Chapter 11 Solutions, Susanna Epp Discrete Math 5th Edition

https://github.com/spamegg1

December 2, 2023

Contents

1 Exercise Set 11.1 . . . **10**
1.1 Exercise 1 . . . 10
1.1.1 (a) . . . 10
1.1.2 (b) . . . 10
1.1.3 (c) . . . 10
1.1.4 (d) . . . 10
1.1.5 (e) . . . 10
1.1.6 (f) . . . 11
1.2 Exercise 2 . . . 11
1.2.1 (a) . . . 11
1.2.2 (b) . . . 11
1.2.3 (c) . . . 11
1.2.4 (d) . . . 11
1.2.5 (e) . . . 11
1.2.6 (f) . . . 12
1.3 Exercise 3 . . . 12
1.4 Exercise 4 . . . 12
1.5 Exercise 5 . . . 13
1.6 Exercise 6 . . . 13
1.7 Exercise 7 . . . 14
1.8 Exercise 8 . . . 14
1.9 Exercise 9 . . . 14
1.10 Exercise 10 . . . 14
1.11 Exercise 11 . . . 15
1.12 Exercise 12 . . . 15
1.13 Exercise 13 . . . 15
1.14 Exercise 14 . . . 16
1.15 Exercise 15 . . . 16
1.16 Exercise 16 . . . 16
1.17 Exercise 17 . . . 17

1.17.1 (a) . . . 17
1.17.2 (b) . . . 17
1.18 Exercise 18 . . . 17
1.18.1 (a) . . . 17
1.18.2 (b) . . . 18
1.19 Exercise 19 . . . 18
1.20 Exercise 20 . . . 18
1.21 Exercise 21 . . . 19
1.21.1 (a) . . . 19
1.21.2 (b) . . . 19
1.22 Exercise 22 . . . 19
1.23 Exercise 23 . . . 20
1.24 Exercise 24 . . . 21
1.25 Exercise 25 . . . 21
1.26 Exercise 26 . . . 21
1.27 Exercise 27 . . . 21
1.28 Exercise 28 . . . 22

2 Exercise Set 11.2 22
2.1 Exercise 1 . . . 22
2.1.1 (a) . . . 22
2.1.2 (b) . . . 23
2.2 Exercise 2 . . . 23
2.2.1 (a) . . . 23
2.2.2 (b) . . . 23
2.3 Exercise 3 . . . 23
2.3.1 (a) . . . 23
2.3.2 (b) . . . 24
2.4 Exercise 4 . . . 24
2.5 Exercise 5 . . . 24
2.6 Exercise 6 . . . 24
2.7 Exercise 7 . . . 24
2.8 Exercise 8 . . . 24
2.9 Exercise 9 . . . 25
2.10 Exercise 10 . . . 25
2.10.1 (a) . . . 25
2.10.2 (b) . . . 25
2.10.3 (c) . . . 25
2.10.4 (d) . . . 25
2.10.5 (e) . . . 26
2.11 Exercise 11 . . . 26
2.11.1 (a) . . . 26
2.11.2 (b) . . . 26
2.11.3 (c) . . . 26
2.11.4 (d) . . . 26
2.11.5 (e) . . . 27

2.12 Exercise 12 . . . 27
2.12.1 (a) . . . 27
2.12.2 (b) . . . 27
2.12.3 (c) . . . 27
2.12.4 (d) . . . 28
2.12.5 (e) . . . 28
2.13 Exercise 13 . . . 28
2.14 Exercise 14 . . . 28
2.15 Exercise 15 . . . 29
2.16 Exercise 16 . . . 29
2.17 Exercise 17 . . . 29
2.18 Exercise 18 . . . 30
2.19 Exercise 19 . . . 30
2.20 Exercise 20 . . . 31
2.21 Exercise 21 . . . 31
2.22 Exercise 22 . . . 32
2.22.1 (a) . . . 32
2.22.2 (b) . . . 32
2.22.3 (c) . . . 32
2.23 Exercise 23 . . . 32
2.23.1 (a) . . . 32
2.23.2 (b) . . . 33
2.23.3 (c) . . . 33
2.24 Exercise 24 . . . 33
2.24.1 (a) . . . 33
2.24.2 (b) . . . 34
2.24.3 (c) . . . 34
2.25 Exercise 25 . . . 34
2.25.1 (a) . . . 34
2.25.2 (b) . . . 35
2.25.3 (c) . . . 35
2.26 Exercise 26 . . . 36
2.27 Exercise 27 . . . 36
2.28 Exercise 28 . . . 36
2.29 Exercise 29 . . . 36
2.30 Exercise 30 . . . 36
2.31 Exercise 31 . . . 36
2.32 Exercise 32 . . . 37
2.33 Exercise 33 . . . 37
2.34 Exercise 34 . . . 37
2.35 Exercise 35 . . . 37
2.36 Exercise 36 . . . 37
2.37 Exercise 37 . . . 38
2.38 Exercise 38 . . . 38
2.39 Exercise 39 . . . 38
2.40 Exercise 40 . . . 38

2.40.1 (a) . . . 38
2.40.2 (b) . . . 39
2.41 Exercise 41 . . . 39
2.42 Exercise 42 . . . 39
2.43 Exercise 43 . . . 39
2.44 Exercise 44 . . . 39
2.45 Exercise 45 . . . 40
2.46 Exercise 46 . . . 40
2.46.1 (a) . . . 40
2.46.2 (b) . . . 40
2.47 Exercise 47 . . . 40
2.47.1 (a) . . . 40
2.47.2 (b) . . . 41
2.47.3 (c) . . . 41
2.47.4 (d) . . . 41
2.48 Exercise 48 . . . 42
2.49 Exercise 49 . . . 42
2.50 Exercise 50 . . . 42
2.50.1 (a) . . . 42
2.50.2 (b) . . . 43
2.50.3 (c) . . . 43
2.51 Exercise 51 . . . 43
2.51.1 (a) . . . 43
2.51.2 (b) . . . 43
2.51.3 (c) . . . 44

3 Exercise Set 11.3 44
3.1 Exercise 1 . . . 44
3.1.1 () . . . 44
3.2 Exercise 2 . . . 44
3.2.1 () . . . 44
3.3 Exercise 3 . . . 44
3.3.1 () . . . 44
3.4 Exercise 4 . . . 44
3.4.1 () . . . 44
3.5 Exercise 5 . . . 45
3.5.1 () . . . 45
3.6 Exercise 6 . . . 45
3.6.1 () . . . 45
3.7 Exercise 7 . . . 45
3.7.1 () . . . 45
3.8 Exercise 8 . . . 45
3.8.1 () . . . 45
3.9 Exercise 9 . . . 45
3.9.1 () . . . 45
3.10 Exercise 10 . . . 45

3.10.1 () 45
3.11 Exercise 11 45
3.11.1 () 45
3.12 Exercise 12 45
3.12.1 () 45
3.13 Exercise 13 46
3.13.1 () 46
3.14 Exercise 14 46
3.14.1 () 46
3.15 Exercise 15 46
3.15.1 () 46
3.16 Exercise 16 46
3.16.1 () 46
3.17 Exercise 17 46
3.17.1 () 46
3.18 Exercise 18 46
3.18.1 () 46
3.19 Exercise 19 46
3.19.1 () 46
3.20 Exercise 20 46
3.20.1 () 46
3.21 Exercise 21 47
3.21.1 () 47
3.22 Exercise 22 47
3.22.1 () 47
3.23 Exercise 23 47
3.23.1 () 47
3.24 Exercise 24 47
3.24.1 () 47
3.25 Exercise 25 47
3.25.1 () 47
3.26 Exercise 26 47
3.26.1 () 47
3.27 Exercise 27 47
3.27.1 () 47
3.28 Exercise 28 47
3.28.1 () 47
3.29 Exercise 29 48
3.29.1 () 48
3.30 Exercise 30 48
3.30.1 () 48
3.31 Exercise 31 48
3.31.1 () 48
3.32 Exercise 32 48
3.32.1 () 48
3.33 Exercise 33 48

3.33.1 () . . . 48
3.34 Exercise 34 . . . 48
3.34.1 () . . . 48
3.35 Exercise 35 . . . 48
3.35.1 () . . . 48
3.36 Exercise 36 . . . 48
3.36.1 () . . . 48
3.37 Exercise 37 . . . 49
3.37.1 () . . . 49
3.38 Exercise 38 . . . 49
3.38.1 () . . . 49
3.39 Exercise 39 . . . 49
3.39.1 () . . . 49
3.40 Exercise 40 . . . 49
3.40.1 () . . . 49
3.41 Exercise 41 . . . 49
3.41.1 () . . . 49
3.42 Exercise 42 . . . 49
3.42.1 () . . . 49
3.43 Exercise 43 . . . 49
3.43.1 () . . . 49

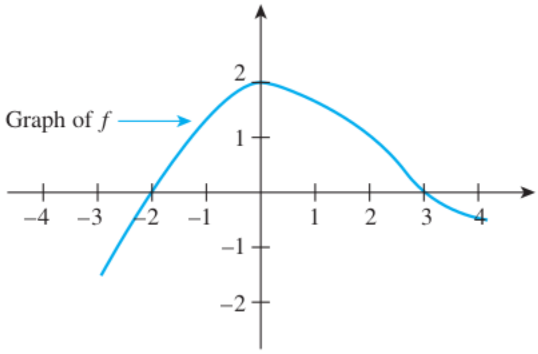
4 Exercise Set 11.4 **49**
4.1 Exercise 1 . . . 49
4.1.1 () . . . 49
4.2 Exercise 2 . . . 50
4.2.1 () . . . 50
4.3 Exercise 3 . . . 50
4.3.1 () . . . 50
4.4 Exercise 4 . . . 50
4.4.1 () . . . 50
4.5 Exercise 5 . . . 50
4.5.1 () . . . 50
4.6 Exercise 6 . . . 50
4.6.1 () . . . 50
4.7 Exercise 7 . . . 50
4.7.1 () . . . 50
4.8 Exercise 8 . . . 50
4.8.1 () . . . 50
4.9 Exercise 9 . . . 50
4.9.1 () . . . 50
4.10 Exercise 10 . . . 51
4.10.1 () . . . 51
4.11 Exercise 11 . . . 51
4.11.1 () . . . 51
4.12 Exercise 12 . . . 51

4.12.1 () . . . 51
4.13 Exercise 13 . . . 51
4.13.1 () . . . 51
4.14 Exercise 14 . . . 51
4.14.1 () . . . 51
4.15 Exercise 15 . . . 51
4.15.1 () . . . 51
4.16 Exercise 16 . . . 51
4.16.1 () . . . 51
4.17 Exercise 17 . . . 51
4.17.1 () . . . 51
4.18 Exercise 18 . . . 52
4.18.1 () . . . 52
4.19 Exercise 19 . . . 52
4.19.1 () . . . 52
4.20 Exercise 20 . . . 52
4.20.1 () . . . 52
4.21 Exercise 21 . . . 52
4.21.1 () . . . 52
4.22 Exercise 22 . . . 52
4.22.1 () . . . 52
4.23 Exercise 23 . . . 52
4.23.1 () . . . 52
4.24 Exercise 24 . . . 52
4.24.1 () . . . 52
4.25 Exercise 25 . . . 52
4.25.1 () . . . 52
4.26 Exercise 26 . . . 53
4.26.1 () . . . 53
4.27 Exercise 27 . . . 53
4.27.1 () . . . 53
4.28 Exercise 28 . . . 53
4.28.1 () . . . 53
4.29 Exercise 29 . . . 53
4.29.1 () . . . 53
4.30 Exercise 30 . . . 53
4.30.1 () . . . 53
4.31 Exercise 31 . . . 53
4.31.1 () . . . 53
4.32 Exercise 32 . . . 53
4.32.1 () . . . 53
4.33 Exercise 33 . . . 53
4.33.1 () . . . 53
4.34 Exercise 34 . . . 54
4.34.1 () . . . 54
4.35 Exercise 35 . . . 54

4.35.1 () . . . 54
4.36 Exercise 36 . . . 54
4.36.1 () . . . 54
4.37 Exercise 37 . . . 54
4.37.1 () . . . 54
4.38 Exercise 38 . . . 54
4.38.1 () . . . 54
4.39 Exercise 39 . . . 54
4.39.1 () . . . 54
4.40 Exercise 40 . . . 54
4.40.1 () . . . 54
4.41 Exercise 41 . . . 54
4.41.1 () . . . 54
4.42 Exercise 42 . . . 55
4.42.1 () . . . 55
4.43 Exercise 43 . . . 55
4.43.1 () . . . 55
4.44 Exercise 44 . . . 55
4.44.1 () . . . 55
4.45 Exercise 45 . . . 55
4.45.1 () . . . 55
4.46 Exercise 46 . . . 55
4.46.1 () . . . 55
4.47 Exercise 47 . . . 55
4.47.1 () . . . 55
4.48 Exercise 48 . . . 55
4.48.1 () . . . 55
4.49 Exercise 49 . . . 55
4.49.1 () . . . 55
4.50 Exercise 50 . . . 56
4.50.1 () . . . 56
4.51 Exercise 51 . . . 56
4.51.1 () . . . 56

5 Exercise Set 11.5 **56**
5.1 Exercise 1 . . . 56
5.1.1 () . . . 56
5.2 Exercise 2 . . . 56
5.2.1 () . . . 56
5.3 Exercise 3 . . . 56
5.3.1 () . . . 56
5.4 Exercise 4 . . . 56
5.4.1 () . . . 56
5.5 Exercise 5 . . . 56
5.5.1 () . . . 56
5.6 Exercise 6 . . . 56

5.6.1 () 56
5.7 Exercise 7 57
5.7.1 () 57
5.8 Exercise 8 57
5.8.1 () 57
5.9 Exercise 9 57
5.9.1 () 57
5.10 Exercise 10 57
5.10.1 () 57
5.11 Exercise 11 57
5.11.1 () 57
5.12 Exercise 12 57
5.12.1 () 57
5.13 Exercise 13 57
5.13.1 () 57
5.14 Exercise 14 57
5.14.1 () 57
5.15 Exercise 15 58
5.15.1 () 58
5.16 Exercise 16 58
5.16.1 () 58
5.17 Exercise 17 58
5.17.1 () 58
5.18 Exercise 18 58
5.18.1 () 58
5.19 Exercise 19 58
5.19.1 () 58
5.20 Exercise 20 58
5.20.1 () 58
5.21 Exercise 21 58
5.21.1 () 58
5.22 Exercise 22 58
5.22.1 () 58
5.23 Exercise 23 59
5.23.1 () 59
5.24 Exercise 24 59
5.24.1 () 59
5.25 Exercise 25 59
5.25.1 () 59
5.26 Exercise 26 59
5.26.1 () 59



1 Exercise Set 11.1

1.1 Exercise 1

The graph of a function f is shown above.

1.1.1 (a)

Is $f(0)$ positive or negative?

Proof. positive □

1.1.2 (b)

For what values of x does $f(x) = 0$?

Proof. $f(x) = 0$ when $x = -2$ and $x = 3$ (approximately) □

1.1.3 (c)

Find approximate values for x_1 and x_2 so that $f(x_1) = f(x_2) = 1$ but $x_1 \neq x_2$.

Proof. $x_1 = -1$ and $x_2 = 2$ (approximately) □

1.1.4 (d)

Find an approximate value for x such that $f(x) = 1.5$.

Proof. $x = 1$ or $x = -1/2$ (approximately) □

1.1.5 (e)

As x increases from -3 to -1, do the values of f increase or decrease?

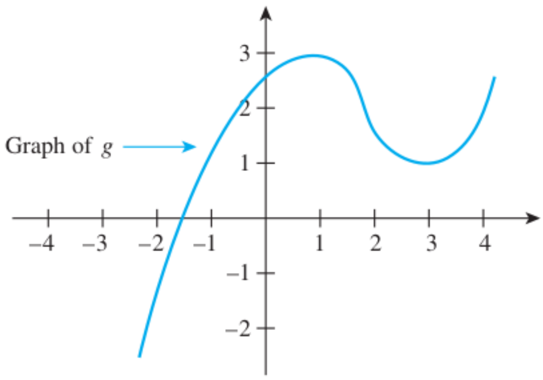
Proof. increase □

1.1.6 (f)

As x increases from 0 to 4, do the values of f increase or decrease?

Proof. decrease □

1.2 Exercise 2



The graph of a function g is shown above.

1.2.1 (a)

Is $g(0)$ positive or negative?

Proof. positive □

1.2.2 (b)

Find an approximate value of x so that $g(x) = 0$.

Proof. -1.5 (approximately) □

1.2.3 (c)

Find approximate values for x_1 and x_2 so that $g(x_1) = g(x_2) = 1$ but $x_1 \neq x_2$.

Proof. $x_1 = -1, x_2 = 3$ (approximately) □

1.2.4 (d)

Find an approximate value for x such that $g(x) = -2$.

Proof. $x = -2.2$ (approximately) □

1.2.5 (e)

As x increases from -2 to 1, do the values of g increase or decrease?

Proof. increase □

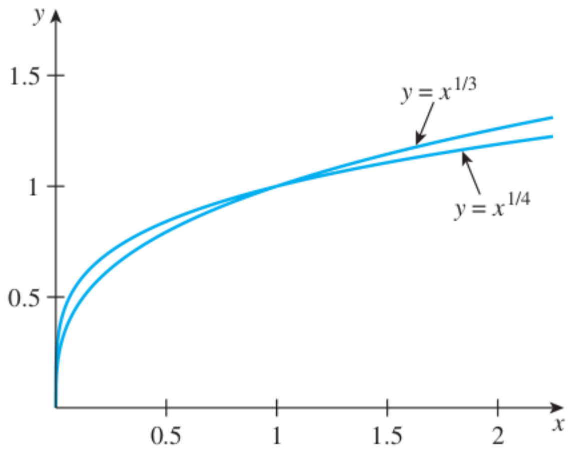
1.2.6 (f)

As x increases from 1 to 3, do the values of g increase or decrease?

Proof. decrease $\square$

1.3 Exercise 3

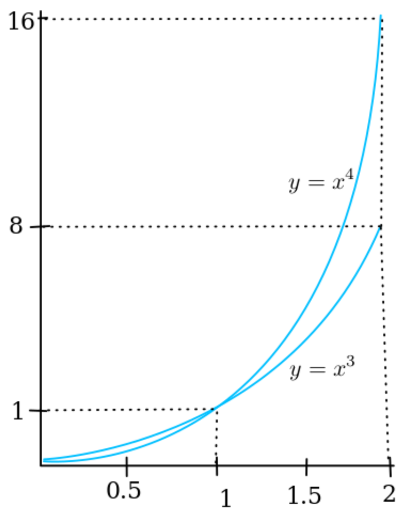
Sketch the graphs of the power functions $p_{1/3}$ and $p_{1/4}$ on the same set of axes. When $0 < x < 1$, which is greater: $x^{1/3}$ or $x^{1/4}$? When $x > 1$, which is greater: $x^{1/3}$ or $x^{1/4}$?



Proof. When $0 < x < 1$, $x^{1/3} < x^{1/4}$. When $1 < x$, $x^{1/4} < x^{1/3}$. $\square$

1.4 Exercise 4

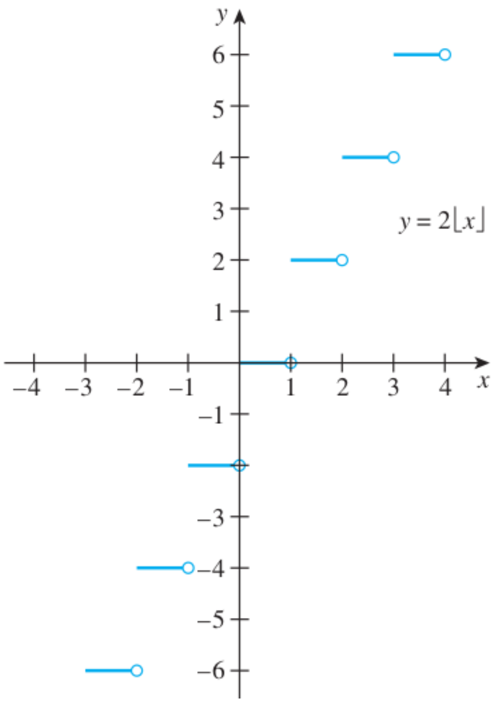
Sketch the graphs of the power functions p_3 and p_4 on the same set of axes. When $0 < x < 1$, which is greater: x^3 or x^4? When $x > 1$, which is greater: x^3 or x^4?

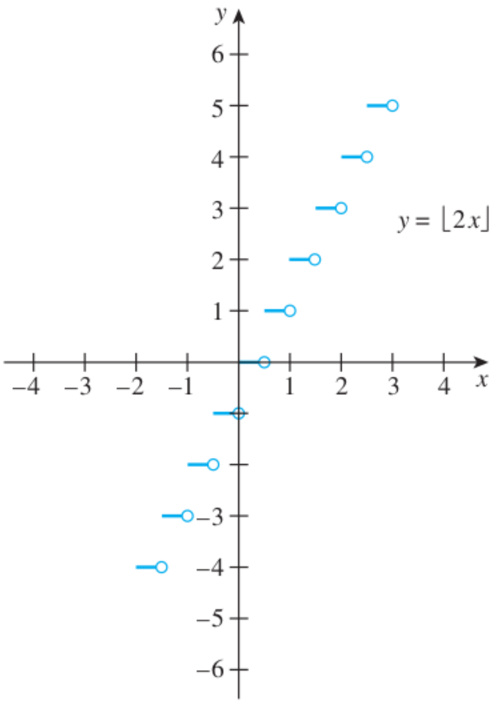


Proof. When $0 < x < 1$, $x^4 < x^3$. When $1 < x$, $x^3 < x^4$. $\square$

1.5 Exercise 5

Sketch the graphs of $y = 2\lfloor x \rfloor$; and $y = \lfloor 2x \rfloor$ for each real number x. What can you conclude from these graphs?





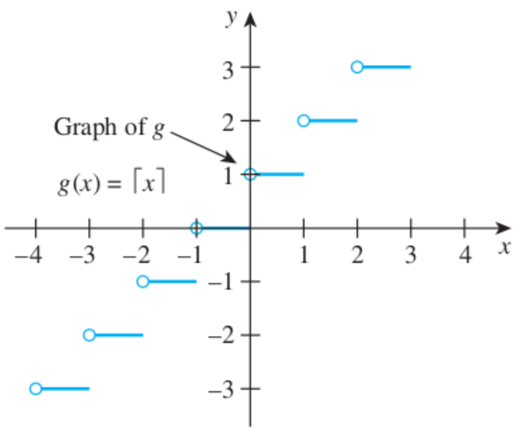
Proof.

The graphs show that $2\lfloor x \rfloor \neq \lfloor 2x \rfloor$ for many values of x. □

Sketch a graph for each of the functions defined in $6-9$ below.

1.6 Exercise 6

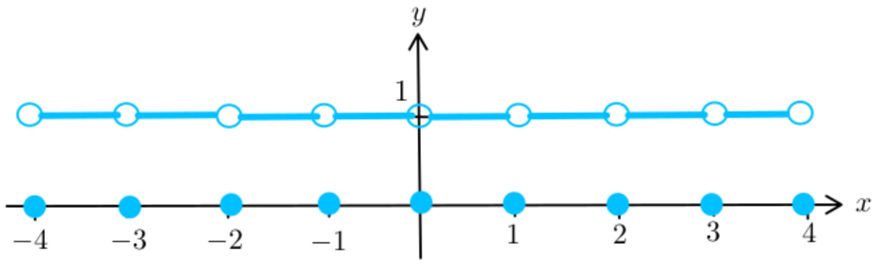
$g(x) = \lceil x \rceil$ for each real number x (Recall that the ceiling of x, $\lceil x \rceil$, is the least integer that is greater than or equal to x. That is, $\lceil x \rceil$ = the unique integer n such that $n - 1 < x \leq n$.



Proof. □

1.7 Exercise 7

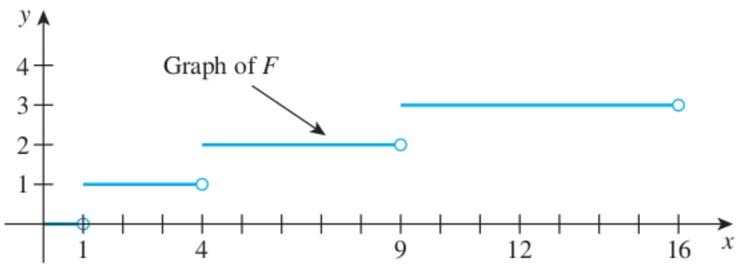
$h(x) = \lceil x \rceil - \lfloor x \rfloor$ for each real number x



Proof. $\square$

1.8 Exercise 8

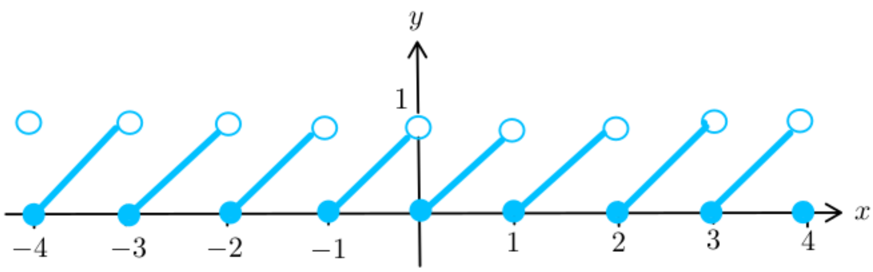
$F(x) = \lfloor x^{1/2} \rfloor$ for each real number x



Proof. $\square$

1.9 Exercise 9

$G(x) = x - \lfloor x \rfloor$ for each real number x



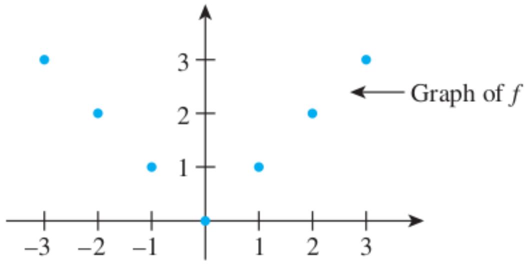
Proof. $\square$

In each of $10 - 13$ a function is defined on a set of integers. Sketch a graph for each function.

1.10 Exercise 10

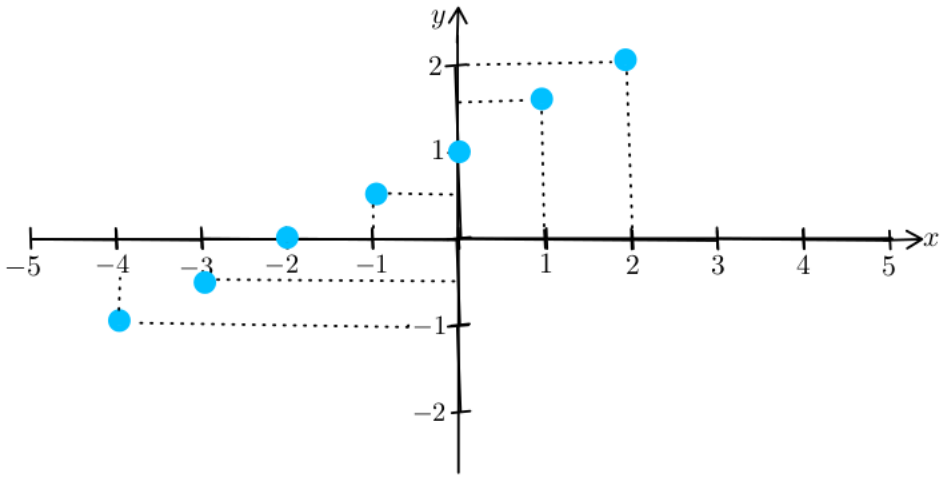
$f(n) = |n|$ for each integer n

Proof. $\square$



1.11 Exercise 11

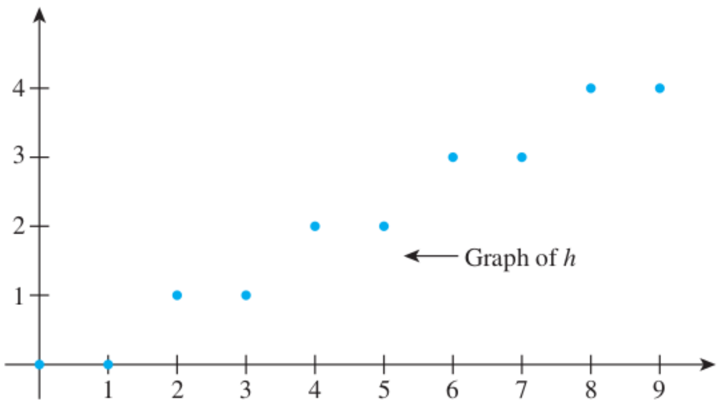
$g(n) = (n/2) + 1$ for each integer n



Proof. □

1.12 Exercise 12

$h(n) = \lfloor n/2 \rfloor$ for each integer $n \geq 0$

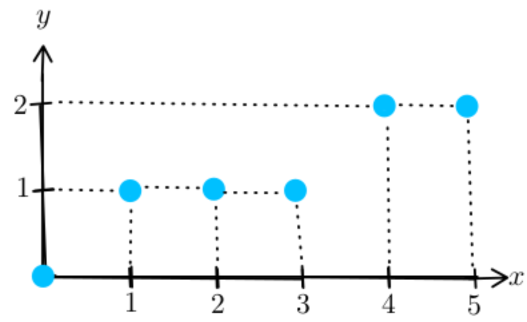


Proof. □

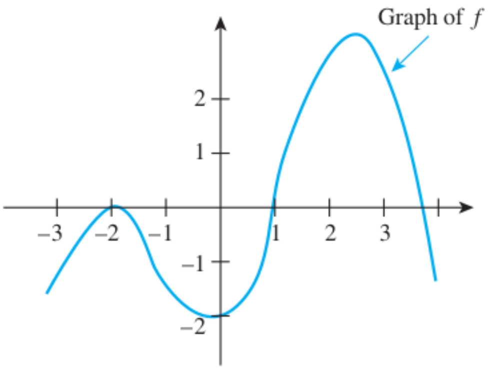
1.13 Exercise 13

$k(n) = \lfloor n^{1/2} \rfloor$ for each integer $n \geq 0$

Proof. □



1.14 Exercise 14



The graph of a function f is shown below. Find the intervals on which f is increasing and the intervals on which f is decreasing.

Proof. f is increasing on the intervals $\{x \in \mathbb{R} \,|\, -3 < x < -2\}$ and $\{x \in \mathbb{R} \,|\, 0 < x < 2.5\}$, and f is decreasing on $\{x \in \mathbb{R} \,|\, -2 < x < 0\}$ and $\{x \in \mathbb{R} \,|\, 2.5 < x < 4\}$ (approximately). □

1.15 Exercise 15

Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = 2x - 3$ is increasing on the set of real numbers.

Proof. Suppose that x_1 and x_2 are particular but arbitrarily chosen real numbers such that $x_1 < x_2$. *[We must show that $f(x_1) < f(x_2)$.]* Since $x_1 < x_2$ then $2x_1 < 2x_2$ and $2x_1 - 3 < 2x_2 - 3$ by basic properties of inequalities. Thus, by definition of f, $f(x_1) < f(x_2)$ *[as was to be shown]*. Hence f is increasing on the set of all real numbers. □

1.16 Exercise 16

Show that the function $g : \mathbb{R} \to \mathbb{R}$ defined by the formula $g(x) = -(x/3)+1$ is decreasing on the set of real numbers.

Proof. □

1.17 Exercise 17

Let h be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by the formula $h(x) = x^2$ for each real number x.

1.17.1 (a)

Show that h is decreasing on the set of real numbers less than zero.

Proof. Suppose that x_1 and x_2 are particular but arbitrarily chosen real numbers such that $x_1 < x_2 < 0$. *[We must show that $h(x_1) > h(x_2)$.]*

Since $x_1 < x_2 < 0$ then $0 < -x_2 < -x_1$ and multiplying by $-x_1$ (which is a positive number) we get $(-x_1)(-x_2) < (-x_1)(-x_1) = x_1^2$ by basic properties of inequalities.

Similarly, since $x_1 < x_2 < 0$ then $0 < -x_2 < -x_1$ and multiplying by $-x_2$ (which is a positive number) we get $(-x_2)(-x_2) = x_2^2 < (-x_1)(-x_2)$ by basic properties of inequalities.

By combining the two results we get $x_2^2 < (-x_1)(-x_2) < x_1^2$ so $x_2^2 < x_1^2$.

Thus, by definition of h, $h(x_1) > h(x_2)$ *[as was to be shown]*. Hence h is increasing on the set of all real numbers. □

1.17.2 (b)

Show that h is increasing on the set of real numbers greater than zero.

Proof. Suppose that x_1 and x_2 are particular but arbitrarily chosen real numbers such that $0 < x_1 < x_2$. *[We must show that $h(x_1) < h(x_2)$.]*

Since $0 < x_1 < x_2$ then multiplying by x_1 (which is a positive number) we get $x_1x_1 = x_1^2 < x_1x_2$ by basic properties of inequalities.

Similarly, since $0 < x_1 < x_2$ then multiplying by x_2 (which is a positive number) we get $x_1x_2 < x_2x_2 = x_2^2$ by basic properties of inequalities.

By combining the two results we get $x_1^2 < x_1x_2 < x_2^2$ so $x_1^2 < x_2^2$.

Thus, by definition of h, $h(x_1) < h(x_2)$ *[as was to be shown]*. Hence h is increasing on the set of all real numbers. □

1.18 Exercise 18

Let $k : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula $k(x) = (x-1)/x$ for each real number $x \neq 0$.

1.18.1 (a)

Show that k is increasing for every real number $x > 0$.

Proof. Suppose that x_1 and x_2 are positive real numbers and $x_1 < x_2$. *[We must show that $k(x_1) < k(x_2)$.]*

$$\begin{array}{lll}
 & x_1 < x_2 & \text{by assumption} \\
\Longrightarrow & -x_2 < -x_1 & \text{by multiplying by } -1 \\
\Longrightarrow & x_1x_2 - x_2 < x_1x_2 - x_1 & \text{by adding } x_1x_2 \text{ to both sides} \\
\Longrightarrow & x_2(x_1 - 1) < x_1(x_2 - 1) & \text{by factoring both sides} \\
\Longrightarrow & \dfrac{x_1 - 1}{x_1} < \dfrac{x_2 - 1}{x_2} & \text{by dividing both sides by } x_1x_2 > 0 \\
\Longrightarrow & k(x_1) < k(x_2) & \text{by definition of } k
\end{array}$$

□

1.18.2 (b)

Is k increasing or decreasing for $x < 0$? Prove your answer.

Proof. It is increasing. The same proof as in part (a) works. Note that the only place in the proof where the signs of x_1 and x_2 matter is when we divide both sides by x_1x_2. For the proof to work, x_1x_2 has to be positive. But if both x_1 and x_2 are negative, then x_1x_2 *is* positive. Therefore the proof still works. □

1.19 Exercise 19

Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is increasing, then f is one-to-one.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is increasing. *[We must show that f is one-to-one. In other words, we must show that for all real numbers x_1 and x_2 , if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.]* Suppose x_1 and x_2 are real numbers and $x_1 \neq x_2$. By the trichotomy law *[Appendix A, T17]* $x_1 < x2$, or $x_1 > x_2$. In case $x_1 < x_2$, then since f is increasing, $f(x_1) < f(x_2)$ and so $f(x_1) \neq f(x_2)$. Similarly, in case $x_1 > x_2$, then $f(x_1) > f(x_2)$ and so $f(x_1) \neq f(x_2)$. Thus in either case, $f(x_1) \neq f(x_2)$ *[as was to be shown]*. □

1.20 Exercise 20

Given real-valued functions f and g with the same domain D, the sum of f and g, denoted $f + g$, is defined as follows: For each real number x, $(f + g)(x) = f(x) + g(x)$. Show that if f and g are both increasing on a set S, then $f + g$ is also increasing on S.

Proof. Assume $x_1, x_2 \in S$ and $x_1 < x_2$. *[We want to show $(f + g)(x_1) < (f + g)(x_2)$.]* Since f is increasing, $f(x_1) < f(x_2)$. Since g is increasing, $g(x_1) < g(x_2)$. By definition of $f + g$ we have $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$, *[as was to be shown.]* □

1.21 Exercise 21

1.21.1 (a)

Let m be any positive integer, and define $f(x) = x^m$ for each nonnegative real number x. Use the binomial theorem to show that f is an increasing function.

Proof. Suppose u and v are nonnegative real numbers with $u < v$. *[We must show that $f(u) < f(v)$.]* Note that $v = u + h$ for some positive real number h. By substitution and the binomial theorem,

$$v^m = (u+h)^m = \sum_{i=0}^{m} \binom{m}{i} u^{m-i} h^i = u^m + \sum_{i=1}^{m} \binom{m}{i} u^{m-i} h^i$$

The last summation is positive because $u \geq 0$ and $h > 0$, and a sum of nonnegative terms that includes at least one positive term is positive. Hence $v^m = u^m +$ a positive number, and so $f(u) = u^m < v^m = f(v)$, *[as was to be shown]*. □

1.21.2 (b)

Let m and n be any positive integers, and let $g(x) = x^{m/n}$ for each nonnegative real number x. Prove that g is an increasing function.

Note: The results of exercise 21 are used in the exercises for Sections 11.2 and 11.4.

Proof. Write $f(x) = x^m$. Then $g(x) = (f(x))^{1/n}$ by the law of exponents.

Now assume $0 \leq x_1 < x_2$. In part (a) we showed that f is increasing. Therefore $f(x_1) < f(x_2)$, in other words $x_1^m < x_2^m$. So we need to show that the function $h(x) = x^{1/n}$ is an increasing function. That will imply $g(x_1) = h(x_1^m) < h(x_2^m) = g(x_2)$, in other words $x_1^{m/n} < x_2^{m/n}$, which is what we want.

To show h is increasing, assume $0 \leq z_1 < z_2$. By definition, $h(z_1) = z_1^{1/n} = y_1$ is the real number with the property that $y_1^n = z_1$. Similarly $h(z_2) = z_2^{1/n} = y_2$ is the real number with the property that $y_2^n = z_2$. *[We want to show $y_1 < y_2$.]*

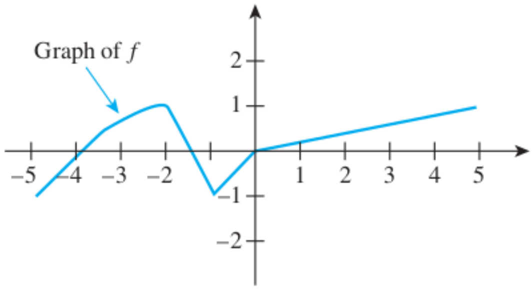
Argue by contradiction and assume $y_2 \leq y_1$. Now consider the function $e(y) = y^n$. This function is also increasing by part (a), since m and n are both any positive integers. Therefore $e(y_2) \leq e(y_1)$, in other words $z_2 \leq z_1$, which is a contradiction!

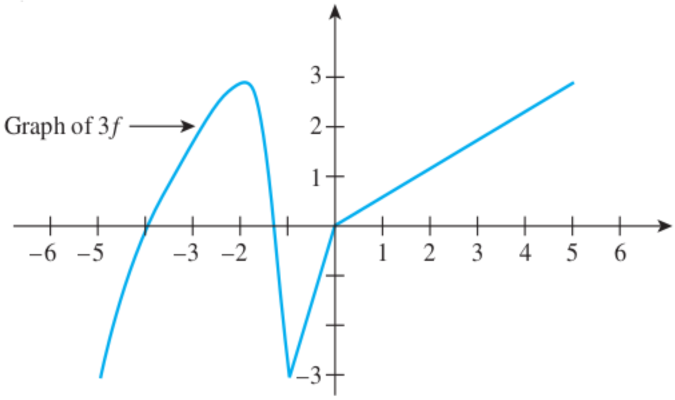
Therefore $y_1 < y_2$ and h is increasing, and thus g is increasing as a consequence. □

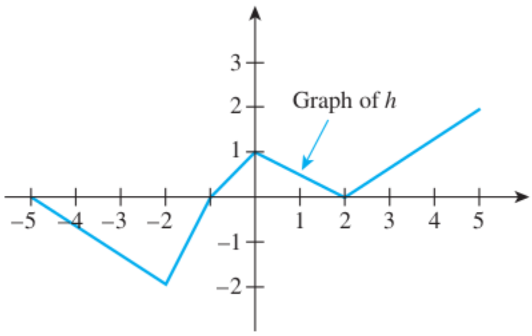
1.22 Exercise 22

Let f be the function whose graph follows. Sketch the graph of $3f$.

Proof. □

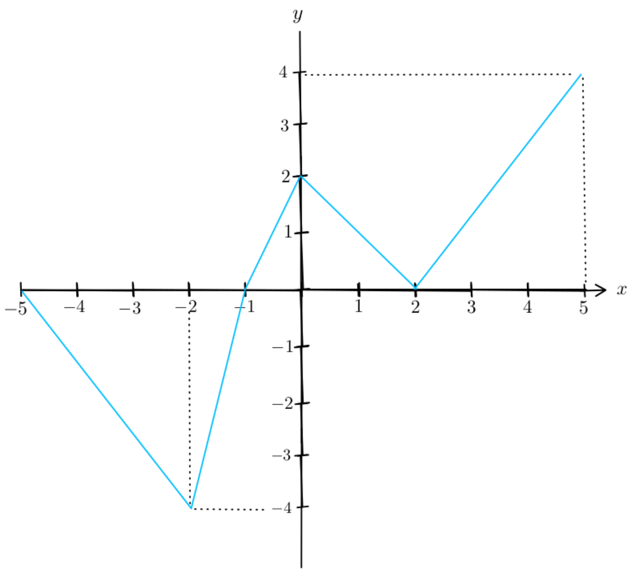






1.23 Exercise 23

Let h be the function whose graph is shown above. Sketch the graph of $2h$.



Proof. □

1.24 Exercise 24

Let f be a real-valued function of a real variable. Show that if f is decreasing on a set S and if M is any positive real number, then Mf is decreasing on S.

Proof. Suppose that f is a real-valued function of a real variable, f is decreasing on a set S, and M is any positive real number. *[We must show that Mf is decreasing on S. In other words, we must show that for all x_1 and x_2 in S, if $x_1 < x_2$ then $(Mf)(x_1) > (Mf)(x_2)$.]* Suppose x_1 and x_2 are in S and $x_1 < x_2$. Since f is decreasing on S, $f(x_1) > f(x_2)$, and since M is positive, $Mf(x_1) > Mf(x_2)$ *[because when both sides of an inequality are multiplied by a positive number, the direction of the inequality is unchanged]*. It follows by definition of Mf that $(Mf)(x_1) > (Mf)(x_2)$, *[as was to be shown]*. □

1.25 Exercise 25

Let f be a real-valued function of a real variable. Show that if f is increasing on a set S and if M is any negative real number, then Mf is decreasing on S.

Proof. The proof is the same as in Exercise 24, except that this time we have $f(x_1) < f(x_2)$ because f is increasing, and multiplying an inequality by a negative number M reverses the direction of the equality, so $Mf(x_1) > Mf(x_2)$. □

1.26 Exercise 26

Let f be a real-valued function of a real variable. Show that if f is decreasing on a set S and if M is any negative real number, then Mf is increasing on S.

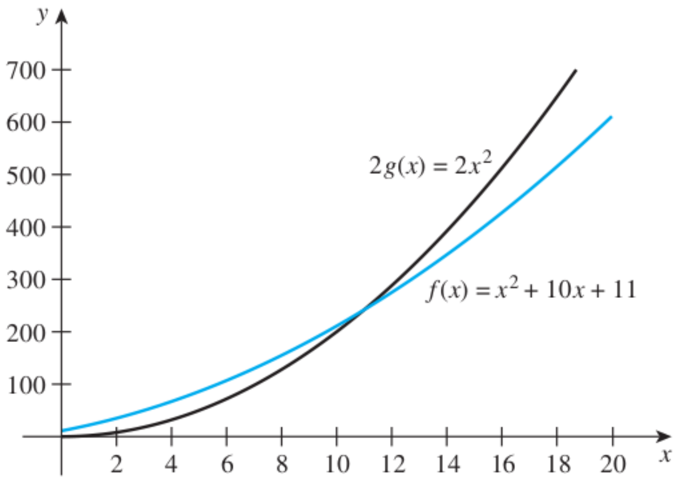
Proof. The proof is the same as in Exercise 24, except that this time multiplying an inequality by a negative number M reverses the direction of the equality, so $Mf(x_1) < Mf(x_2)$. □

In 27 and 28, functions f and g are defined. In each case sketch the graphs of f and $2g$ on the same set of axes and find a number x_0 so that $f(x) \leq 2g(x)$ for all $x > x_0$. You can find an exact value for x_0 by solving a quadratic equation, or you can find an approximate value for x_0 by using a graphing calculator or computer.

1.27 Exercise 27

$f(x) = x^2 + 10x + 11$ and $g(x) = x^2$ for each real number $x \geq 0$

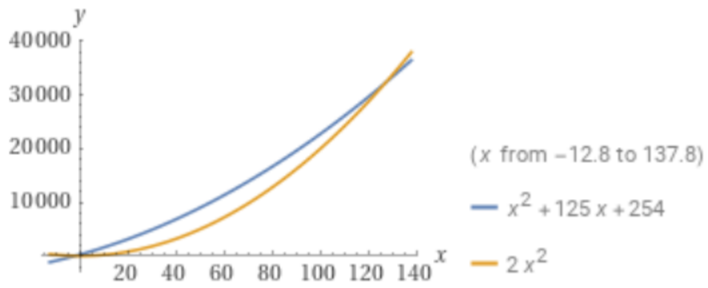
Proof. To find the answer algebraically, solve the equation $2x^2 = x^2 + 10x + 11$ for x. Subtracting x^2 from both sides gives $x^2 - 10x - 11 = 0$, and either using the quadratic formula or factoring $x^2 - 10x - 11 = (x - 11)(x + 1)$ gives $x = 11$ (since $x > 0$). To find an approximate answer with a graphing calculator, plot both $f(x) = x^2 + 10x + 11$ and $2g(x) = 2x^2$ for $x > 0$, as shown in the figure, and find that $2g(x) > f(x)$ when



$x > 11$ (approximately). You can obtain only an approximate answer from a graphing calculator because the calculator computes values only to an accuracy of a finite number of decimal places. □

1.28 Exercise 28

$f(x) = x^2 + 125x + 254$ and $g(x) = x^2$ for each real number $x \geq 0$



Proof. If we set $f(x) = 2g(x)$ and solve, we get $x^2 + 125x + 254 = 2x^2$ which gives $x^2 - 125x - 254 = 0$ which factors as $(x-127)(x+2) = 0$ which has solutions $x = -2, 127$. So let $x_0 = 127$, so that $f(x) < g(x)$ for all $x > x_0 = 127$. □

2 Exercise Set 11.2

2.1 Exercise 1

The following is a formal definition for Ω-notation, written using quantifiers and variables: $f(n)$ is $\Omega(g(n))$ if, and only if, $\exists$ positive real numbers a and A such that $\forall n \geq a, Ag(n) \leq f(n)$.

2.1.1 (a)

Write the formal negation for the definition using the symbols $\forall$ and $\exists$.

Proof. Formal version of negation: $f(n)$ is not $\Omega(g(n))$ if, and only if, $\forall$ positive real numbers a and A, $\exists$ an integer $n \geq a$ such that $Ag(n) > f(n)$. □

2.1.2 (b)

Restate the negation less formally without using the symbols $\forall$ and $\exists$ or the words "for any," "for every," or "there exists."

Proof. Informal version of negation: $f(n)$ is not $\Omega(g(n))$ if, and only if, no matter what positive real numbers a and A might be chosen, it is possible to find an integer n greater than or equal to a with the property that $Ag(n) > f(n)$. $\square$

2.2 Exercise 2

The following is a formal definition for O-notation, written using quantifiers and variables: $f(n)$ is $O(g(n))$ if, and only if, $\exists$ positive real numbers b and B such that $\forall n \geq b$, $0 \leq f(n) \leq Bg(n)$.

2.2.1 (a)

Write the formal negation for the definition using the symbols $\forall$ and $\exists$.

Proof. $f(n)$ is not $O(g(n))$ if, and only if, $\forall$ positive real numbers b and B, $\exists n \geq b$ such that $0 > f(n)$ or $f(n) > Bg(n)$. $\square$

2.2.2 (b)

Restate the negation less formally without using the symbols $\forall$ and $\exists$ or the words "for any," "for every," or "there exists."

Proof. $f(n)$ is not $O(g(n))$ if, and only if, no matter what positive real numbers b and B are chosen, it is possible to choose an integer n greater than b with the property that either $0 > f(n)$ or $f(n) > Bg(n)$. $\square$

2.3 Exercise 3

The following is a formal definition for Θ-notation, written using quantifiers and variables: $f(n)$ is $\Theta(g(n))$ if, and only if, $\exists$ positive real numbers k, A and B such that $\forall n \geq b, Ag(n) \leq f(n) \leq Bg(n)$.

2.3.1 (a)

Write the formal negation for the definition using the symbols $\forall$ and $\exists$.

Proof. $f(n)$ is not $\Theta(g(n))$ if, and only if, $\forall$ positive real numbers k, A and B, $\exists n \geq b$ such that $Ag(n) > f(n)$ or $f(n) > Bg(n)$. $\square$

2.3.2 (b)

Restate the negation less formally without using the symbols $\forall$ and $\exists$ or the words "for any," "for every," or "there exists."

Proof. $f(n)$ is not $\Theta(g(n))$ if, and only if, no matter what positive real numbers k, A and B are chosen, it is possible to choose an integer n greater than b with the property that either $Ag(n) > f(n)$ or $f(n) > Bg(n)$. $\square$

In $4-9$, express each statement using Ω-, O-, or Θ-notation.

2.4 Exercise 4

$\frac{1}{2}n \leq n - \left\lfloor \frac{n}{2} \right\rfloor + 1$ for every integer $n \geq 1$. (Use Ω-notation).

Proof. $n - \left\lfloor \frac{n}{2} \right\rfloor + 1$ is $\Omega(n)$ $\square$

2.5 Exercise 5

$0 \leq n - \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq n$ for every integer $n \geq 3$. (Use O-notation).

Proof. $n - \left\lfloor \frac{n}{2} \right\rfloor + 1$ is $O(n)$ $\square$

2.6 Exercise 6

$n^2 \leq 3n(n-2) \leq 4n^2$ for every integer $n \geq 3$. (Use Θ-notation.)

Proof. $3n(n-2)$ is $\Theta(n^2)$ $\square$

2.7 Exercise 7

$\frac{1}{2}n^2 \leq \frac{n(3n-2)}{2}$ for every integer $n \geq 3$. (Use Ω-notation).

Proof. $\frac{n(3n-2)}{2}$ is $\Omega(n^2)$ $\square$

2.8 Exercise 8

$0 \leq \frac{n(3n-2)}{2} \leq n^2$ for every integer $n \geq 1$. (Use O-notation).

Proof. $\frac{n(3n-2)}{2}$ is $O(n^2)$ $\square$

2.9 Exercise 9

$\frac{n^3}{6} \le n^2\left(\left\lceil \frac{n}{3} \right\rceil - 1\right) \le n^3$ for every integer $n \ge 2$. (Use Θ-notation.)

Proof. $n^2\left(\left\lceil \frac{n}{3} \right\rceil - 1\right)$ is $\Theta(n^3)$ □

2.10 Exercise 10

2.10.1 (a)

Show that for any integer $n \ge 1, 0 \le 2n^2 + 15n + 4 \le 21n^2$.

Proof. For each integer $n \ge 1, 0 \le 2n^2 + 15n + 4$ because all terms in $2n^2 + 15n + 4$ are positive. Moreover, $2n^2 + 15n + 4 \le 2n^2 + 15n^2 + 4n^2$ because when $n \ge 1$, $15n \le 15n^2$ and $4 \le 4n^2$, which add up to $21n^2$ by combining like terms. Therefore, by transitivity of equality and order, $0 \le 2n^2 + 15n + 4 \le 21n^2$ for each integer $n \ge 1$. □

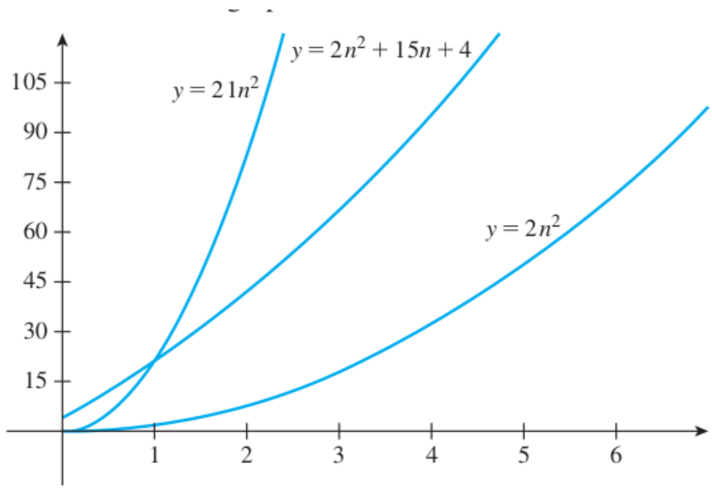
2.10.2 (b)

Show that for any integer $n \ge 1, 2n^2 \le 2n^2 + 15n + 4$.

Proof. For each integer $n \ge 1, 2n^2 \le 2n^2 + 15n + 4$ because $15n + 4 > 0$ since n is positive. □

2.10.3 (c)

Sketch a graph to illustrate the results of parts (a) and (b).



Proof. □

2.10.4 (d)

Use the O- and Ω-notations to express the results of parts (a) and (b).

Proof. Let $A = 2$ and $a = 1$. Then, by substitution from the result of part (b), $An^2 < 2n^2+15n+4$ for each integer $n \geq a$, and hence, by definition of Ω-notation, $2n^2+15n+4$ is $\Omega(n^2)$. Let $B = 21$ and $b = 1$. Then, by substitution from the result of part (a), $0 < 2n^2 + 15n + 4 \leq Bn^2$ for each integer $n \geq b$, and hence by definition of O-notation, $2n^2 + 15n + 4$ is $O(n^2)$. ☐

2.10.5 (e)

What can you deduce about the order of $2n^2 + 15n + 4$?

Proof. *Solution 1:* Let $A = 2, B = 21$, and $k = 1$. By the results of parts (a) and (b), $An^2 \leq 2n^2 + 15n + 4 \leq Bn^2$ for each integer $n \geq k$, and hence, by definition of Θ-notation, $2n^2 + 15n + 4$ is $\Theta(n^2)$.

Solution 2: By part (d), $2n^2 + 15n + 4$ is both $\Omega(n^2)$ and $O(n^2)$. Hence, by Theorem 11.2.1, $2n^2 + 15n + 4$ is $\Theta(n^2)$. ☐

2.11 Exercise 11

2.11.1 (a)

Show that for any integer $n \geq 1, 0 \leq 23n^4 + 8n^2 + 4n \leq 35n^4$.

Proof. For each integer $n \geq 1, 0 \leq 23n^4 + 8n^2 + 4n$ because all terms in $23n^4 + 8n^2 + 4n$ are positive. Moreover, $23n^4 + 8n^2 + 4n \leq 23n^4 + 8n^4 + 4n^4$ because when $n \geq 1$, $8n^2 \leq 8n^4$ and $4n \leq 4n^4$, which add up to $35n^4$ by combining like terms. Therefore, by transitivity of equality and order, $0 \leq 23n^4+8n^2+4n \leq 35n^4$ for each integer $n \geq 1$. ☐

2.11.2 (b)

Show that for any integer $n \geq 1, 23n^4 \leq 23n^4 + 8n^2 + 4n$.

Proof. For each integer $n \geq 1, 23n^4 \leq 23n^4 + 8n^2 + 4n$ because $8n^2 + 4n > 0$ since n is positive. ☐

2.11.3 (c)

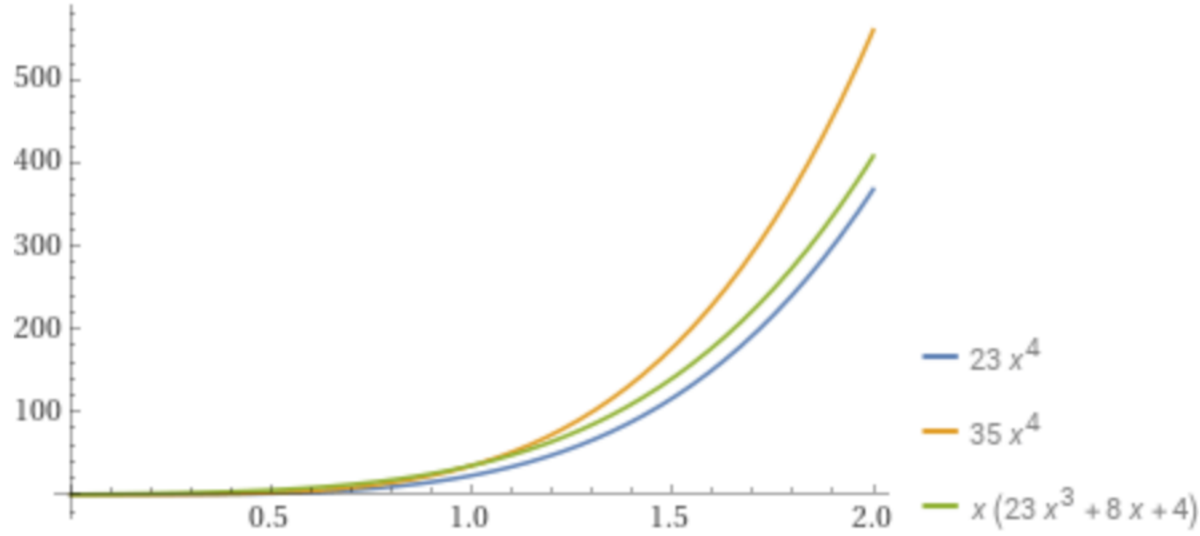
Sketch a graph to illustrate the results of parts (a) and (b).

Proof. ☐

2.11.4 (d)

Use the O- and Ω-notations to express the results of parts (a) and (b).

Proof. Let $A = 23$ and $a = 1$. Then, by substitution from the result of part (b), $An^4 < 23n^4 + 8n^2 + 4n$ for each integer $n \geq a$, and hence, by definition of Ω-notation, $23n^4 + 8n^2 + 4n$ is $\Omega(n^4)$. Let $B = 35$ and $b = 1$. Then, by substitution from the result



of part (a), $0 < 23n^4 + 8n^2 + 4n \leq Bn^4$ for each integer $n \geq b$, and hence by definition of O-notation, $23n^4 + 8n^2 + 4n$ is $O(n^4)$. $\square$

2.11.5 (e)

What can you deduce about the order of $23n^4 + 8n^2 + 4n$?

Proof. By part (d), $23n^4+8n^2+4n$ is both $\Omega(n^4)$ and $O(n^4)$. Hence, by Theorem 11.2.1, $23n^4 + 8n^2 + 4n$ is $\Theta(n^4)$. $\square$

2.12 Exercise 12

2.12.1 (a)

Show that for any integer $n \geq 1, 0 \leq 7n^3 + 10n^2 + 3 \leq 20n^3$.

Proof. For each integer $n \geq 1, 0 \leq 7n^3+10n^2+3$ because all terms in $7n^3+10n^2+3$ are positive. Moreover, $7n^3+10n^2+3 \leq 7n^3+10n^3+3n^3$ because when $n \geq 1$, $10n^2 \leq 10n^3$ and $3 \leq 3n^3$, which add up to $20n^3$ by combining like terms. Therefore, by transitivity of equality and order, $0 \leq 7n^3 + 10n^2 + 3 \leq 20n^3$ for each integer $n \geq 1$. $\square$

2.12.2 (b)

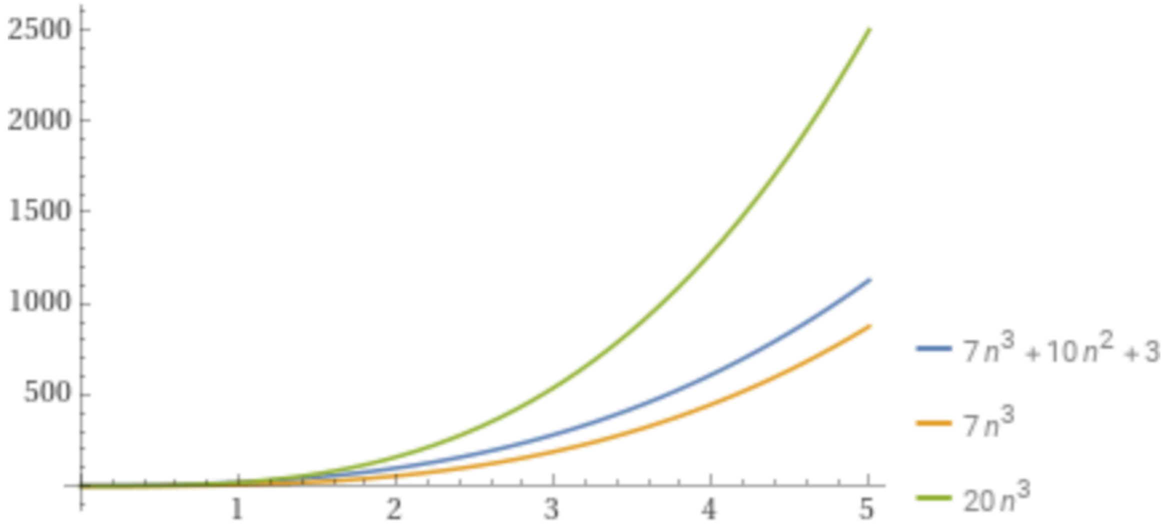
Show that for any integer $n \geq 1, 7n^3 \leq 7n^3 + 10n^2 + 3$.

Proof. For each integer $n \geq 1, 7n^3 \leq 7n^3 + 10n^2 + 3$ because $10n^2 + 3 > 0$ since n is positive. $\square$

2.12.3 (c)

Sketch a graph to illustrate the results of parts (a) and (b).

Proof. $\square$



2.12.4 (d)

Use the O- and Ω-notations to express the results of parts (a) and (b).

Proof. Let $A = 7$ and $a = 1$. Then, by substitution from the result of part (b), $An^3 < 7n^3+10n^2+3$ for each integer $n \geq a$, and hence, by definition of Ω-notation, $7n^3+10n^2+3$ is $\Omega(n^3)$. Let $B = 20$ and $b = 1$. Then, by substitution from the result of part (a), $0 < 7n^3 + 10n^2 + 3 \leq Bn^3$ for each integer $n \geq b$, and hence by definition of O-notation, $7n^3 + 10n^2 + 3$ is $O(n^3)$. □

2.12.5 (e)

What can you deduce about the order of $7n^3 + 10n^2 + 3$?

Proof. By part (d), $7n^3 + 10n^2 + 3$ is both $\Omega(n^3)$ and $O(n^3)$. Hence, by Theorem 11.2.1, $7n^3 + 10n^2 + 3$ is $\Theta(n^3)$. □

2.13 Exercise 13

Use the definition of Θ-notation to show that $5n^3 + 65n + 30$ is $\Theta(n^3)$.

Proof. For each integer $n \geq 1$, $5n^3 \leq 5n^3 + 65n + 30$ because $65n + 30 > 0$ since n is positive. Moreover, $5n^3 + 65n + 30 \leq 5n^3 + 65n^3 + 30n^3$ because when $n \geq 1$, then $65n < 65n^3$ and $30 < 30n^3$, which add up to $100n^3$ by combining like terms. Therefore, by transitivity of order and equality, $5n^3 \leq 5n^3 + 65n + 30 \leq 100n^3$. Thus, let $A = 5, B = 100$, and $k = 1$. Then $An^3 \leq 5n^3 + 65n + 30 \leq Bn^3$ for each integer $n \geq k$, and hence, by definition of Θ-notation, $5n^3 + 65n + 30$ is $\Theta(n^3)$. □

2.14 Exercise 14

Use the definition of Θ-notation to show that $n^2 + 100n + 88$ is $\Theta(n^2)$.

Proof. For each integer $n \geq 1$, $n^2 \leq n^2 + 100n + 88$ because $100n + 88 > 0$ since n is positive. Moreover, $n^2 + 100n + 88 \leq n^2 + 100n^2 + 88n^2$ because when $n \geq 1$, then $100n < 100n^2$ and $88 < 88n^2$, which add up to $189n^2$ by combining like terms. Therefore, by transitivity of order and equality, $n^2 \leq n^2 + 100n + 88 \leq 189n^2$. Thus, let $A = 1, B = 189$, and $k = 1$. Then $An^2 \leq n^2 + 100n + 88 \leq Bn^2$ for each integer $n \geq k$, and hence, by definition of Θ-notation, $n^2 + 100n + 88$ is $\Theta(n^2)$. $\square$

2.15 Exercise 15

Use the definition of Θ-notation to show that $\left\lfloor n + \frac{1}{2} \right\rfloor$ is $\Theta(n)$.

Proof. For each integer $n \geq 1$, $n \leq n + \frac{1}{2} < n+1$, and so by definition of floor, $\left\lfloor n + \frac{1}{2} \right\rfloor = n$, and $\left\lfloor n + \frac{1}{2} \right\rfloor$ is nonnegative. In addition, when $n \geq 1$, then $n + 1 \leq n + n = 2n$, and thus, by transitivity of equality and order, $n \leq \left\lfloor n + \frac{1}{2} \right\rfloor \leq 2n$. Let $A = 1, B = 2$, and $k = 1$. Then $An \leq \left\lfloor n + \frac{1}{2} \right\rfloor \leq Bn$ for every integer $n \geq k$, and hence, by definition of Θ-notation, $\left\lfloor n + \frac{1}{2} \right\rfloor$ is $\Theta(n)$. $\square$

2.16 Exercise 16

Use the definition of Θ-notation to show that $\left\lceil n + \frac{1}{2} \right\rceil$ is $\Theta(n)$.

Proof. For each integer $n \geq 1$, $n < n + \frac{1}{2} \leq n + 1$, and so by definition of ceiling, $\left\lceil n + \frac{1}{2} \right\rceil = n + 1$, and $\left\lceil n + \frac{1}{2} \right\rceil$ is nonnegative. In addition, when $n \geq 1$, then $n + 1 \leq n + n = 2n$, and thus, by transitivity of equality and order, $n < \left\lceil n + \frac{1}{2} \right\rceil \leq 2n$. Let $A = 1, B = 2$, and $k = 1$. Then $An \leq \left\lceil n + \frac{1}{2} \right\rceil \leq Bn$ for every integer $n \geq k$, and hence, by definition of Θ-notation, $\left\lceil n + \frac{1}{2} \right\rceil$ is $\Theta(n)$. $\square$

2.17 Exercise 17

Use the definition of Θ-notation to show that $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$. (*Hint:* Show that if $n \geq 4$, then $\frac{n}{2} - 1 \geq \frac{1}{4}n$.)

Proof. Assume $n \geq 2$ is even.

Then $n = 2k$ for some integer k and thus $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = \frac{n}{2}$. Then notice that $\frac{1}{4}n \leq \frac{n}{2} \leq n$. So $\frac{1}{4}n \leq \left\lfloor \frac{n}{2} \right\rfloor \leq n$.

Now assume $n \geq 2$ is odd.

Then $n = 2k+1$ for some integer k and thus $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor = k = \frac{n-1}{2}$. Now $n - 1 \leq n \leq 2n$, so $\frac{n-1}{2} \leq n$. And $2 \leq n$ implies $0 \leq n - 2$ so $n \leq 2n - 2$ then $\frac{1}{4}n \leq \frac{2n-2}{4} = \frac{n-1}{2}$. Thus $\frac{1}{4}n \leq \left\lfloor \frac{n}{2} \right\rfloor \leq n$.

So, in all cases, for $n \geq 2$ we have $\frac{1}{4}n \leq \left\lfloor \frac{n}{2} \right\rfloor \leq n$. Let $A = \frac{1}{4}, B = 1, k = 2$. Then for all $n \geq k$, $An \leq \left\lfloor \frac{n}{2} \right\rfloor \leq Bn$. So by definition of Θ notation, $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$. $\square$

2.18 Exercise 18

Prove Theorem 11.2.7(b): If f and g are real-valued functions defined on the same set of nonnegative integers and if $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, where r is a positive real number, then if $f(n)$ is $\Theta(g(n))$, then $g(n)$ is $\Theta(f(n))$.

Proof. Suppose f and g are real-valued functions defined on the same set of nonnegative integers, suppose $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, where r is a positive real number, and suppose $f(n)$ is $\Theta(g(n))$. *[We must show that $g(n)$ is $\Theta(f(n))$.]* By definition of Θ-notation, there exist positive real numbers A, B, and k with $k \geq r$ such that for each integer $n \geq k$, $Ag(n) \leq f(n) \leq Bg(n)$. Dividing the left-hand inequality by A and the right-hand inequality by B gives that $g(n) \leq \frac{1}{A}f(n)$ and $\frac{1}{B}f(n) \leq g(n)$, and combining the resulting inequalities produces $\frac{1}{B}f(n) \leq g(n) \leq \frac{1}{A}f(n)$ for each integer $n \geq k$. Now both $f(n) \geq 0$ and $g(n) \geq 0$ for each integer $n \geq k$. Also, since both A and B are positive real numbers, so are $1/A$ and $1/B$. Thus, by definition of Θ-notation, $g(n)$ is $\Theta(f(n))$. $\square$

2.19 Exercise 19

Prove Theorem 11.2.1: If f and g are real-valued functions defined on the same set of nonnegative integers and if $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, where r is a positive real number, then $f(n)$ is $\Theta(g(n))$ if, and only if, $f(n)$ is $\Omega(g(n))$ and $f(n)$ is $O(g(n))$.

Proof. Assume $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r > 0$.

($\implies$) 1. Assume $f(n)$ is $\Theta(g(n))$.

2. By definition of Θ-notation, there exist positive real numbers A, B, and $k \geq r$ such that $Ag(n) \leq f(n) \leq Bg(n)$ for every integer $n \geq k$.

3. By 2 and assumption, $0 \leq f(n) \leq Bg(n)$ for all $n \geq k$, so by definition of O-notation, $f(n)$ is $O(g(n))$.

4. By 2, $Ag(n) \leq f(n)$ for all $n \geq k$, so by definition of Ω-notation, $f(n)$ is $\Omega(g(n))$.

($\Longleftarrow$) 1. Assume $f(n)$ is $\Omega(g(n))$ and $f(n)$ is $O(g(n))$.

2. By 1 and definition of Ω-notation, there exist positive real numbers A and $a \geq r$ such that $Ag(n) \leq f(n)$ for every integer $n \geq a$.

3. By 1 and definition of O-notation, there exist positive real numbers B and $b \geq r$ such that $0 \leq f(n) \leq Bg(n)$ for every integer $n \geq b$.

4. Let $c = max(a, b)$. Then by 2 and 3, for every $n \geq c$, $Ag(n) \leq f(n) \leq Bg(n)$. So by definition of Θ-notation, $f(n)$ is $\Theta(g(n))$. □

2.20 Exercise 20

Without using Theorem 11.2.4 prove that n^5 is not $O(n^2)$.

Proof. Suppose not. That is, suppose n^5 is $O(n^2)$. *[We must show that this supposition leads to a contradiction.]* By definition of O-notation, there exist positive real numbers B and b such that $0 \leq n^5 \leq Bn^2$ for each integer $n \geq b$. Dividing the inequalities by n^2 and taking the cube root of both sides gives $0 \leq n \leq \sqrt[3]{B}$ for each integer $n \geq b$. These two conditions are contradictory because on the one hand n can be any integer greater than or equal to b, but when n is greater than b, then n is less than $\sqrt[3]{B}$, which is a fixed integer. Thus the supposition leads to a contradiction, and hence the supposition is false. □

2.21 Exercise 21

Prove Theorem 11.2.4: If f is a real-valued function defined on a set of nonnegative integers and $f(n)$ is $\Omega(n^m)$, where m is a positive integer, then $f(n)$ is not $O(n^p)$ for any positive real number $p < m$.

Proof. Assume m is a positive integer, p is a positive real number, $p < m$ and $f(n)$ is $\Omega(n^m)$.

By definition of Ω-notation there exist positive real numbers A and $a \geq 0$ such that $An^m \leq f(n)$ for every integer $n \geq a$. (We are taking $r = 0$ since $n^m \geq 0$ for all $n \geq 0$.)

Argue by contradiction and assume $f(n)$ is $O(n^p)$. By definition of O-notation, there exist positive real numbers B and $b \geq r$ such that $0 \leq f(n) \leq Bn^p$ for every integer $n \geq b$.

Let $c = max(a, b)$. Then for all $n \geq c$ we have $An^m \leq f(n) \leq Bn^p$. In particular, $An^m \leq Bn^p$ for all $n \geq c$. Dividing by An^p we get $n^{m-p} \leq \dfrac{B}{A}$ for all $n \geq c$. Since $m - p > 0$, this is a contradiction: the left hand side is a function that grows without bound as n gets larger, and the right hand side is a positive constant.

So our supposition was false, and $f(n)$ is not $O(n^p)$. $\square$

2.22 Exercise 22

2.22.1 (a)

Use one of the methods of Example 11.2.4 to show that $2n^4 - 90n^3 + 3$ is $\Omega(n^4)$.

Proof. To use the general procedure from Example 11.2.4 to show that $2n^4 - 90n^3 + 3$ is $\Omega(n^4)$, let $A = \frac{1}{2} \cdot 2 = 1$ and $a = \frac{2}{2}(|-90| + |3|) = 93$ and note that $a \geq 1$. We will show that $n^4 \leq 2n^4 - 90n^3 + 3$ for every integer $n \geq a$. Now $n \geq a$ means that $n \geq 90 + 3$. Multiplying both sides by n^3 gives $n^4 \geq 90n^3 + 3n^3$ and subtracting first $3n^3$ and then 3 from the right-hand side gives that $n^4 \geq 90n^3 \geq 90n^3 - 3$ for every integer $n \geq a$. Subtracting the right-hand side from the left-hand side and adding n^4 to both sides gives $2n^4 - 90n^3 + 3 \geq n^4$ for every integer $n \geq a$. Thus since $A = 1$, $2n^4 - 90n^3 + 3 \geq An^4$ for every integer $n \geq a$, and so, by definition of Ω-notation, $2n^4 - 90n^3 + 3$ is $\Omega(n^4)$. $\square$

2.22.2 (b)

Show that $2n^4 - 90n^3 + 3$ is $O(n^4)$.

Proof. To show that $2n^4 - 90n^3 + 3$ is $O(n^4)$, observe that for every integer $n \geq 1$, $2n^4 - 90n^3 + 3 \leq 2n^4 + 90n^3 + 3$ because when $n \geq 1$, then $90n^3$ is positive,

$\leq 2n^4 + 90n^4 + 3n^4$ by Theorem 11.2.2 (since $n \geq 1$, $n^3 \leq n^4$ and $1 \leq n^4$, $90n^3 \leq 90n^4$ and $3 \leq 3n^4$),

and so $= 95n^4$ because $2 + 90 + 3 = 95$. Thus, by transitivity of order and equality, for every integer $n \geq 1$, $2n^4 - 90n^3 + 3 \leq 95n^4$.

In addition, by part (a), for every integer $n \geq 60$, $\frac{1}{2}n^4 \leq 2n^4 - 90n^3 + 3$ so since $0 \leq 12n^4$, transitivity of order gives that for every integer $n \geq 60$, $0 \leq 2n^4 - 90n^3 + 3 \leq 95n^4$.

Let $B = 14$ and $b = 60$. Then, for every integer $n \geq b$, $0 \leq 2n^4 - 90n^3 + 3 \leq Bn^4$ and hence, by definition of O-notation, $2n^4 - 90n^3 + 3$ is $O(n^4)$. $\square$

2.22.3 (c)

Justify the conclusion that $2n^4 - 90n^3 + 3$ is $\Theta(n^4)$.

Proof. By parts (a) and (b), $2n^4 - 90n^3 + 3$ is both $\Omega(n^4)$ and $O(n^4)$. Hence, by Theorem 11.2.1, $2n^4 - 90n^3 + 3$ is $\Theta(n^4)$. $\square$

2.23 Exercise 23

2.23.1 (a)

Use one of the methods of Example 11.2.4 to show that $\frac{1}{5}n^2 - 42n - 8$ is $\Omega(n^2)$.

Proof. Let $f(n) = \frac{1}{5}n^2 - 42n - 8$.

To find the lower bound, let us follow the procedure. Let $A = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$. Let $a = \frac{2}{1/5}(|-42| + |-8|) = 500$. Now we need to show that $\frac{1}{10}n^2 \leq f(n)$ for $n \geq 500$.

Assume $n \geq 500$, which means $n \geq 10(42 + 8)$. Divide by 10 and multiply by n to get $\frac{1}{10}n^2 \geq 42n + 8n$. Subtract $8n - 8$ from the right hand side to get $42n + 8n \geq 42n + 8$ (because when $n \geq 500$, $8n - 8 \geq 0$, so subtracting a positive number makes it smaller). So by transitivity of order, $\frac{1}{10}n^2 \geq 42n + 8$. Subtract right hand side from left hand side to get $\frac{1}{10}n^2 - 42n - 8 \geq 0$. Now add $\frac{1}{10}n^2$ to both sides to get $\frac{1}{5}n^2 - 42n - 8 \geq \frac{1}{10}n^2$ for all $n \geq 500$. So by definition of Ω-notation, $f(n)$ is $\Omega(n^2)$. □

2.23.2 (b)

Show that $\frac{1}{5}n^2 - 42n - 8$ is $O(n^2)$.

Proof. Setting $f(n) = 0$ we find

$$n = \frac{42 \pm \sqrt{(-42)^2 - 4(1/5)(-8)}}{2/5} = 105 \pm \sqrt{11065} \approx 0 \text{ and } 210,$$

so $f(n) \geq 0$ for all $n \geq 211$.

To find the upper bound, we can replace $\frac{1}{5}n^2 - 42n - 8$ with the bigger $n^2 + 42n^2 + 8n^2 = 51n^2$. So $0 \leq f(n) \leq 51n^2$ for all $n \geq 211$, so $f(n)$ is $O(n^2)$. □

2.23.3 (c)

Justify the conclusion that $\frac{1}{5}n^2 - 42n - 8$ is $\Theta(n^2)$.

Proof. By parts (a) and (b), $f(n)$ is both $\Omega(n^2)$ and $O(n^2)$. By Theorem 11.2.1, $f(n)$ is $\Theta(n^2)$. □

2.24 Exercise 24

2.24.1 (a)

Use one of the methods of Example 11.2.4 to show that $\frac{1}{4}n^5 - 50n^3 + 3n + 12$ is $\Omega(n^5)$.

Proof. $A = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$, $a = \frac{2}{1/4}(|-50| + |3| + |12|) = 8(50 + 3 + 12) = 520$.

Assume $n \geq 520 = 8(50 + 3 + 12)$.

Divide by 8 and multiply by n^4 to get $\frac{1}{8}n^5 \geq 50n^4 + 3n^4 + 12n^4$.

From the right hand side, subtract $50n^4 - 50n^3$ to get $50n^4 + 3n^4 + 12n^4 \geq 50n^3 + 3n^4 + 12n^4$ (because when $n \geq 520$ we have $12n^4 - 12n^3 = 12n^3(n-1) \geq 0$ so subtracting something positive makes it smaller).

From the right hand side, subtract $3n^4 + 3n$ to get $50n^3 + 3n^4 + 12n^4 \geq 50n^3 - 3n + 12n^4$ (because when $n \geq 520$ we have $3n^4 + 3n = 3n(n^3 + 1) \geq 0$ so subtracting something positive makes it smaller).

From the right hand side, subtract $12n^4 + 12$ to get $50n^3 - 3n + 12n^4 \geq 50n^3 - 3n - 12$ (because when $n \geq 520$ we have $12n^4 + 12n = 12(n^4 + 1) \geq 0$ so subtracting something positive makes it smaller).

By transitivity of order we get $\dfrac{1}{8}n^5 \geq 50n^3 - 3n - 12$. Moving everything to the left hand side, we get $\dfrac{1}{8}n^5 - 50n^3 + 3n + 12 \geq 0$. Now add $\dfrac{1}{8}n^5$ to both sides to finally get $\dfrac{1}{4}n^5 - 50n^3 + 3n + 12 \geq \dfrac{1}{8}n^5$ for all $n \geq 520$.

So by definition of Ω-notation, $f(n)$ is $\Omega(n^5)$. □

2.24.2 (b)

Show that $\frac{1}{4}n^5 - 50n^3 + 3n + 12$ is $O(n^5)$.

Proof. Setting $f(n) = 0$ we find $n \approx -14, 1, 14$. So $f(n) \geq 0$ for all $n \geq 15$.

$\frac{1}{4}n^5 - 50n^3 + 3n + 12 \leq n^5 + 50n^5 + 3n^5 + 12n^5 = 66n^5$ for all $n \geq 1$.

Therefore $0 \leq f(n) \leq 66n^5$ for all $n \geq 15$. By definition of O-notation, $f(n)$ is $O(n^5)$. □

2.24.3 (c)

Justify the conclusion that $\frac{1}{4}n^5 - 50n^3 + 3n + 12$ is $\Theta(n^5)$.

Proof. By parts (a) and (b), $f(n)$ is both $\Omega(n^5)$ and $O(n^5)$. By Theorem 11.2.1, $f(n)$ is $\Theta(n^5)$. □

2.25 Exercise 25

Suppose $P(n) = a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$, where all the coefficients $a_0, a_1, \ldots, a_m$ are real numbers and $a_m > 0$.

2.25.1 (a)

Prove that $P(n)$ is $\Omega(n^m)$ by using the general procedure described in Example 11.2.4.

Proof. Let $A = \dfrac{1}{2}a_m$, $d = \dfrac{2}{a_m}(|a_{m-1}| + \cdots + |a_0|)$ and $a = max(d, 1)$. Then $n \geq a$ means that $n \geq \dfrac{2}{a_m}(|a_{m-1}| + \cdots + |a_0|)$. Multiplying both sides by $\dfrac{1}{2}a_m n^{m-1}$ gives

$\dfrac{1}{2}a_m n^m \geq (|a_{m-1}|+\cdots+|a_0|)n^{m-1} = |a_{m-1}|n^{m-1}+|a_{m-2}|n^{m-1}+\cdots+|a_1|n^{m-1}+|a_0|n^{m-1}$

which is $\geq |a_{m-1}|n^{m-1} + |a_{m-2}|n^{m-2} + \cdots + |a_1|n^1 + |a_0|n^0$. So by transitivity of order

$$\frac{1}{2}a_m n^m \geq |a_{m-1}|n^{m-1} + |a_{m-2}|n^{m-2} + \cdots + |a_1|n^1 + |a_0|n^0.$$

Subtracting the right hand side gives

$$\frac{1}{2}a_m n^m - |a_{m-1}|n^{m-1} - |a_{m-2}|n^{m-2} - \cdots - |a_1|n^1 - |a_0|n^0 \geq 0.$$

Since each $a_i \geq -|a_i|$, we have

$$\frac{1}{2}a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1 n + a_0 \geq \frac{1}{2}a_m n^m - |a_{m-1}|n^{m-1} - \cdots - |a_1|n^1 - |a_0|n^0.$$

By transitivity of order $\frac{1}{2}a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1 n + a_0 \geq 0$. Add $\frac{1}{2}a_m n^m$ to both sides to get $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1 n + a_0 \geq \frac{1}{2}a_m n^m$. So by definition of Ω notation, $P(n)$ is $\Omega(n^m)$. □

2.25.2 (b)

Prove that $P(n)$ is $O(n^m)$.

Proof. For all $n \geq 1$ we have $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$

$\leq |a_m|n^m + |a_{m-1}|n^m + \cdots + |a_2|n^m + |a_1|n^m + |a_0|n^m = (|a_m| + \cdots + |a_0|)n^m$.

Let $B = |a_m| + \cdots + |a_0|$. Then, by transitivity of order and equality, for each integer $n \geq 1$, $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0 \geq Bn^m$.

In addition, by part (a), there exists a positive real number a such that for each integer $n \geq a$, $\frac{a_m}{2}n^m \leq a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$.

Now $\frac{a_n}{2}n^m > 0$ because $a_m > 0$, and thus, transitivity of order gives that for each integer $n \geq a$, $0 \leq a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$.

And hence, by definition of O-notation, $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$ is $O(n^m)$. □

2.25.3 (c)

Justify the conclusion that $P(n)$ is $\Theta(n^m)$.

Proof. By parts (a) and (b), $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$ is both $\Omega(n^m)$ and $O(n^m)$. Hence, by Theorem 11.2.1, it is $\Theta(n^m)$. □

Use the theorem on polynomial orders to prove each of the statements in $26 - 31$.

2.26 Exercise 26

$\dfrac{(n+1)(n-2)}{4}$ is $\Theta(n^2)$

Proof. $\dfrac{(n+1)(n-2)}{4} = \dfrac{n^2-n-2}{4} = \dfrac{1}{4}n^2 - \dfrac{1}{4}n - \dfrac{1}{2}$, which is $\Theta(n^2)$ by the theorem on polynomial orders. $\square$

2.27 Exercise 27

$\dfrac{n}{3}(4n^2-1)$ is $\Theta(n^3)$

Proof. $\dfrac{n}{3}(4n^2-1) = \dfrac{4}{3}n^3 - \dfrac{1}{3}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders. $\square$

2.28 Exercise 28

$\dfrac{n(n-1)}{2} + 3n$ is $\Theta(n^2)$

Proof. $\dfrac{n(n-1)}{2} + 3n = \dfrac{n^2-n}{2} + 3n = \dfrac{1}{2}n^2 + \dfrac{5}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. $\square$

2.29 Exercise 29

$\dfrac{n(n-1)(2n+1)}{6}$ is $\Theta(n^3)$

Proof. $\dfrac{n(n-1)(2n+1)}{6} = \dfrac{(n^2-n)(2n+1)}{6} = \dfrac{2n^3-n^2-n}{6} = \dfrac{1}{3}n^3 - \dfrac{1}{6}n^2 - \dfrac{1}{6}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders. $\square$

2.30 Exercise 30

$\left[\dfrac{n(n+1)}{2}\right]^2$ is $\Theta(n^4)$

Proof. $\left[\dfrac{n(n+1)}{2}\right]^2 = \dfrac{n^2(n+1)^2}{4} = \dfrac{n^2(n^2+2n+1)}{4} = \dfrac{n^4+2n^3+n^2}{4} = \dfrac{1}{4}n^4 + \dfrac{1}{2}n^3 + \dfrac{1}{4}n^2$, which is $\Theta(n^4)$ by the theorem on polynomial orders. $\square$

2.31 Exercise 31

$2(n-1) + \dfrac{n(n+1)}{2} + 4\left(\dfrac{n(n-1)}{2}\right)$ is $\Theta(n^2)$

Proof. $2(n-1)+\dfrac{n(n+1)}{2}+4\left(\dfrac{n(n-1)}{2}\right)=2n-2+\dfrac{1}{2}n^2+\dfrac{1}{2}n+2n^2-2n=\frac{5}{2}n^2+\frac{1}{2}n-2$, which is $\Theta(n^2)$ by the theorem on polynomial orders. $\square$

Prove each of the statements in 32–39. Use the theorem on polynomial orders and results from the theorems and exercises in Section 5.2 as appropriate.

2.32 Exercise 32

$1^2+2^2+3^2+\cdots+n^2$ is $\Theta(n^3)$

Proof. By exercise 10 of Section 5.2, this sum equals $\dfrac{n(n-1)(2n+1)}{6}$, which is $\Theta(n^3)$ by Exercise 29 above. $\square$

2.33 Exercise 33

$1^3+2^3+3^3+\cdots+n^3$ is $\Theta(n^4)$

Proof. By exercise 11 of Section 5.2, this sum equals $\left[\dfrac{n(n+1)}{2}\right]^2$, which is $\Theta(n^4)$ by Exercise 30 above. $\square$

2.34 Exercise 34

$2+4+6+\cdots+2n$ is $\Theta(n^2)$

Proof. $2+4+6+\cdots+2n=2(1+2+3+\cdots+n)=2\cdot\frac{n(n+1)}{2}=n^2+n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. $\square$

2.35 Exercise 35

$5+10+15+20+25+\cdots+5n$ is $\Theta(n^2)$

Proof. $5+10+15+20+25+\cdots+5n=5(1+2+3+\cdots+n)=5\cdot\frac{n(n+1)}{2}=\frac{5}{2}n^2+\frac{5}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. $\square$

2.36 Exercise 36

$\displaystyle\sum_{i=1}^{n}(4i-9)$ is $\Theta(n^2)$

Proof. $\displaystyle\sum_{i=1}^{n}(4i-9)=4\sum_{i=1}^{n}i-9\sum_{i=1}^{n}1=4\cdot\frac{n(n+1)}{2}-9n=2n^2+2n-9n=2n^2-7n$ which is $\Theta(n^2)$ by the theorem on polynomial orders. $\square$

2.37 Exercise 37

$\sum_{k=1}^{n}(k+3)$ is $\Theta(n^2)$

Proof. $\sum_{k=1}^{n}(k+3) = \sum_{k=1}^{n} k + 3\sum_{k=1}^{n} 1 = \frac{n(n+1)}{2} + 3n = \frac{1}{2}n^2 + \frac{7}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. □

2.38 Exercise 38

$\sum_{i=1}^{n} i(i+1)$ is $\Theta(n^3)$

Proof. $\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{2n^3+3n^2+n}{6} + \frac{n^2+n}{2} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders. □

2.39 Exercise 39

$\sum_{k=3}^{n}(k^2-2k)$ is $\Theta(n^3)$

Proof. $\sum_{k=3}^{n}(k^2-2k) = \sum_{k=1}^{n}(k^2-2k) - (1^2 - 2\cdot 1 + 2^2 - 2\cdot 2) = \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k - (-1) = \frac{n(n+1)(2n+1)}{6} - 2\cdot\frac{n(n+1)}{2} + 1 = \frac{2n^3+3n^2+n}{6} + n^2 + n + 1 = \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{7}{6}n + 1$, which is $\Theta(n^3)$ by the theorem on polynomial orders. □

2.40 Exercise 40

2.40.1 (a)

Prove: If c is a positive real number and if f is a real-valued function defined on a set of nonnegative integers with $f(n) \geq 0$ for every integer n greater than or equal to some positive real number, then $cf(n)$ is $\Theta(f(n))$.

Proof. Suppose c is a positive real number and f is a real-valued function defined on a set of nonnegative integers with $f(n) \geq 0$ for each integer n greater than or equal to a positive real number k. Now if we let $A = B = c$, we have that for each integer $n \geq k$, $Af(n) \leq cf(n) \leq Bf(n)$ and so, by definition of Θ-notation, $cf(n)$ is $\Theta(f(n))$. □

2.40.2 (b)

Use part (a) to show that $3n$ is $\Theta(n)$.

Proof. Let $c = 3$ and $f(n) = n$. Then f is a real-valued function and $f(n) \geq 0$ for each integer $n \geq 0$. So by part (a), $cf(n)$ is $\Theta(f(n))$, or, by substitution, $3n$ is $\Theta(n)$. □

2.41 Exercise 41

Prove: If c is a positive real number and $f(n) = c$ for every integer $n \geq 1$, then $f(n)$ is $\Theta(1)$.

Proof. Assume c is a positive real number and $f(n) = c$ for every integer $n \geq 1$. Then let $A = B = c$ and $k = 1$. Then $A \cdot 1 \leq f(n) \leq B \cdot 1$ for all $n \geq k$, so by definition, $f(n)$ is $\Theta(1)$. □

2.42 Exercise 42

What can you say about a function f with the property that $f(n)$ is $\Theta(1)$?

Proof. If $f(n)$ is $\Theta(1)$ then by definition, there are positive reals A, B and a positive integer k such that $A \leq f(n) \leq B$ for all $n \geq k$. So the graph of f is trapped between the two horizontal lines $y = A$ and $y = B$ for $n \geq k$. □

Use Theorems 11.2.5 – 11.2.9 and the results of exercises 15 – 17, 40, and 41 to justify the statements in 43 – 45.

2.43 Exercise 43

$\left\lfloor \dfrac{n+1}{2} \right\rfloor + 3n$ is $\Theta(n)$

Proof. By exercise 15 $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ is $\Theta(n)$ and by exercise 40 (b) $3n$ is $\Theta(n)$. Thus $\left\lfloor \dfrac{n+1}{2} \right\rfloor +$ $3n$ is $\Theta(n)$ by Theorem 11.2.9(a). □

2.44 Exercise 44

$\dfrac{n(n-1)}{2} + \left\lfloor \dfrac{n}{2} \right\rfloor + 1$ is $\Theta(n^2)$

Proof. By exercise 28 $\dfrac{n(n-1)}{2}$ is $\Theta(n^2)$, by exercise 17 $\left\lfloor \dfrac{n}{2} \right\rfloor$ is $\Theta(n)$ and by exercise 41 (with $f(n) = 1$), 1 is $\Theta(1)$. So $\dfrac{n(n-1)}{2} + \left\lfloor \dfrac{n}{2} \right\rfloor + 1$ is $\Theta(n^2)$ by Theorem 11.2.9(c). □

2.45 Exercise 45

$\left\lfloor \frac{n}{2} \right\rfloor + 4n + 3$ is $\Theta(n)$

Proof. By exercise 17 $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$, by exercise 40 (b) $4n$ is $\Theta(n)$, and by exercise 41 (with $f(n) = 3$), 3 is $\Theta(1)$. So $\left\lfloor \frac{n}{2} \right\rfloor + 4n + 3$ is $\Theta(n)$ by Theorem 11.2.9(c). □

2.46 Exercise 46

2.46.1 (a)

Use mathematical induction to prove that if n is any integer with $n > 1$, then for every integer $m \geq 1$, $n^m > 1$.

Proof. Let the property $P(m)$ be the sentence: "If n is any integer with $n > 1$, then $n^m > 1$".

Show that $P(1)$ is true: We must show that if n is any integer with $n > 1$, then $n^1 > 1$. But this is true because $n^1 = n$. So $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be a particular but arbitrarily chosen integer with $k \geq 1$, and suppose that if n is any integer with $n > 1$, then $n^k > 1$.

We must show that if n is any integer with $n > 1$, then $n^{k+1} > 1$.

So suppose n is any integer with $n > 1$. By inductive hypothesis, $n^k > 1$, and multiplying both sides by the positive number n gives $n \cdot n^k > n \cdot 1$, or, equivalently, $n^{k+1} > n$. Thus $n^{k+1} > n$ and $n > 1$, and so, by transitivity of order, $n^{k+1} > 1$, *[as was to be shown]*. □

2.46.2 (b)

Prove that if n is any integer with $n > 1$, then $n^r < n^s$ for all integers r and s with $r < s$.

Proof. Suppose n is any integer with $n > 1$ and r and s are integers with $r < s$. Then $s - r$ is an integer with $s - r \geq 1$, and so, by part (a), $n^{s-r} > 1$. Multiplying both sides by n^r gives $n^r \cdot n^{s-r} > n^r \cdot 1$, and so, by the laws of exponents, $n^s > n^r$ *[as was to be shown]*. □

2.47 Exercise 47

2.47.1 (a)

Let x be any positive real number. Use mathematical induction to prove that for every integer $m \geq 1$, if $x \leq 1$ then $x^m \leq 1$.

Proof. Let the property $P(m)$ be the sentence "If $0 < x \leq 1$, then $x^m \leq 1$".

Show that P(1) is true: We must show that if $0 < x \leq 1$, then $x^1 \leq 1$. But $x \leq 1$ by assumption and $x^1 = x$. So $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that if $0 < x \leq 1$, then $x^k \leq 1$ (inductive hypothesis). We must show that if $0 < x \leq 1$, then $x^{k+1} \leq 1$.

So let x be any number with $0 < x \leq 1$. By inductive hypothesis, $x^k \leq 1$, and multiplying both sides of this inequality by the nonnegative number x gives $x \cdot x^k \leq x^1$. Thus, by the laws of exponents, $x^{k+1} \leq x$. Then $x^{k+1} \leq x$ and $x \leq 1$, and hence, by the transitive property of order (T18 in Appendix A), $x^{k+1} \leq 1$. □

2.47.2 (b)

Explain how it follows from part (b) that if x is any positive real number, then for every integer $m \geq 1$, if $x^m > 1$ then $x > 1$.

Proof. This is the contrapositive of the statement in part (a), therefore it's true. □

2.47.3 (c)

Explain how it follows from part (b) that if x is any positive real number, then for every integer $m \geq 1$, if $x > 1$ then $x^{1/m} > 1$.

Proof. Let $y = x^{1/m}$. Then by part (b), with y replacing x, we have: if $y^m > 1$ then $y > 1$. Now substitute $y = x^{1/m}$ to get: if $(x^{1/m})^m > 1$ then $x^{1/m} > 1$. In other words: if $x > 1$ then $x^{1/m} > 1$. □

2.47.4 (d)

Let p, q, r, and s be positive integers, and suppose $p/q > r/s$. Use part (c) and the result of exercise 46 to prove Theorem 11.2.2. In other words, show that for any integer n, if $n > 1$ then $n^{p/q} > n^{r/s}$.

Proof. 1. Assume n is any integer with $n > 1$, p, q, r, and s are positive integers with $p/q > r/s$.

2. Notice $ps > qr$, so by part exercise 46 (b), $n^{ps} > n^{qr}$. By algebra, $\dfrac{n^{ps}}{n^{qr}} > 1$.

3. Let $x = \dfrac{n^{ps}}{n^{qr}}$. By 2, $x > 1$. Therefore by part (c), $x^{1/s} > 1$.

4. Rewriting 3, $\left(\dfrac{n^{ps}}{n^{qr}}\right)^{1/s} > 1$. So by law of exponents $\dfrac{n^p}{n^{qr/s}} > 1$.

5. Let $y = \dfrac{n^p}{n^{qr/s}}$. By 4, $y > 1$. So by part (c) $y^{1/q} > 1$.

6. Rewriting 5, $\left(\dfrac{n^p}{n^{qr/s}}\right)^{1/q} > 1$. So by law of exponents $\dfrac{n^{p/q}}{n^{r/s}} > 1$.

7. By 6 and algebra, $n^{p/q} > n^{r/s}$. $\square$

2.48 Exercise 48

Prove Theorem 11.2.6(b): If f and g are real-valued functions defined on the same set of nonnegative integers, and if there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, and if $g(n)$ is $O(f(n))$, then $f(n)$ is $\Omega(g(n))$.

Proof. Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for each integer $n \geq r$. Suppose also that $g(n)$ is $O(f(n))$. We will show that $f(n)$ is $\Omega(g(n))$. By definition of O-notation, there are positive real numbers B and b such that $b \geq r$, and, for each integer $n \geq b$, $0 \leq g(n) \leq Bf(n)$. Divide the right-hand inequality by B to obtain $\frac{1}{B}g(n) \leq f(n)$, for each integer $n \geq b$. Let $A = 1/B$ and $a = b$. Then for each integer $n \geq a$, $Ag(n) \leq f(n)$ and so $f(n)$ is $\Omega(g(n))$ by definition of Ω-notation. $\square$

2.49 Exercise 49

Prove Theorem 11.2.7(a): If f is a real-valued function defined on a set of nonnegative integers and there is a real number r such that $f(n) \geq 0$ for every integer $n \geq r$, then $f(n)$ is $\Theta(f(n))$.

Proof. Since $f(n) \geq 0$ for all $n \geq r$, and since $f(n) \leq f(n) \leq f(n)$, we can let $g(n) = f(n), A = B = 1$ and $k = r$ in the definition of Θ-notation to obtain that $f(n)$ is $\Theta(f(n))$. $\square$

2.50 Exercise 50

Prove Theorem 11.2.8:

2.50.1 (a)

Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) > 0$ for every integer $n \geq r$. If $f(n)$ is $\Omega(g(n))$ and c is any positive real number, then $cf(n)$ is $\Omega(g(n))$.

Proof. Assume r is a positive real number such that $f(n) \geq 0$ and $g(n) > 0$ for every integer $n \geq r$, and $f(n)$ is $\Omega(g(n))$. Assume c is any positive real number.

By definition of Ω-notation, there exist positive real numbers A and $a \geq r$ such that $Ag(n) \leq f(n)$ for every integer $n \geq a$.

Let $A' = cA$ and $a' = a$. Then A' and $a' \geq r$ are positive real numbers, and by 2 $A'g(n) = cAg(n) \leq cf(n)$ for every integer $n \geq a'$. So by definition of Ω-notation, $cf(n)$ is $\Omega(g(n))$. $\square$

2.50.2 (b)

Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$. If $f(n)$ is $O(g(n))$ and c is any positive real number, then $cf(n)$ is $O(g(n))$.

Proof. The proof is almost identical to part (a), except start with $0 \leq f(n) \leq Bg(n)$ for every integer $n \geq b$, let $B' = cB, b' = b$ and end with $0 \leq cf(n) \leq cBg(n) = B'g(n)$ for every integer $n \geq b'$. □

2.50.3 (c)

Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$. If $f(n)$ is $\Theta(g(n))$ and c is any positive real number, then $cf(n)$ is $\Theta(g(n))$.

Proof. This follows from parts (a) and (b) and Theorem 11.2.1. □

2.51 Exercise 51

Prove Theorem 11.2.9:

2.51.1 (a)

Let f_1, f_2 and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0, f_2(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$. If $f_1(n)$ is $\Theta(g(n))$ and $f_2(n)$ is $\Theta(g(n))$, then $(f_1(n) + f_2(n))$ is $\Theta(g(n))$.

Proof. Let f_1, f_2, and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0, f_2(n) \geq 0$, and $g(n) \geq 0$ for each integer $n \geq r$. Suppose also that $f_1(n)$ is $\Theta(g(n))$ and $f_2(n)$ is $\Theta(g(n))$. *[We will show that $(f_1(n) + f_2(n))$ is $\Theta(g(n))$.]* By definition of Θ-notation, there are positive real numbers A, B, A', B', k, and k' such that $k \geq r, k' \geq r$ and, for each integer n such that $n \geq k$ and $n \geq k'$, $Ag(n) \leq f_1(n) \leq Bg(n)$ and $A'g(n) \leq f_2(n) \leq B'g(n)$.

Notice that $Ag(n) + A'g(n) \leq f_1(n) + f_2(n) \leq Bg(n) + B'g(n)$ for every integer $n \geq max(k, k')$. Let $k'' = max(k, k')$, $A'' = A + A'$ and $B'' = B + B'$. So $A''g(n) \leq f_1(n) + f_2(n) \leq B''g(n)$ for every integer $n \geq k''$. Then by definition of Θ-notation, $(f_1(n) + f_2(n))$ is $\Theta(g(n))$. □

2.51.2 (b)

Let f_1, f_2, g_1, and g_2 be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0, f_2(n) \geq$

$0, g_1(n) \geq 0$, and $g_2(n) \geq 0$ for every integer $n \geq r$. If $f_1(n)$ is $\Theta(g_1(n))$ and $f_2(n)$ is $\Theta(g_2(n))$, then $(f_1(n)f_2(n))$ is $\Theta(g_1(n)g_2(n))$.

Proof. The proof is almost identical to part (a), except in the crucial step we have $AA'g_1(n)g_2(n) \leq f_1(n)f_2(n) \leq BB'g_1(n)g_2(n)$ for every integer $n \geq max(k, k')$. □

2.51.3 (c)

Let f_1, f_2, g_1, and g_2 be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0, f_2(n) \geq 0, g_1(n) \geq 0$, and $g_2(n) \geq 0$ for every integer $n \geq r$. If $f_1(n)$ is $\Theta(g_1(n))$ and $f_2(n)$ is $\Theta(g_2(n))$ and if there is a real number s so that $g_1(n) \leq g_2(n)$ for every integer $n \geq s$, then $(f_1(n) + f_2(n))$ is $\Theta(g_2(n))$.

Proof. The proof is almost identical to part (a), except in the crucial step we have:

$$A'g_2(n) \leq Ag_1(n) + A'g_2(n) \leq f_1(n) + f_2(n) \leq Bg_1(n) + B'g_2(n) \leq Bg_2(n) + B'g_2(n)$$

for all $n \geq max(s, k, k')$. So we can let $A'' = A', B'' = B + B'$ and $k'' = max(s, k, k')$. Then for every integer $n \geq k'$, we have $A''g_2(n) \leq f_1(n) + f_2(n) \leq B''g_2(n)$. □

3 Exercise Set 11.3

3.1 Exercise 1

3.1.1 ()

Proof. □

3.2 Exercise 2

3.2.1 ()

Proof. □

3.3 Exercise 3

3.3.1 ()

Proof. □

3.4 Exercise 4

3.4.1 ()

Proof. □

3.5 Exercise 5

3.5.1 ()

Proof. □

3.6 Exercise 6

3.6.1 ()

Proof. □

3.7 Exercise 7

3.7.1 ()

Proof. □

3.8 Exercise 8

3.8.1 ()

Proof. □

3.9 Exercise 9

3.9.1 ()

Proof. □

3.10 Exercise 10

3.10.1 ()

Proof. □

3.11 Exercise 11

3.11.1 ()

Proof. □

3.12 Exercise 12

3.12.1 ()

Proof. □

3.13 Exercise 13

3.13.1 ()

Proof. □

3.14 Exercise 14

3.14.1 ()

Proof. □

3.15 Exercise 15

3.15.1 ()

Proof. □

3.16 Exercise 16

3.16.1 ()

Proof. □

3.17 Exercise 17

3.17.1 ()

Proof. □

3.18 Exercise 18

3.18.1 ()

Proof. □

3.19 Exercise 19

3.19.1 ()

Proof. □

3.20 Exercise 20

3.20.1 ()

Proof. □

3.21 Exercise 21

3.21.1 ()

Proof. □

3.22 Exercise 22

3.22.1 ()

Proof. □

3.23 Exercise 23

3.23.1 ()

Proof. □

3.24 Exercise 24

3.24.1 ()

Proof. □

3.25 Exercise 25

3.25.1 ()

Proof. □

3.26 Exercise 26

3.26.1 ()

Proof. □

3.27 Exercise 27

3.27.1 ()

Proof. □

3.28 Exercise 28

3.28.1 ()

Proof. □

3.29 Exercise 29

3.29.1 ()

Proof. □

3.30 Exercise 30

3.30.1 ()

Proof. □

3.31 Exercise 31

3.31.1 ()

Proof. □

3.32 Exercise 32

3.32.1 ()

Proof. □

3.33 Exercise 33

3.33.1 ()

Proof. □

3.34 Exercise 34

3.34.1 ()

Proof. □

3.35 Exercise 35

3.35.1 ()

Proof. □

3.36 Exercise 36

3.36.1 ()

Proof. □

3.37 Exercise 37

3.37.1 ()

Proof. □

3.38 Exercise 38

3.38.1 ()

Proof. □

3.39 Exercise 39

3.39.1 ()

Proof. □

3.40 Exercise 40

3.40.1 ()

Proof. □

3.41 Exercise 41

3.41.1 ()

Proof. □

3.42 Exercise 42

3.42.1 ()

Proof. □

3.43 Exercise 43

3.43.1 ()

Proof. □

4 Exercise Set 11.4

4.1 Exercise 1

4.1.1 ()

Proof. □

4.2 Exercise 2

4.2.1 ()

Proof. □

4.3 Exercise 3

4.3.1 ()

Proof. □

4.4 Exercise 4

4.4.1 ()

Proof. □

4.5 Exercise 5

4.5.1 ()

Proof. □

4.6 Exercise 6

4.6.1 ()

Proof. □

4.7 Exercise 7

4.7.1 ()

Proof. □

4.8 Exercise 8

4.8.1 ()

Proof. □

4.9 Exercise 9

4.9.1 ()

Proof. □

4.10 Exercise 10

4.10.1 ()

Proof. □

4.11 Exercise 11

4.11.1 ()

Proof. □

4.12 Exercise 12

4.12.1 ()

Proof. □

4.13 Exercise 13

4.13.1 ()

Proof. □

4.14 Exercise 14

4.14.1 ()

Proof. □

4.15 Exercise 15

4.15.1 ()

Proof. □

4.16 Exercise 16

4.16.1 ()

Proof. □

4.17 Exercise 17

4.17.1 ()

Proof. □

4.18 Exercise 18

4.18.1 ()

Proof. □

4.19 Exercise 19

4.19.1 ()

Proof. □

4.20 Exercise 20

4.20.1 ()

Proof. □

4.21 Exercise 21

4.21.1 ()

Proof. □

4.22 Exercise 22

4.22.1 ()

Proof. □

4.23 Exercise 23

4.23.1 ()

Proof. □

4.24 Exercise 24

4.24.1 ()

Proof. □

4.25 Exercise 25

4.25.1 ()

Proof. □

4.26 Exercise 26

4.26.1 ()

Proof. □

4.27 Exercise 27

4.27.1 ()

Proof. □

4.28 Exercise 28

4.28.1 ()

Proof. □

4.29 Exercise 29

4.29.1 ()

Proof. □

4.30 Exercise 30

4.30.1 ()

Proof. □

4.31 Exercise 31

4.31.1 ()

Proof. □

4.32 Exercise 32

4.32.1 ()

Proof. □

4.33 Exercise 33

4.33.1 ()

Proof. □

4.34 Exercise 34

4.34.1 ()

Proof. □

4.35 Exercise 35

4.35.1 ()

Proof. □

4.36 Exercise 36

4.36.1 ()

Proof. □

4.37 Exercise 37

4.37.1 ()

Proof. □

4.38 Exercise 38

4.38.1 ()

Proof. □

4.39 Exercise 39

4.39.1 ()

Proof. □

4.40 Exercise 40

4.40.1 ()

Proof. □

4.41 Exercise 41

4.41.1 ()

Proof. □

4.42 Exercise 42

4.42.1 ()

Proof. □

4.43 Exercise 43

4.43.1 ()

Proof. □

4.44 Exercise 44

4.44.1 ()

Proof. □

4.45 Exercise 45

4.45.1 ()

Proof. □

4.46 Exercise 46

4.46.1 ()

Proof. □

4.47 Exercise 47

4.47.1 ()

Proof. □

4.48 Exercise 48

4.48.1 ()

Proof. □

4.49 Exercise 49

4.49.1 ()

Proof. □

4.50 Exercise 50

4.50.1 ()

Proof. □

4.51 Exercise 51

4.51.1 ()

Proof. □

5 Exercise Set 11.5

5.1 Exercise 1

5.1.1 ()

Proof. □

5.2 Exercise 2

5.2.1 ()

Proof. □

5.3 Exercise 3

5.3.1 ()

Proof. □

5.4 Exercise 4

5.4.1 ()

Proof. □

5.5 Exercise 5

5.5.1 ()

Proof. □

5.6 Exercise 6

5.6.1 ()

Proof. □

5.7 Exercise 7

5.7.1 ()

Proof. □

5.8 Exercise 8

5.8.1 ()

Proof. □

5.9 Exercise 9

5.9.1 ()

Proof. □

5.10 Exercise 10

5.10.1 ()

Proof. □

5.11 Exercise 11

5.11.1 ()

Proof. □

5.12 Exercise 12

5.12.1 ()

Proof. □

5.13 Exercise 13

5.13.1 ()

Proof. □

5.14 Exercise 14

5.14.1 ()

Proof. □

5.15 Exercise 15

5.15.1 ()

Proof. □

5.16 Exercise 16

5.16.1 ()

Proof. □

5.17 Exercise 17

5.17.1 ()

Proof. □

5.18 Exercise 18

5.18.1 ()

Proof. □

5.19 Exercise 19

5.19.1 ()

Proof. □

5.20 Exercise 20

5.20.1 ()

Proof. □

5.21 Exercise 21

5.21.1 ()

Proof. □

5.22 Exercise 22

5.22.1 ()

Proof. □

5.23 Exercise 23

5.23.1 ()

Proof. □

5.24 Exercise 24

5.24.1 ()

Proof. □

5.25 Exercise 25

5.25.1 ()

Proof. □

5.26 Exercise 26

5.26.1 ()

Proof. □