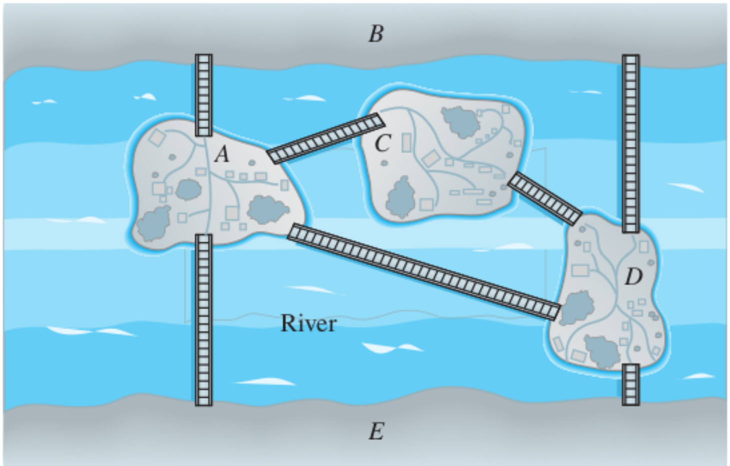
Proof. No Euler circuit (graph is not connected). □



1.17 Exercise 17

Proof. No Euler circuit (C has odd degree 3). □

1.18 Exercise 18

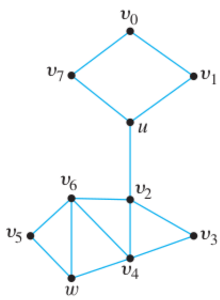


Is it possible to take a walk around the city whose map is shown above, starting and ending at the same point and crossing each bridge exactly once? If so, how can this be done?

Proof. DCADBAED □

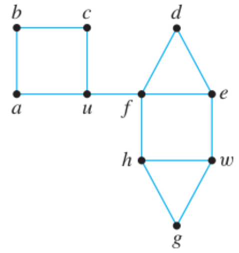
For each of the graphs in 19 – 21, determine whether there is an Euler trial from u to w. If there is, find such a trail.

1.19 Exercise 19



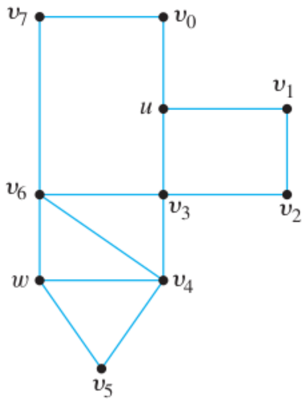
Proof. There is an Euler trail since deg(u) and deg(w) are odd, all other vertices have positive even degree, and the graph is connected. $uv_1v_0v_7uv_2v_3v_4v_2v_6v_4wv_5v_6w$ is one Euler trail. □

1.20 Exercise 20



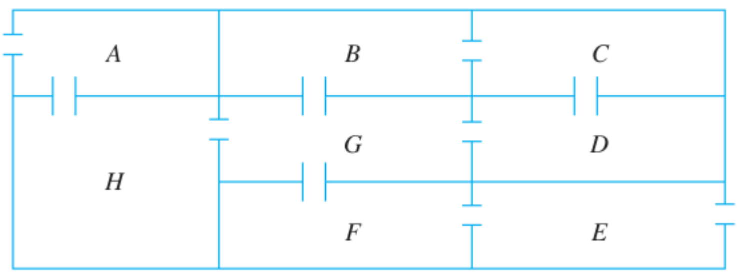
Proof. No Euler trail, since there are 4 vertices with odd degrees: u, e, h, w. □

1.21 Exercise 21



Proof. One Euler trail is $uv_0v_7v_6v_3uv_1v_2v_3v_4v_6wv_4v_5w$. □

1.22 Exercise 22



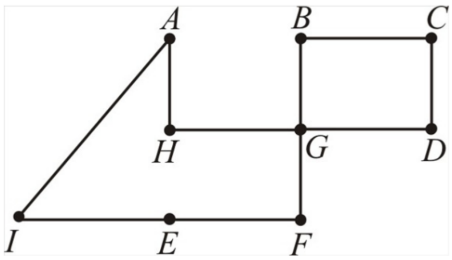
The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E? If so, how can this be done?

Proof. Let rooms be vertices, let the outside also be a vertex (I). Then

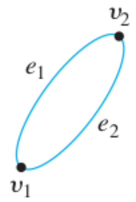
One Euler circuit is: IAHGBCDGFEI □

1.23 Exercise 23

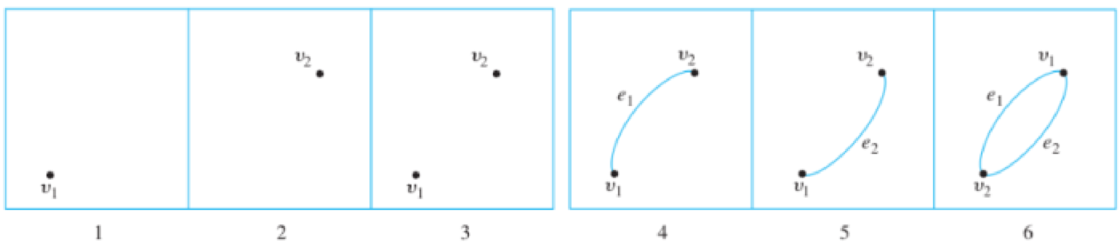
Find all subgraphs of each of the following graphs.



1.23.1 (a)



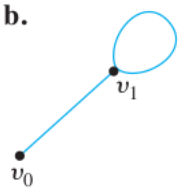
Proof. The nonempty subgraphs are as follows:



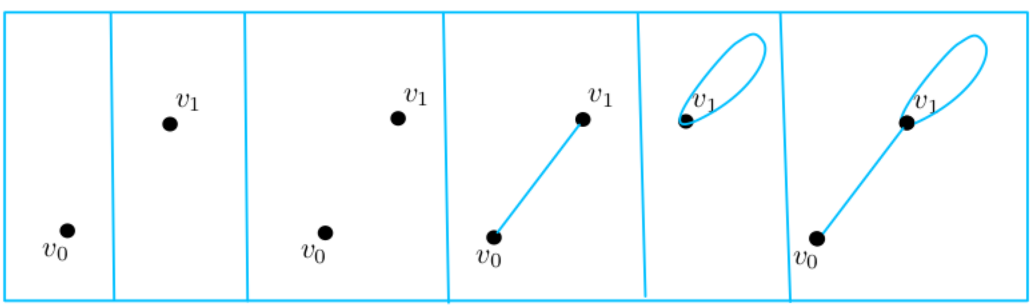
□

1.23.2 (b)

b.

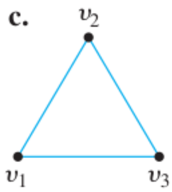


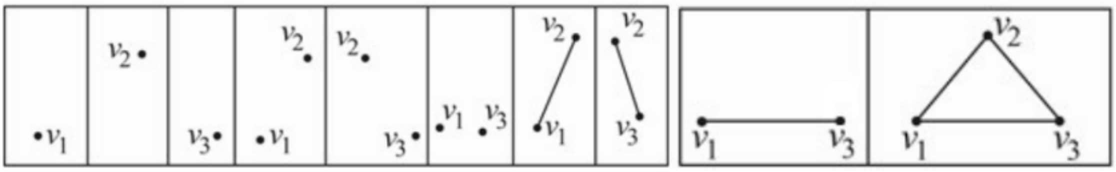
Proof. The nonempty subgraphs are as follows:



□

c.



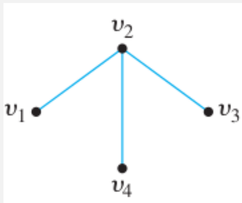


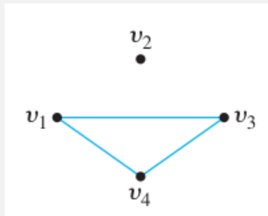
1.23.3 (c)

Proof. The nonempty subgraphs are as follows:

□

Definition: If G is a simple graph, the complement of G, denoted G', is obtained as follows: The vertex set of G' is identical to the vertex set of G. However, two distinct vertices v and w of G' are connected by an edge if, and only if, v and w are not connected by an edge in G.

For example, if G is the graph 

then G' is the graph 

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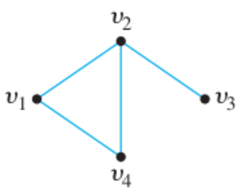
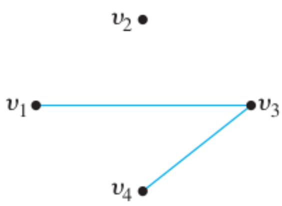
1.24 Exercise 24

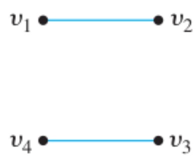
Find the complement of each of the following graphs.

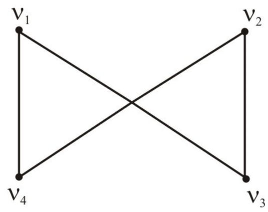
1.24.1 (a)





Proof. □





1.24.2 (b)

Proof. □

1.25 Exercise 25

1.25.1 (a)

Find the complement of the graph K_4, the complete graph on four vertices.

Proof. This is the graph with four vertices and no edges. □

1.25.2 (b)

Find the complement of the graph $K_{3,2}$, the complete bipartite graph on (3, 2) vertices.

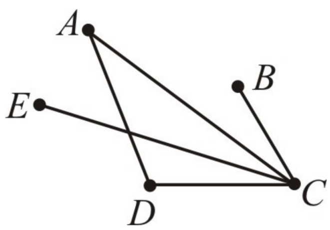
Proof. This is the union of separate graphs K_3 (a triangle) and K_2 (a line). □

1.26 Exercise 26

Suppose that in a group of five people A, B, C, D, and E the following pairs of people are acquainted with each other. A and C, A and D, B and C, C and D, C and E.

1.26.1 (a)

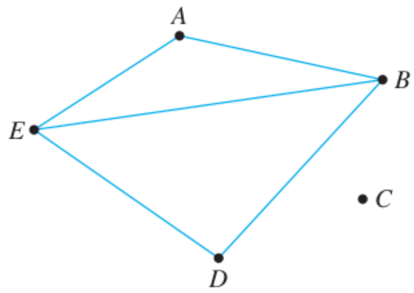
Draw a graph to represent this situation.



Proof. □

1.26.2 (b)

Draw a graph that illustrates who among these five people are not acquainted. That is, draw an edge between two people if, and only if, they are not acquainted.



Proof. □

1.27 Exercise 27

Let G be a simple graph with n vertices. What is the relation between the number of edges of G and the number of edges of the complement G'?

Hint: Consider the graph obtained by taking the vertices and edges of G plus all the edges of G'.

Proof. G' has all the edges that G does not between any two vertices. Therefore, if we put all the edges of G and G' together, it should give us all possible edges between any two vertices, in other words, the complete graph K_n on n vertices. We know K_n has $\frac{n(n-1)}{2}$ edges total, therefore: "the number of edges of G plus the number of edges of G' equals $\frac{n(n-1)}{2}$". □

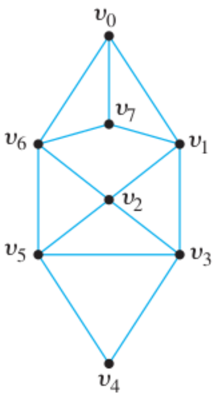
1.28 Exercise 28

Show that at a party with at least two people, there are at least two mutual acquaintances or at least two mutual strangers.

Proof. Since there are at least 2 people at the party, consider person 1 and person 2. Either they know each other (so they are mutually acquainted), or they do not (so they are mutual strangers).

(This problem is stated weirdly! I have to assume that acquaintance is a symmetric relation, given Exercise 26 above: if person A is acquainted with person B, then person B is also acquainted with person A; in other words A cannot know B non-mutually without B knowing A. Otherwise, the acquaintance graph would have to be a directed graph, which has not been yet covered in this section.) □

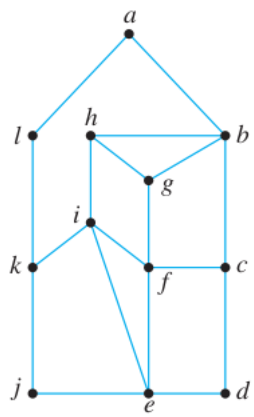
Find Hamiltonian circuits for each of the graphs in 29 and 30.



1.29 Exercise 29

Proof. $v_0v_7v_1v_2v_3v_4v_5v_6v_0$ □

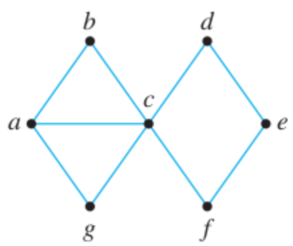
1.30 Exercise 30



Proof. $alkjedcfihgba$ □

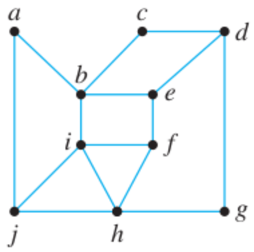
Show that none of the graphs in $31 - 33$ has a Hamiltonian circuit.

1.31 Exercise 31



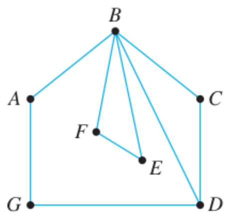
Proof. Argue by contradiction and assume G has a Hamiltonian circuit. By Proposition 10.1.6 G has a subgraph H that is connected, contains every vertex of G, has the same number of edges as vertices, and every vertex has degree 2. Since vertex c has degree 5, in the subgraph H 3 of these edges must be removed. But 4 of these 5 edges cannot be removed, otherwise b, d, g, f would have degree less than 2. This is a contradiction! So G has no Hamiltonian circuit. □

1.32 Exercise 32



Proof. Call the given graph G, and suppose G has a Hamiltonian circuit. Then G has a subgraph H that satisfies conditions (1)–(4) of Proposition 10.1.6. Since the degree of b in G is 4 and every vertex in H has degree 2, two edges incident on b must be removed from G to create H. Edge $\{a, b\}$ cannot be removed because doing so would result in vertex d having degree less than 2 in H. Similar reasoning shows that edge $\{b, c\}$ cannot be removed either. So edges $\{b, i\}$ and $\{b, e\}$ must be removed from G to create H. Because vertex e must have degree 2 in H and because edge $\{b, e\}$ is not in H, both edges $\{e, d\}$ and $\{e, f\}$ must be in H. Similarly, since both vertices c and g must have degree 2 in H, edges $\{c, d\}$ and $\{g, d\}$ must also be in H. But then three edges incident on d, namely, $\{e, d\}, \{c, d\}$, and $\{g, d\}$, must all be in H, which contradicts the fact that vertex d must have degree 2 in H. □

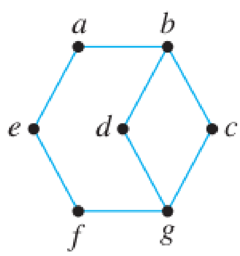
1.33 Exercise 33



Proof. Similar argument to Exercises 31, 32 above. In H every vertex has degree 2. Since B has degree 5, 3 edges must be removed, but 4 of these 5 edges cannot be removed, otherwise A, F, E, C would have degree less than 2. □

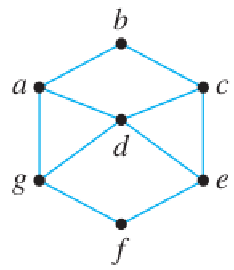
In 34 − 37, find Hamiltonian circuits for those graphs that have them. Explain why the other graphs do not.

1.34 Exercise 34



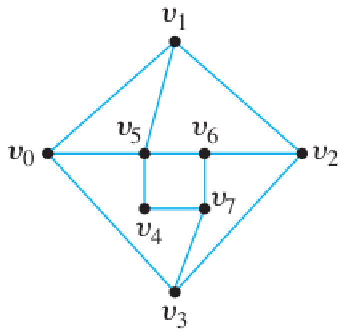
Proof. This graph does not have a Hamiltonian circuit. Argue like Exercises 31, 32, 33 above. In H every vertex has degree 2. Since b has degree 3, it must have one edge removed in H. But none of the 3 edges incident on b can be removed: otherwise one of a, d, c would have degree less than 2. □

1.35 Exercise 35



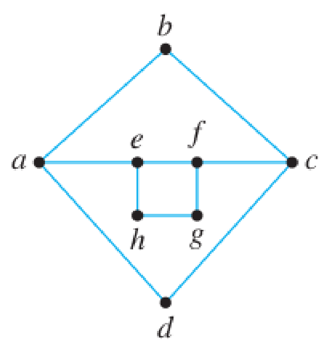
Proof. $adgfecba$ □

1.36 Exercise 36



Proof. $v_4v_5v_1v_0v_3v_2v_6v_7v_4$ □

1.37 Exercise 37

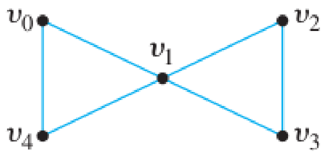


Proof. This graph has no Hamiltonian circuits. Similar argument as before: in H every vertex has degree 2. Since a has degree 3, we must remove one edge. That has to be $\{a, e\}$ otherwise b or d would have degree 1. Similarly c has degree 3, we must remove one edge, it has to be $\{f, c\}$ otherwise b or d would have degree 1. But after removing those 2 edges, we are left with a disconnected graph: a "diamond" $(a - b - c - d)$ with a "square" inside it $(e - f - g - h)$, which contradicts the fact that H must be connected. □

1.38 Exercise 38

Give two examples of graphs that have Euler circuits but not Hamiltonian circuits.

Proof. One example: This graph has an Euler circuit $v_0v_1v_2v_3v_1v_4v_0$ but no Hamiltonian circuit.

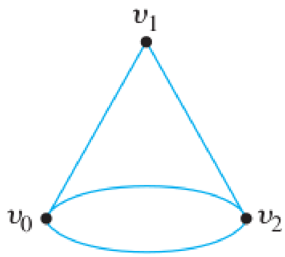


□

1.39 Exercise 39

Give two examples of graphs that have Hamiltonian circuits but not Euler circuits.

Proof. One example: This graph has a Hamiltonian circuit $v_0v_1v_2v_0$ but no Euler circuit.

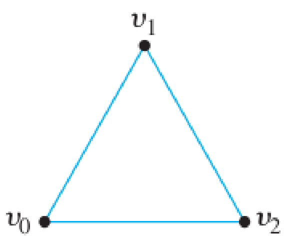


□

1.40 Exercise 40

Give two examples of graphs that have circuits that are both Euler circuits and Hamiltonian circuits.

Proof. One example: The walk $v_0v_1v_2v_0$ is both an Euler circuit and a Hamiltonian circuit for this graph.

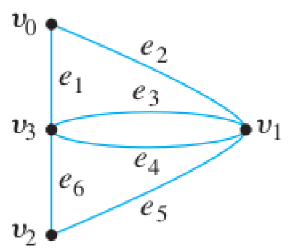


□

1.41 Exercise 41

Give two examples of graphs that have Euler circuits and Hamiltonian circuits that are not the same.

Proof. One example: This graph has the Euler circuit $e_1e_2e_3e_4e_5e_6$ and the Hamiltonian circuit $v_0v_1v_2v_3v_0$. These are not the same.



□

1.42 Exercise 42

	Berlin	Brussels	Düsseldorf	Luxembourg	Munich
Brussels	783				
Düsseldorf	564	223			
Luxembourg	764	219	224		
Munich	585	771	613	517	
Paris	1,057	308	497	375	832

A traveler in Europe wants to visit each of the cities shown on the map exactly once, starting and ending in Brussels. The distance (in kilometers) between each pair of cities is given in the table. Find a Hamiltonian circuit that minimizes the total distance traveled. (Use the map to narrow the possible circuits down to just a few. Then use the table to find the total distance for each of those.)

Proof. Find a Hamiltonian circuit that minimizes the total distance traveled. Some possible Hamiltonian circuits are as follows:

1. Brussels-Luxembourg-Dusseldorf-Paris-Munich-Berlin-Brussels. Total distance traveled by the visitor is $219 + 224 + 497 + 832 + 585 + 783 = 3140$ km.

2. Brussels-Luxembourg-Paris-Munich-Berlin-Dusseldorf-Brussels. Total distance traveled by the visitor is $219 + 375 + 832 + 585 + 564 + 223 = 2798$ km.

3. Brussels-Paris-Luxembourg-Munich-Berlin-Dusseldorf-Brussels. Total distance traveled by the visitor is $308 + 375 + 517 + 585 + 564 + 223 = 2572$ km.

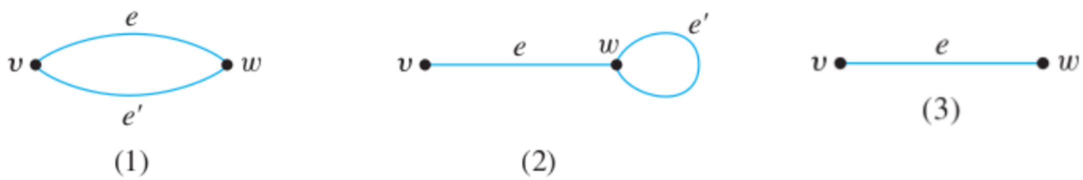
Therefore, the Hamiltonian circuit is Brussels-Paris-Luxembourg-Munich-Berlin-Dusseldorf-Brussels in which total distance is minimum. □

1.43 Exercise 43

1.43.1 (a)

Prove that if a walk in a graph contains a repeated edge, then the walk contains a repeated vertex.

Proof. Suppose G is a graph and W is a walk in G that contains a repeated edge e. Let v and w be the endpoints of e. In case $v = w$, then v is a repeated vertex of W. In case $v \neq w$, then one of the following must occur: (1) W contains two copies of vew or of wev (for instance, W might contain a section of the form $vewe'vew$, as illustrated below);



(2) W contains separate sections of the form vew and wev (for instance, W might contain a section of the form $vewe'wev$, as illustrated below); or (3) W contains a section of the form $vewev$ or of the form $wevew$ (as illustrated below). In cases (1) and (2), both vertices v and w are repeated, and in case (3), one of v or w is repeated. In all cases, there is at least one vertex in W that is repeated. □

1.43.2 (b)

Explain how it follows from part (a) that any walk with no repeated vertex has no repeated edge.

Proof. This statement is the contrapositive of part (a). □

1.44 Exercise 44

Prove Lemma 10.1.1(a): If G is a connected graph, then any two distinct vertices of G can be connected by a path. (You may use the result stated in exercise 43.)

Proof. Suppose G is a connected graph and v and w are any particular but arbitrarily chosen vertices of G. *[We must show that u and v can be connected by a path.]* Since G is connected, there is a walk from v to w. If the walk contains a repeated vertex, then delete the portion of the walk from the first occurrence of the vertex to its next occurrence. (For example, in the walk $ve_1v_2e_5v_7e_6v_2e_3w$, the vertex v_2 occurs twice. Deleting the portion of the walk from one occurrence to the next gives $ve_1v_2e_3w$.) If the resulting walk still contains a repeated vertex, do the above deletion process another time. Then check again for a repeated vertex. Continue in this way until all repeated vertices have been deleted. (This must occur eventually, since the total number of vertices is finite.) The resulting walk connects v to w but has no repeated vertex. By exercise 43(b), it has no repeated edge either. Hence it is a path from v to w. □

1.45 Exercise 45

Prove Lemma 10.1.1(b): If vertices v and w are part of a circuit in a graph G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

Proof. Write down the circuit as:

$$ve_1v_1e_2v_2 \dots v_ne_nwf_1w_1f_2w_2 \dots f_{m-1}w_{m-1}f_mv$$

If one of the edges $e_1, \dots, e_n$ is removed then there is still a trail

$$wf_1w_1f_2w_2 \dots f_{m-1}w_{m-1}f_mv$$

between w and v, written backwards it's a trail from v to w, and similarly if one of the edges $f_1, \dots, f_m$ is removed then there is still a trail

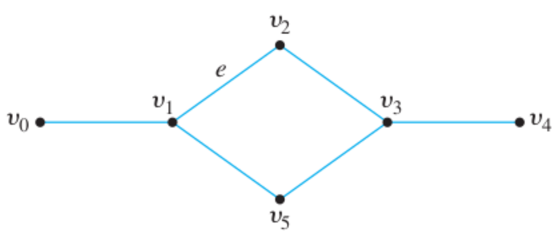
$$ve_1v_1e_2v_2 \dots v_ne_nw$$

from v to w. □

1.46 Exercise 46

Draw a picture to illustrate Lemma 10.1.1(c): If a graph G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

Proof. The graph below contains a circuit, any edge of which can be removed without disconnecting the graph. For instance, if edge e is removed, then the following walk can be used to go from v_1 to v_2: $v_1v_5v_3v_2$.



□

1.47 Exercise 47

Prove that if there is a trail in a graph G from a vertex v to a vertex w, then there is a trail from w to v.

Proof. Simply traverse the trail from v to w but starting at w and going backwards. This is also a trail from w to v (no repeated edges). □

1.48 Exercise 48

If a graph contains a circuit that starts and ends at a vertex v, does the graph contain a simple circuit that starts and ends at v? Why?

Proof. Yes, suppose a graph contains a circuit that starts and ends at a vertex v. Successively delete sections of this circuit as follows. For each repeated vertex w in the circuit (excluding the first vertex if its only repetition is at the end of the circuit, but including the first vertex if it is repeated in the middle of the circuit), there is a section of the circuit of the form $we_1v_1 \ldots e_{n-1}v_{n-1}e_nw$. Replace this section of the circuit by the single vertex w. Because the circuit has finite length, only a finite number of such deletions can be made, after which a simple circuit starting and ending at will remain. □

1.49 Exercise 49

Prove that if there is a circuit in a graph that starts and ends at a vertex v and if w is another vertex in the circuit, then there is a circuit in the graph that starts and ends at w.

Proof. If a circuit starts and ends at v and contains w then the circuit has the form

$$ve_1v_1e_2v_2 \ldots v_ne_nwf_1w_1f_2w_2 \ldots f_{m-1}w_{m-1}f_mv.$$

Then the following is a circuit that starts and ends at w:

$$wf_1w_1f_2w_2 \ldots f_{m-1}w_{m-1}f_mve_1v_1e_2v_2 \ldots v_ne_nw$$

□

1.50 Exercise 50

Let G be a connected graph, and let C be any circuit in G that does not contain every vertex of C. Let G' be the subgraph obtained by removing all the edges of C from G and also any vertices that become isolated when the edges of C are removed. Prove that there exists a vertex v such that v is in both C and G'.

Proof. Let G be a connected graph and let C be a circuit in G. Let G' be the subgraph obtained by removing all the edges of C from G and also any vertices that become isolated when the edges of C are removed. *[We must show that there exists a vertex v such that v is in both C and G'.]* Pick any vertex v of C and any vertex w of G'. Since G is connected, there is a path from v to w (by Lemma 10.1.1(a)):

$$\begin{array}{cccccccccccccccc} v & = & v_0 & e_1 & v_1 & \cdots & e_i & v_i & e_{i+1} & v_{i+1} & \cdots & v_{n-1} & e_n & v_n & = & w \\ \uparrow &&&&&&& \uparrow && \uparrow &&&&&& \uparrow \\ \text{in } C &&&&&&& \text{in } C && \text{not in } C &&&&&& \text{in } G' \end{array}$$

Let i be the largest subscript such that v_i is in C. If $i = n$, then $v_n = w$ is in C and also in G', and we are done. If $i < n$, then v_i is in C and $v_i + 1$ is not in C. This implies that $e_i + 1$ is not in C (for if it were, both endpoints would be in C by definition of circuit). Hence when G' is formed by removing the edges and resulting isolated vertices from G, then $e_i + 1$ is not removed. That means that v_i does not become an isolated vertex, so v_i is not removed either. Hence v_i is in G'. Consequently, v_i is in both C and G' *[as was to be shown]*. □

1.51 Exercise 51

Prove that any graph with an Euler circuit is connected.

Proof. Suppose G is a graph with an Euler circuit. If G has only one vertex, then G is automatically connected. If v and w are any two vertices of G, then v and w each appear at least once in the Euler circuit (since an Euler circuit contains every vertex of the graph). The section of the circuit between the first occurrence of one of v or w and the first occurrence of the other is a walk from one of the two vertices to the other. Since the choice of v and w was arbitrary, given any two vertices in G there is a walk from one to the other. So, by definition, G is connected. □

1.52 Exercise 52

Prove Corollary 10.1.5.

Corollary: Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler trail from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

Proof. ($\implies$) Assume there is an Euler trail from v to w.

Then this trail passes through every vertex of G at least once and traverses every edge of G exactly once, therefore G is connected. Adding an extra edge e between v and w to this trail gives us an Euler circuit. Then every vertex in this new graph has even degree. Removing e shows that v and w have odd degree and all other vertices have even degree in the original graph G.

($\impliedby$) Assume G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

Add an extra edge e between v and w. Now every vertex in this new graph has even degree, and it is connected. So it has an Euler circuit by Theorem 10.1.4. This Euler circuit contains e. Removing e from the circuit gives us an Euler trail from v to w in the original graph G. □

1.53 Exercise 53

For what values of n does the complete graph K_n with n vertices have (a) an Euler circuit? (b) a Hamiltonian circuit? Justify your answers.

Proof. K_n has a circuit if and only if $n \geq 3$.

(a) Euler circuits require a connected graph where every vertex has even degree. Every vertex in K_n has degree $n-1$. So this means that for all n, K_n has an Euler circuit if and only if $n \geq 3$ and n is odd.

(b) Hamiltonian circuits require a connected subgraph H with the same number of edges and vertices, which contains every vertex of G, such that every vertex has degree 2 in H. This is possible in every K_n with $n \geq 3$. If the vertices are $v_1, \ldots, v_n$ then let H be $v_1 v_2 \cdots v_{n-1} v_n v_1$. This is a connected circuit with n vertices and n edges, where every vertex has degree 2. □

1.54 Exercise 54

For what values of m and n does the complete bipartite graph on (m, n) vertices have (a) an Euler circuit? (b) a Hamiltonian circuit? Justify your answers.

Proof. (a) Euler circuit requires every vertex to have even degree, and in $K_{m,n}$ the left m vertices all have degree n and the right n vertices all have degree m. So $K_{m,n}$ has an Euler circuit if and only if both m and n are even (and ≥ 2).

(b) Hamiltonian circuit requires every vertex to have degree 2. The total degree of the left m vertices is $2m$, so there are $2m$ edges going from the left m vertices to the right n vertices, covering a total degree of $2n$ on the right. This forces $2m = 2n$ so $m = n$. So $K_{m,n}$ has a Hamiltonian circuit if and only if $m = n$ (and ≥ 2). □

1.55 Exercise 55

What is the maximum number of edges a simple disconnected graph with n vertices can have? Prove your answer.

Proof. Simple means no loops or parallel edges. Leave out 1 vertex by itself so it's not connected to any of the other $n-1$. Now we try to find the maximum number of edges we can have among the $n-1$ vertices without introducing any loops. We can add $n-2$ edges between them in a chain. Adding 1 more edge to this would create a loop. So the maximum is $n-2$. □

1.56 Exercise 56

1.56.1 (a)

Prove that if G is any bipartite graph, then every circuit in G has an even number of edges.

Proof. Let $G = (V_1, V_2)$ be a bipartite graph where V_1 is the set of vertices on the left and V_2 is the set of vertices on the right. Consider any circuit. Say, it starts and ends at a vertex $v \in V_1$. Since v can only be connected to vertices in V_2, one edge brings us to V_2. Similarly, since vertices in V_2 can only be connected to vertices in V_1, the second edge of the circuit brings us back to V_1. And so on. Any odd number of edges starting from v will end in V_2, unable to complete the circuit back at v. Hence, no circuit can have an odd number of edges, but must have an even number of edges. $\square$

1.56.2 (b)

Prove that if G is any graph with at least two vertices and if G does not have a circuit with an odd number of edges, then G is bipartite.

Hint: Divide the proof into three parts. (1) Show that if G is any graph containing a closed walk with an odd number of edges, then G contains a circuit with an odd number of edges. (2) Show that if G is any connected graph that does not have a circuit with an odd number of edges, then G is bipartite. (3) Show that if G is any graph with at least two vertices and is such that G does not have a circuit with an odd number of edges, then G is bipartite.

Proof. (1) Let W be a closed walk with an odd number of edges. If no vertices are repeated in W then W is already a circuit with an odd number of edges. So suppose W has a repeated vertex. Write $W = v_1 v_2 \cdots v_j v_{j+1} \cdots v_k v_{k+1} \cdots v_n v_1$ where $v_j = v_k$ is the repeated vertex. Now W is the union of two closed walks: $v_1 v_2 \cdots v_j v_{k+1} \cdots v_n$ and $v_j v_{j+1} \cdots v_k$. Since the number of edges of W is odd, and the sum of the numbers of edges of these two walks equals the number of edges of W, one of these two walks must have an odd number of edges and the other must have an even number of edges (because odd + odd = even, even + even = even, and only odd + even = odd). Take the walk among the two walks that has the odd number of edges. Either this smaller walk is a circuit, or it also has a repeated vertex. Then repeat the procedure to obtain an even smaller closed walk with an odd number of edges. The process eventually terminates since W is finite. So there is a circuit with an odd number of edges.

(2) Assume G is connected and has no circuit with an odd number of edges. Let v be a vertex in G. Since G is connected, there is a walk between v and any other vertex. For any vertex w let $d(w, v)$ denote the length of the shortest walk between v and w. Let's define a bipartition of G as follows:

$$V_1 = \{w \in V(G) \mid d(w, v) \text{ is odd}\}, \;\; V_2 = \{w \in V(G) \mid d(w, v) \text{ is even}\}$$

We need to prove (V_1, V_2) is a bipartition of G. Argue by contradiction and assume $x, y \in V_1$ and assume there is an edge e between x and y. Now there is a closed walk with an odd number of edges from v to v as follows: start with a walk from v to x which has odd length (since $x \in V_1$), followed by e (add 1), followed by the walk from y to v which has odd length. This is a closed walk of odd length, which, by (1), implies there is a circuit of odd length from v to v, which is a contradiction to the assumption that G has no circuits of odd length. Similar argument for V_2.

(3) Assume G is any graph with at least two vertices and has no circuit with an odd number of edges. Consider all the connected components $G_1, \dots, G_n$ of G. By (2) each G_i is bipartite: $G_i = (V_{1,i}, V_{2,i})$. Then we can create a bipartition of G: $(\bigcup_{i=1}^n V_{1,i}, \bigcup_{i=1}^n V_{2,i})$. So G is bipartite. □

1.57 Exercise 57

An alternative proof for Theorem 10.1.3 has the following outline. Suppose G is a connected graph in which every vertex has even degree. Suppose the trail $C : v_1 e_1 v_2 e_2 v_3 \dots e_n v_{n+1}$ has maximum length in G. That is, C has at least as many vertices and edges as any other trail in G. First derive a contradiction from the assumption that $v_1 \neq v_{n+1}$. Next let H be the subgraph of G that contains all the vertices and edges in C. Then derive a contradiction from the assumption that $H \neq G$. Show that H contains every vertex of G, and show that H contains every edge of G.

Proof. Argue by contradiction and assume $v_1 \neq v_{n+1}$. Note that v_1 is connected to only one other vertex, v_2, on C, and v_{n+1} is connected to only one other vertex, v_n, in C. Since both v_1 and v_{n+1} have even degree, they have degree at least 2. So they are connected to at least one more vertex. Then there is a trail in G longer than C, contradiction. So $v_1 = v_{n+1}$.

Let H be the subgraph of G that contains all the vertices and edges in C. Argue by contradiction and assume $H \neq G$. First assume G has a vertex w that H does not. Since w is different than all $v_1, \dots, v_n$, w is not on C. Since G is connected there is a trail from v_1 to w. Since w is not on C, this trail is not part of C. Adding this trail to C gives us a longer trail, contradiction. So H contains every vertex of G.

Now assume G has an edge e that H does not. This edge must be connecting two of the vertices $v_1, \dots, v_n$. Say v_i and v_j. But then v_i and v_j have degree 3, contradiction. So H contains every edge of G. □

2 Exercise Set 10.2

2.1 Exercise 1

Find real numbers a, b, and c such that the following are true.

2.1.1 (a)

$$\begin{bmatrix} a+b & a-c \\ c & b-a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$$

Proof. $c = -1, a = -1, b = 2$ □

2.1.2 (b)

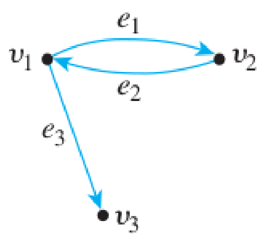
$$\begin{bmatrix} 2a & b+c \\ c-a & 2b-a \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$$

Proof. $a = 2, c = 3, b = 0$ $\square$

2.2 Exercise 2

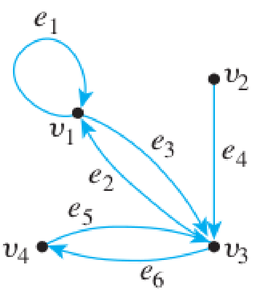
Find the adjacency matrices for the following directed graphs.

2.2.1 (a)



Proof.
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

2.2.2 (b)



Proof.

	v_1	v_2	v_3	v_4
v_1	1	0	1	0
v_2	0	0	1	0
v_3	1	0	0	1
v_4	0	0	1	0

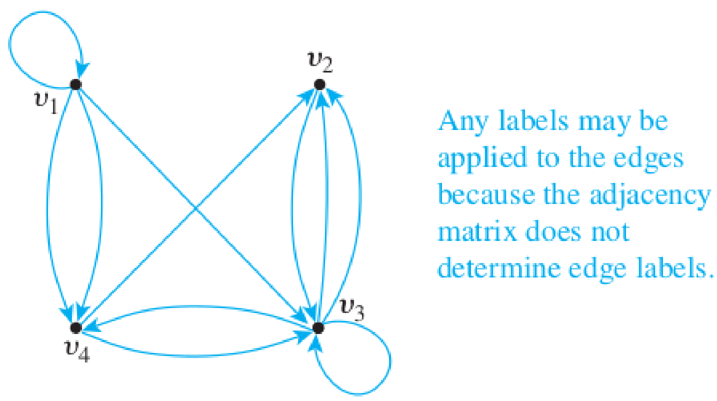
2.3 Exercise 3

Find directed graphs that have the following adjacency matrices:

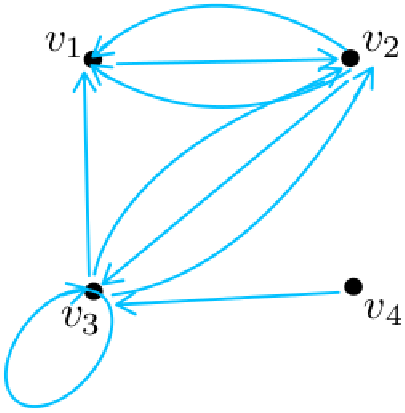
2.3.1 (a)

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Proof.



2.3.2 (b)



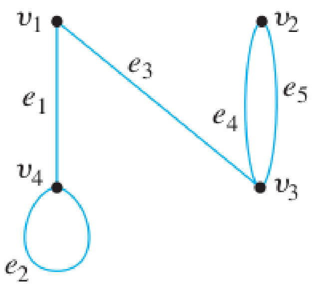
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Proof.

2.4 Exercise 4

Find adjacency matrices for the following (undirected) graphs.

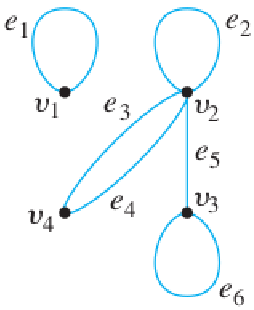
2.4.1 (a)



Proof.

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

2.4.2 (b)



Proof.

$$\begin{array}{ccccc} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 1 & 0 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 2 \\ v_3 & 0 & 1 & 1 & 0 \\ v_4 & 0 & 2 & 0 & 0 \end{array}$$

2.4.3 (c)

K_4, the complete graph on four vertices *Proof.*

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2.4.4 (d)

$K_{2,3}$, the complete bipartite graph on (2, 3) vertices *Proof.*

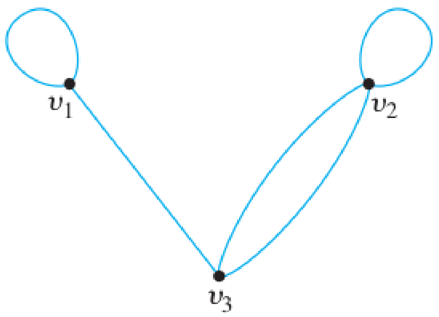
$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & w_1 & w_2 \\ \hline v_1 & 0 & 0 & 0 & 1 & 1 \\ v_2 & 0 & 0 & 0 & 1 & 1 \\ v_3 & 0 & 0 & 0 & 1 & 1 \\ w_1 & 1 & 1 & 1 & 0 & 0 \\ w_2 & 1 & 1 & 1 & 0 & 0 \end{array}$$

2.5 Exercise 5

Find graphs that have the following adjacency matrices.

2.5.1 (a)

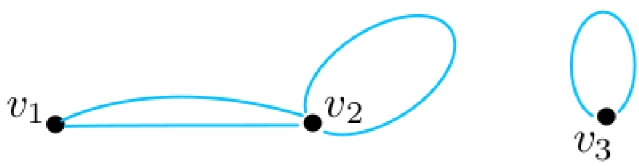
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$ *Proof.*



Any labels may be applied to the edges because the adjacency matrix does not determine edge labels.

2.5.2 (b)

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ *Proof.*



2.6 Exercise 6

The following are adjacency matrices for graphs. In each case determine whether the graph is connected by analyzing the matrix without drawing the graph.

2.6.1 (a)

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Proof. From the first row, we see that v_1 is adjacent to v_2 and v_3. Therefore the graph is connected. □

2.6.2 (b)

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Proof. From the first two rows, we can see there are 2 edges between v_1 and v_2. From the last two rows, we can see that v_3 and v_4 are adjacent. However, there are no edges between the $v_1 + v_2$ connected component and the $v_3 + v_4$ connected component. Thus the graph is disconnected. □

2.7 Exercise 7

Suppose that for every positive integer i, all the entries in the ith row and ith column of the adjacency matrix of a graph are 0. What can you conclude about the graph?

Proof. The ith row and column defines the edges between v_i and the other vertices. If the entries are all 0, that means there are no edges between v_i and the other vertices. So the graph is not connected; v_i is disconnected from the rest of the graph. □

2.8 Exercise 8

Find each of the following products.

2.8.1 (a)

$$\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Proof. $2 \cdot 1 + (-1) \cdot 3 = -1$ □

2.8.2 (b)

$$\begin{bmatrix} 4 & -1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Proof. $4 \cdot 1 + (-1) \cdot 2 + 7 \cdot 0 = 2$ □

2.9 Exercise 9

Find each of the following products.

2.9.1 (a)

$$\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$$

Proof. $\begin{bmatrix} 3 & -3 & 12 \\ 1 & -5 & 2 \end{bmatrix}$ □

2.9.2 (b)

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & -4 \\ -2 & 2 \end{bmatrix}$$

Proof. $\begin{bmatrix} 0 & 8 \\ -5 & 4 \end{bmatrix}$ □

2.9.3 (c)

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$$

Proof. $\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$ □

2.9.4 (d)

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^2$$

Proof. $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ □

2.10 Exercise 10

Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 0 & -2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix}$. For each of the following, determine whether the indicated product exists, and compute it if it does.

2.10.1 (a)

AB

Proof. No product. (**A** has three columns, and **B** has two rows.) □

2.10.2 (b)

BA

Proof. $\mathbf{BA} = \begin{bmatrix} -2 & -2 & 2 \\ 1 & -5 & 2 \end{bmatrix}$ □

2.10.3 (c)

$\mathbf{A}^2$

Proof. No product. (**A** has different numbers of rows and columns.) □

2.10.4 (d)

BC

Proof. No product. (**B** has 2 columns, **C** has 3 rows.) □

2.10.5 (e)

CB

Proof. $\mathbf{CB} = \begin{bmatrix} -2 & -6 \\ -5 & 3 \\ -2 & 0 \end{bmatrix}$ □

2.10.6 (f)

$\mathbf{B}^2$

Proof. $\mathbf{B}^2 = \begin{bmatrix} 4 & 0 \\ 1 & 9 \end{bmatrix}$ □

2.10.7 (g)

$\mathbf{B}^3$

Proof. $\mathbf{B}^3 = \begin{bmatrix} -8 & 0 \\ 7 & 27 \end{bmatrix}$ □

2.10.8 (h)

$\mathbf{C}^2$

Proof. No product. (**C** has different numbers of rows and columns.) □

2.10.9 (i)

AC

Proof. $\mathbf{AC} = \begin{bmatrix} 2 & -1 \\ -5 & -2 \end{bmatrix}$ □

2.10.10 (j)

CA

Proof. $\mathbf{CA} = \begin{bmatrix} 0 & 4 & -2 \\ 3 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix}$ □

2.11 Exercise 11

Give an example different from that in the text to show that matrix multiplication is not commutative. That is, find 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB}$ and $\mathbf{BA}$ both exist but $\mathbf{AB} \neq \mathbf{BA}$.

Proof. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}, \mathbf{AB} = \begin{bmatrix} 8 & 16 \\ 18 & 36 \end{bmatrix}, \mathbf{BA} = \begin{bmatrix} 14 & 20 \\ 21 & 30 \end{bmatrix}, \mathbf{AB} \neq \mathbf{BA}.$ □

2.12 Exercise 12

Let O denote the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Find 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \neq \mathrm{O}$ and $\mathbf{B} \neq \mathrm{O}$ but $\mathbf{AB} = \mathrm{O}$.

Proof. One among many possible examples is $\mathbf{A} = \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. □

2.13 Exercise 13

Let O denote the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Find 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \neq \mathbf{B}$, $\mathbf{B} \neq \mathrm{O}$ and $\mathbf{AB} \neq \mathrm{O}$ but $\mathbf{BA} = \mathrm{O}$.

Proof. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\mathbf{A} \neq \mathbf{B} \neq \mathrm{O}$, $\mathbf{AB} = \mathbf{B} \neq \mathrm{O}$, but $\mathbf{BA} = \mathrm{O}$. □

In 14 – 18, assume the entries of all matrices are real numbers.

2.14 Exercise 14

Prove that if $\mathbf{I}$ is the $m \times m$ identity matrix and $\mathbf{A}$ is any $m \times n$ matrix, then $\mathbf{IA} = \mathbf{A}$.

Proof. We need to show that for all $1 \le i \le m$ and all $1 \le j \le m$, $(\mathbf{IA})_{ij} = \mathbf{A}_{ij}$. In other words, we need to show that $\mathbf{IA}$ and $\mathbf{A}$ have the same entries.

Denote the entries of $\mathbf{I}$ by δ_{ik}. The identity matrix has 1's only on the diagonal and 0's elsewhere. Thus $\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$. Now by the multiplication formula, $(\mathbf{IA})_{ij} = \sum_{k=1}^{m} \mathbf{I}_{ik}\mathbf{A}_{kj} = \sum_{k=1}^{m} \delta_{ik}\mathbf{A}_{kj} = \mathbf{A}_{ij}$ because all the terms in the sum are 0 except when $k = i$, *[as was to be shown]*. □

2.15 Exercise 15

Prove that if $\mathbf{A}$ is an $m \times m$ symmetric matrix, then $\mathbf{A^2}$ is symmetric.

Proof. Suppose $\mathbf{A}$ is an $m \times m$ symmetric matrix. Then for all integers i and j with $1 \le i, j \le m$,

$$\mathbf{A}^2_{ij} = \sum_{k=1}^{m} \mathbf{A}_{ik}\mathbf{A}_{kj} \text{ and } \mathbf{A}^2_{ji} = \sum_{k=1}^{m} \mathbf{A}_{jk}\mathbf{A}_{ki}$$

But since $\mathbf{A}$ is symmetric, $\mathbf{A}_{ik} = \mathbf{A}_{ki}$ and $\mathbf{A}_{kj} = \mathbf{A}_{jk}$ for all i, j, and k, and thus $\mathbf{A}_{ik}\mathbf{A}_{kj} = \mathbf{A}_{jk}\mathbf{A}_{ki}$ *[by the commutative law for multiplication of real numbers]*. Hence $(\mathbf{A}^2)_{ij} = (\mathbf{A}^2)_{ji}$ for all integers i and j with $1 \le i, j \le m$. □

2.16 Exercise 16

Prove that matrix multiplication is associative: If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any $m \times k, k \times r$, and $r \times n$ matrices, respectively, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. (*Hint:* Summation notation is helpful.)

Proof. $\mathbf{AB}$ is an $m \times r$ matrix, where the ij^{th} entry is $\mathbf{AB}_{ij} = \sum_{x=1}^{k} \mathbf{A}_{ix}\mathbf{B}_{xj}$.

$\mathbf{BC}$ is an $k \times n$ matrix, where the ij^{th} entry is $\mathbf{BC}_{ij} = \sum_{y=1}^{r} \mathbf{B}_{iy}\mathbf{C}_{yj}$.

$(\mathbf{AB})\mathbf{C}$ is an $m \times n$ matrix, where the ij^{th} entry is $[(\mathbf{AB})\mathbf{C}]_{ij} = \sum_{y=1}^{r} (\mathbf{AB})_{iy}\mathbf{C}_{yj} =$

$$\sum_{y=1}^{r}\left(\sum_{x=1}^{k} \mathbf{A}_{ix}\mathbf{B}_{xy}\right)\mathbf{C}_{yj} = \sum_{y=1}^{r}\left(\sum_{x=1}^{k} \mathbf{A}_{ix}\mathbf{B}_{xy}\mathbf{C}_{yj}\right) = \sum_{x=1}^{k} \mathbf{A}_{ix}\left(\sum_{y=1}^{r} \mathbf{B}_{xy}\mathbf{C}_{yj}\right)$$

(notice we are allowed to switch the order of the summation since this is a finite sum).

$\mathbf{A}(\mathbf{BC})$ is an $m \times n$ matrix, where the ij^{th} entry is $[\mathbf{A}(\mathbf{BC})]_{ij} = \sum_{x=1}^{k} \mathbf{A}_{ix}(\mathbf{BC})_{xj} =$

$$\sum_{x=1}^{k} \mathbf{A}_{ix}\left(\sum_{y=1}^{r} \mathbf{B}_{xy}\mathbf{C}_{yj}\right)$$

[as was to be shown.] □

2.17 Exercise 17

Use mathematical induction and the result of exercise 16 to prove that if $\mathbf{A}$ is any $m \times m$ matrix, then $\mathbf{A^n A} = \mathbf{A A^n}$ for each integer $n \geq 1$.

Proof. Let the property $P(n)$ be the equation $\mathbf{A}^n\mathbf{A} = \mathbf{A}\mathbf{A}^n$.

Show that $P(1)$ **is true:** We must show that $\mathbf{A}^1\mathbf{A} = \mathbf{A}\mathbf{A}^1$. But this is true because $\mathbf{A}^1 = \mathbf{A}$ and $\mathbf{AA} = \mathbf{AA}$.

Show that for every integer $k \geq 1$**, if** $P(k)$ **is true, then** $P(k+1)$ **is true:** Let k be any integer such that $k \geq 1$, and suppose that $\mathbf{A}^k\mathbf{A} = \mathbf{A}\mathbf{A}^k$. *[This is the inductive hypothesis.]* We must show that $\mathbf{A}^{k+1}\mathbf{A} = \mathbf{A}\mathbf{A}^{k+1}$. But

$$\begin{aligned}
\mathbf{A}^{k+1}\mathbf{A} &= (\mathbf{A}\mathbf{A}^k)\mathbf{A} && \text{by definition of matrix power} \\
&= \mathbf{A}(\mathbf{A^k}\mathbf{A}) && \text{by exercise 16} \\
&= \mathbf{A}(\mathbf{A}\mathbf{A}^k) && \text{by inductive hypothesis} \\
&= \mathbf{A}\mathbf{A}^{k+1} && \text{by definition of matrix power}
\end{aligned}$$

□

2.18 Exercise 18

Use mathematical induction to prove that if $\mathbf{A}$ is an $m \times m$ symmetric matrix, then for any integer $n \geq 1$, $\mathbf{A}^n$ is also symmetric.

Proof. Let the property $P(n)$ be the statement "$\mathbf{A}^n$ is symmetric".

Show that $P(1)$ **is true:** We must show that $\mathbf{A}^1$ is symmetric. But this is true because $\mathbf{A}^1 = \mathbf{A}$ and $\mathbf{A}$ is symmetric by assumption.

Show that for every integer $k \geq 1$**, if** $P(k)$ **is true, then** $P(k+1)$ **is true:** Let k be any integer such that $k \geq 1$, and suppose that $\mathbf{A}^k$ is symmetric. *[This is the inductive hypothesis.]* We must show that $\mathbf{A}^{k+1}$ is symmetric.

Since $\mathbf{A}^{k+1} = \mathbf{A}\mathbf{A}^k$, the ij^{th} entry in $\mathbf{A}^{k+1}$ is, by definition of matrix multiplication $\sum_{l=1}^{m} \mathbf{A}_{il}(\mathbf{A}^k)_{lj}$. Similarly the ji^{th} entry in $\mathbf{A}^{k+1}$ is $\sum_{l=1}^{m} \mathbf{A}_{jl}(\mathbf{A}^k)_{li}$. We need to show these two are equal.

By exercise 17, $\mathbf{A}\mathbf{A}^k = \mathbf{A}^k\mathbf{A}$. This means that the ij^{th} entries in both of these products are the same. So $\sum_{l=1}^{m} \mathbf{A}_{il}(\mathbf{A}^k)_{lj}$ is the same as $\sum_{l=1}^{m} (\mathbf{A}^k)_{il}\mathbf{A}_{lj}$. By the inductive hypothesis $(\mathbf{A}^k)_{il} = (\mathbf{A}^k)_{li}$, and since $\mathbf{A}$ is symmetric, $\mathbf{A}_{lj} = \mathbf{A}_{jl}$. Putting all these facts together, we see that $\sum_{l=1}^{m} \mathbf{A}_{il}(\mathbf{A}^k)_{lj}$ is the same as $\sum_{l=1}^{m} \mathbf{A}_{jl}(\mathbf{A}^k)_{li}$, *[as was to be shown.]* □

2.19 Exercise 19

2.19.1 (a)

Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^2$ and $\mathbf{A}^3$.

Proof. $\mathbf{A}^2 = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix}$ □

2.19.2 (b)

Let G be the graph with vertices v_1, v_2, and v_3 and with A as its adjacency matrix. Use the answers to part (a) to find the number of walks of length 2 from v_1 to v_3 and the number of walks of length 3 from v_1 to v_3. Do not draw G to solve this problem.

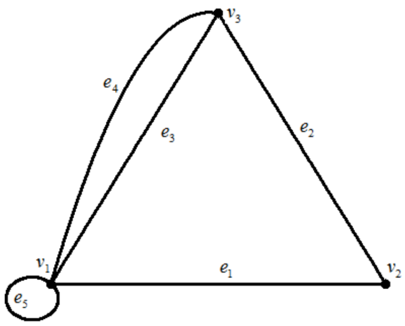
Proof. By Theorem 10.2.2, there are 3 walks of length 2 and 15 walks of length 3. (top-right or bottom-left entry) □

2.19.3 (c)

Examine the calculations you performed in answering part (a) to find five walks of length 2 from v_3 to v_3. Then draw G and find the walks by visual inspection.

Proof. In $\mathbf{A}$ looking at the third row we see there are 2 edges between v_1 and v_3, call them e_3, e_4, and 1 edge between v_2 and v_3, call it e_2. We can use these to form length 2 walks from v_3 to v_3: $v_3e_3v_1e_3v_3$, $v_3e_3v_1e_4v_3$, $v_3e_4v_1e_3v_3$, $v_3e_4v_1e_4v_3$, $v_3e_2v_2e_2v_3$.

If we draw G we can see these walks:



□

2.20 Exercise 20

The following is an adjacency matrix for a graph:

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

2.20.1 (a)

How many walks of length 2 are there from v_2 to v_3?

Proof. $\mathbf{A}^2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 3 \\ 2 & 2 & 6 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$. 2 since $(\mathbf{A}^2)_{23} = 2$ ☐

2.20.2 (b)

How many walks of length 2 are there from v_3 to v_4?

Proof. 3 since $(\mathbf{A}^2)_{34} = 3$ ☐

2.20.3 (c)

How many walks of length 3 are there from v_1 to v_4?

Proof. $\mathbf{A}^3 = \begin{bmatrix} 4 & 8 & 8 & 6 \\ 8 & 9 & 17 & 11 \\ 8 & 17 & 14 & 11 \\ 6 & 11 & 11 & 9 \end{bmatrix}$. 6 since $(\mathbf{A}^3)_{14} = 6$ ☐

2.20.4 (d)

How many walks of length 3 are there from v_2 to v_3?

Proof. 17 since $(\mathbf{A}^3)_{23} = 17$ ☐

2.21 Exercise 21

Let $\mathbf{A}$ be the adjacency matrix for K_3, the complete graph on three vertices. Use mathematical induction to prove that for each positive integer n, all the entries along the main diagonal of $\mathbf{A}^n$ are equal to each other and all the entries that do not lie along the main diagonal are equal to each other.

Proof. Let $P(n)$ be the statement "all the entries along the main diagonal of $\mathbf{A}^n$ are equal to each other and all the entries that do not lie along the main diagonal are equal to each other."

Show that $P(1)$ is true: $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, so $P(1)$ is true.

Show that for any integer $k \geq 1$ if $P(k)$ is true then $P(k+1)$ is true: Assume $k \geq 1$ is any integer and assume $P(k)$ is true. So all the entries along the main diagonal of $\mathbf{A}^k$ are equal to each other, say, to some constant B, and all the entries that do not lie along the main diagonal of $\mathbf{A}^k$ are equal to each other, say, to some constant C. *[This is the inductive hypothesis.]*

[We want to show that all the entries along the main diagonal of $\mathbf{A}^{k+1}$ are equal to each other, and all the entries that do not lie along the main diagonal of $\mathbf{A}^{k+1}$ are equal to each other.]

By definition of matrix multiplication, $\mathbf{A}^{k+1}_{ij} = \displaystyle\sum_{l=1}^{3} \mathbf{A}_{il}\mathbf{A}^k_{lj}$ for all $1 \leq i, j \leq 3$.

The diagonal entries of $\mathbf{A}^{k+1}$ are $\mathbf{A}^{k+1}_{11} = \displaystyle\sum_{l=1}^{3} \mathbf{A}_{1l}\mathbf{A}^k_{l1}$, $\mathbf{A}^{k+1}_{22} = \displaystyle\sum_{l=1}^{3} \mathbf{A}_{2l}\mathbf{A}^k_{l2}$, and

$\mathbf{A}^{k+1}_{33} = \displaystyle\sum_{l=1}^{3} \mathbf{A}_{3l}\mathbf{A}^k_{l3}$.

Let's look closer at $\mathbf{A}^{k+1}_{11}$: $\mathbf{A}^{k+1}_{11} = \mathbf{A}_{11}\mathbf{A}^k_{11} + \mathbf{A}_{12}\mathbf{A}^k_{21} + \mathbf{A}_{13}\mathbf{A}^k_{31}$. We know $\mathbf{A}_{11} = 0$, $\mathbf{A}_{12} = \mathbf{A}_{13} = 1$. So $\mathbf{A}^{k+1}_{11} = \mathbf{A}^k_{21} + \mathbf{A}^k_{31}$. By the inductive hypothesis, $\mathbf{A}^k_{21} = \mathbf{A}^k_{31} = C$, so $\mathbf{A}^{k+1}_{11} = 2C$.

Doing the same calculation for $\mathbf{A}^{k+1}_{22} = \mathbf{A}_{21}\mathbf{A}^k_{12} + \mathbf{A}_{22}\mathbf{A}^k_{22} + \mathbf{A}_{23}\mathbf{A}^k_{32} = \mathbf{A}^k_{12} + \mathbf{A}^k_{32} = 2C$.

Doing the same calculation for $\mathbf{A}^{k+1}_{33} = \mathbf{A}_{31}\mathbf{A}^k_{13} + \mathbf{A}_{32}\mathbf{A}^k_{23} + \mathbf{A}_{33}\mathbf{A}^k_{33} = \mathbf{A}^k_{13} + \mathbf{A}^k_{23} = 2C$.

So all the diagonal entries of $\mathbf{A}^{k+1}$ are equal to $2C$.

Similarly, let's take a look at one of the off-diagonal entries. For example $\mathbf{A}^{k+1}_{12} = \displaystyle\sum_{l=1}^{3} \mathbf{A}_{1l}\mathbf{A}^k_{l2} = \mathbf{A}_{11}\mathbf{A}^k_{12} + \mathbf{A}_{12}\mathbf{A}^k_{22} + \mathbf{A}_{13}\mathbf{A}^k_{32} = \mathbf{A}^k_{22} + \mathbf{A}^k_{32} = B + C$.

Another one is $\mathbf{A}^{k+1}_{23} = \displaystyle\sum_{l=1}^{3} \mathbf{A}_{2l}\mathbf{A}^k_{l3} = \mathbf{A}_{21}\mathbf{A}^k_{13} + \mathbf{A}_{22}\mathbf{A}^k_{23} + \mathbf{A}_{23}\mathbf{A}^k_{33} = \mathbf{A}^k_{13} + \mathbf{A}^k_{33} = C + B$.

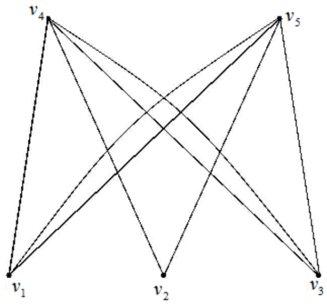
We can repeat similar calculations for the other 4 off-diagonal entries with the same result. So, all the off- diagonal entries of $\mathbf{A}^{k+1}$ are equal to $B + C$. □

2.22 Exercise 22

2.22.1 (a)

Draw a graph that has $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}$ as its adjacency matrix. Is this graph bipartite?

Hint: If G is bipartite, then its vertices can be partitioned into two sets V_1 and V_2 so that no vertices in V_1 are connected to each other by an edge and no vertices in V_2 are connected to each other by an edge. Label the vertices in V_1 as $v_1, v_2, \ldots, v_k$ and label the vertices in V_2 as $v_{k+1}, v_{k+2}, \ldots, v_n$. Now look at the matrix of G formed according to the given vertex labeling.



Proof. Yes, G is bipartite: label the vertices v_1, v_2, v_3, v_4, v_5 and let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$. From the adjacency matrix we can see that v_1, v_2, v_3 have no edges between them, and v_4, v_5 have no edges between them. □

Definition: Given an $m \times n$ matrix $\mathbf{A}$ whose ijth entry is denoted a_{ij}, the **transpose of A** is the matrix $\mathbf{A}^t$ whose ijth entry is a_{ji}, for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Note that the first row of $\mathbf{A}$ becomes the first column of $\mathbf{A}^t$, the second row of $\mathbf{A}$ becomes the second column of $\mathbf{A}^t$, and so forth. For instance, if $\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ then $\mathbf{A}^t = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$.

2.22.2 (b)

Show that a graph with n vertices is bipartite if, and only if, for some labeling of its vertices, its adjacency matrix has the form

$$\begin{bmatrix} O & \mathbf{A} \\ \mathbf{A}^t & O \end{bmatrix}$$

where $\mathbf{A}$ is a $k \times (n-k)$ matrix for some integer k such that $0 < k < n$, the top left O represents a $k \times k$ matrix all of whose entries are 0, $\mathbf{A}^t$ is the transpose of A, and the bottom right O represents an $(n-k) \times (n-k)$ matrix all of whose entries are 0.

Proof. **$(\Longleftarrow)$:** If the adjacency matrix of G has the above form, then due to the $k \times k$ matrix O on the top-left, there are no edges among $v_1, \ldots, v_k$; and due to the $(n-k) \times (n-k)$ matrix O on the bottom-right there are no edges among $v_{k+1}, \ldots, v_n$. So let $V_1 = \{v_1, \ldots, v_k\}, V_2 = \{v_{k+1}, \ldots, v_n\}$ and therefore G is bipartite because (V_1, V_2) is a bipartition of G.

$(\Longrightarrow)$: If G is bipartite then its vertices can be split into two disjoint sets $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$ where there are no edges among the vertices of V_1 and there are no edges among the vertices of V_2. So we can order the vertices like that,

$v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$, and the adjacency matrix with respect to this ordering of the vertices will have zeros in the top-left $k \times k$ and in the bottom-right $(n-k) \times (n-k)$ sub-matrices. The other two sub-matrices will be transposes of each other, since, if there is an edge between $v_i \in V_1$ and $v_j \in V_2$ where $1 \le i \le k < j \le n$, this will be in the ijth entry in the bottom-left sub-matrix, and the same edge can be viewed as an edge between v_j and v_i, so it will be in the jith entry in the top-right sub-matrix, which is the transpose of the bottom-left sub-matrix. $\square$

2.23 Exercise 23

2.23.1 (a)

Let G be a graph with n vertices, and let v and w be distinct vertices of G. Prove that if there is a walk from v to w, then there is a walk from v to w that has length less than or equal to $n-1$.

Proof. If the walk contains repeated vertices, we can remove all the edges between the repeated vertices (which are circuits), to obtain a walk that has no repeated vertices. Now this circuit-free walk can use at most all the vertices of G. Since there are n vertices, this walk uses up at most $n-1$ edges. $\square$

2.23.2 (b)

If $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are any $m \times n$ matrices, the matrix $\mathbf{A} + \mathbf{B}$ is the $m \times n$ matrix whose ijth entry is $a_{ij} + b_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Let G be a graph with n vertices where $n > 1$, and let $\mathbf{A}$ be the adjacency matrix of G. Prove that G is connected if, and only if, every entry of $\mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$ is positive.

Hint: Consider the ijth entry of $\mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$. If G is connected, then given the vertices v_i and v_j, there is a walk connecting v_i and v_j. If this walk has length k, then by Theorem 10.2.2, the ijth entry of $\mathbf{A}^k$ is not equal to 0. Use the facts that all entries of each power of A are nonnegative and that a sum of nonnegative numbers is positive provided that at least one of the numbers is positive.

Proof. G is connected if and only if there is a walk between every pair of its vertices.

By part (a), this is true if and only if there is a walk of length at most $n-1$ between every pair of its vertices.

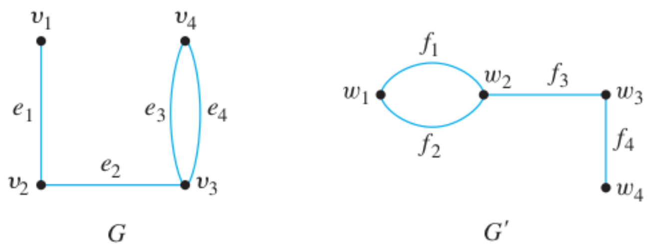
By Theorem 10.2.2, this is true if and only if, for every pair of distinct vertices v_i, v_j, there is an integer $1 \le k \le n-1$ (the length of the walk) such that the ijth entry of $\mathbf{A}^k$ is at least 1.

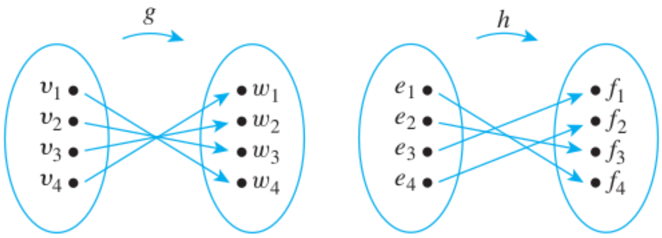
This is true if and only if the ijth entry of $\mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$ is positive, for every pair $1 \le i, j \le n$. $\square$

3 Exercise Set 10.3

For each pair of graphs G and G' in $1-5$, determine whether G and G' are isomorphic. If they are, give functions $g : V(G) \to V(G')$ and $h : E(G) \to E(G')$ that define the isomorphism. If they are not, give an invariant for graph isomorphism that they do not share.

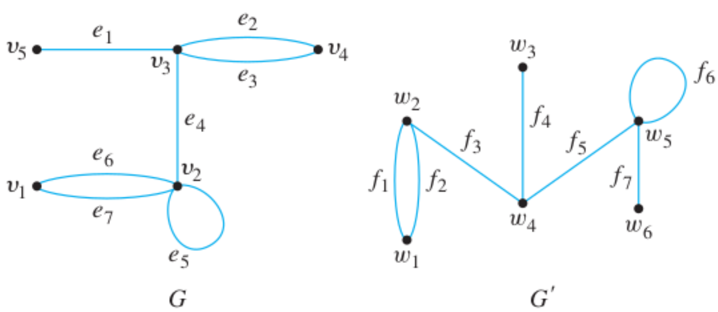
3.1 Exercise 1



Proof. 

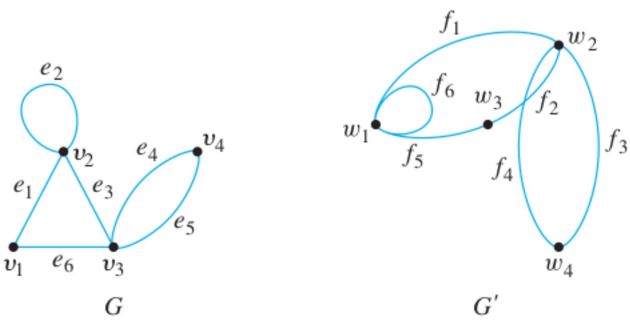
□

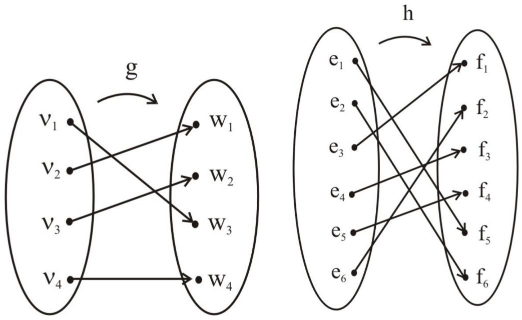
3.2 Exercise 2



Proof. Not isomorphic. (5 vertices vs. 6 vertices) □

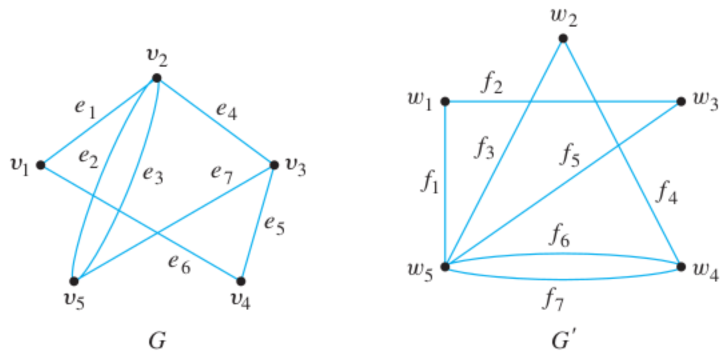
3.3 Exercise 3



Proof. 

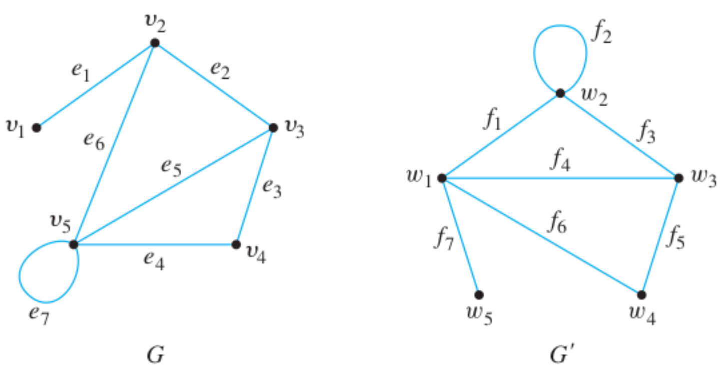
□

3.4 Exercise 4



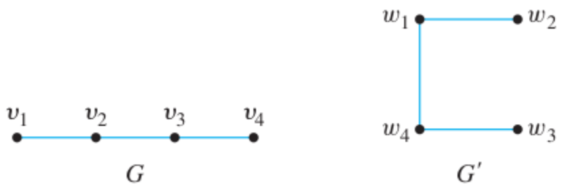
Proof. Not isomorphic. The degrees of vertices in G are, in increasing order: 2, 2, 3, 3, 4; whereas the degrees in G' are 2, 2, 2, 3, 5. □

3.5 Exercise 5

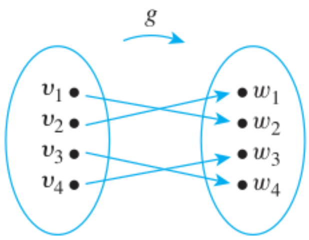


Proof. Not isomorphic. The degrees of vertices in G are, in increasing order: 1, 2, 3, 3, 5; whereas the degrees in G' are 1, 2, 3, 4, 4. □

For each pair of simple graphs G and G' in $6-13$, determine whether G and G' are isomorphic. If they are, give a function $g : V(G) \to V(G')$ that defines the isomorphism. If they are not, give an invariant for graph isomorphism that they do not share.

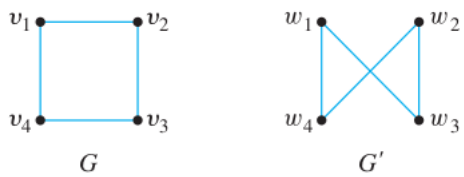


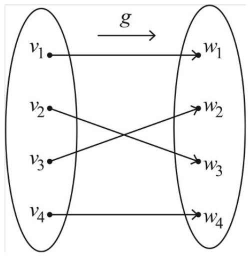
3.6 Exercise 6

Proof. The graphs are isomorphic. One is the following: 

□

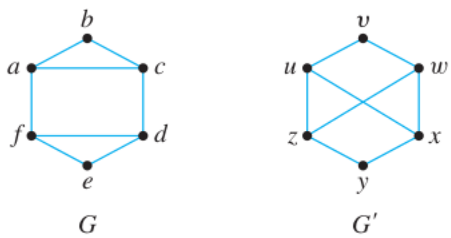
3.7 Exercise 7



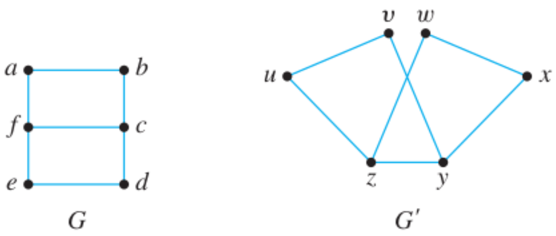
Proof. The graphs are isomorphic. One is the following: 

□

3.8 Exercise 8

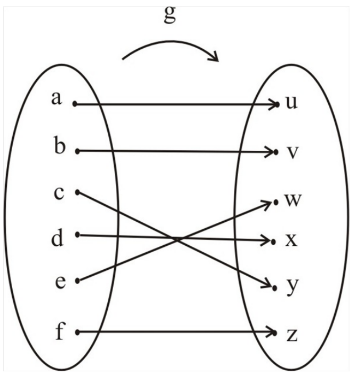


Proof. Not isomorphic. G has a circuit of length 3 but the shortest circuit in G' has length 4. □



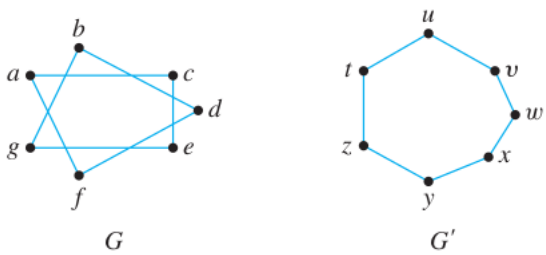
3.9 Exercise 9

Proof. They are isomorphic:

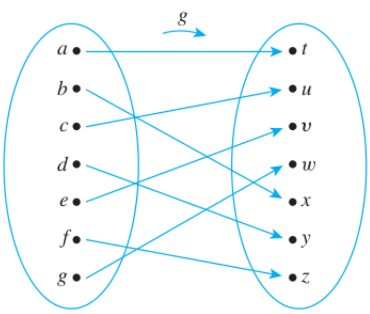


□

3.10 Exercise 10

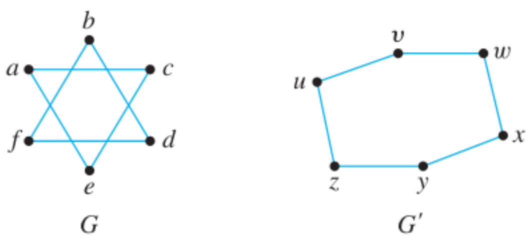


Proof. They are isomorphic:



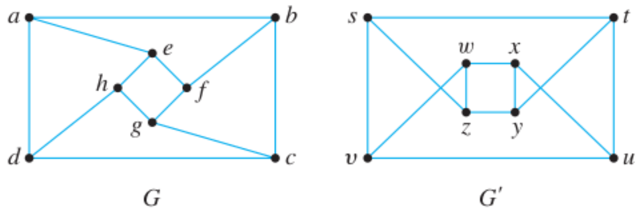
□

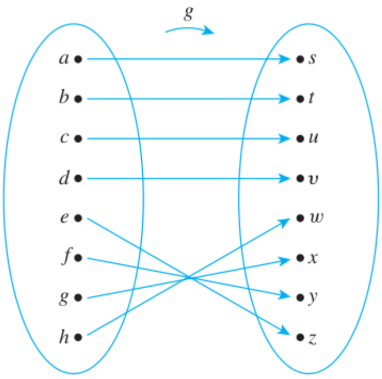
3.11 Exercise 11



Proof. Not isomorphic. G is not connected, whereas G' is. □

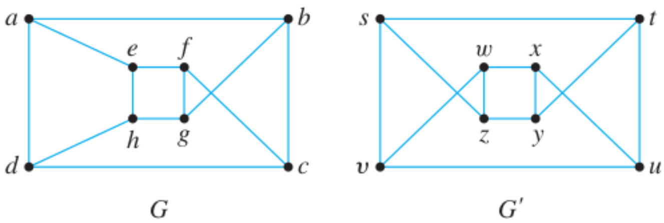
3.12 Exercise 12





Proof. They are isomorphic: □

3.13 Exercise 13

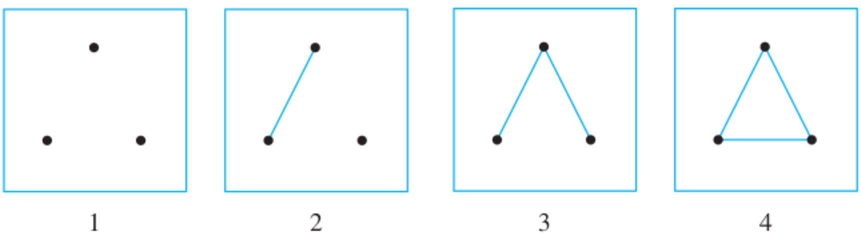


Proof. Not isomorphic. G has 4 circuits of length 4: $abcd$, $efgh$, $aehd$, $bcfg$; whereas G' has 6 circuits of length 4: $stuv$, $wxyz$, $svwz$, $tuxy$, $styz$ and $wxuv$. □

3.14 Exercise 14

Draw all nonisomorphic simple graphs with three vertices.

Proof.

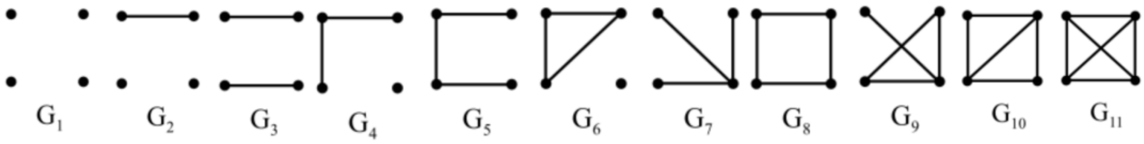


□

3.15 Exercise 15

Draw all nonisomorphic simple graphs with four vertices.

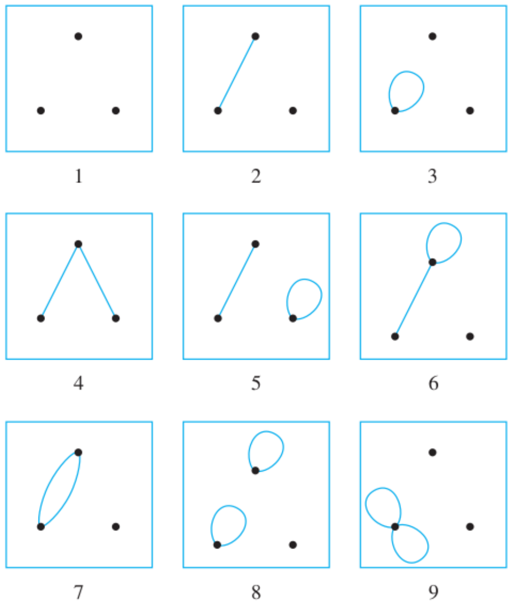
Proof.



□

3.16 Exercise 16

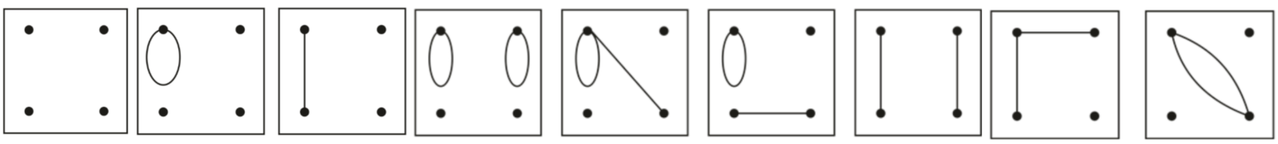
Draw all nonisomorphic graphs with three vertices and no more than two edges.



Proof. □

3.17 Exercise 17

Draw all nonisomorphic graphs with four vertices and no more than two edges.



Proof. □

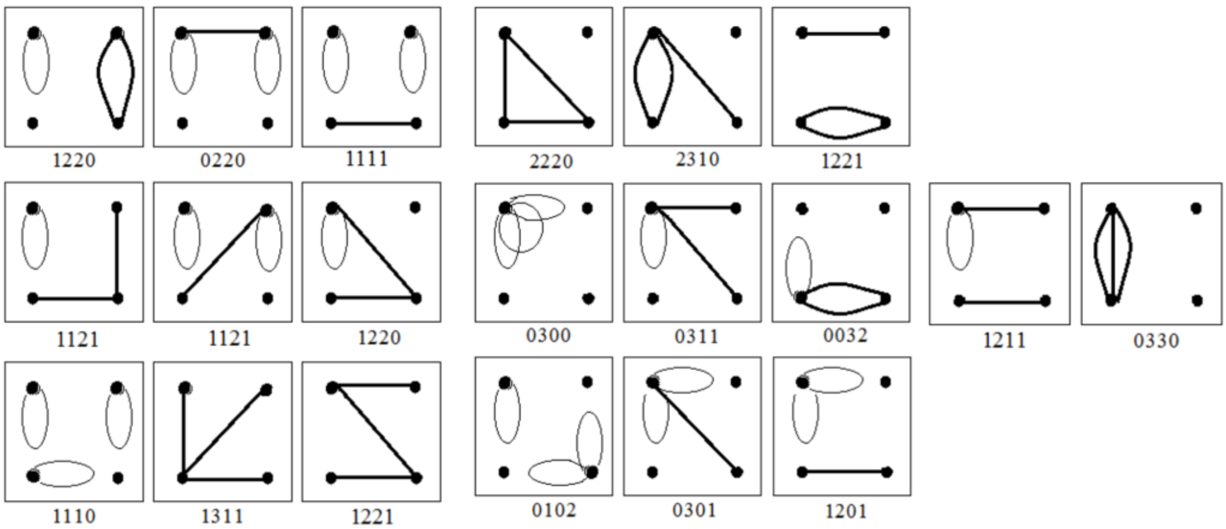
3.18 Exercise 18

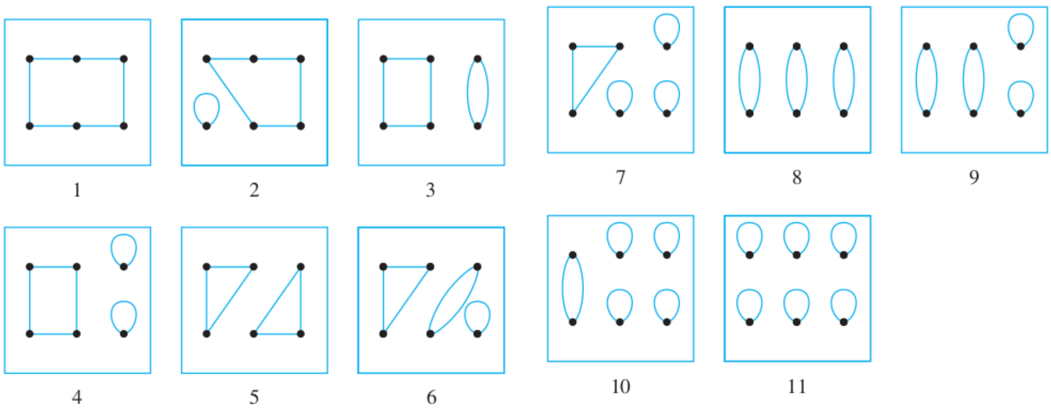
Draw all nonisomorphic graphs with four vertices and three edges.

Proof. □

3.19 Exercise 19

Draw all nonisomorphic graphs with six vertices, all having degree 2.

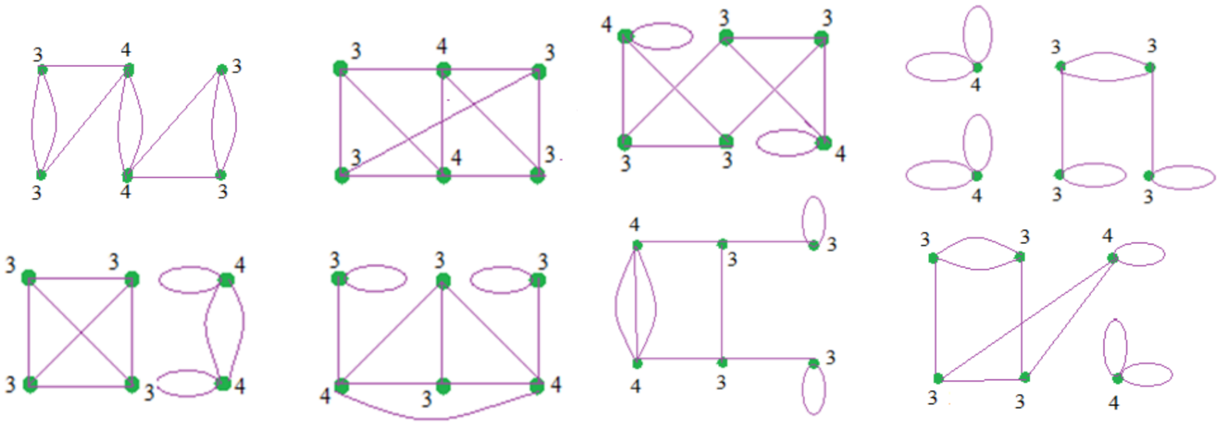




Proof. □

3.20 Exercise 20

Draw four nonisomorphic graphs with six vertices, two of degree 4 and four of degree 3.



Proof. □

Prove that each of the properties in $21 - 29$ is an invariant for graph isomorphism. assume that n, m, and k are all nonnegative integers.

3.21 Exercise 21

Has n vertices

Proof. Suppose G and G' are isomorphic graphs and G has n vertices, where n is a nonnegative integer. *[We must show that G' has n vertices.]* By definition of graph isomorphism, there is a one-to-one correspondence $g : V(G) \to V(G')$ sending vertices of G to vertices of G'. Since $V(G)$ is a finite set and g is a one-to- one correspondence, the number of vertices in $V(G')$ equals the number of vertices in $V(G)$. Hence G' has n vertices *[as was to be shown].* □

3.22 Exercise 22

Has m edges

Proof. Suppose G and G' are isomorphic graphs and G has m edges, where m is a nonnegative integer. *[We must show that G' has m edges.]* By definition of graph isomorphism, there is a one-to-one correspondence $h : E(G) \to E(G')$ sending edges of G to edges of G'. Since $E(G)$ is a finite set and h is a one-to-one correspondence, the number of edges in $E(G')$ equals the number of edges in $E(G)$. Hence G' has m edges *[as was to be shown].* □

3.23 Exercise 23

Has a circuit of length k

Proof. Suppose G and G' are isomorphic graphs and suppose G has a circuit C of length k, where k is a nonnegative integer. Let C be $v_0 e_1 v_1 e_2 \dots e_k v_k (= v_0)$. By definition of graph isomorphism, there are one-to-one correspondences $g : V(G) \to V(G')$ and $h : E(G) \to E(G')$ that preserve the edge-endpoint functions in the sense that for each v in $V(G)$ and each e in $E(G)$, v is an endpoint of $e \iff g(v)$ is an endpoint of $h(e)$. Let C' be $g(v_0)h(e_1)g(v_1)h(e_2)\dots h(e_k)g(v_k)(= g(v_0))$. Then C' is a circuit of length k in G'. The reasons are that (1) because g and h preserve the edge-endpoint functions, both $g(v_i)$ and $g(v_{i+1})$ are incident on $h(e_{i+1})$ for each $i = 0, 1, \dots, k-1$, and so C' is a walk from $g(v_0)$ to $g(v_0)$, and (2) since C is a circuit, then $e_1, e_2, \dots, e_k$ are distinct, and since h is a one-to- one correspondence, $h(e_1), h(e_2), \dots, h(e_k)$ are also distinct, which implies that C' has k distinct edges. Therefore, G' has a circuit C of length k. □

3.24 Exercise 24

Has a simple circuit of length k

Proof. The proof is like in Exercise 23, except that additionally the simplicity is also preserved. If C does not have repeated vertices in G except the first and last, then neither does C' in G' because g is a one-to-one correspondence. □

3.25 Exercise 25

Has m vertices of degree k

Proof. Suppose G and G' are isomorphic and G has m vertices of degree k; call them $v_1, v_2, \ldots, v_m$. Since G and G' are isomorphic, there are one-to-one correspondences $g : V(G) \to V(G')$ and $h : E(G) \to E(G')$. We want to show that $g(v_1), g(v_2), \ldots, g(v_m)$ are m distinct vertices of G', each of which has degree k.

$g(v_1), g(v_2), \ldots, g(v_m)$ are m distinct vertices since g is one-to-one.

Each v_i is incident to k edges in G, say $e^i_1, \ldots, e^i_k$, then by definition of graph isomorphism, each $g(v_i)$ is incident to k edges $h(e^i_1), \ldots, h(e^i_k)$ in G', thus they each have degree k. □

3.26 Exercise 26

Has m simple circuits of length k

Proof. This follows from Exercise 24. We can define a function K from the set of all simple circuits of length k in G to the set of all simple circuits of length k in G' by letting $K(C) = C'$ as in the proof of Exercise 23. Then K is a one-to-one correspondence: for each simple circuit C of length k there is a unique simple circuit C' of length k, because g and h are one-to-one correspondences that preserve the edge relations. Since K is a one-to-one correspondence and G has m simple circuits of length k, so does G'. □

3.27 Exercise 27

Is connected

Proof. Suppose G and G' are isomorphic and G is connected. To show that G' is connected, suppose w and x are any two vertices of G'. Since $g : V(G) \to V(G')$ is a one-to-one correspondence, there are vertices $u, v \in G$ such that $g(u) = w, g(v) = x$. Since G is connected, there is a walk from u to v in G, say $u = v_0 e_1 v_1 e_2 \ldots e_n v_n = v$. Then since $h : E(G) \to E(G')$ is a one-to-one correspondence, $w = g(u) = g(v_0)h(e_1)g(v_1)h(e_2)\ldots h(e_n)g(v_n) = g(v) = x$ is a walk in G' because of the definition of graph isomorphism. Therefore G' is connected. □

3.28 Exercise 28

Has an Euler circuit

Proof. Suppose G and G' are isomorphic graphs and G has an Euler circuit C. Let m be the number of edges in G. Then C has length m as it includes every edge of G. By Exercise 23, G' has a corresponding circuit C' of length m. Also G' has m edges by Exercise 22. As all the edges of the Euler circuit C are distinct, and the graph isomorphism sends distinct edges in G to distinct edges in G', C' includes all of the m edges of G'. Thus, G' has an Euler circuit. □

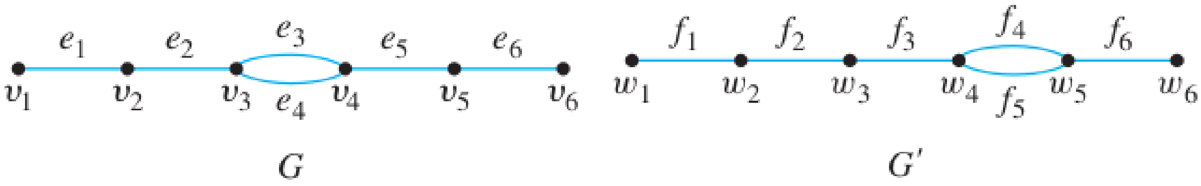
3.29 Exercise 29

Has a Hamiltonian circuit

Proof. A Hamiltonian circuit for G is a simple circuit that includes every vertex of G. Therefore this follows from Exercises 21 and 24. □

3.30 Exercise 30

Show that the following two graphs are not isomorphic by supposing they are isomorphic and deriving a contradiction.



Proof. 1. Assume G and G' are isomorphic with one-to-one correspondences g and h.

2. v_1 has degree 1, so $g(v_1) = w_1$ or $g(v_1) = w_6$.

3. **Case 1: $g(v_1) = w_1$.** v_1 is connected to v_2, so w_1 is connected to $g(v_2)$ in G'. This forces $g(v_2) = w_2$.

3.1. v_2 is connected to v_3, so w_2 is connected to $g(v_3)$ in G'. This forces $g(v_3) = w_3$. But this is a contradiction, since v_3 has degree 3 and w_3 has degree 2.

4. **Case 2: $g(v_1) = w_6$.** v_1 is connected to v_2, so w_1 is connected to $g(v_2)$ in G'. This forces $g(v_2) = w_5$. But this is a contradiction, since v_2 has degree 2 and w_5 has degree 3.

5. In both cases we get a contradiction. Therefore our assumption was false and G and G' are not isomorphic. □

4 Exercise Set 10.4

4.1 Exercise 1

Read the tree in Example 10.4.2 from left to right to answer the following questions.

4.1.1 (a)

A student scored 12 on part I and 4 on part II. What course should the student take?

Proof. Math 110 □

4.1.2 (b)

A student scored 8 on part I and 9 on part II. What course should the student take?

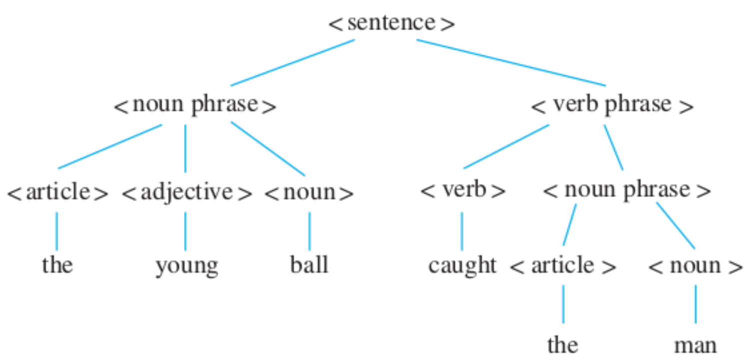
Proof. Math 110 □

4.2 Exercise 2

Draw trees to show the derivations of the following sentences from the rules given in Example 10.4.3.

4.2.1 (a)

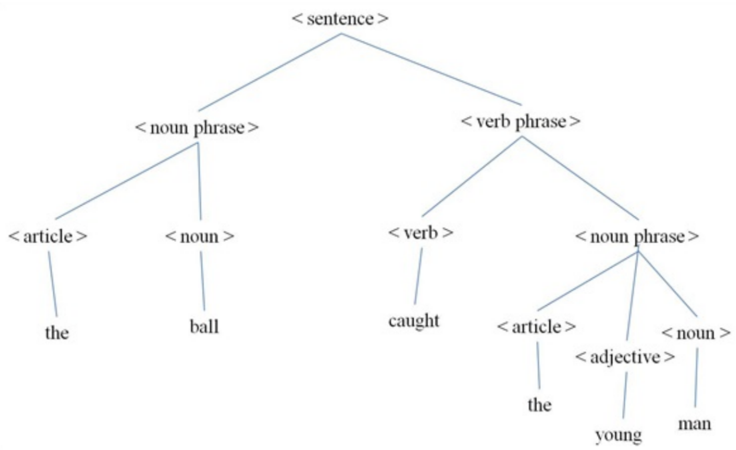
The young ball caught the man.



Proof. □

4.2.2 (b)

The man caught the young ball.



Proof. □

4.3 Exercise 3

What is the total degree of a tree with n vertices? Why?

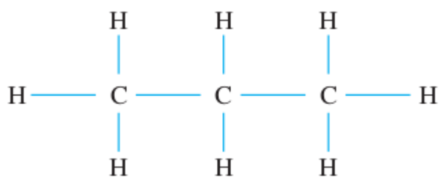
Proof. The answer is $2n - 2$. The total degree of a graph is two times the number of edges. If a tree has n vertices, then it has $n - 1$ edges, therefore it has total degree $2(n - 1) = 2n - 2$. □

4.4 Exercise 4

Let G be the graph of a hydrocarbon molecule with the maximum number of hydrogen atoms for the number of its carbon atoms.

4.4.1 (a)

Draw the graph of G if G has three carbon atoms and eight hydrogen atoms.

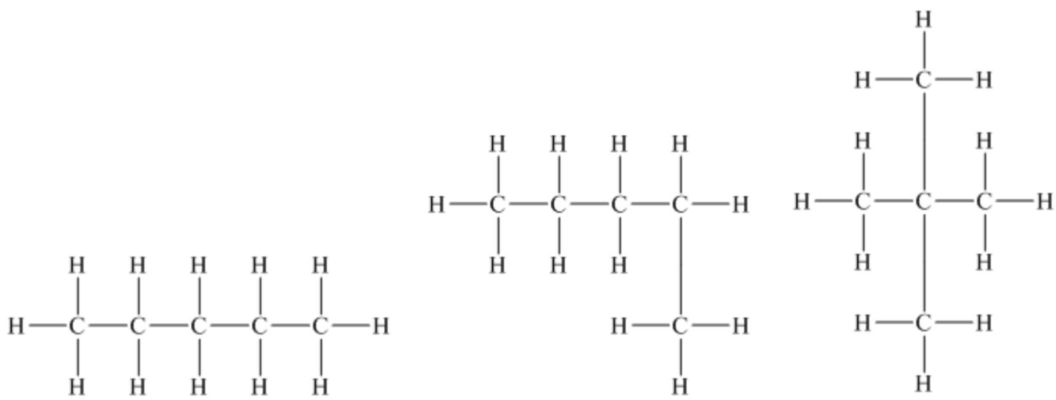


Proof. □

4.4.2 (b)

Draw the graphs of three isomers of C_5H_{12}.

Proof.



□

4.4.3 (c)

Use Example 10.4.4 and exercise 3 to prove that if the vertices of G consist of k carbon atoms and m hydrogen atoms, then G has a total degree of $2k + 2m - 2$.

Proof. G has a total of $k + m$ vertices. Since G is a tree, it has $k + m - 1$ edges. So the total degree is $2(k + m - 1)$. □

4.4.4 (d)

Prove that if the vertices of G consist of k carbon atoms and m hydrogen atoms, then G has a total degree of $4k + m$.

Proof. Each carbon atom in G is bonded to four other atoms in G, because otherwise an additional hydrogen atom could be bonded to it, and this would contradict the assumption that G has the maximum number of hydrogen atoms for its number of carbon atoms. Also each hydrogen atom is bonded to exactly one carbon atom in G, because otherwise G would not be connected. So G has a total degree of $4k + m$. □

4.4.5 (e)

Equate the results of (c) and (d) to prove Cayley's result that a saturated hydrocarbon molecule with k carbon atoms and a maximum number of hydrogen atoms has $2k + 2$ hydrogen atoms.

Proof. $4k + m = 2k + 2m - 2$ implies $m = 2k + 2$, so if a molecule has k carbon atoms, then it has $2k + 2$ hydrogen atoms. □

4.5 Exercise 5

Extend the argument given in the proof of Lemma 10.4.1 to show that a tree with more than one vertex has at least two vertices of degree 1.

Proof. Revise the algorithm given in the proof of Lemma 10.4.1 to keep track of which vertex and edge were chosen in step 1 (by, say, labeling them v_0 and e_0). Then after one vertex of degree 1 is found, return to v_0 and search for another vertex of degree 1 by moving along a path outward from v_0 starting with another edge incident on v_0. Such an edge exists because v_0 has degree at least 2. □

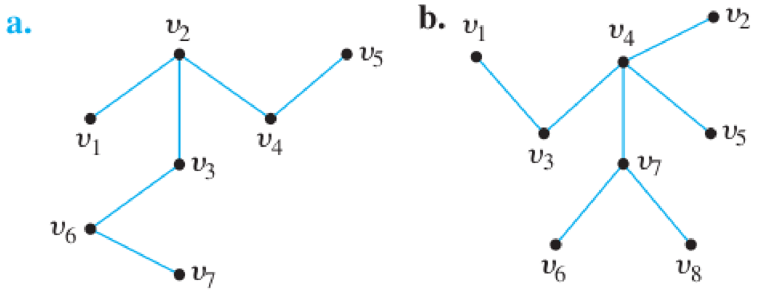
4.6 Exercise 6

If graphs are allowed to have an infinite number of vertices and edges, then Lemma 10.4.1 is false. Give a counterexample that shows this. In other words, give an example of an "infinite tree" (a connected, circuit-free graph with an infinite number of vertices and edges) that has no vertex of degree 1.

Proof. Consider an infinite chain that grows in both directions. It is an (infinite) tree since it's connected and circuit-free. But every vertex has degree 2 (connected to a vertex to the left and to a vertex to the right). □

4.7 Exercise 7

Find all leaves (or terminal vertices) and all internal (or branch) vertices for the following trees.



4.7.1 (a)

Proof. Leaves: v_1, v_5, v_7 Branches: v_2, v_3, v_4, v_6 □

4.7.2 (b)

Proof. Leaves: v_1, v_2, v_5, v_6, v_8 Branches: v_3, v_4, v_7 □

In each of $8 - 21$, either draw a graph with the given specifications or explain why no such graph exists.

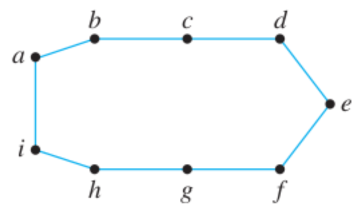
4.8 Exercise 8

Tree, nine vertices, nine edges

Proof. Any tree with nine vertices has eight edges, not nine. Thus there is no tree with nine vertices and nine edges. □

4.9 Exercise 9

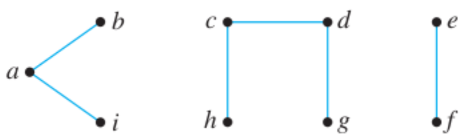
Graph, connected, nine vertices, nine edges



Proof. □

4.10 Exercise 10

Graph, circuit-free, nine vertices, six edges



Proof. □

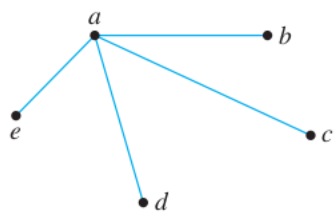
4.11 Exercise 11

Tree, six vertices, total degree 14

Proof. There is no tree with six vertices and a total degree of 14. Any tree with six vertices has five edges and hence, by the handshake theorem (Theorem 4.9.1) it has a total degree of 10, not 14. □

4.12 Exercise 12

Tree, five vertices, total degree 8



Proof. □

4.13 Exercise 13

Graph, connected, six vertices, five edges, has a circuit

Proof. No such graph exists. By Theorem 10.4.4, a connected graph with six vertices and five edges is a tree. Hence such a graph cannot have a nontrivial circuit. □

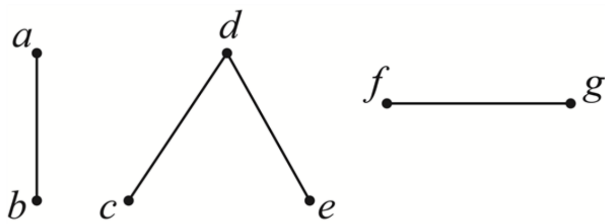
4.14 Exercise 14

Graph, two vertices, one edge, not a tree

Proof. □

4.15 Exercise 15

Graph, circuit-free, seven vertices, four edges



Proof. □

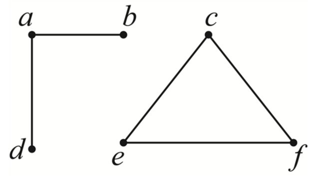
4.16 Exercise 16

Tree, twelve vertices, fifteen edges

Proof. For a positive integer n, any tree with n vertices has $n-1$ edges. Thus, for a tree having twelve vertices, the possible number of edges are: $12 - 1 = 11$. We have total fifteen edges but the total number of all possible edges in the tree is eleven. Therefore, no tree can exist which has twelve vertices and fifteen edges. □

4.17 Exercise 17

Graph, six vertices, five edges, not a tree



Proof. □

4.18 Exercise 18

Tree, five vertices, total degree 10

Proof. No such graph exists. We know that a tree with n vertices has $n-1$ edges. Also, a graph with $n-1$ edges has a total number of $2(n-1)$ degrees. Therefore, a tree with 5 vertices has a total degree of 8. Thus, there is no such tree with 5 vertices and a total degree of 10. □

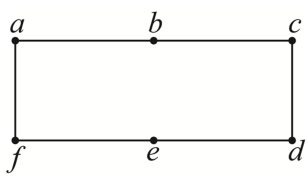
4.19 Exercise 19

Graph, connected, ten vertices, nine edges, has a circuit

Proof. A graph with n vertices and $n-1$ edges is a tree by Theorem 10.4.4. Trees are circuit-free. Thus such a graph does not exist. □

4.20 Exercise 20

Simple graph, connected, six vertices, six edges



Proof. □

4.21 Exercise 21

Tree, ten vertices, total degree 24

Proof. A tree with 10 vertices has 9 edges and therefore $9 \cdot 2 = 18$ total degree; thus this graph does not exist. □

4.22 Exercise 22

A connected graph has twelve vertices and eleven edges. Does it have a vertex of degree 1? Why?

Proof. Yes. Since it is connected and has 12 vertices and 11 edges, by Theorem 10.4.4 it is a tree. It follows from Lemma 10.5.1 that it has a vertex of degree 1. □

4.23 Exercise 23

A connected graph has nine vertices and twelve edges. Does it have a circuit? Why?

Proof. Assume it has no circuit. Since it's connected, it's a tree. But a tree with 9 vertices must have 8 edges, contradiction. Therefore the graph has a circuit. □

4.24 Exercise 24

Suppose that v is a vertex of degree 1 in a connected graph G and that e is the edge incident on v. Let G' be the subgraph of G obtained by removing v and e from G. Must G' be connected? Why?

Proof. Write $V(G) = \{v, v_1, \dots, v_n\}$ where v is adjacent to v_1 via the edge e.

Since G is connected, there is a walk in G between every pair of vertices.

We claim that G' is connected, in other words, there is a walk in G' between every pair of vertices in $V(G') = \{v_1, \dots, v_n\}$.

To see this, consider any two vertices v_i and v_j in G', where $1 \leq i, j \leq n$. There is a walk from v_i to v_j in G. If this walk contains the edge e, then it must visit v, then traverse e again and come back to v_1, because v is only adjacent to v_1 and no other vertex. Then we can remove v and the two occurrences of e from this walk, to obtain a walk in G', *[as was to be shown.]* □

4.25 Exercise 25

A graph has eight vertices and six edges. Is it connected? Why?

Proof. Suppose there were a connected graph with eight vertices and six edges. Either the graph itself would be a tree or edges could be eliminated from its circuits to obtain a tree. In either case, there would be a tree with eight vertices and six or fewer edges.

But by Theorem 10.4.2, a tree with eight vertices has seven edges, not six or fewer. This contradiction shows that the supposition is false, so there is no connected graph with eight vertices and six edges. □

4.26 Exercise 26

If a graph has n vertices and $n-2$ or fewer edges, can it be connected? Why?

Proof. No. Same argument as Exercise 25, replacing 8 and 6 with n and $n-2$. □

4.27 Exercise 27

A circuit-free graph has ten vertices and nine edges. Is it connected? Why?

Proof. Yes. Suppose G is a circuit-free graph with ten vertices and nine edges. Let $G_1, G_2, \ldots, G_k$ be the connected components of G. *[To show that G is connected, we will show that $k = 1$.]* Each G_i is a tree since each G_i is connected and circuit-free. For each $i = 1, 2, \ldots, k$, let G_i have n_i vertices. Note that since G has ten vertices in all, $n_1 + n_2 + \cdots n_k = 10$. By Theorem 10.4.2, G_1 has $n_1 - 1$ edges, G_2 has $n_2 - 1$ edges, …, G_k has $n_k - 1$ edges. So the number of edges of G equals

$$(n_1 - 1) + \cdots + (n_k - 1) = (n_1 + \cdots + n_k) - k = 10 - k.$$

But we are given that G has nine edges. Hence $10 - k = 9$, and so $k = 1$. Thus G has just one connected component, G_1, and so G is connected. □

4.28 Exercise 28

Is a circuit-free graph with n vertices and at least $n-1$ edges connected? Why?

Proof. If it has exactly $n-1$ edges, then it is connected: simply repeat the proof of Exercise 27 above (replacing 10 with n and 9 with $n-1$).

If it has n or more edges, then by Corollary 10.4.5 it must have a circuit, so no. □

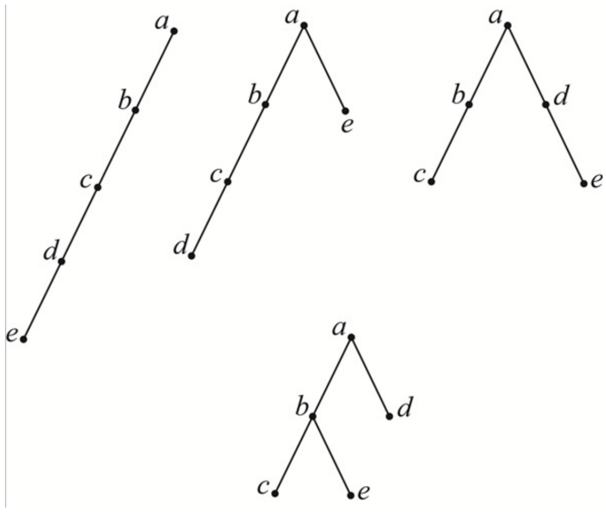
4.29 Exercise 29

Prove that every nontrivial tree has at least two circuit vertices of degree 1 by filling in the details and completing the following argument: Let T be a nontrivial tree and let S be the set of all paths from one vertex to another in T. Among all the paths in S, choose a path P with a maximum number of edges. (Why is it possible to find such a P?) What can you say about the initial and final vertices of P? Why?

Proof. It is possible to find such a P because paths do not have repeated edges or vertices, and since T is finite, there are finitely many paths from one vertex to another. The initial and final vertices of P must have degree 1; otherwise if they had degree 2 or higher, we can add one more edge to P and obtain a longer path, which contradicts the maximality of P. □

4.30 Exercise 30

Find all nonisomorphic trees with five vertices.



Proof. □

4.31 Exercise 31

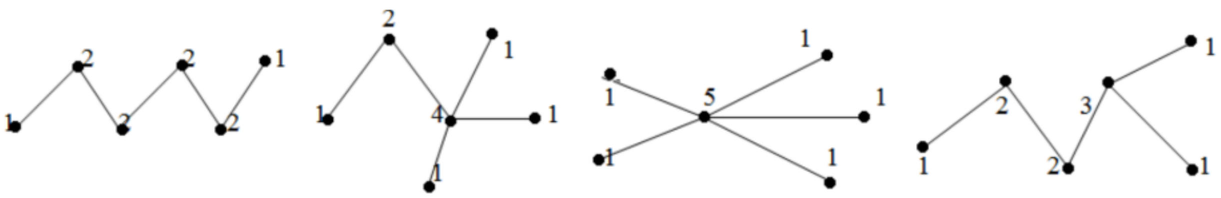
4.31.1 (a)

Prove that the following is an invariant for graph isomorphism: A vertex of degree i is adjacent to a vertex of degree j.

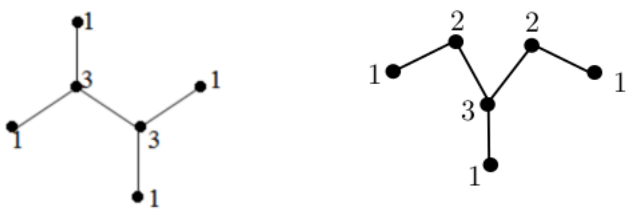
Proof. A graph isomorphism preserves adjacency, and vertex degrees (by exercises in section 10.3). Therefore it also preserves "a vertex of degree i being adjacent to a vertex of degree j": if v has degree i and is adjacent to w which has degree j then $g(v)$ must have degree i and must be adjacent to $g(w)$ which must have degree j. □

4.31.2 (b)

Find all nonisomorphic trees with six vertices.



Proof.



□

5 Exercise Set 10.5

5.1 Exercise 1

5.1.1 ()

Proof. □

5.2 Exercise 2

5.2.1 ()

Proof. □

5.3 Exercise 3

5.3.1 ()

Proof. □

5.4 Exercise 4

5.4.1 ()

Proof. □

5.5 Exercise 5

5.5.1 ()

Proof. □

5.6 Exercise 6

5.6.1 ()

Proof. □

5.7 Exercise 7

5.7.1 ()

Proof. □

5.8 Exercise 8

5.8.1 ()

Proof. □

5.9 Exercise 9

5.9.1 ()

Proof. □

5.10 Exercise 10

5.10.1 ()

Proof. □

5.11 Exercise 11

5.11.1 ()

Proof. □

5.12 Exercise 12

5.12.1 ()

Proof. □

5.13 Exercise 13

5.13.1 ()

Proof. □

5.14 Exercise 14

5.14.1 ()

Proof. □

5.15 Exercise 15

5.15.1 ()

Proof. □

5.16 Exercise 16

5.16.1 ()

Proof. □

5.17 Exercise 17

5.17.1 ()

Proof. □

5.18 Exercise 18

5.18.1 ()

Proof. □

5.19 Exercise 19

5.19.1 ()

Proof. □

5.20 Exercise 20

5.20.1 ()

Proof. □

5.21 Exercise 21

5.21.1 ()

Proof. □

5.22 Exercise 22

5.22.1 ()

Proof. □

5.23 Exercise 23

5.23.1 ()

Proof. □

5.24 Exercise 24

5.24.1 ()

Proof. □

5.25 Exercise 25

5.25.1 ()

Proof. □

6 Exercise Set 10.6

6.1 Exercise 1

6.1.1 ()

Proof. □

6.2 Exercise 2

6.2.1 ()

Proof. □

6.3 Exercise 3

6.3.1 ()

Proof. □

6.4 Exercise 4

6.4.1 ()

Proof. □

6.5 Exercise 5

6.5.1 ()

Proof. □

6.6 Exercise 6

6.6.1 ()

Proof. □

6.7 Exercise 7

6.7.1 ()

Proof. □

6.8 Exercise 8

6.8.1 ()

Proof. □

6.9 Exercise 9

6.9.1 ()

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6.10 Exercise 10

6.10.1 ()

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6.11 Exercise 11

6.11.1 ()

Proof. □

6.12 Exercise 12

6.12.1 ()

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6.13 Exercise 13

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6.14 Exercise 14

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6.17 Exercise 17

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6.26 Exercise 26

6.26.1 ()

Proof. □

6.27 Exercise 27

6.27.1 ()

Proof. □

6.28 Exercise 28

6.28.1 ()

Proof. □

6.29 Exercise 29

6.29.1 ()

Proof. □

6.30 Exercise 30

6.30.1 ()

Proof. □

6.31 Exercise 31

6.31.1 ()

Proof. □