Solutions to Chapter 7, Susanna Epp Discrete Math 5th Edition

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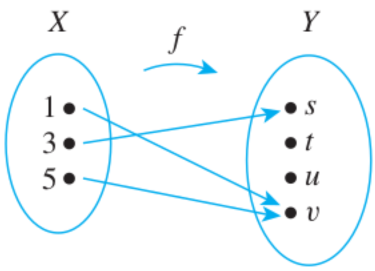
Contents

1 Exercise Set 7.1

1.1 Exercise 1

Let $X = \{1, 3, 5\}$ and $Y = \{s, t, u, v\}$. Define $f : X \to Y$ by the following arrow diagram.



1.1.1 (a)

Write the domain of f and the co-domain of f.

Proof. domain of $f = \{1, 3, 5\}$, co-domain of $f = \{s, t, u, v\}$ □

1.1.2 (b)

Find $f(1), f(3)$, and $f(5)$.

Proof. $f(1) = v, f(3) = s, f(5) = v$ □

1.1.3 (c)

What is the range of f?

Proof. range of $f = \{s, v\}$ □

1.1.4 (d)

Is 3 an inverse image of s? Is 1 an inverse image of u?

Proof. yes, no □

1.1.5 (e)

What is the inverse image of s? of u? of v?

Proof. inverse image of $s = \{3\}$, inverse image of $u = \varnothing$, inverse image of $v = \{1, 5\}$ ☐

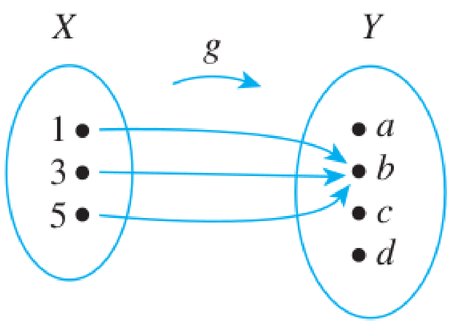
1.1.6 (f)

Represent f as a set of ordered pairs.

Proof. $\{(1, v), (3, s), (5, v)\}$ ☐

1.2 Exercise 2

Let $X = \{1, 3, 5\}$ and $Y = \{a, b, c, d\}$. Define $g : X \to Y$ by the following arrow diagram.



1.2.1 (a)

Write the domain of g and the co-domain of g.

Proof. domain: $\{1, 3, 5\}$ co-domain: $\{a, b, c, d\}$ ☐

1.2.2 (b)

Find $g(1), g(3)$, and $g(5)$.

Proof. $g(1) = b, g(3) = b, g(5) = b$ ☐

1.2.3 (c)

What is the range of g?

Proof. $\{b\}$ ☐

1.2.4 (d)

Is 3 an inverse image of a? Is 1 an inverse image of b?

Proof. no, yes ☐

1.2.5 (e)

What is the inverse image of b? of c?

Proof. $\{1, 3, 5\}$ and $\varnothing$ □

1.2.6 (f)

Represent g as a set of ordered pairs.

Proof. $\{(1, b), (3, b), (5, b)\}$ □

1.3 Exercise 3

Indicate whether the statements in parts (a)–(d) are true or false for all functions. Justify your answers.

1.3.1 (a)

If two elements in the domain of a function are equal, then their images in the co-domain are equal.

Proof. True. The definition of function says that for any input there is one and only one output, so if two inputs are equal, their outputs must also be equal. □

1.3.2 (b)

If two elements in the co-domain of a function are equal, then their preimages in the domain are also equal.

Proof. Not necessarily true. A function can have the same output for more than one input. □

1.3.3 (c)

A function can have the same output for more than one input.

Proof. True. The definition of function does not prohibit this occurrence. □

1.3.4 (d)

A function can have the same input for more than one output.

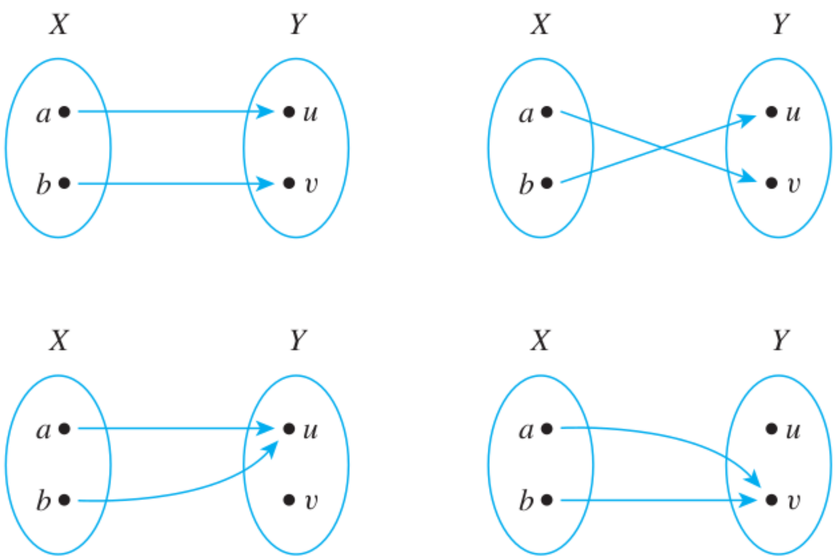
Proof. False, this is ruled out by the definition of a function. Functions are single valued. Every input corresponds to only one output. □

1.4 Exercise 4

1.4.1 (a)

Find all functions from $X = \{a, b\}$ to $Y = \{u, v\}$.

Proof. There are four functions from X to Y as shown *on the next page.*



□

1.4.2 (b)

Find all functions from $X = \{a, b, c\}$ to $Y = \{u\}$.

Proof. There is only one function $f : X \to Y$ given by the set $\{(a, u), (b, u), (c, u)\}$. □

1.4.3 (c)

Find all functions from $X = \{a, b, c\}$ to $Y = \{u, v\}$.

Proof. There are 8 functions:

$\{(a, u), (b, u), (c, u)\}$

$\{(a, u), (b, u), (c, v)\}$

$\{(a, u), (b, v), (c, u)\}$

$\{(a, u), (b, v), (c, v)\}$

$\{(a, v), (b, u), (c, u)\}$

$\{(a, v), (b, u), (c, v)\}$

$\{(a, v), (b, v), (c, u)\}$

$\{(a, v), (b, v), (c, v)\}$ □

1.5 Exercise 5

Let $I_{\mathbb{Z}}$ be the identity function defined on the set of all integers, and suppose that $e, b_i^{jk}, K(t)$, and u_{kj} all represent integers. Find the following:

1.5.1 (a)

$I_{\mathbb{Z}}(e)$

Proof. e (because $I_{\mathbb{Z}}$ is the identity function). □

1.5.2 (b)

$I_{\mathbb{Z}}(b_i^{jk})$

Proof. b_i^{jk} (because $I_{\mathbb{Z}}$ is the identity function). □

1.5.3 (c)

$I_{\mathbb{Z}}(K(t))$

Proof. $K(t)$ (because $I_{\mathbb{Z}}$ is the identity function). □

1.5.4 (d)

$I_{\mathbb{Z}}(u_{kj})$

Proof. u_{kj} (because $I_{\mathbb{Z}}$ is the identity function). □

1.6 Exercise 6

Find functions defined on the set of nonnegative integers that can be used to define the sequences whose first six terms are given below.

1.6.1 (a)

$1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}$

Proof. The sequence is given by the function $f : \mathbb{Z}^{nonneg} \to \mathbb{R}$ defined by the rule $f(n) = \frac{(-1)^n}{2n+1}$ for each nonnegative integer n. □

1.6.2 (b)

$0, -2, 4, -6, 8, -10$

Proof. The sequence is given by the function $f : \mathbb{Z}^{nonneg} \to \mathbb{Z}$ defined by the rule $f(n) = (-1)^n \cdot (2n)$ for each nonnegative integer n. □

1.7 Exercise 7

Let $A = \{1, 2, 3, 4, 5\}$, and define a function $F : \mathscr{P}(A) \to \mathbb{Z}$ as follows: For each set X in $\mathscr{P}(A)$,

$$F(x) = \begin{cases} 0 & \text{if } X \text{ has an even number of elements} \\ 1 & \text{if } X \text{ has an odd number of elements} \end{cases}$$

Find the following:

1.7.1 (a)

$F(\{1, 3, 4\})$

Proof. $F(\{1, 3, 4\}) = 1$ *[because* $\{1, 3, 4\}$ *has an odd number of elements]*

□

1.7.2 (b)

$F(\varnothing)$

Proof. $F(\{\varnothing\}) = 0$ *[because* $\{\varnothing\}$ *has an even number of elements]* □

1.7.3 (c)

$F(\{2, 3\})$

Proof. $F(\{2, 3\}) = 0$ *[because* $\{2, 3\}$ *has an even number of elements]* □

1.7.4 (d)

$F(\{2, 3, 4, 5\})$

Proof. $F(\{2, 3, 4, 5\}) = 0$ *[because* $\{2, 3, 4, 5\}$ *has an even number of elements]* □

1.8 Exercise 8

Let $J_5 = \{0, 1, 2, 3, 4\}$, and define a function $F : J_5 \to J_5$ as follows: For each $x \in J_5$, $F(x) = (x^3 + 2x + 4) \mod 5$. Find the following:

1.8.1 (a)

$F(0)$

Proof. $F(0) = (0^3 + 2 \cdot 0 + 4) \mod 5 = 4 \mod 5 = 4$

□

1.8.2 (b)

$F(1)$

Proof. $F(1) = (1^3 + 2 \cdot 1 + 4) \mod 5 = 7 \mod 5 = 2$ $\square$

1.8.3 (c)

$F(2)$

Proof. $F(2) = (2^3 + 2 \cdot 2 + 4) \mod 5 = 16 \mod 5 = 1$ $\square$

1.8.4 (d)

$F(3)$

Proof. $F(3) = (3^3 + 2 \cdot 3 + 4) \mod 5 = 37 \mod 5 = 2$ $\square$

1.8.5 (e)

$F(4)$

Proof. $F(4) = (4^3 + 2 \cdot 4 + 4) \mod 5 = 76 \mod 5 = 1$ $\square$

1.9 Exercise 9

Define a function $S : \mathbb{Z}^+ \to \mathbb{Z}^+$ as follows: For each positive integer $n, S(n) =$ the sum of the positive divisors of n. Find the following:

1.9.1 (a)

$S(1)$

Proof. $S(1) = 1$

$\square$

1.9.2 (b)

$S(15)$

Proof. $S(15) = 1 + 3 + 5 + 15 = 24$ $\square$

1.9.3 (c)

$S(17)$

Proof. $S(15) = 1 + 17 = 18$ $\square$

1.9.4 (d)

$S(5)$

Proof. $S(5) = 1 + 5 = 6$ □

1.9.5 (e)

$S(18)$

Proof. $S(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39$ □

1.9.6 (f)

$S(21)$

Proof. $S(21) = 1 + 3 + 7 + 21 = 32$ □

1.10 Exercise 10

Let D be the set of all finite subsets of positive integers. Define a function $T : \mathbb{Z}^+ \to D$ as follows: For each positive integer $n, T(n) =$ the set of positive divisors of n. Find the following:

1.10.1 (a)

$T(1)$

Proof. $T(1) = \{1\}$

□

1.10.2 (b)

$T(15)$

Proof. $T(15) = \{1, 3, 5, 15\}$ □

1.10.3 (c)

$T(17)$

Proof. $T(17) = \{1, 17\}$ □

1.10.4 (d)

$T(5)$

Proof. $T(1) = \{1\}$ □

1.10.5 (e)

$T(18)$

Proof. $T(18) = \{1, 2, 3, 6, 9, 18\}$ □

1.10.6 (f)

$T(21)$

Proof. $T(21) = \{1, 3, 7, 21\}$ □

1.11 Exercise 11

Define $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ as follows: For every ordered pair (a, b) of integers, $F(a, b) = (2a + 1, 3b - 2)$. Find the following:

1.11.1 (a)

$F(4, 4)$

Proof. $F(4, 4) = (2 \cdot 4 + 1, 3 \cdot 4 - 2) = (9, 10)$ □

1.11.2 (b)

$F(2, 1)$

Proof. $F(2, 1) = (2 \cdot 2 + 1, 3 \cdot 1 - 2) = (5, 1)$ □

1.11.3 (c)

$F(3, 2)$

Proof. $F(3, 3) = (2 \cdot 3 + 1, 3 \cdot 3 - 2) = (7, 7)$ □

1.11.4 (d)

$F(1, 5)$

Proof. $F(1, 5) = (2 \cdot 1 + 1, 3 \cdot 5 - 2) = (3, 13)$ □

1.12 Exercise 12

Let $J_5 = \{0, 1, 2, 3, 4\}$, and define $G : J_5 \times J_5 \to J_5 \times J_5$ as follows: For each $(a, b) \in J_5 \times J_5, G(a, b) = ((2a + 1) \mod 5, (3b - 2) \mod 5)$. Find the following:

1.12.1 (a)

$G(4,4)$

Proof. $G(4,4) = ((2 \cdot 4 + 1) \mod 5, (3 \cdot 4 - 2) \mod 5) = (9 \mod 5, 10 \mod 5) = (4,0)$ □

1.12.2 (b)

$G(2,1)$

Proof. $G(2,1) = ((2 \cdot 2 + 1) \mod 5, (3 \cdot 1 - 2) \mod 5) = (5 \mod 5, 1 \mod 5) = (0,1)$ □

1.12.3 (c)

$G(3,2)$

Proof. $G(3,2) = ((2 \cdot 3 + 1) \mod 5, (3 \cdot 2 - 2) \mod 5) = (7 \mod 5, 4 \mod 5) = (2,4)$ □

1.12.4 (d)

$G(1,5)$

Proof. $G(1,5) = ((2 \cdot 1 + 1) \mod 5, (3 \cdot 5 - 2) \mod 5) = (3 \mod 5, 13 \mod 5) = (3,3)$ □

1.13 Exercise 13

Let $J_5 = \{0, 1, 2, 3, 4\}$, and define functions $f : J_5 \to J_5$ and $g : J_5 \to J_5$ as follows: For each $x \in J_5, f(x) = (x+4)^2 \mod 5$ and $g(x) = (x^2 + 3x + 1) \mod 5$. Is $f = g$? Explain.

Proof.

x	$f(x)$	$g(x)$
0	$4^2 \mod 5 = 1$	$(0^2 + 3 \cdot 0 + 1) \mod 5 = 1$
1	$5^2 \mod 5 = 0$	$(1^2 + 3 \cdot 1 + 1) \mod 5 = 0$
2	$6^2 \mod 5 = 1$	$(2^2 + 3 \cdot 2 + 1) \mod 5 = 1$
3	$7^2 \mod 5 = 4$	$(3^2 + 3 \cdot 3 + 1) \mod 5 = 4$
4	$8^2 \mod 5 = 4$	$(4^2 + 3 \cdot 4 + 1) \mod 5 = 4$

The table shows that $f(x) = g(x)$ for every x in J_5. Thus, by definition of equality of functions, $f = g$. □

1.14 Exercise 14

Define functions H and K from $\mathbb{R}$ to $\mathbb{R}$ by the following formulas: For every $x \in \mathbb{R}, H(x) = \lfloor x \rfloor + 1$ and $K(x) = \lceil x \rceil$. Does $H = K$? Explain.

Proof. No, because $H(2) = \lfloor 2 \rfloor + 1 = 3 \neq 2 = \lceil 2 \rceil = K(2)$. $\square$

1.15 Exercise 15

Let F and G be functions from the set of all real numbers to itself. Define new functions $F \cdot G : \mathbb{R} \to \mathbb{R}$ and $G \cdot F : \mathbb{R} \to \mathbb{R}$ as follows: For every $x \in \mathbb{R}$, $(F \cdot G)(x) = F(x) \cdot G(x)$, $(G \cdot F)(x) = G(x) \cdot F(x)$. Does $F \cdot G = G \cdot F$? Explain.

Proof.

$$\begin{aligned} (F \cdot G)(x) &= F(x) \cdot G(x) && \text{by definition of } F \cdot G \\ &= G(x) \cdot F(x) && \text{by commutative law for real numbers} \\ &= (G \cdot F)(x) && \text{by definition of } G \cdot F \end{aligned}$$

for every real number x. Therefore $F \cdot G$ and $G \cdot F$ are equal. $\square$

1.16 Exercise 16

Let F and G be functions from the set of all real numbers to itself. Define new functions $F - G : \mathbb{R} \to \mathbb{R}$ and $G - F : \mathbb{R} \to \mathbb{R}$ as follows: For every $x \in \mathbb{R}$, $(F - G)(x) = F(x) - G(x)$, $(G - F)(x) = G(x) - F(x)$. Does $F - G = G - F$? Explain.

Proof. Counterexample: Let $F(x) = 2x, G(x) = 3x$. Then

$$(F - G)(1) = F(1) - G(1) = 2 - 3 = -1 \neq 1 = 3 - 2 = G(1) - F(1) = (G - F)(1).$$

Therefore $F - G$ does not equal $G - F$. $\square$

1.17 Exercise 17

Use the definition of logarithm to fill in the blanks below.

1.17.1 (a)

$\log_2 8 = 3$ because ____ .

Proof. $2^3 = 8$ $\square$

1.17.2 (b)

$\log_5(\frac{1}{25}) = -2$ because ____ .

Proof. $5^{-2} = \dfrac{1}{25}$ $\square$

1.17.3 (c)

$\log_4 4 = 1$ because ____ .

Proof. $4^1 = 4$ □

1.17.4 (d)

$\log_3(3^n) = n$ because ____ .

Proof. $3^n = 3^n$ □

1.17.5 (e)

$\log_4 1 = 0$ because ____ .

Proof. $4^0 = 1$ □

1.18 Exercise 18

Find exact values for each of the following quantities without using a calculator.

1.18.1 (a)

$\log_3 81$

Proof. 4 □

1.18.2 (b)

$\log_2 1024$

Proof. 10 □

1.18.3 (c)

$\log_3 \frac{1}{27}$

Proof. -3 □

1.18.4 (d)

$\log_2 1$

Proof. 0 □

1.18.5 (e)

$\log_{10}(\frac{1}{10})$

Proof. -1 □

1.18.6 (f)

$\log_3 3$

Proof. 1 □

1.18.7 (g)

$\log_2 2^k$

Proof. k □

1.19 Exercise 19

Use the definition of logarithm to prove that for any positive real number b with $b \neq 1, \log_b b = 1$.

Proof. Let b be any positive real number with $b \neq 1$. Since $b^1 = b$, then $\log_b b = 1$ by definition of logarithm. □

1.20 Exercise 20

Use the definition of logarithm to prove that for any positive real number b with $b \neq 1, \log_b 1 = 0$.

Proof. Let b be any positive real number with $b \neq 1$. Since $b^0 = 1$, then $\log_b 1 = 0$ by definition of logarithm. □

1.21 Exercise 21

If b is any positive real number with $b \neq 1$ and x is any real number, b^{-x} is defined as follows: $b^{-x} = \dfrac{1}{b^x}$. Use this definition and the definition of logarithm to prove that $\log_b\left(\dfrac{1}{u}\right) = -\log_b(u)$ for all positive real numbers u and b, with $b \neq 1$.

Proof. Suppose b and u are any positive real numbers with $b \neq 1$. *[We must show that* $\log_b(\frac{1}{u}) = -\log_b(u)$*.]* Let $v = \log_b(\frac{1}{u})$. By definition of logarithm, $b^v = \frac{1}{u}$. Multiplying both sides by u and dividing by b^v gives $u = b^{-v}$, and thus, by definition of logarithm, $-v = \log_b(u)$. When both sides of this equation are multiplied by -1, the result is $v = -\log_b(u)$. Therefore, $\log_b(\frac{1}{u}) = -\log_b(u)$ because both expressions equal v. *[This is what was to be shown.]* □

1.22 Exercise 22

Use the unique factorization for the integers theorem (Section 4.4) and the definition of logarithm to prove that $\log_3(7)$ is irrational.

Proof. 1. Argue by contradiction and assume $r = \log_3(7)$ is rational.

2. By 1 and definition of rational, $r = a/b$ for some integers a, b where $b \neq 0$.

3. We may assume $b > 0$. (If $b < 0$ then $a/b = (-a)/(-b)$ therefore we can replace a/b with $-a/(-b)$ where $-b > 0$.)

4. By 1, 2 and definition of logarithm, $7 = 3^{a/b}$.

5. By 4, taking the bth powers of both sides, we get $7^b = 3^a$.

6. Since b is a positive integer, 7^b is a positive integer. Therefore 3^a is the same positive integer.

7. By 6, we have two different prime factorizations of the same positive integer. By the uniqueness part of the prime factorization theorem, this is only possible if the positive integer is equal to 1.

8. By 7, $7^b = 3^a = 1$ so $a = b = 0$. This is a contradiction since $b > 0$.

9. So our supposition in 1 is false by 8, thus $\log_3(7)$ is irrational. $\square$

1.23 Exercise 23

If b and y are positive real numbers such that $\log_b y = 3$, what is $\log_{1/b}(y)$? Explain.

Proof. By definition of logarithm with base b, $b^3 = y$. So

$$y = b^3 = \frac{1}{\frac{1}{b^3}} = \frac{1}{\left(\frac{1}{b}\right)^3} = \left(\frac{1}{b}\right)^{-3}$$

So by definition of logarithm with base $1/b$, $\log_{1/b}(y) = -3$. $\square$

1.24 Exercise 24

If b and y are positive real numbers such that $\log_b y = 2$, what is $\log_{b^2}(y)$? Explain.

Proof. By definition of logarithm with base b, $b^2 = y$. So by definition of logarithm with base b^2, $\log_{b^2}(y) = 1$ because $(b^2)^1 = y$. $\square$

1.25 Exercise 25

Let $A = \{2, 3, 5\}$ and $B = \{x, y\}$. Let p_1 and p_2 be the projections of $A \times B$ onto the first and second coordinates. That is, for each pair $(a, b) \in A \times B, p_1(a, b) = a$ and $p_2(a, b) = b$.

1.25.1 (a)

Find $p_1(2, y)$ and $p_1(5, x)$. What is the range of p_1?

Proof. $p_1(2, y) = 2, p_1(5, x) = 5$, range of $p_1 = \{2, 3, 5\}$ □

1.25.2 (b)

Find $p_2(2, y)$ and $p_2(5, x)$. What is the range of p_2?

Proof. $p_2(2, y) = y, p_2(5, x) = x$, range of $p_2 = \{x, y\}$ □

1.26 Exercise 26

Observe that mod and div can be defined as functions from $\mathbb{Z}^{nonneg} \times \mathbb{Z}^{+}$ to $\mathbb{Z}$. For each ordered pair (n, d) consisting of a nonnegative integer n and a positive integer d, let

$mod(n, d) = n \mod d$ (the nonnegative remainder obtained when n is divided by d).

$div(n, d) = n$ div d (the integer quotient obtained when n is divided by d).

Find each of the following:

1.26.1 (a)

$mod(67, 10)$ and $div(67, 10)$

Proof. $mod(67, 10) =, div(67, 10) =$ □

1.26.2 (b)

$mod(59, 8)$ and $div(59, 8)$

Proof. $mod(67, 10) = 7, div(67, 10) = 6$ □

1.26.3 (c)

$mod(30, 5)$ and $div(30, 5)$

Proof. $mod(30, 5) = 0, div(30, 5) = 6$ □

1.27 Exercise 27

Let S be the set of all strings of a's and b's.

1.27.1 (a)

Define $f : S \to \mathbb{Z}$ as follows: For each string s in S

$$f(s) = \begin{cases} \text{the number of } b\text{'s to the left of the left most } a \text{ in } s & \text{if } s \text{ contains some } a\text{'s} \\ 0 & \text{if } s \text{ contains no } a\text{'s} \end{cases}$$

Find $f(aba)$, $f(bbab)$, and $f(b)$. What is the range of f?

Proof. $f(aba) = 0$ *[because there are no b's to the left of the leftmost a in aba]*

$f(bbab) = 2$ *[because there are two b's to the left of the leftmost a in bbab]*

$f(b) = 0$ *[because the string b contains no a's]*

range of $f = \mathbb{Z}^{nonneg}$ □

1.27.2 (b)

Define $g : S \to S$ as follows: For each string s in S, $g(s) =$ the string obtained by writing the characters of s in reverse order. Find $g(aba)$, $g(bbab)$, and $g(b)$. What is the range of g?

Proof. $g(aba) = aba, g(bbab) = babb, g(b) = b$, range of $g = S$ □

1.28 Exercise 28

Consider the coding and decoding functions E and D defined in Example 7.1.9.

1.28.1 (a)

Find $E(0110)$ and $D(111111000111)$.

Proof. $E(0110) = 000111111000$ and $D(111111000111) = 1101$ □

1.28.2 (b)

Find $E(1010)$ and $D(000000111111)$.

Proof. $E(1010) = 111000111000$ and $D(000000111111) = 0011$ □

1.29 Exercise 29

Consider the Hamming distance function defined in Example 7.1.10.

1.29.1 (a)

Find $H(10101, 00011)$.

Proof. $H(10101, 00011) = 3$ □

1.29.2 (b)

Find $H(00110, 10111)$.

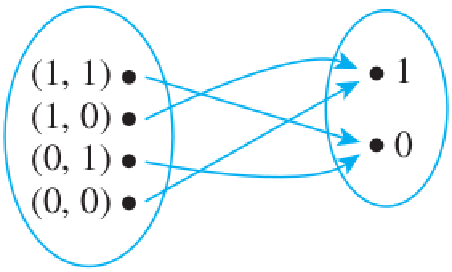
Proof. $H(00110, 10111) = 2$ □

1.30 Exercise 30

Draw arrow diagrams for the Boolean functions defined by the following input/output tables.

1.30.1 (a)

Input		Output
P	Q	R
1	1	0
1	0	1
0	1	0
0	0	1

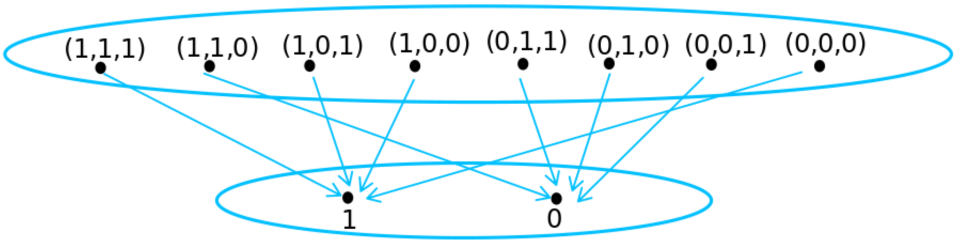


Proof. □

1.30.2 (b)

Input			Output
P	Q	R	S
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

Proof. □



1.31 Exercise 31

Fill in the following table to show the values of all possible two-place Boolean functions.

Proof.

Input	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
1 1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
1 0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0 1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0 0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

□

1.32 Exercise 32

Consider the three-place Boolean function f defined by the following rule: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (4x_1 + 3x_2 + 2x_3) \mod 2.$$

1.32.1 (a)

Find $f(1, 1, 1)$ and $f(0, 0, 1)$.

Proof. $f(1, 1, 1) = (4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1) \mod 2 = 9 \mod 2 = 1$

$f(0, 0, 1) = (4 \cdot 0 + 3 \cdot 0 + 2 \cdot 1) \mod 2 = 2 \mod 2 = 0$ □

1.32.2 (b)

Describe f using an input/output table.

Proof.

Input			Output
x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	0

□

1.33 Exercise 33

Student A tries to define a function $g : \mathbb{Q} \to \mathbb{Z}$ by the rule

$$g\left(\frac{m}{n}\right) = m - n \text{ for all integers } m \text{ and } n \text{ with } n \neq 0.$$

Student B claims that g is not well defined. Justify student B's claim.

Proof. If g were well defined, then $g(1/2) = g(2/4)$ because $1/2 = 2/4$. However, $g(1/2) = 1 - 2 = -1$ and $g(2/4) = 2 - 4 = -2$. Since $-1 \neq -2, g(1/2) \neq g(2/4)$. Thus g is not well defined. □

1.34 Exercise 34

Student C tries to define a function $h : \mathbb{Q} \to \mathbb{Q}$ by the rule

$$h\left(\frac{m}{n}\right) = \frac{m^2}{n} \text{ for all integers } m \text{ and } n \text{ with } n \neq 0.$$

Student D claims that h is not well defined. Justify student D's claim.

Proof.

$$h(2) = h\left(\frac{4}{2}\right) = \frac{4^2}{2} = 8 \neq 4 = \frac{2^2}{1} = h\left(\frac{2}{1}\right) = h(2).$$

□

1.35 Exercise 35

Let $U = \{1, 2, 3, 4\}$. Student A tries to define a function $R : U \to \mathbb{Z}$ as follows: For each $x \in U$, $R(x)$ is the integer y so that $(xy) \mod 5 = 1$. Student B claims that R is not well defined. Who is right: student A or student B? Justify your answer.

Proof. Student B is correct. If R were well defined, then $R(3)$ would have a uniquely determined value. However, on the one hand, $R(3) = 2$ because $(3 \cdot 2) \mod 5 = 1$, and, on the other hand, $R(3) = 7$ because $(3 \cdot 7) \mod 5 = 1$. Hence $R(3)$ does not have a uniquely determined value, and so R is not well defined. □

1.36 Exercise 36

Let $V = \{1, 2, 3\}$. Student C tries to define a function $S : V \to V$ as follows: For each $x \in V$, $S(x)$ is the integer y in V so that $(xy) \mod 4 = 1$. Student D claims that S is not well defined. Who is right: student C or student D? Justify your answer.

Proof. Student D is right, because $S(2)$ is not defined. $2 \cdot 1 \mod 4 = 2, 2 \cdot 2 \mod 4 = 0$, and $2 \cdot 3 \mod 4 = 2$. So when $x = 2$ there is no y in V such that $xy \mod 4 = 1$. □

1.37 Exercise 37

On certain computers the integer data type goes from -2,147,483,648 to 2,147,483,647. Let S be the set of all integers from -2,147,483,648 through 2,147,483,647. Try to define a function $f : S \to S$ by the rule $f(n) = n^2$ for each n in S. Is f well defined? Explain.

Proof. No, $2, 147, 483, 647 = 2^{31} - 1$ so for values of n greater than, say, 2^{16}, $f(n) = n^2$ will be greater than 2^{32} which falls outside of S.

Computers handle this by using 2's complement and looping the overshoot around back to -2^{31} and onward toward the positive values again. Here is an example from Scala:

```
$ scala
// Welcome to Scala 3.3.0 (17.0.7, Java OpenJDK 64-Bit Server VM).
// Type in expressions for evaluation. Or try :help.
scala> def f(n: Int): Int = n*n
def f(n: Int): Int
scala> f(2147483647)
val res0: Int = 1
scala>
```

□

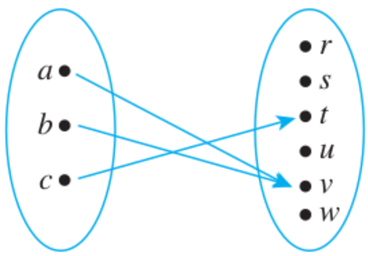
1.38 Exercise 38

Let $X = \{a, b, c\}$ and $Y = \{r, s, t, u, v, w\}$. Define $f : X \to Y$ as follows: $f(a) = b, f(b) = v, f(c) = t$.

1.38.1 (a)

Draw an arrow diagram for f.

Proof. □



1.38.2 (b)

Let $A = \{a, b\}, C = \{t\}, D = \{u, v\}, E = \{r, s\}$.

Find $f(A), f(X), f^{-1}(C), f^{-1}(D), f^{-1}(E), f^{-1}(Y)$.

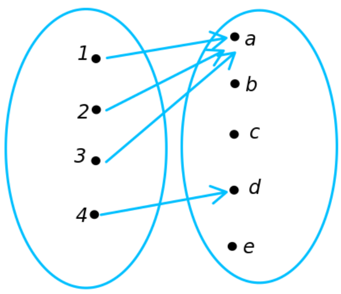
Proof. $f(A) = \{v\}, f(X) = \{t, v\}, f^{-1}(C) = \{c\}, f^{-1}(D) = \{a, b\}, f^{-1}(E) = \varnothing$, $f^{-1}(Y) = \{a, b, c\}$ $\square$

1.39 Exercise 39

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$. Define $g : X \to Y$ as follows: $g(1) = a, g(2) = a, g(3) = a, g(4) = d$.

1.39.1 (a)

Draw an arrow diagram for g.



Proof. $\square$

1.39.2 (b)

Let $A = \{2, 3\}, C = \{a\}$, and $D = \{b, c\}$. Find $g(A), g(X), g^{-1}(C), g^{-1}(D)$, and $g^{-1}(Y)$.

Proof. $g(A) = \{a\}, g(X) = \{a, d\}, g^{-1}(C) = \{1, 2, 3\}, g^{-1}(D) = \varnothing, g^{-1}(Y) = \{1, 2, 3, 4\}$. $\square$

1.40 Exercise 40

Let X and Y be sets, let A and B be any subsets of X, and let F be a function from X to Y. Fill in the blanks in the following proof that $F(A) \cup F(B) \subseteq F(A \cup B)$.

Proof: Let y be any element in $F(A) \cup F(B)$. *[We must show that y is in $F(A \cup B)$.]* By definition of union, (i) ____ .

Case 1, $y \in F(A)$: In this case, by definition of $F(A)$, $y = F(x)$ for (ii) ____ $x \in A$. Since $A \subseteq A \cup B$, it follows from the definition of union that $x \in$ (iii) ____ . Hence, $y = F(x)$ for some $x \in A \cup B$, and thus, by definition of $F(A \cup B), y \in$ (iv) ____ .

Case 2, $y \in F(B)$: In this case, by definition of $F(B)$, (v) ____ for some $x \in B$. Since $B \subseteq A \cup B$ it follows from the definition of union that (vi) ____ . Thus $y \in F(A \cup B)$.

Therefore, regardless of whether $y \in F(A)$ or $y \in F(B)$, we have that $y \in F(A \cup B)$ *[as was to be shown]*.

Proof. (i) $y \in F(A)$ or $y \in F(B)$ (ii) some (iii) $A \cup B$ (iv) $F(A \cup B)$ (v) $y = F(x)$ (vi) $x \in A \cup B$ □

In $41 - 49$ let X and Y be sets, let A and B be any subsets of X, and let C and D be any subsets of Y. Determine which of the properties are true for every function F from X to Y and which are false for at least one function F from X to Y. Justify your answers.

1.41 Exercise 41

If $A \subseteq B$ then $F(A) \subseteq F(B)$.

Proof. Let F be a function from X to Y, and suppose $A \subseteq X, B \subseteq X$, and $A \subseteq B$. Let $y \in F(A)$. *[We must show that $y \in F(B)$.]* By definition of image of a set, $y = F(x)$ for some $x \in A$. Thus since $A \subseteq B, x \in B$, and so $y = F(x)$ for some $x \in B$. Hence $y \in F(B)$ *[as was to be shown]*. □

1.42 Exercise 42

$F(A \cap B) \subseteq F(A) \cap F(B)$

Proof. 1. Assume $y \in F(A \cap B)$. *[We want to show $y \in F(A) \cap F(B)$.]*

2. By 1 and definition of $F(A \cap B)$, $y = F(x)$ for some $x \in A \cap B$.

3. By 2 and definition of intersection, $x \in A$ and $x \in B$.

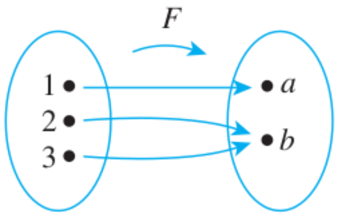
4. By 3 and definition of $F(A)$ and $F(B)$, $y = F(x)$ is in $F(A)$ and in $F(B)$.

5. By 4 and definition of intersection, $y \in F(A) \cap F(B)$.

6. By 1, 5 and definition of subset, $F(A \cap B) \subseteq F(A) \cap F(B)$. □

1.43 Exercise 43

$F(A) \cap F(B) \subseteq F(A \cap B)$



Proof. Counterexample: Let $X = \{1, 2, 3\}$, let $Y = \{a, b\}$, and define a function $F : X \to Y$ by the arrow diagram shown above.

Let $A = \{1, 2\}$ and $B = \{1, 3\}$. Then $F(A) = \{a, b\} = F(B)$, and so $F(A) \cap F(B) = \{a, b\}$. But $F(A \cap B) = F(\{1\}) = \{a\} \neq \{a, b\}$. And so $F(A) \cap F(B) \not\subseteq F(A \cap B)$. (This is just one of many possible counterexamples.) □

1.44 Exercise 44

For all subsets A and B of X, $F(A - B) = F(A) - F(B)$.

Proof. Counterexample: Let $X = \{1, 2\}, Y = \{a\}, A = \{1\}, B = \{2\}$, define $F : X \to Y$ by $F(1) = F(2) = a$. Then $A - B = \{1\}, F(A) = \{a\}, F(B) = \{a\}$. So $F(A - B) = \{a\} \neq \varnothing = F(A) - F(B)$. □

1.45 Exercise 45

For all subsets C and D of Y, if $C \subseteq D$ then $F^{-1}(C) \subseteq F^{-1}(D)$.

Proof. Let F be a function from a set X to a set Y, and suppose $C \subseteq Y, D \subseteq Y$, and $C \subseteq D$. *[We must show that $F^{-1}(C) \subseteq F^{-1}(D)$.]* Suppose $x \in F^{-1}(C)$. Then $F(x) \in C$. Since $C \subseteq D, F(x) \in D$ also. Hence, by definition of inverse image, $x \in F^{-1}(D)$. *[So $F^{-1}(C) \subseteq F^{-1}(D)$.]* □

1.46 Exercise 46

For all subsets C and D of Y, $F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$.

Proof. 1. Assume $x \in F^{-1}(C \cup D)$ and let $y = F(x)$. *[Want to show $x \in F^{-1}(C) \cup F^{-1}(D)$.]*

2. By 1 and definition of $F^{-1}(C \cup D)$, $y \in C \cup D$.

3. By 2 and definition of union, $y \in C$ or $y \in D$.

4. **Case 1 ($\boldsymbol{y \in C}$):** By definition of $F^{-1}(C)$, $x \in F^{-1}(C)$.

By definition of union, $x \in F^{-1}(C) \cup F^{-1}(D)$.

5. **Case 2 ($\boldsymbol{y \in D}$):** By definition of $F^{-1}(D)$, $x \in F^{-1}(D)$.

By definition of union, $x \in F^{-1}(C) \cup F^{-1}(D)$.

6. By 4 and 5, $x \in F^{-1}(C) \cup F^{-1}(D)$.

7. By 1, 6 and definition of subset, $F^{-1}(C \cup D) \subseteq F^{-1}(C) \cup F^{-1}(D)$.

The proof of the reverse direction $F^{-1}(C) \cup F^{-1}(D) \subseteq F^{-1}(C \cup D)$ is similar. □

1.47 Exercise 47

For all subsets C and D of Y, $F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$.

Proof. True, the proof is extremely similar to exercise 46. □

1.48 Exercise 48

For all subsets C and D of Y, $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.

Proof. 1. Assume $x \in F^{-1}(C - D)$ and let $y = F(x)$. *[Want to show $x \in F^{-1}(C) - F^{-1}(D)$.]*

2. By 1 and definition of $F^{-1}(C - D), y \in C - D$.

3. By 2 and definition of difference, $y \in C$ and $y \notin D$.

4. By 3 and definition of $F^{-1}(C)$, $x \in F^{-1}(C)$. Similarly, since $y = F(x)$ and $y \notin D$, $x \notin F^{-1}(D)$.

5. By 4 and definition of difference, $x \in F^{-1}(C) - F^{-1}(D)$.

6. By 1, 5 and definition of subset, $F^{-1}(C - D) \subseteq F^{-1}(C) - F^{-1}(D)$.

Now the reverse part:

7. Assume $x \in F^{-1}(C) - F^{-1}(D)$ and let $y = F(x)$. *[Want to show $x \in F^{-1}(C - D)$.]*

8. By 7 and definition of difference, $x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$.

9. By 8 and definition of $F^{-1}(C), y \in C$. Similarly $y \notin D$.

10. By 9 and definition of difference $y \in C - D$.

11. Since $y = F(x)$, by 10 and definition of $F^{-1}(C - D)$, $x \in F^{-1}(C - D)$.

12. By 7, 11 and definition of subset, $F^{-1}(C) - F^{-1}(D) \subseteq F^{-1}(C - D)$.

Conclusion:

13. By 6, 12 and definition of set equality $F^{-1}(C) - F^{-1}(D) = F^{-1}(C - D)$. □

1.49 Exercise 49

$F(F^{-1}(C)) \subseteq C$.

Proof. 1. Assume $y \in F(F^{-1}(C))$. *[Want to show $y \in C$.]*

2. By 1 and definition of $F(F^{-1}(C))$, there exists some $x \in F^{-1}(C)$ such that $y = F(x)$.

3. By 2 and definition of $F^{-1}(C)$, $F(x) \in C$. So $y \in C$ because $y = F(x)$.

4. By 1, 3 and definition of subset, $F(F^{-1}(C)) \subseteq C$. □

1.50 Exercise 50

Given a set S and a subset A, the characteristic function of A, denoted χ_A, is the function defined from S to $\mathbb{Z}$ with the property that for each $u \in S$,

$$\chi_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

Show that each of the following holds for all subsets A and B of S and every $u \in S$.

1.50.1 (a)

$\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$

Proof. Assume A, B are any subsets of S and u is any element in S. There are 4 cases:

Case 1 ($\boldsymbol{u \in A, u \in B}$): Then $\chi_A(u) = 1$ and $\chi_B(u) = 1$.

By definition of intersection $u \in A \cap B$. Thus $\chi_{A\cap B}(u) = 1$ also. Since $1 = 1 \cdot 1$, $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$.

Case 2 ($\boldsymbol{u \in A, u \notin B}$): Then $\chi_A(u) = 1$ and $\chi_B(u) = 0$.

By definition of intersection $u \notin A \cap B$. Thus $\chi_{A\cap B}(u) = 0$ also. Since $0 = 1 \cdot 0$, $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$.

Case 3 ($\boldsymbol{u \notin A, u \in B}$): Then $\chi_A(u) = 0$ and $\chi_B(u) = 1$.

By definition of intersection $u \notin A \cap B$. Thus $\chi_{A\cap B}(u) = 0$ also. Since $0 = 0 \cdot 1$, $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$.

Case 4 ($\boldsymbol{u \notin A, u \notin B}$): Then $\chi_A(u) = 0$ and $\chi_B(u) = 0$.

By definition of intersection $u \notin A \cap B$. Thus $\chi_{A\cap B}(u) = 0$ also. Since $0 = 0 \cdot 0$, $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$. □

1.50.2 (b)

$\chi_{A\cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$

Proof. Assume A, B are any subsets of S and u is any element in S. There are 4 cases:

Case 1 ($\boldsymbol{u \in A, u \in B}$): Then $\chi_A(u) = 1$ and $\chi_B(u) = 1$.

By definition of union $u \in A \cup B$. Thus $\chi_{A\cup B}(u) = 1$ also. Since $1 = 1 + 1 - (1 \cdot 1)$, $\chi_{A\cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$.

Case 2 ($\boldsymbol{u \in A, u \notin B}$): Then $\chi_A(u) = 1$ and $\chi_B(u) = 0$.

By definition of union $u \in A \cup B$. Thus $\chi_{A\cup B}(u) = 1$ also. Since $1 = 1 + 0 - (1 \cdot 0)$, $\chi_{A\cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$.

Case 3 ($u \notin A, u \in B$): Then $\chi_A(u) = 0$ and $\chi_B(u) = 1$.

By definition of union $u \in A \cup B$. Thus $\chi_{A\cup B}(u) = 1$ also. Since $1 = 0 + 1 - (0 \cdot 1)$, $\chi_{A\cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$.

Case 4 ($u \notin A, u \notin B$): Then $\chi_A(u) = 0$ and $\chi_B(u) = 0$.

By definition of union $u \notin A \cup B$. Thus $\chi_{A\cup B}(u) = 0$ also. Since $0 = 0 + 0 - (0 \cdot 0)$, $\chi_{A\cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$. □

Each of exercises $51 - 53$ refers to the Euler phi function, denoted ϕ, which is defined as follows: For each integer $n \geq 1$, $\phi(n)$ is the number of positive integers less than or equal to n that have no common factors with n except ± 1. For example, $\phi(10) = 4$ because there are four positive integers less than or equal to 10 that have no common factors with 10 except ± 1, namely, 1, 3, 7, and 9.

1.51 Exercise 51

Find each of the following:

1.51.1 (a)

$\phi(15)$

Proof. $\phi(15) = 8$ *[because* $1, 2, 4, 7, 8, 11, 13,$ *and* 14 *have no common factors with* 15 *other than* ± 1*]* □

1.51.2 (b)

$\phi(2)$

Proof. $\phi(2) = 1$ *[because the only positive integer less than or equal to* 2 *having no common factors with* 2 *other than* ± 1 *is* 1*]* □

1.51.3 (c)

$\phi(5)$

Proof. $\phi(5) = 4$ *[because* $1, 2, 3,$ *and* 4 *have no common factors with* 5 *other than* ± 1*]* □

1.51.4 (d)

$\phi(12)$

Proof. $\phi(12) = 4$ (1, 5, 7, 11) □

1.51.5 (e)

$\phi(11)$

Proof. $\phi(11) = 10$ (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) □

1.51.6 (f)

$\phi(1)$

Proof. $\phi(1) = 1$ □

1.52 Exercise 52

Prove that if p is a prime number and n is an integer with $n \geq 1$, then $\phi(p^n) = p^n - p^{n-1}$.

Proof. Let p be any prime number and n any integer with $n \geq 1$. There are p^{n-1} positive integers less than or equal to p^n that have a common factor other than ± 1 with p^n, namely, $p, 2p, 3p, \ldots, (p^{n-1})p$. Hence, there are $p^n - p^{n-1}$ positive integers less than or equal to p^n that do not have a common factor with p^n except for ± 1. □

1.53 Exercise 53

Prove that there are infinitely many integers n for which $\phi(n)$ is a perfect square.

Proof. By exercise 52, for any integer n with $n \geq 1, \phi(2^n) = 2^n - 2^{n-1} = 2^{n-1}$.

So, for all integers k with $k \geq 1$, we have that $\phi(2^{2k+1}) = 2^{2k} = (2^k)^2$ is a perfect square. □

2 Exercise Set 7.2

2.1 Exercise 1

The definition of one-to-one is stated in two ways:

$$\forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2$$

and

$$\forall x_1, x_2 \in X, \text{ if } x_1 \neq x_2 \text{ then } F(x_1) \neq F(x_2).$$

Why are these two statements logically equivalent?

Proof. The second statement is the contrapositive of the first. □

2.2 Exercise 2

Fill in each blank with the word most or least.

2.2.1 (a)

A function F is one-to-one if, and only if, each element in the co-domain of F is the image of at ____ one element in the domain of F.

Proof. most □

2.2.2 (b)

A function F is onto if, and only if, each element in the co-domain of F is the image of at ____ one element in the domain of F.

Proof. least □

2.3 Exercise 3

When asked to state the definition of one-to-one, a student replies, "A function f is one-to-one if, and only if, every element of X is sent by f to exactly one element of Y." Give a counterexample to show that the student's reply is incorrect.

Proof. One counterexample is given and explained below. Give a different counterexample and accompany it with an explanation.

Counterexample: Consider $X = \{a, b\}, Y = \{u, v\}$ and the function f defined by $f(a) = f(b) = u$. Observe that a is sent to exactly one element of Y, namely, u, and b is also sent to exactly one element of Y, namely, u also. So it is true that every element of X is sent to exactly one element of Y. But f is not one-to-one because $f(a) = f(b)$ whereas $a \neq b$. *[Note that to say, "Every element of X is sent to exactly one element of Y" is just another way of saying that in the arrow diagram for the function there is only one arrow coming out of each element of X. But this statement is part of the definition of any function, not just of a one-to-one function.]* □

2.4 Exercise 4

Let $f : X \to Y$ be a function. True or false? A sufficient condition for f to be one-to-one is that for every element y in Y, there is at most one x in X with $f(x) = y$. Explain your answer.

Proof. True. Assume $x_1, x_2 \in X$ and assume $f(x_1) = f(x_2)$. *[We want to show $x_1 = x_2$].* Let $y = f(x_1) = f(x_2)$. By assumption there is at most one x in X such that $y = f(x)$. Thus $x = x_1 = x_2$, *[as was to be shown.]* □

2.5 Exercise 5

All but two of the following statements are correct ways to express the fact that a function f is onto. Find the two that are incorrect.

2.5.1 (a)

f is onto $\iff$ every element in its co-domain is the image of some element in its domain.

Proof. Correct. $\square$

2.5.2 (b)

f is onto $\iff$ every element in its domain has a corresponding image in its co-domain.

Proof. Incorrect. $\square$

2.5.3 (c)

f is onto $\iff$ $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

Proof. Correct. $\square$

2.5.4 (d)

f is onto $\iff$ $\forall x \in X, \exists y \in Y$ such that $f(x) = y$.

Proof. Incorrect. $\square$

2.5.5 (e)

f is onto $\iff$ the range of f is the same as the co-domain of f.

Proof. Correct. $\square$

2.6 Exercise 6

Let $X = \{1, 5, 9\}, Y = \{3, 4, 7\}$.

2.6.1 (a)

Define $f : X \to Y$ by specifying that $f(1) = 4, f(5) = 7, f(9) = 4$. Is f one-to-one? Is f onto? Explain your answers.

Proof. Not 1-1: $f(1) = f(9)$ but $1 \neq 9$. Not onto: no $x \in X$ such that $f(x) = 3$. $\square$

2.6.2 (b)

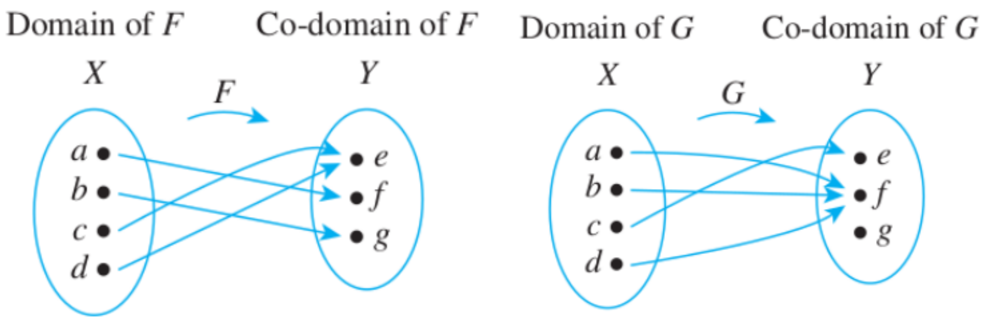
Define $g : X \to Y$ by specifying that $g(1) = 7, g(5) = 3, g(9) = 4$. Is g one-to-one? Is g onto? Explain your answers.

Proof. g is 1-1, because $g(1) \neq f(5), g(1) \neq g(9)$ and $g(1) \neq g(9)$.

g is onto because each element of Y is the image of some $x \in X$: $3 = g(5), 4 = g(9), 7 = g(1)$. □

2.7 Exercise 7

Let $X = \{a, b, c, d\}$ and $Y = \{e, f, g\}$. Define functions F and G by the arrow diagrams below.



2.7.1 (a)

Is F one-to-one? Why or why not? Is it onto? Why or why not?

Proof. F is not 1-1 because $F(c) = F(d)$ but $c \neq d$.

F is onto, because each element in Y is the image of some element in X. □

2.7.2 (b)

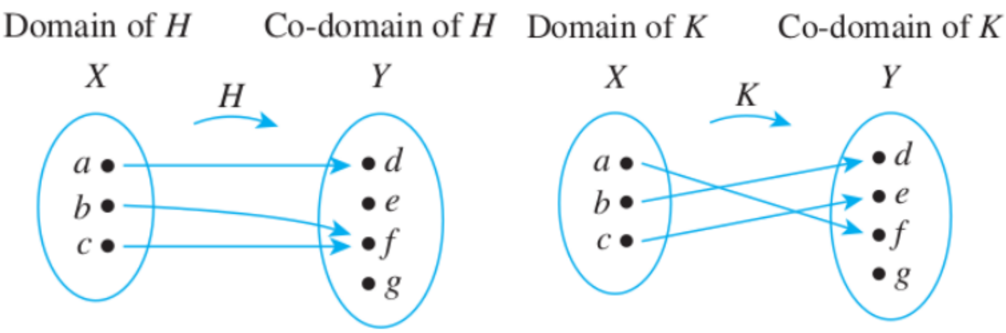
Is G one-to-one? Why or why not? Is it onto? Why or why not?

Proof. G is not 1-1 because $G(a) = G(b) = f$.

G is not onto, because there is no $x \in X$ such that $G(x) = g$. □

2.8 Exercise 8

Let $X = \{a, b, c\}$ and $Y = \{d, e, f, g\}$. Define functions H and K by the arrow diagrams below.



2.8.1 (a)

Is H one-to-one? Why or why not? Is it onto? Why or why not?

Proof. H is not 1-1 because $H(b) = H(c)$ but $b \neq c$.

H is not onto, because there is no $x \in X$ such that $H(x) = g$. □

2.8.2 (b)

Is K one-to-one? Why or why not? Is it onto? Why or why not?

Proof. K is 1-1 because $K(a) \neq K(b), K(a) \neq K(c)$ and $K(b) \neq K(c)$.

K is not onto, because there is no $x \in X$ such that $K(x) = g$. □

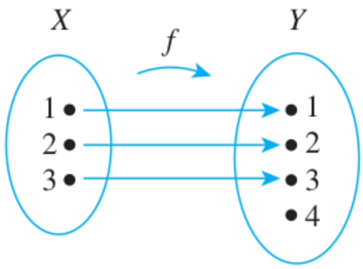
2.9 Exercise 9

Let $X = \{1, 2, 3\}, Y = \{1, 2, 3, 4\}$, and $Z = \{1, 2\}$.

2.9.1 (a)

Define a function $f : X \to Y$ that is one-to-one but not onto.

Proof. One example of many is the following: □



2.9.2 (b)

Define a function $g : X \to Z$ that is onto but not one-to-one.

Proof. Let $g(1) = 1, g(2) = 2, g(3) = 2$. □

2.9.3 (c)

Define a function $h : X \to X$ that is neither one-to-one nor onto.

Proof. Let $h(1) = 1, h(2) = 1, h(3) = 1, h(4) = 1$. □

2.9.4 (d)

Define a function $k : X \to X$ that is one-to-one and onto but is not the identity function on X.

Proof. Let $k(1) = 2, k(2) = 3, k(3) = 4, k(4) = 1$. □

2.10 Exercise 10

2.10.1 (a)

Define $f : \mathbb{Z} \to \mathbb{Z}$ by the rule $f(n) = 2n$, for every integer n.

(i) Is f one-to-one? Prove or give a counterexample.

(ii) Is f onto? Prove or give a counterexample.

Proof. (i) f is one-to-one. Suppose $f(n_1) = f(n_2)$ for some integers n_1 and n_2. *[We must show that $n_1 = n_2$.]* By definition of $f, 2n_1 = 2n_2$, and dividing both sides by 2 gives $n_1 = n_2$ *[as was to be shown]*.

(ii) f is not onto. Counterexample: Consider $1 \in \mathbb{Z}$. We claim that $1 \neq f(n)$, for any integer n, because if there were an integer n such that $1 = f(n)$, then, by definition of $f, 1 = 2n$. Dividing both sides by 2 would give $n = 1/2$. But $1/2$ is not an integer. Hence $1 \neq f(n)$ for any integer n, and so f is not onto. □

2.10.2 (b)

Let $2\mathbb{Z}$ denote the set of all even integers. That is,

$$2\mathbb{Z} = \{n \in \mathbb{Z} \,|\, n = 2k, \text{ for some integer } k\}.$$

Define $h : \mathbb{Z} \to 2\mathbb{Z}$ by the rule $h(n) = 2n$, for each integer n. Is h onto? Prove or give a counterexample.

Proof. h is onto. Suppose $m \in 2\mathbb{Z}$. *[We must show that there exists an integer such that h of that integer equals m.]* Since $m \in 2\mathbb{Z}, m = 2k$ for some integer k. Then $h(k) = 2k = m$. Hence there exists an integer (namely, k) such that $h(k) = m$ *[as to be shown]*. □

2.11 Exercise 11

2.11.1 (a)

Define $g : \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 4n - 5$, for every integer n.

(i) Is g one-to-one? Prove or give a counterexample.

(ii) Is g onto? Prove or give a counterexample.

Proof. g is 1-1: assume $n_1, n_2 \in \mathbb{Z}$ and $g(n_1) = g(n_2)$. *[We want to show $n_1 = n_2$].* By g's definition $4n_1 - 5 = 4n_2 - 5$. Adding 5 to both sides and dividing both sides by 4 we get $n_1 = n_2$.

g is not onto: there is no $n \in \mathbb{Z}$ such that $g(n) = 0$. Argue by contradiction and assume $g(n) = 0$ for some $n \in \mathbb{Z}$. Then $4n - 5 = 0$ so $n = 5/4$ which is not an integer, a contradiction. □

2.11.2 (b)

Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 4x - 5$, for every real number x. Is G onto? Prove or give a counterexample.

Proof. G is onto. Suppose y is any element of $\mathbb{R}$. *[We must show that there is an element x in $\mathbb{R}$ such that $G(x) = y$. What would x be if it exists? Scratch work shows that x would have to equal $(y+5)/4$. The proof must then show that x has the necessary properties.]* Let $x = (y+5)/4$. Then (1) $x \in \mathbb{R}$, and (2) $G(x) = G((y+5)/4) = 4[(y+5)/4] - 5 = (y+5) - 5 = y$ *[as was to be shown].* □

2.12 Exercise 12

2.12.1 (a)

Define $F : \mathbb{Z} \to \mathbb{Z}$ by the rule $F(n) = 2 - 3n$, for each integer n.

(i) Is F one-to-one? Prove or give a counterexample.

(ii) Is F onto? Prove or give a counterexample.

Proof. F is 1-1: assume $n_1, n_2 \in \mathbb{Z}$ and $F(n_1) = F(n_2)$. *[We want to show $n_1 = n_2$.]* By F's definition, $2 - 3n_1 = 2 - 3n_2$. Subtracting 2 from both sides and dividing by -3 we get $n_1 = n_2$.

F is not onto: there is no $n \in \mathbb{Z}$ such that $F(n) = 0$. Because otherwise $2 - 3n = 0$ for some integer n, but then $n = 2/3$ which is not an integer, a contradiction. □

2.12.2 (b)

Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 2 - 3x$, for each real number x. Is G onto? Prove or give a counterexample.

Proof. G is 1-1 just like F above. Same proof applies.

G is also onto: for every $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that $G(x) = y$: let $x = (y-2)/(-3)$. Then $G(x) = G((y-2)/(-3)) = 2 - 3 \cdot \dfrac{y-2}{-3} = 2 + (y-2) = y$. □

2.13 Exercise 13

2.13.1 (a)

Define $H : \mathbb{R} \to \mathbb{R}$ by the rule $H(x) = x^2$, for each real number x.

(i) Is H one-to-one? Prove or give a counterexample.

(ii) Is H onto? Prove or give a counterexample.

Proof. (i) H is not one-to-one. Counterexample: $H(1) = 1 = H(-1)$ but $1 \neq -1$.

(ii) H is not onto. Counterexample: $H(x) \neq -1$ for any real number x because $H(x) = x^2$ and no real numbers have negative squares. □

2.13.2 (b)

Define $K : \mathbb{R}^{nonneg} \to \mathbb{R}^{nonneg}$ by the rule $K(x) = x^2$, for each nonnegative real number x. Is K onto? Prove or give a counterexample.

Proof. K is onto. Given any $y \in \mathbb{R}^{nonneg}$ let $x = \sqrt{y}$. Then $K(x) = K(\sqrt{y}) = (\sqrt{y})^2 = y$. □

2.14 Exercise 14

Explain the mistake in the following "proof."

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by the formula $f(n) = 4n + 3$, for each integer n, is one-to-one.

"Proof: Suppose any integer n is given. Then by definition of f, there is only one possible value for $f(n)$, namely, $4n + 3$. Hence f is one-to-one."

Proof. The "proof" claims that f is one-to-one because for each integer n there is only one possible value for $f(n)$. But to say that for each integer n there is only one possible value for $f(n)$ is just another way of saying that f satisfies one of the conditions necessary for it to be a function. To show that f is one-to-one, one must show that any integer n has a different function value from that of the integer m whenever $n \neq m$. □

In each of $15 - 18$ a function f is defined on a set of real numbers. Determine whether or not f is one- to-one and justify your answer.

2.15 Exercise 15

$f(x) = \dfrac{x+1}{x}$, for each real number $x \neq 0$

Proof. f is 1-1: Suppose $f(x_1) = f(x_2)$ where x_1 and x_2 are nonzero real numbers. *[We must show that $x_1 = x_2$.]* By definition of f,

$$\frac{x_1 + 1}{x_1} = \frac{x_2 + 1}{x_2}$$

Cross-multiplying gives $x_1x_2 + x_2 = x_1x_2 + x_1$ and subtracting x_1x_2 gives $x_1 = x_2$. $\square$

2.16 Exercise 16

$f(x) = \dfrac{x}{x^2+1}$, for each real number x

Proof. f is not one-to-one. Counterexample: Note that

$$\begin{aligned} \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1} &\implies & x_1x_2^2 + x_1 &= x_2x_1^2 + x_2 \\ &\implies & x_1x_2^2 - x_2x_1^2 &= x_2 - x_1 \\ &\implies & x_1x_2(x_2 - x_1) &= x_2 - x_1 \\ &\implies & x_1 = x_2 \quad &\text{or} \quad x_1x_2 = 1. \end{aligned}$$

Thus take any x_1 and x_2 with $x_1 \neq x_2$ but $x_1x_2 = 1$. For instance, take $x_1 = 2$ and $x_2 = 1/2$. Then $f(x_1) = f(2) = 2/5$ and $f(x_2) = f(1/2) = 2/5$, but $2 \neq 1/2$. $\square$

2.17 Exercise 17

$f(x) = \dfrac{3x-1}{x}$, for each real number $x \neq 0$

Proof. f is 1-1: Assume $x_1 \neq 0 \neq x_2$ and assume $\frac{3x_1-1}{x_1} = \frac{3x_2-1}{x_2}$. *[We want to show $x_1 = x_2$.]* Cross-multiplying gives $(3x_1-1)x_2 = (3x_2-1)x_1$. So $3x_1x_2 - x_2 = 3x_1x_2 - x_1$. Canceling $3x_1x_2$ gives $-x_2 = -x_1$, so $x_1 = x_2$. $\square$

2.18 Exercise 18

$f(x) = \dfrac{x+1}{x-1}$, for each real number $x \neq 1$

Proof. f is 1-1: Assume $x_1 \neq 1 \neq x_2$ and assume $\frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$. *[We want to show $x_1 = x_2$.]* Cross-multiplying gives $(x_1+1)(x_2-1) = (x_2+1)(x_1-1)$. So $x_1x_2 - x_1 + x_2 - 1 = x_1x_2 + x_1 - x_2 - 1$. Canceling $x_1x_2 - 1$ gives $-x_1 + x_2 = x_1 - x_2$, so $2x_2 = 2x_1$ and dividing by 2 gives $x_2 = x_1$. $\square$

2.19 Exercise 19

Referring to Example 7.2.3, assume that records with the following ID numbers are to be placed in sequence into Table 7.2.1. Find the position into which each record is placed.

2.19.1 (a)

417302072

Proof. When 417302072 is divided by 11, the remainder is 0. So, $417302072 \mod 11 = H(417302072) = 0$. Since position 0 is unoccupied, the record is placed there. $\square$

TABLE 7.2.1

	0
356633102	1
223799061	2
	3
	4
	5
	6
513408716	7
328343419	8
	9
	10

2.19.2 (b)

364981703

Proof. When 364981703 is divided by 11, the remainder is 9. So, $364981703 \mod 11 = H(364981703) = 9$. Since position 9 is unoccupied, the record is placed there. □

2.19.3 (c)

283090787

Proof. When 283090787 is divided by 11, the remainder is 1. So, $283090787 \mod 11 = H(283090787) = 1$. Since position 1 is occupied, the record is placed in position 3. □

2.20 Exercise 20

Define Floor: $\mathbb{R} \to \mathbb{Z}$ by the formula $\text{Floor}(x) = \lfloor x \rfloor$, for every real number x.

2.20.1 (a)

Is Floor one-to-one? Prove or give a counterexample.

Proof. Floor is not one-to-one. Counterexample: $\text{Floor}(0) = 0 = \text{Floor}(1/2)$ but $0 \neq 1/2$. □

2.20.2 (b)

Is Floor onto? Prove or give a counterexample.

Proof. Floor is onto. Suppose $m \in \mathbb{Z}$. *[We must show that there exists a real number y such that Floor(y) $= m$.]* Let $y = m$. Then $\text{Floor}(y) = \text{Floor}(m) = m$ since m is an integer. (Actually, Floor takes the value m for all real numbers in the interval $m \leq x < m+1$.) Hence there exists a real number y such that $\text{Floor}(y) = m$ *[as was to be shown]*. □

2.21 Exercise 21

Let S be the set of all strings of 0's and 1's, and define $L : S \to \mathbb{Z}^{nonneg}$ by $L(s) =$ the length of s, for every string s in S.

2.21.1 (a)

Is L one-to-one? Prove or give a counterexample.

Proof. L is not one-to-one. Counterexample: $L(0) = L(1) = 1$ but $1 \neq 0$. $\square$

2.21.2 (b)

Is L onto? Prove or give a counterexample.

Proof. L is onto. Suppose n is a nonnegative integer. *[We must show that there exists a string s in S such that $L(s) = n$.]* Let

$$s = \begin{cases} \lambda \text{ (the null string)} & \text{if } n = 0 \\ 00\ldots0 \text{ (with } n \text{ 0's)} & \text{if } n > 0 \end{cases}$$

Then $L(s) =$ the length of $s = n$ *[as was to be shown]*. $\square$

2.22 Exercise 22

Let S be the set of all strings of 0's and 1's, and define $D : S \to \mathbb{Z}$ as follows: for every string s in S, $D(s) =$ the number of 1's in s minus the number of 0's in s.

2.22.1 (a)

Is D one-to-one? Prove or give a counterexample.

Proof. No. Counterexample: $D(10) = 0 = D(01)$ but $10 \neq 01$. $\square$

2.22.2 (b)

Is D onto? Prove or give a counterexample.

Proof. Yes. Given any $n \in \mathbb{Z}$, if $n < 0$ then let s be the string of $-n$ consecutive 0's. Then $D(s) = n$. Similarly if $n > 0$ then let s be the string of n consecutive 1's. Then $D(s) = n$. If $n = 0$ then $D(\lambda) = n$. $\square$

2.23 Exercise 23

Define $F : \mathscr{P}(\{a, b, c\}) \to \mathbb{Z}$ as follows: For every A in $\mathscr{P}(\{a, b, c\})$, $F(A) =$ the number of elements in A.

2.23.1 (a)

Is F one-to-one? Prove or give a counterexample.

Proof. F is not one-to-one. Counterexample: Let $A = \{a\}, B = \{b\}$. Then $F(A) = F(B) = 1$ but $A \neq B$. $\square$

2.23.2 (b)

Is F onto? Prove or give a counterexample.

Proof. No. Counterexample: There are no subsets of A with a negative number of, say, -1 elements. $\square$

2.24 Exercise 24

Let S be the set of all strings of a's and b's, and define $N : S \to \mathbb{Z}$ by $N(s) =$ the number of a's in s, for each $s \in S$.

2.24.1 (a)

Is N one-to-one? Prove or give a counterexample.

Proof. No. Counterexample: $N(a) = 1 = N(ab)$ but $a \neq ab$. $\square$

2.24.2 (b)

Is N onto? Prove or give a counterexample.

Proof. No. Counterexample: There are no strings with -1 a's in it. $\square$

2.25 Exercise 25

Let S be the set of all strings in a's and b's, and define $C : S \to S$ by $C(s) = as$, for each $s \in S$. (C is called concatenation by a on the left.)

2.25.1 (a)

Is C one-to-one? Prove or give a counterexample.

Proof. Yes. Assume $C(s_1) = C(s_2)$. So $as_1 = as_2$. Then s_1 and s_2 are the same string. $\square$

2.25.2 (b)

Is C onto? Prove or give a counterexample.

Proof. No. Counterexample: there is no string s such that $C(s) = b$ because b has no a's in it. $\square$

2.26 Exercise 26

Define $S : \mathbb{Z}^+ \to \mathbb{Z}^+$ by the rule: For each integer n, $S(n) =$ the sum of the positive divisors of n.

2.26.1 (a)

Is S one-to-one? Prove or give a counterexample.

Proof. No. Counterexample: $S(6) = 1 + 2 + 3 + 6 = 12$ and $S(11) = 1 + 11 = 12$ but $6 \neq 11$. $\square$

2.26.2 (b)

Is S onto? Prove or give a counterexample.

Proof. No. Counterexample: In order for there to be a positive integer n such that $S(n) = 5$, n would have to be less than 5. But $S(1) = 1, S(2) = 3, S(3) = 4$, and $S(4) = 7$. Hence there is no positive integer n such that $S(n) = 5$. $\square$

2.27 Exercise 27

Let D be the set of all finite subsets of positive integers, and define $T : \mathbb{Z}^+ \to D$ by the following rule: For every integer n, $T(n) =$ the set of all of the positive divisors of n.

2.27.1 (a)

Is T one-to-one? Prove or give a counterexample.

Proof. Yes. Assume $T(n_1) = T(n_2)$. *[We want to show* $n_1 = n_2$*].* Argue by contradiction and assume $n_1 \neq n_2$. Then either $n_1 < n_2$ or $n_1 > n_2$. In the first case, n_2 is a positive divisor of n_2 so $n_2 \in T(n_2)$. But since $T(n_1) = T(n_2), n_2 \in T(n_1)$ too. So n_2 is a positive divisor of n_1, contradiction. The other case is similar. $\square$

2.27.2 (b)

Is T onto? Prove or give a counterexample.

Proof. No. Counterexample: There is no $n \in \mathbb{Z}^+$ such that $T(n) = \{1, 2, 3\}$, any such n would also be divisible by 6, so $T(n)$ would have to include 6 too. $\square$

2.28 Exercise 28

Define $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $G(x, y) = (2y, -x)$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}$.

2.28.1 (a)

Is G one-to-one? Prove or give a counterexample.

Proof. Yes. Suppose (x_1, y_1) and (x_2, y_2) are any elements of $\mathbb{R}\times\mathbb{R}$ such that $G(x_1, y_1) = G(x_2, y_2)$. *[We must show that $(x_1, y_1) = (x_2, y_2)$.]* Then, by definition of G, $(2y_1, -x_1) = (2y_2, -x_2)$, and, by definition of ordered pair, $2y_1 = 2y_2$ and $-x_1 = -x_2$. Dividing both sides of the equation on the left by 2 and both sides of the equation on the right by -1 gives that $y_1 = y_2$ and $x_1 = x_2$, and so, by definition of ordered pair, $(x_1, y_1) = (x_2, y_2)$ *[as was to be shown].* □

2.28.2 (b)

Is G onto? Prove or give a counterexample.

Proof. Yes. Suppose (u, v) is any element of $\mathbb{R} \times \mathbb{R}$. *[We must show that there is an element (x, y) in $\mathbb{R} \times \mathbb{R}$ such that $G(x, y) = (u, v)$.]* Let $(x, y) = (-v, u/2)$. Then (1) $(x, y) \in \mathbb{R} \times \mathbb{R}$ and (2) $G(x, y) = (2y, -x) = (2(u/2), -(-v)) = (u, v)$ *[as was to be shown].* □

2.29 Exercise 29

Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $H(x, y) = (x + 1, 2 - y)$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}$.

2.29.1 (a)

Is H one-to-one? Prove or give a counterexample.

Proof. Yes. Assume $H(x_1, y_1) = H(x_2, y_2)$. *[We want to show that $x_1 = x_2$ and $y_1 = y_2$).]* We have $(x_1 + 1, 2 - y_1) = (x_2 + 1, 2 - y_2)$. By definition of a tuple, $x_1 + 1 = x_2 + 1$ and $2 - y_1 = 2 - y_2$. Solving, we get $x_1 = x_2$ and $y_1 = y_2$). □

2.29.2 (b)

Is H onto? Prove or give a counterexample.

Proof. Yes. Assume $(x, y) \in \mathbb{R} \times \mathbb{R}$ is any pair of real numbers. *[We want to show there exists a pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $H(a, b) = (x, y)$.]* Let $a = x - 1, b = 2 - y$. Then $H(a, b) = (a + 1, 2 - b) = (x - 1 + 1, 2 - (2 - y)) = (x, y)$, *[as was to be shown.]* □

2.30 Exercise 30

Define $J : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ as follows: $J(r, s) = r + \sqrt{2}s$ for every $(r, s) \in \mathbb{Q} \times \mathbb{Q}$.

2.30.1 (a)

Is J one-to-one? Prove or give a counterexample.

Proof. Yes. Assume $J(r_1, s_1) = J(r_2, s_2)$. *[We want to show that $r_1 = r_2$ and $s_1 = s_2$).]* We have $r_1 + \sqrt{2}s_1 = r_2 + \sqrt{2}s_2$. Moving terms around we get (*) $r_1 - r_2 = \sqrt{2}(s_2 - s_1)$. For the moment assume $s_1 \neq s_2$. Then $s_2 - s_1 \neq 0$. Dividing, we get

$$\frac{r_1 - r_2}{s_2 - s_1} = \sqrt{2}.$$

This is a contradiction, since the right hand side is an irrational number, and the left hand side is a rational number (being a ratio of differences of rational numbers). Thus our supposition was false and $s_1 = s_2$. Going back to the equation (*), this gives $r_1 - r_2 = \sqrt{2} \cdot 0 = 0$ and thus $r_1 = r_2$, as was to be shown. $\square$

2.30.2 (b)

Is J onto? Prove or give a counterexample.

Proof. No. Counterexample: There are no rationals (r, s) such that $J(r, s) = \sqrt{3}$. Argue by contradiction and assume $J(r, s) = \sqrt{3}$ for some rationals r, s. Then $r + \sqrt{2}s = \sqrt{3}$. Squaring both sides, we get

$$(r + \sqrt{2}s)^2 = \sqrt{3}^2 \implies r^2 + 2rs\sqrt{2} + 2s^2 = 3 \implies \sqrt{2} = \frac{3 - r^2 - 2s^2}{2rs}.$$

This is a contradiction, since the left hand side is irrational, and the right side is rational (being a ratio of differences of products of rationals). Thus our supposition was false, and J is not onto. $\square$

2.31 Exercise 31

Define $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ and $G : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ as follows: for each $(n, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $F(n, m) = 3^n 5^m$ and $G(n, m) = 3^n 6^m$.

2.31.1 (a)

Is F one-to-one? Prove or give a counterexample.

Proof. Yes. Assume $F(n_1, m_1) = F(n_2, m_2)$. Then $3^{n_1} 5^{m_1} = 3^{n_2} 5^{n_2}$. Call this positive integer z. So we have two different prime factorizations of the same integer z. By the uniqueness part of the factorization theorem, the exponents are unique, therefore $n_1 = n_2$ and $m_1 = m_2$. $\square$

2.31.2 (b)

Is G one-to-one? Prove or give a counterexample.

Proof. Yes. We can rewrite $G(n,m) = 3^n 6^m = 3^n(2\cdot 3)^m = 3^n \cdot 2^m \cdot 3^m = 2^m \cdot 3^{m+n}$. Now assume $G(n_1,m_1) = G(n_2,m_2)$. Then $2^{m_1}\cdot 3^{m_1+n_1} = 2^{m_2}\cdot 3^{m_2+n_2}$. Similar to part (a), by the uniqueness of prime factorizations, the exponents must be equal, so $m_1 = m_2$ and $m_1 + n_1 = m_2 + n_2$. Using $m_1 = m_2$ in the second equation we can cancel the ms to get $n_1 = n_2$. □

2.32 Exercise 32

2.32.1 (a)

Is $\log_8 27 = \log_2 3$? Why or why not?

Proof. 1. Let $x = \log_8 27, y = \log_2 3$.

2. By 1 and definition of logarithm, $8^x = 27$ and $2^y = 3$.

3. By 2 and laws of exponents, $(2^3)^x = 2^{3x} = (2^x)^3$ and $27 = 3^3$. So $(2^x)^3 = 3^3$.

4. By 3, taking the cube root of both sides, we get $2^x = 3$.

5. By 2 and 4, $2^x = 2^y$. Applying $\log_2$ to both sides, we get $x = y$. □

2.32.2 (b)

Is $\log_{16} 9 = \log_4 3$? Why or why not?

Proof. 1. Let $x = \log_{16} 9, y = \log_4 3$.

2. By 1 and definition of logarithm, $16^x = 9$ and $4^y = 3$.

3. By 2 and laws of exponents, $(2^4)^x = 2^{4x} = (2^{2x})^2$ and $9 = 3^2$. So $(2^{2x})^2 = 3^2$. Similarly $2^{2y} = 3$.

4. By 3, taking the square root of both sides, we get $2^{2x} = 3$.

5. By 2 and 4, $2^{2x} = 2^{2y}$. Applying $\log_2$ to both sides, we get $2x = 2y$ so $x = y$. □

The properties of logarithm established in 33 – 35 are used in Sections 11.4 and 11.5.

2.33 Exercise 33

Prove that for all positive real numbers b, x, and y with $b \neq 1$,

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y.$$

Proof. Suppose that b, x, and y are any positive real numbers such that $b \neq 1$. Let $u = \log_b(x)$ and $v = \log_b(y)$. By definition of logarithm, $b^u = x$ and $b_v = y$. By substitution, $\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v}$ *[by (7.2.3) and the fact that* $b^{-v} = \frac{1}{b^v}$*]*. Translating $\frac{x}{y} = b^{u-v}$ into logarithmic form gives $\log_b \frac{x}{y} = u - v$, and so, by substitution, $\log_b \frac{x}{y} = \log_b(x) - \log_b(y)$, *[as was to be shown]*. □

2.34 Exercise 34

Prove that for all positive real numbers b, x, and y with $b \neq 1$,

$$\log_b(xy) = \log_b x + \log_b y.$$

Proof. 1. Let $z = \log_b(xy), m = \log_b x, n = \log_b y$.

2. By 1 and definition of log, $b^z = xy$.

3. By 1 and definition of log, $b^m = x$.

4. By 1 and definition of log, $b^n = y$.

5. By 2, 3, 4, and law of exponents, $b^z = xy = b^m \cdot b^n = b^{m+n}$.

6. By 5, applying $\log_b$ to both sides, we get $\log_b(b^z) = \log_b(b^{m+n})$, which gives $z = m+n$.

7. By 1 and 6, $\log_b(xy) = \log_b x + \log_b y$. □

2.35 Exercise 35

Prove that for all real numbers a, b and x with b and x positive and $b \neq 1$,

$$\log_b(x^a) = a \log_b x.$$

Proof. 1. Let $z = \log_b(x^a), m = \log_b x$.

2. By 1 and definition of log, $b^z = x^a$.

3. By 1 and definition of log, $b^m = x$.

4. By 3 and law of exponents, $(b^m)^a = x^a$ so $b^{am} = x^a$.

5. By 2 and 4, $b^z = x^a = b^{am}$.

6. By 5, applying $\log_b$ to both sides, we get $\log_b(b^z) = \log_b(b^{am})$, which gives $z = am$.

7. By 1 and 6, $\log_b(x^a) = a \log_b x$. □

Exercises 36 and 37 use the following definition: If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions, then the function $(f+g) : \mathbb{R} \to \mathbb{R}$ is defined by the formula $(f+g)(x) = f(x) + g(x)$ for every real number x.

2.36 Exercise 36

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both one-to-one, is $f+g$ also one-to-one? Justify your answer.

Proof. No. Counterexample: Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as follows: $f(x) = x$ and $g(x) = -x$ for every real number x. Then f and g are both one-to-one *[because for all real numbers x_1 and x_2, if $f(x_1) = f(x_2)$ then $x_1 = x_2$, and if $g(x_1) = g(x_2)$ then $-x_1 = -x_2$, so $x_1 = x_2$ in this case as well]*. But $f+g$ is not one-to-one *[because $f+g$*

satisfies the equation $(f+g)(x) = x + (-x) = 0$ *for every real number* x*, and so, for instance,* $(f+g)(1) = (f+g)(2)$ *but* $1 \neq 2$*].* $\square$

2.37 Exercise 37

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both onto, is $f + g$ also onto? Justify your answer.

Proof. No. Same counterexample as exercise 36 works. $f(x) = x$ and $g(x) = -x$ are both onto, but $(f + g)(x) = 0$ is not. $\square$

Exercises 38 and 39 use the following definition: If $f : \mathbb{R} \to \mathbb{R}$ is a function and c is a nonzero real number, then the function $(c \cdot f) : \mathbb{R} \to \mathbb{R}$ is defined by the formula $(c \cdot f)(x) = c \cdot (f(x))$ for every real number x.

2.38 Exercise 38

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and c a nonzero real number. If f is one-to-one, is $c \cdot f$ also one-to- one? Justify your answer.

Proof. Yes. Let f be a one-to-one function from $\mathbb{R}$ to $\mathbb{R}$, and let c be any nonzero real number. Suppose $(c \cdot f)(x_1) = (c \cdot f)(x_2)$. *[We must show that* $x_1 = x_2$*.]* It follows by definition of $(c \cdot f)$ that $c \cdot (f(x1)) = c \cdot (f(x_2))$. Since $c \neq 0$, we may divide both sides of the equation by c to obtain $f(x_1) = f(x_2)$. And since f is one-to-one, this implies that $x_1 = x_2$, *[as was to be shown].* $\square$

2.39 Exercise 39

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and c a nonzero real number. If f is onto, is $c \cdot f$ also onto? Justify your answer.

Proof. Yes. Suppose $y \in \mathbb{R}$. *[We want to show there exists* $x \in \mathbb{R}$ *such that* $(c \cdot f)(x) = y$*].* Since $c \neq 0$ and f is onto, there exists $z \in \mathbb{R}$ such that $f(z) = y/c$. Let $x = z$. Then $(c \cdot f)(x) = c \cdot f(x) = c \cdot f(z) = c \cdot (y/c) = y$, *[as was to be shown].* $\square$

2.40 Exercise 40

Suppose $F : X \to Y$ is one-to-one.

2.40.1 (a)

Prove that for every subset $A \subseteq X, F^{-1}(F(A)) = A$.

Proof. Assume $A \subseteq X$.

1. Assume $x \in F^{-1}(F(A))$. *[Want to show* $x \in A$*.]*

2. By 1 and definition of inverse image $F^{-1}(F(A)) = \{t \in X \mid F(t) \in F(A)\}$ applied to $t = x$, we have $x \in X$ and $F(x) \in F(A)$.

3. By 2 and definition of $F(A)$, there exists $r \in A$ such that $F(r) = F(x)$.

4. By 3 and since F is 1-1, $r = x$, hence $x \in A$.

5. By 1, 4 and definition of subset, $F^{-1}(F(A)) \subseteq A$.

Now the reverse direction:

6. Assume $x \in A$. *[Want to show $x \in F^{-1}(F(A))$.]*

7. By 6 and definition of $F(A)$, we have $F(x) \in F(A)$.

8. By 6 and 7, $x \in X$ and $F(x) \in F(A)$. So x satisfies the definition of being a member of the inverse image $F^{-1}(F(A)) = \{t \in X \mid F(t) \in F(A)\}$ (with $t = x$). Therefore we have $x \in F^{-1}(F(A))$.

9. By 6, 8 and definition of subset, $A \subseteq F^{-1}(F(A))$.

Conclusion:

10. By 5, 9 and definition of set equality, $F^{-1}(F(A)) = A$. □

2.40.2 (b)

Prove that for all subsets A_1 and A_2 in X, $F(A_1 \cap A_2) = F(A_1) \cap F(A_2)$.

Proof. Assume $A_1 \subseteq X, A_2 \subseteq X$.

1. Assume $y \in F(A_1 \cap A_2)$. *[Want to show $y \in F(A_1) \cap F(A_2)$.]*

2. By 1 and definition of $F(A_1 \cap A_2)$, there exists $x \in A_1 \cap A_2$ such that $y = F(x)$.

3. By 2 and definition of intersection, $x \in A_1$ and $x \in A_2$.

4. By 3, and since $y = F(x)$, and by definitions of $F(A_1)$ and $F(A_2)$, $y \in F(A_1)$ and $y \in F(A_2)$.

5. By 4 and definition of intersection, $y \in F(A_1) \cap F(A_2)$.

6. By 1, 5 and definition of subset, $F(A_1 \cap A_2) \subseteq F(A_1) \cap F(A_2)$.

Now the reverse direction:

7. Assume $y \in F(A_1) \cap F(A_2)$. *[Want to show $y \in F(A_1 \cap A_2)$.]*

8. By 7 and definition of intersection, $y \in F(A_1)$ and $y \in F(A_2)$.

9. By 8 and definitions of $F(A_1)$ and $F(A_2)$, there exist $x_1 \in A_1$ and $x_2 \in A_2$ such that $y = F(x_1)$ and $y = F(x_2)$.

10. By 9 and since F is 1-1, $x_1 = x_2$.

11. By 9, 10 and definition of intersection, $x_1 \in A_1 \cap A_2$.

12. By 11 and since $y = F(x_1), y \in F(A_1 \cap A_2)$.

13. By 7, 12 and definition of subset, $F(A_1) \cap F(A_2) \subseteq F(A_1 \cap A_2)$.

Conclusion:

14. By 6, 13 and definition of set equality, $F(A_1 \cap A_2) = F(A_1) \cap F(A_2)$. $\square$

2.41 Exercise 41

Suppose $F : X \to Y$ is onto. Prove that for every subset $B \subseteq Y, F(F^{-1}(B)) = B$.

Proof. Assume $B \subseteq Y$.

1. Assume $y \in F(F^{-1}(B))$. *[Want to show $y \in B$.]*

2. By 1 and definition of $F(F^{-1}(B))$, there exists $x \in F^{-1}(B)$ such that $F(x) = y$.

3. By 2 and definition of inverse image $F^{-1}(B) = \{t \in X \mid F(t) \in B\}$, we have $F(x) \in B$.

4. By 2 and 3, $y = F(x) \in B$. So $y \in B$.

5. By 1, 4 and definition of subset, $F(F^{-1}(B)) \subseteq B$.

Now the reverse direction:

6. Assume $y \in B$. *[Want to show $y \in F(F^{-1}(B))$.]*

7. By 6 and since F is onto, there exists $x \in X$ such that $F(x) = y$.

8. By 6 and 7, $x \in X$ and $F(x) \in B$. So x satisfies the definition of being a member of the inverse image $F^{-1}(B) = \{t \in X \mid F(t) \in B\}$ (with $t = x$). So $x \in F^{-1}(B)$.

9. By 7 and 8, $y = F(x) \in F(F^{-1}(B))$.

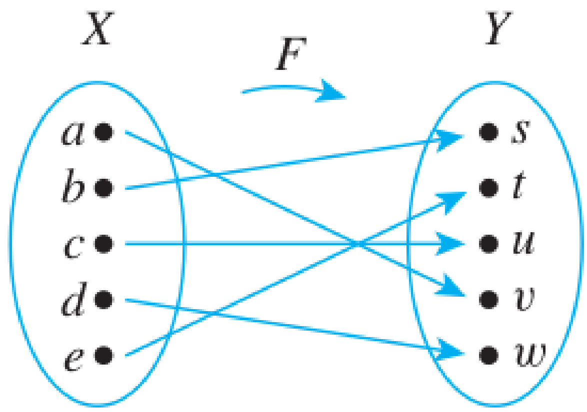
10. By 6, 9 and definition of subset, $B \subseteq F(F^{-1}(B))$.

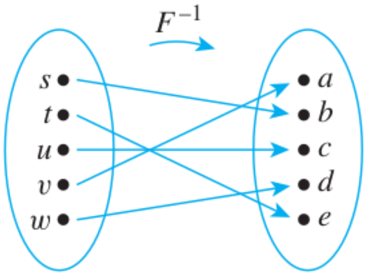
Conclusion:

11. By 5, 10 and definition of set equality, $F(F^{-1}(B)) = B$. $\square$

Let $X = \{a, b, c, d, e\}$ and $Y = \{s, t, u, v, w\}$. In each of 42 and 43 a one-to-one correspondence $F : X \to Y$ is defined by an arrow diagram. In each case draw an arrow diagram for F^{-1}.

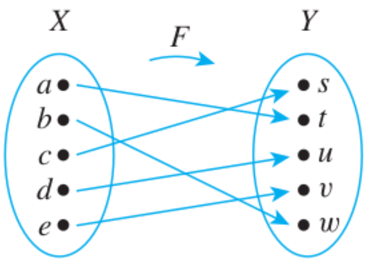
2.42 Exercise 42

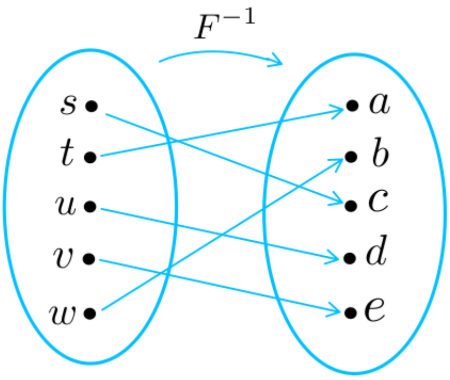




Proof. □

2.43 Exercise 43





Proof. □

In 44 − 55 **indicate which of the functions in the referenced exercise are one-to-one correspondences. For each function that is a one-to-one correspondence, find the inverse function.**

2.44 Exercise 44

Exercise 10a

Proof. The function is not a one-to-one correspondence because it is not onto. □

2.45 Exercise 45

Exercise 10b

Proof. The answer to exercise 10(b) shows that h is onto. To show that h is one-to-one, suppose $h(n_1) = h(n_2)$. By definition of h, this implies that $2n_1 = 2n_2$. Dividing both sides by 2 gives $n_1 = n_2$. Hence h is one-to-one, and so h is a one-to-one correspondence. Given any even integer m, if $m = h(n)$, then by definition of $h, m = 2n$, and so $n = m/2$. Thus $h^{-1}(m) = m/2$ for every $m \in 2\mathbb{Z}$. □

2.46 Exercise 46

Exercise 11a

Proof. The function g is not a one-to-one correspondence because it is not onto. For instance, if $m = 2$, it is impossible to find an integer n such that $g(n) = m$. (This is because if $g(n) = m$, then $4n - 5 = 2$, which implies that $n = 7/4$. Thus the only number n with the property that $g(n) = m$ is $7/4$. But $7/4$ is not an integer.) □

2.47 Exercise 47

Exercise 11b

Proof. The answer to exercise 11b shows that G is onto. In addition, G is one-to-one. To prove this, suppose $G(x_1) = G(x_2)$ for some x_1 and x_2 in $\mathbb{R}$. *[We must show that $x_1 = x_2$.]* By definition of G, $4x_1 - 5 = 4x_2 - 5$. Add 5 to both sides of this equation and divide both sides by 4 to obtain $x_1 = x_2$ *[as was to be shown]*. We claim that $G^{-1}(y) = (y+5)/4$ for each y in $\mathbb{R}$. By definition of inverse function, this is true if, and only if, $G((y+5)/4) = y$. But $G((y+5)/4) = 4((y+5)/4) - 5 = (y+5) - 5 = y$, and so it is the case that $G^{-1}(y) = (y+5)/4$ for each y in $\mathbb{R}$. □

2.48 Exercise 48

Exercise 12a

Proof. The function F is not a one-to-one correspondence because it is not onto. □

2.49 Exercise 49

Exercise 12b

Proof. The function G is a one-to-one correspondence because it is both one-to-one and onto. As shown in the solution to exercise 12b, the inverse function is $G^{-1}(y) = \frac{y-2}{-3}$. □

2.50 Exercise 50

Exercise 21

Proof. The function L is not a one-to-one correspondence because it is not 1-1. □

2.51 Exercise 51

Exercise 22

Proof. The function D is not a one-to-one correspondence because it is not 1-1. $\square$

2.52 Exercise 52

Exercise 15 with the co-domain taken to be the set of all real numbers not equal to 1

Proof. The answer to exercise 15 shows that f is one-to-one, and if the co-domain is taken to be the set of all real numbers not equal to 1, then f is also onto. The reason is that given any real number $y \neq 1$, if we take $x = \frac{1}{y-1}$, then x is a real number and

$$f(x) = f\left(\frac{1}{y-1}\right) = \frac{\frac{1}{y-1} + 1}{\frac{1}{y-1}} = \frac{1 + (y-1)}{1} = y.$$

Thus $f^{-1}(y) = \frac{1}{y-1}$ for each real number $y \neq 1$. $\square$

2.53 Exercise 53

Exercise 16 with the co-domain taken to be the set of all real numbers

Proof. The answer to exercise 16 shows that f is not one-to-one. Therefore, it is not a one-to-one correspondence. $\square$

2.54 Exercise 54

Exercise 17 with the co-domain taken to be the set of all real numbers not equal to 3

Proof. The answer to exercise 17 shows that f is 1-1. If the co-domain excludes 3, then it also becomes onto. Assume y is any real number with $y \neq 3$. Then letting $y = \frac{3x-1}{x}$ and solving, we get $x = \frac{1}{3-y}$ which is defined since $y \neq 3$. Thus f is a one-to-one correspondence and $f^{-1}(y) = \frac{1}{3-y}$. $\square$

2.55 Exercise 55

Exercise 18 with the co-domain taken to be the set of all real numbers not equal to 1

Proof. The answer to exercise 18 shows that f is 1-1. If the co-domain excludes 1, then it also becomes onto. Assume y is any real number with $y \neq 1$. Then letting $y = \frac{x+1}{x-1}$ and solving, we get $x = \frac{y+1}{y-1}$ which is defined since $y \neq 1$. Thus f is a one-to-one correspondence and $f^{-1}(y) = \frac{y+1}{y-1}$. $\square$

2.56 Exercise 56

In Example 7.2.8 a one-to-one correspondence was defined from the power set of $\{a, b\}$ to the set of all strings of 0's and 1's that have length 2. Thus the elements of these two sets can be matched up exactly, and so the two sets have the same number of elements.

2.56.1 (a)

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set with n elements. Use Example 7.2.8 as a model to define a one-to-one correspondence from $\mathscr{P}(X)$, the set of all subsets of X, to the set of all strings of 0's and 1's that have length n.

Proof. Let S_n be the set of all strings of 0's and 1's that have length n.

Define $f : \mathscr{P}(X) \to S_n$ as follows. For any subset A of X, define $f(A)$ to be the string where, the nth character in the string is 0 if $x_i \notin A$, and 1 if $x_i \in A$.

For example, if $n = 5$ and $A = \{x_2, x_4\}$ then $f(A) = 01010$. □

2.56.2 (b)

In Section 9.2 we show that there are 2^n strings of 0's and 1's that have length n. What does this allow you to conclude about the number of elements of $\mathscr{P}(X)$? (This provides an alternative proof of Theorem 6.3.1.)

Proof. Since $f : \mathscr{P}(X) \to S_n$ is a one-to-one correspondence, $\mathscr{P}(X)$ has the same number of elements as S_n. Since S_n has 2^n elements, $\mathscr{P}(X)$ also has 2^n elements. □

2.57 Exercise 57

Write a computer algorithm to check whether a function from one finite set to another is one-to-one. Assume the existence of an independent algorithm to compute values of the function.

Proof. Let a function F be given and suppose the domain of F is represented as a one-dimensional array $a[1], a[2], \ldots, a[n]$.

$answer :=$ "one-to-one"
$i := 1$
while ($i \leq n - 1$ and $answer =$ "one-to-one")
    $j := i + 1$
    while ($j \leq n$ and $answer =$ "one-to-one")
        if ($F(a[i]) = F(a[j])$ and $a[i] \neq a[j]$)
        then $answer :=$ "not one-to-one"
        $j := j + 1$
    end while
$i := i + 1$
end while
return $answer$

□

2.58 Exercise 58

Write a computer algorithm to check whether a function from one finite set to another is onto. Assume the existence of an independent algorithm to compute values of the function.

Proof. Let a function F be given and suppose the domain and the co-domain of F are represented as one-dimensional arrays $a[1], a[2], \ldots, a[n]$ and $b[1], b[2], \ldots, b[m]$.

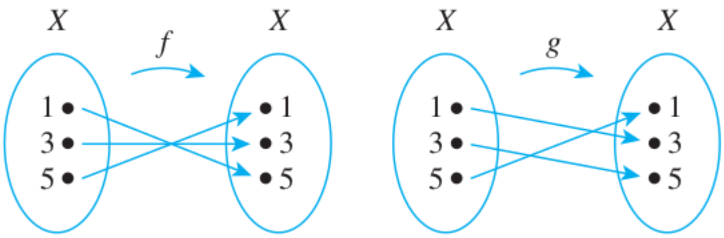
$answer :=$ "onto"
$i := 1$
while ($i \leq m$ and $answer =$ "onto")
    $found :=$ "false"
    $j := 1$
    while ($j \leq n$ and $found =$ "false")
        if ($b[i] = F(a[j])$)
        then $found :=$ "true"
        $j := j + 1$
    end while
    if $found =$ false
    then $answer =$ "not onto"
$i := i + 1$
end while
return $answer$

□

3 Exercise Set 7.3

In each of 1 and 2, functions f and g are defined by arrow diagrams. Find $g \circ f$ and $f \circ g$ and determine whether $g \circ f$ equals $f \circ g$.

3.1 Exercise 1



Proof. $g \circ f$ is defined as follows:

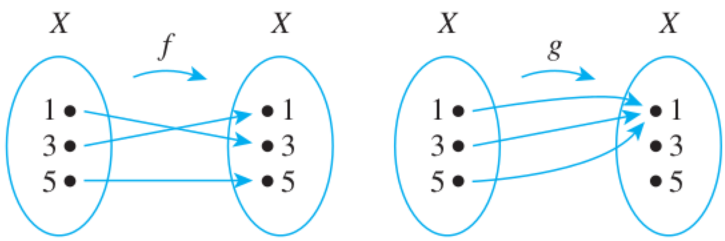
$$\begin{array}{lllllll}(g\circ f)(1) & = & g(f(1)) & = & g(5) & = & 1\\ (g\circ f)(3) & = & g(f(3)) & = & g(3) & = & 5\\ (g\circ f)(5) & = & g(f(5)) & = & g(1) & = & 3\end{array}$$

$f \circ g$ is defined as follows:

$$\begin{array}{lllllll}(f\circ g)(1) & = & f(g(1)) & = & f(3) & = & 3\\ (f\circ g)(3) & = & f(g(3)) & = & f(5) & = & 1\\ (f\circ g)(5) & = & f(g(5)) & = & f(1) & = & 5\end{array}$$

Then $g \circ f \neq f \circ g$ because, for example, $(g \circ f)(1) \neq (f \circ g)(1)$. □

3.2 Exercise 2



Proof. $g \circ f$ is defined as follows:

$$\begin{array}{lllllll}(g\circ f)(1) & = & g(f(1)) & = & g(3) & = & 1\\ (g\circ f)(3) & = & g(f(3)) & = & g(1) & = & 1\\ (g\circ f)(5) & = & g(f(5)) & = & g(5) & = & 1\end{array}$$

$f \circ g$ is defined as follows:

$$\begin{array}{lllllll}(f\circ g)(1) & = & f(g(1)) & = & f(1) & = & 3\\ (f\circ g)(3) & = & f(g(3)) & = & f(1) & = & 3\\ (f\circ g)(5) & = & f(g(5)) & = & f(1) & = & 3\end{array}$$

Then $g \circ f \neq f \circ g$ because, for example, $(g \circ f)(1) \neq (f \circ g)(1)$. □

In 3 and 4, functions F and G are defined by formulas. Find $G \circ F$ and $F \circ G$ and determine whether $G \circ F$ equals $F \circ G$.

3.3 Exercise 3

$F(x) = x^3$ and $G(x) = x - 1$, for each real number x.

Proof. $(G \circ F)(x) = G(F(x)) = G(x^3) = x^3 - 1$ for every real number x.

$(F \circ G)(x) = F(G(x)) = F(x-1) = (x-1)^3$ for every real number x.

$G \circ F \neq F \circ G$ because, for instance, $(G \circ F)(2) = 2^3 - 1 = 7$, whereas $(F \circ G)(2) = (2-1)^3 = 1$. □

3.4 Exercise 4

$F(x) = x^5$ and $G(x) = x^{1/5}$, for each real number x.

Proof. $(G \circ F)(x) = G(F(x)) = G(x^5) = (x^5)^{1/5} = x$ for every real number x.

$(F \circ G)(x) = F(G(x)) = F(x^{1/5}) = (x^{1/5})^5 = x$ for every real number x.

So $G \circ F = F \circ G$. □

3.5 Exercise 5

Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = -x$ for every real number x. Find $(f \circ f)(x)$.

Proof. $(f \circ f)(x) = f(f(x)) = f(-x) = -(-x) = x$. □

3.6 Exercise 6

Define $F : \mathbb{Z} \to \mathbb{Z}$ and $G : \mathbb{Z} \to \mathbb{Z}$ by the rules $F(a) = 7a$ and $G(a) = a \mod 5$ for each integer a. Find $(G \circ F)(0), (G \circ F)(1), (G \circ F)(2), (G \circ F)(3)$, and $(G \circ F)(4)$.

Proof. $(G \circ F)(0) = G(F(0)) = G(7 \cdot 0) = 0 \mod 5 = 0$,

$(G \circ F)(1) = G(F(1)) = G(7 \cdot 1) = 7 \mod 5 = 2$,

$(G \circ F)(2) = G(F(2)) = G(7 \cdot 2) = 14 \mod 5 = 4$,

$(G \circ F)(3) = G(F(3)) = G(7 \cdot 3) = 21 \mod 5 = 1$,

$(G \circ F)(4) = G(F(4)) = G(7 \cdot 4) = 28 \mod 5 = 3$. □

3.7 Exercise 7

Define $L : \mathbb{Z} \to \mathbb{Z}$ and $M : \mathbb{Z} \to \mathbb{Z}$ by the rules $L(a) = a^2$ and $M(a) = a \mod 5$ for each integer a.

3.7.1 (a)

Find $(L \circ M)(12), (M \circ L)(12), (L \circ M)(9)$, and $(M \circ L)(9)$.

Proof. $(L \circ M)(12) = L(M(12)) = L(12 \mod 5) = L(2) = 2^2 = 4$.

$(M \circ L)(12) = M(L(12)) = M(12^2) = M(144) = 144 \mod 5 = 4$.

$(L \circ M)(9) = L(M(9)) = L(9 \mod 5) = L(4) = 4^2 = 16$.

$(M \circ L)(9) = M(L(9)) = M(9^2) = M(81) = 81 \mod 5 = 1.$ □

3.7.2 (b)

Is $L \circ M = M \circ L$?

Proof. No, because $(L \circ M)(9) \neq (M \circ L)(9)$. □

3.8 Exercise 8

Let S be the set of all strings in a's and b's and let $L : S \to \mathbb{Z}$ be the length function: For all strings $s \in S, L(s) =$ the number of characters in s. Let $T : \mathbb{Z} \to \{0, 1, 2\}$ be the mod 3 function: For every integer $n, T(n) = n \mod 3$.

3.8.1 (a)

$(T \circ L)(abaa) =?$

Proof. $(T \circ L)(abaa) = T(L(abaa)) = T(4) = 4 \mod 3 = 1$ □

3.8.2 (b)

$(T \circ L)(baaab) =?$

Proof. $(T \circ L)(baaab) = T(L(baaab)) = T(5) = 5 \mod 3 = 2$ □

3.8.3 (c)

$(T \circ L)(aaa) =?$

Proof. $(T \circ L)(aaa) = T(L(aaa)) = T(3) = 3 \mod 3 = 0$ □

3.9 Exercise 9

Define $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{Z}$ by the following formulas: $F(x) = x^2 y^3$ and $G(x) = \lfloor x \rfloor$ for every $x \in \mathbb{R}$.

3.9.1 (a)

$(G \circ F)(2) =?$

Proof. $(G \circ F)(2) = G(F(2)) = G(2^2/3) = G(4/3) = \lfloor 4/3 \rfloor = 1$ □

3.9.2 (b)

$(G \circ F)(-3) =?$

Proof. $(G \circ F)(-3) = G(F(-3)) = G((-3)^2/3) = G(9/3) = \lfloor 3 \rfloor = 3$ □

3.9.3 (c)

$(G \circ F)(5) = ?$

Proof. $(G \circ F)(5) = G(F(5)) = G(5^2/3) = G(25/3) = \lfloor 25/3 \rfloor = 8$ □

3.10 Exercise 10

Define $F : \mathbb{Z} \to \mathbb{Z}$ and $G : \mathbb{Z} \to \mathbb{Z}$ by the following formulas: $F(n) = 2n$ and $G(n) = \lfloor n/2 \rfloor$ for every integer n.

3.10.1 (a)

Find $(G \circ F)(8)$, $(F \circ G)(8)$, $(G \circ F)(3)$ and $(F \circ G)(3)$.

Proof. $(G \circ F)(8) = G(F(8)) = G(16) = \lfloor 16/2 \rfloor = 8$

$(F \circ G)(8) = F(G(8)) = F(\lfloor 8/2 \rfloor) = 2 \cdot 4 = 8$

$(G \circ F)(3) = G(F(3)) = G(6) = \lfloor 6/2 \rfloor = 3$

$(F \circ G)(3) = F(G(3)) = F(\lfloor 3/2 \rfloor) = 2 \cdot 1 = 2$ □

3.10.2 (b)

Is $G \circ F = F \circ G$? Explain.

Proof. No, because $(G \circ F)(3) \neq (F \circ G)(3)$. □

3.11 Exercise 11

Define $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ by the rules $F(x) = 3x$ and $G(x) = \lfloor x/3 \rfloor$ for every real number x.

3.11.1 (a)

Find $(G \circ F)(6)$, $(F \circ G)(6)$, $(G \circ F)(1)$ and $(F \circ G)(1)$.

Proof. $(G \circ F)(6) = G(F(6)) = G(18) = \lfloor 18/3 \rfloor = 6$

$(F \circ G)(6) = F(G(6)) = F(\lfloor 6/3 \rfloor) = 3 \cdot 2 = 6$

$(G \circ F)(1) = G(F(1)) = G(3) = \lfloor 3/3 \rfloor = 1$

$(F \circ G)(1) = F(G(1)) = F(\lfloor 1/3 \rfloor) = 3 \cdot 0 = 0$ □

3.11.2 (b)

Is $G \circ F = F \circ G$? Explain.

Proof. No, because $(G \circ F)(1) \neq (F \circ G)(1)$. □

The functions of each pair in $12 - 14$ are inverse to each other. For each pair, check that both compositions give the identity function.

3.12 Exercise 12

$F : \mathbb{R} \to \mathbb{R}$ and $F^{-1} : \mathbb{R} \to \mathbb{R}$ are defined by

$$F(x) = 3x + 2 \text{ and } F^{-1}(y) = \frac{y-2}{3}, \text{ for every } x, y \in \mathbb{R}.$$

Proof. $(F^{-1} \circ F)(x) = F^{-1}(F(x)) = F^{-1}(3x + 2) = \frac{(3x+2)-2}{3} = x = I_{\mathbb{R}}(x)$ for every $x \in \mathbb{R}$. Hence $F^{-1} \circ F = I_{\mathbb{R}}$ by definition of equality of functions.

$(F \circ F^{-1})(y) = F(F^{-1}(y)) = F(\frac{y-2}{3}) = 3 \cdot \frac{y-2}{3} + 2 = (y - 2) + 2 = y = I_{\mathbb{R}}(y)$ for every $y \in \mathbb{R}$. Hence $F \circ F^{-1} = I_{\mathbb{R}}$ by definition of equality of functions. □

3.13 Exercise 13

$G : \mathbb{R}^+ \to \mathbb{R}^+$ and $G^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ are defined by

$$G(x) = x^2 \text{ and } G^{-1}(y) = \sqrt{y}, \text{ for every } x, y \in \mathbb{R}.$$

Proof. $(G^{-1} \circ G)(x) = G^{-1}(G(x)) = G^{-1}(x^2) = \sqrt{x^2} = x = I_{\mathbb{R}}(x)$ for every $x \in \mathbb{R}$. Hence $G^{-1} \circ G = I_{\mathbb{R}}$ by definition of equality of functions.

$(G \circ G^{-1})(y) = G(G^{-1}(y)) = G(\sqrt{y}) = (\sqrt{y})^2 = y = I_{\mathbb{R}}(y)$ for every $y \in \mathbb{R}$. Hence $G \circ G^{-1} = I_{\mathbb{R}}$ by definition of equality of functions. □

3.14 Exercise 14

H and H^{-1} are both defined from $\mathbb{R} - \{1\}$ to $\mathbb{R} - \{1\}$ by the formula

$$H(x) = H^{-1}(x) = \frac{x+1}{x-1}, \text{ for every } x \in \mathbb{R} - \{1\}.$$

Proof.

$$(H^{-1} \circ H)(x) = H^{-1}(H(x)) = H^{-1}\left(\frac{x+1}{x-1}\right) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x = I_{\mathbb{R}}(x)$$

for every $x \in \mathbb{R}$. Hence $H^{-1} \circ H = I_{\mathbb{R}}$ by definition of equality of functions. The calculation for $(H \circ H^{-1})(x)$ is the same as above. Hence $H \circ H^{-1} = I_{\mathbb{R}}$ by definition of equality of functions. □

3.15 Exercise 15

Explain how it follows from the definition of logarithm that

3.15.1 (a)

$\log_b(b^x) = x$, for every real number x.

Proof. By definition of logarithm with base b, for each real number x, $\log_b(b^x)$ is the exponent to which b must be raised to obtain b^x. But this exponent is just x. So $\log_b(b^x) = x$. □

3.15.2 (b)

$b^{\log_b x} = x$, for every positive real number x.

Proof. By definition of logarithm with base b, for each real number x, $\log_b(x)$ is the exponent to which b must be raised to obtain x. So $b^{\log_b(x)} = x$. □

3.16 Exercise 16

Prove Theorem 7.3.1(b): If f is any function from a set X to a set Y, then $I_Y \circ f = f$, where I_Y is the identity function on Y.

Proof. Suppose f is any function from a set X to a set Y. *[We want to show* $(I_Y \circ f)(x) = f(x)$*]* for every $x \in X$. Indeed $(I_Y \circ f)(x) = I_Y(f(x)) = f(x)$ by definition of I_Y, *[as was to be shown.]* □

3.17 Exercise 17

Prove Theorem 7.3.2(b): If $f : X \to Y$ is a one-to-one and onto function with inverse function $f^{-1} : Y \to X$, then $f \circ f^{-1} = I_Y$, where I_Y is the identity function on Y.

Proof. [We want to show that for every $y \in Y, I_Y(y) = (f \circ f^{-1})(y)$*].* Assume $y \in Y$. Since f is 1-1 and onto, there is a unique $x \in X$ such that $f(x) = y$, and therefore, $f^{-1}(y) = x$. Then

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y = I_Y(y),$$

[as was to be shown.] □

3.18 Exercise 18

Suppose Y and Z are sets and $g : Y \to Z$ is a one-to-one function. This means that if g takes the same value on any two elements of Y, then those elements are equal. Thus, for example, if a and b are elements of Y and $g(a) = g(b)$, then it can be inferred that $a = b$. What can be inferred in the following situations?

3.18.1 (a)

s_k and s_m are elements of Y and $g(s_k) = g(s_m)$.

Proof. We can infer that $s_k = s_m$. □

3.18.2 (b)

$z/2$ and $t/2$ are elements of Y and $g(z/2) = g(t/2)$.

Proof. We can infer that $z/2 = t/2$. □

3.18.3 (c)

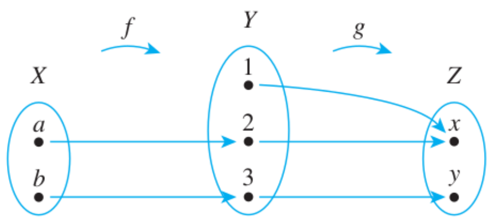
$f(x_1)$ and $f(x_2)$ are elements of Y and $g(f(x_1)) = g(f(x_2))$.

Proof. We can infer that $f(x_1) = f(x_2)$. □

3.19 Exercise 19

If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is one-to-one, must g be one-to-one? Prove or give a counterexample.

Proof. The answer is no. Counterexample: Define f and g by the arrow diagrams below.

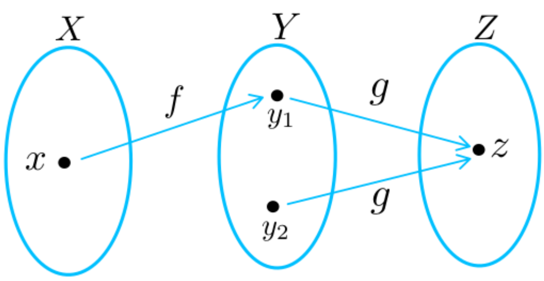


Then $g \circ f$ is one-to-one but g is not one-to-one. (This is one counterexample among many. Can you construct a different one?) □

3.20 Exercise 20

If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is onto, must f be onto? Prove or give a counterexample.

Proof. The answer is no. Counterexample: Define f and g by the arrow diagrams below.



Then $g \circ f$ is onto, because $(g \circ f)(x) = z$. But f is not onto, because there is no $e \in X$ such that $f(e) = y_2$. □

3.21 Exercise 21

If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is one-to-one, must f be one-to-one? Prove or give a counterexample.

Proof. Yes. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is one-to-one. Assume $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. *[We want to show $x_1 = x_2$.]* Since $f(x_1) = f(x_2)$, apply g to both sides to get $g(f(x_1)) = g(f(x_2))$, so by definition of $g \circ f$, $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is 1-1, $x_1 = x_2$, *[as was to be shown.]*

□

3.22 Exercise 22

If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is onto, must g be onto? Prove or give a counterexample.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is onto. Assume $z \in Z$. *[We want to show there exists $y \in Y$ such that $g(y) = z$.]* Since $g \circ f$ is onto, there exists $x \in X$ such that $(g \circ f)(x) = z$. So $g(f(x)) = z$. Let $y = f(x)$. Then $y \in Y$ and $g(y) = z$, *[as was to be shown.]* □

3.23 Exercise 23

Let $f : W \to X, g : X \to Y$, and $h : Y \to Z$ be functions. Must $h \circ (g \circ f) = (h \circ g) \circ f$? Prove or give a counterexample.

Proof. True. *[We want to show $(h \circ (g \circ f))(w) = ((h \circ g) \circ f)(w)$ for all $w \in W$.]* Assume $w \in W$. Then by definition of $h \circ (g \circ f)$

$$(h \circ (g \circ f))(w) = h((g \circ f)(w)) = h(g(f(w)))$$

and by definition of $(h \circ g) \circ f$

$$((h \circ g) \circ f)(w) = (h \circ g)(f(w)) = h(g(f(w)))$$

thus $(h \circ (g \circ f))(w) = ((h \circ g) \circ f)(w)$ *[as was to be shown.]* □

3.24 Exercise 24

True or False? Given any set X and given any functions $f : X \to X, g : X \to X$, and $h : X \to X$, if h is one-to-one and $h \circ f = h \circ g$, then $f = g$. Justify your answer.

Proof. True. Suppose X is any set and f, g, and h are functions from X to X such that h is one-to-one and $h \circ f = h \circ g$. *[We must show that for every x in X, $f(x) = g(x)$.]* Suppose x is any element in X. Because $h \circ f = h \circ g$, we have that $(h \circ f)(x) = (h \circ g)(x)$ by definition of equality of functions. Then, by definition of composition of functions, $h(f(x)) = h(g(x))$. And since h is one- to-one, this implies that $f(x) = g(x)$ *[as was to be shown]*. □

3.25 Exercise 25

True or False? Given any set X and given any functions $f : X \to X, g : X \to X$, and $h : X \to X$, if h is one-to-one and $f \circ h = g \circ h$, then $f = g$. Justify your answer.

Proof. Interestingly, if X is a finite set then this statement is true, because if $h : X \to X$ is 1-1, and X is finite, then h is also onto, but this is not necessarily true if X is infinite.

False. Counterexample: Let $X = \mathbb{Z}$, and for all $n \in \mathbb{Z}, h(n) = 2n$,

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd} \end{cases}, \text{ and } g(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}.$$

Then h is one-to-one: if $h(n_1) = h(n_2)$ then $2n_1 = 2n_2$ therefore $n_1 = n_2$.

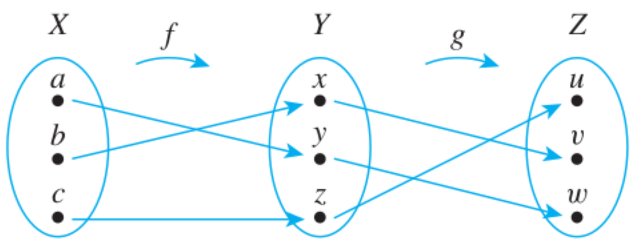
Also $f \circ h = g \circ h$: for all $n \in \mathbb{Z}$ we have $(f \circ h)(n) = f(h(n)) = f(2n) = 2n$ (since $2n$ is even), and similarly $(g \circ h)(n) = g(h(n)) = g(2n) = 2n$, so $(f \circ h)(n) = (g \circ h)(n)$.

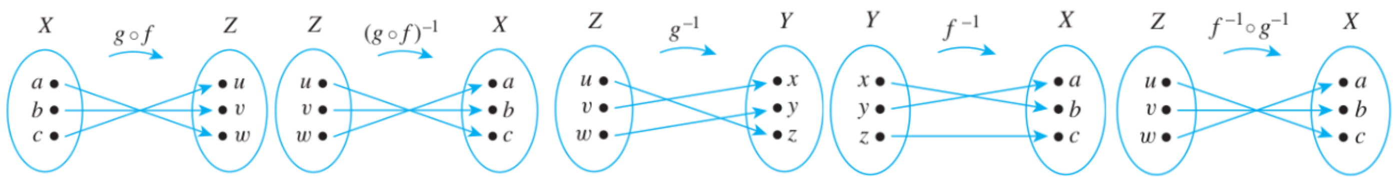
But $f \neq g$, because $f(1) = 5 \neq 3 = g(1)$. □

In 26 and 27 find $(g \circ f)^{-1}, g^{-1}, f^{-1}$, and $f^{-1} \circ g^{-1}$, and state how $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are related.

3.26 Exercise 26

Let $X = \{a, b, c\}, Y = \{x, y, z\}$, and $Z = \{u, v, w\}$. Define $f : X \to Y$ and $g : Y \to Z$ by the arrow diagrams below.





Proof. The functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are equal. □

3.27 Exercise 27

Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by the formulas $f(x) = x + 3$ and $g(x) = -x$ for each $x \in R$.

Proof. $(g \circ f)(x) = g(f(x)) = g(x + 3) = -x - 3, (g \circ f)^{-1}(y) = -y - 3$

$g^{-1}(y) = -y, f^{-1}(y) = y - 3$

$(f^{-1} \circ g^{-1})(y) = f^{-1}(g^{-1}(y)) = f^{-1}(-y) = -y - 3$

The functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are equal. □

3.28 Exercise 28

Prove or give a counterexample: If $f : X \to Y$ and $g : Y \to X$ are functions such that $g \circ f = I_X$ and $f \circ g = I_Y$, then f and g are both one-to-one and onto and $g = f^{-1}$.

Proof. ***f* is 1-1:** Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. *[Want to show $x_1 = x_2$.]* Since $f(x_1) = f(x_2)$, we have $g(f(x_1)) = g(f(x_2))$. So $(g \circ f)(x_1) = (g \circ f)(x_2)$. So $I_X(x_1) = I_X(x_2)$, therefore $x_1 = x_2$. *[as was to be shown.]*

In a parallel way, we can prove g is 1-1.

***f* is onto:** Assume $y \in Y$. *[Want to show there exists $x \in X$ such that $y = f(x)$.]* Let $x = g(y)$. Then $f(x) = f(g(y)) = (f \circ g)(y) = I_Y(y) = y$, *[as was to be shown.]*

In a parallel way, we can prove g is onto. □

3.29 Exercise 29

Suppose $f : X \to Y$ and $g : Y \to Z$ are both one-to-one and onto. Prove that $(g \circ f)^{-1}$ exists and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. By Theorem 7.3.3 $g \circ f$ is 1-1. By Theorem 7.3.4 $g \circ f$ is onto. By Theorem 7.2.2 $(g \circ f)^{-1}$ exists and has the inverse function property: for all $x \in X, z \in Z, (g \circ f)^{-1}(z) = x$ if and only if $(g \circ f)(x) = z$.

[We want to show that for all $z \in Z, (g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$.] Assume $z \in Z$. Let $x = (g \circ f)^{-1}(z)$. By the inverse function property of $(g \circ f)^{-1}$, $(g \circ f)(x) = z$. So $g(f(x)) = z$. By the inverse function property of g^{-1}, $g^{-1}(z) = f(x)$. Then by the inverse function property of f^{-1}, $f^{-1}(g^{-1}(z)) = x$. So $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$, *[as was to be shown.]* □

3.30 Exercise 30

Let $f : X \to Y$ and $g : Y \to Z$. Is the following property true or false? For every subset C in Z, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$. Justify your answer.

Proof. 1. By definition of inverse image, $g^{-1}(C) = \{y \in Y \mid g(y) \in C\}$.

2. By 1 and definition of inverse image,

$$f^{-1}(g^{-1}(C)) = \{x \in X \mid f(x) \in g^{-1}(C)\} = \{x \in X \mid g(f(x)) \in C\}.$$

3. By definition of inverse image and $\circ$,

$$(g \circ f)^{-1}(C) = \{x \in X \mid (g \circ f)(x) \in C\} = \{x \in X \mid g(f(x)) \in C\}.$$

4. By 2 and 3, $f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C)$. □

4 Exercise Set 7.4

4.1 Exercise 1

When asked what it means to say that set A has the same cardinality as set B, a student replies, "A and B are one-to-one and onto." What should the student have replied? Why?

Proof. The student should have replied that for A to have the same cardinality as B means that there is a function from A to B that is one-to-one and onto. A set cannot have the property of being onto or one-to-one another set; only a function can have these properties. □

4.2 Exercise 2

Show that "there are as many squares as there are numbers" by exhibiting a one-to-one correspondence from the positive integers, $\mathbb{Z}^+$, to the set S of all squares of positive integers: $S = \{n \in \mathbb{Z}^+ \mid n = k^2, \text{ for some positive integer } k\}$.

Proof. Define a function $f : \mathbb{Z}^+ \to S$ as follows: For every positive integer k, $f(k) = k^2$.

f is one-to-one: [We must show that for all k_1 and $k_2 \in \mathbb{Z}^+$, if $f(k_1) = f(k_2)$ then $k_1 = k_2$.] Suppose k_1 and k_2 are positive integers and $f(k_1) = f(k_2)$. By definition of $f, (k_1)^2 = (k_2)^2$, so $k_1 = \pm k_2$. But k_1 and k_2 are positive. Hence $k_1 = k_2$.

f is onto: [We must show that for each $n \in S$, there exists $k \in \mathbb{Z}^+$ such that $n = f(k)$.] Suppose $n \in S$. By definition of $S, n = k^2$ for some positive integer k. Then by definition of $f, n = f(k)$.

Since there is a one-to-one, onto function (namely, f) from $\mathbb{Z}^+$ to S, the two sets have the same cardinality. □

4.3 Exercise 3

Let $3\mathbb{Z} = \{n \in \mathbb{Z} \mid n = 3k, \text{ for some integer } k\}$. Prove that $\mathbb{Z}$ and $3\mathbb{Z}$ have the same cardinality.

Proof. Define $f : \mathbb{Z} \to 3\mathbb{Z}$ by the rule $f(n) = 3n$ for each integer n. The function f is one-to-one because for any integers n_1 and n_2, if $f(n_1) = f(n_2)$ then $3n_1 = 3n_2$ and so $n_1 = n_2$. Also f is onto because if m is any element in $3\mathbb{Z}$, then $m = 3k$ for some integer k. Then $f(k) = 3k = m$ by definition of f. Thus, since there is a function $f : \mathbb{Z} \to 3\mathbb{Z}$ that is one-to-one and onto, $\mathbb{Z}$ has the same cardinality as $3\mathbb{Z}$. □

4.4 Exercise 4

Let $\mathbf{O}$ be the set of all odd integers. Prove that $\mathbf{O}$ has the same cardinality as $2\mathbb{Z}$, the set of all even integers.

Proof. Define $f : \mathbf{O} \to 2\mathbb{Z}$ with $f(n) = n - 1$. (Notice that f is well-defined because, subtracting 1 from an odd integer always gives an even integer.) f is 1-1 because, if $n_1 - 1 = n_2 - 1$ then $n_1 = n_2$. f is onto because, given any even integer $2n \in 2\mathbb{Z}$ there is an odd integer $o = 2n + 1$ such that $f(o) = f(2n + 1) = 2n + 1 - 1 = 2n$. □

4.5 Exercise 5

Let $25\mathbb{Z}$ be the set of all integers that are multiples of 25. Prove that $25\mathbb{Z}$ has the same cardinality as $2\mathbb{Z}$, the set of all even integers.

Proof. Define $f : 25\mathbb{Z} \to 2\mathbb{Z}$ as follows: given any $t \in 25\mathbb{Z}, t = 25n$ for some integer n. Then define $f(t) = 2n$. In other words, $f(t) = \frac{2}{25} \cdot t$. (Notice f is well-defined because $\frac{2}{25} \cdot t$ is always an integer for all $t \in 25\mathbb{Z}$). f is 1-1 because, if $\frac{2}{25}t_1 = \frac{2}{25}t_2$ then by canceling $\frac{2}{25}$ we get $t_1 = t_2$. f is onto because, given any even integer $2n \in 2\mathbb{Z}$ there exists $t \in 25\mathbb{Z}$, namely $t = \frac{25}{2} \cdot (2n) = 25n$ such that $f(t) = f(25n) = \frac{2}{25}(25n) = 2n$. □

4.6 Exercise 6

Use the functions I and J defined in the paragraph following Example 7.4.1 to show that even though there is a one-to-one correspondence, H, from $2\mathbb{Z}$ to $\mathbb{Z}$, there is also a function from $2\mathbb{Z}$ to Z that is one-to-one but not onto and a function from Z to $2\mathbb{Z}$ that is onto but not one-to-one. In other words, show that I is one-to-one but not onto, and show that J is onto but not one-to-one.

Proof. Recall $I : 2\mathbb{Z} \to \mathbb{Z}, I(n) = n$ and $J : \mathbb{Z} \to 2\mathbb{Z}, J(n) = 2\lfloor n/2 \rfloor$ for each $n \in \mathbb{Z}$.

I is 1-1: Assume $I(n_1) = I(n_2)$. Then by definition of $I, n_1 = n_2$.

I is not onto: Let $n = 3$. There is no $m \in 2\mathbb{Z}$ such that $I(m) = n$. Why? Argue by contradiction and assume otherwise. Then $I(m) = m = n$ so $m = 3$. But $3 \notin 2\mathbb{Z}$, contradiction.

J is not 1-1: because $J(2) = 2\lfloor 2/2 \rfloor = 2 \cdot 1 = 2$ and $J(3) = 2\lfloor 3/2 \rfloor = 2 \cdot 1 = 2$. So $J(2) = J(3)$ but $2 \neq 3$.

J is onto: Let $m \in 2\mathbb{Z}$ be any even integer. So $m = 2n$ for some integer n. Then there exists an integer $k \in \mathbb{Z}$, namely $k = m$, such that $J(k) = J(m) = J(2n) = 2\lfloor (2n)/2 \rfloor = 2\lfloor n \rfloor = 2n = m$. □

4.7 Exercise 7

4.7.1 (a)

Check that the formula for F given at the end of Example 7.4.2 produces the correct values for $n = 1, 2, 3, 4$.

Proof. Recall that the formula is:

$$F(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is positive and even} \\ -\frac{n-1}{2} & \text{if } n \text{ is positive and odd} \end{cases}$$

and it is supposed to give us the values $F(1) = 0, F(2) = 1, F(3) = -1, F(4) = 2$.

Let's check: 1 is odd, so $F(1) = -\frac{1-1}{2} = 0$. 2 is even, so $F(2) = \frac{2}{2} = 1$. 3 is odd, so $F(3) = -\frac{3-1}{2} = -1$. 4 is even, so $F(4) = \frac{4}{2} = 2$. □

4.7.2 (b)

Use the floor function to write a formula for F as a single algebraic expression for each positive integer n.

Proof. For each positive integer n, $F(n) = (-1)^n \left\lfloor \dfrac{n}{2} \right\rfloor$. □

4.8 Exercise 8

Use the result of exercise 3 to prove that $3\mathbb{Z}$ is countable.

Proof. It was shown in Example 7.4.2 that $\mathbb{Z}$ is countably infinite, which means that $\mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}$. By exercise 3, $\mathbb{Z}$ has the same cardinality as $3\mathbb{Z}$. It follows by the transitive property of cardinality (Theorem 7.4.1 (c)) that $\mathbb{Z}^+$ has the same cardinality as $3\mathbb{Z}$. Thus $3\mathbb{Z}$ is countably infinite *[by definition of countably infinite]*, and hence $3\mathbb{Z}$ is countable *[by definition of countable]*. □

4.9 Exercise 9

Show that the set of all nonnegative integers is countable by exhibiting a one-to-one correspondence between $\mathbb{Z}^+$ and $\mathbb{Z}^{nonneg}$.

Proof. Define $f : \mathbb{Z}^+ \to \mathbb{Z}^{nonneg}$ by $f(n) = n - 1$. f is 1-1 because if $n_1 - 1 = n_2 - 1$ then $n_1 = n_2$. f is onto because, for every $n \in \mathbb{Z}^{nonneg}$ there is $m \in \mathbb{Z}^+$, namely $m = n + 1$ such that $f(m) = f(n + 1) = (n + 1) - 1 = n$. □

In $10-14$ S denotes the set of real numbers strictly between 0 and 1. That is, $S = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$.

4.10 Exercise 10

Let $U = \{x \in \mathbb{R} \,|\, 0 < x < 2\}$. Prove that S and U have the same cardinality.

Proof. Define $f : S \to U$ by the rule $f(x) = 2x$ for each real number x in S. Then f is one-to-one by the same argument as in exercise 10a of Section 7.2 with $\mathbb{R}$ in place of $\mathbb{Z}$. Furthermore, f is onto because if y is any element in U, then $0 < y < 2$ and so $0 < y/2 < 1$. Consequently, $y/2 \in S$ and $f(y/2) = 2(y/2) = y$. Hence f is a one-to-one correspondence, and so S and U have the same cardinality. □

4.11 Exercise 11

Let $V = \{x \in \mathbb{R} \,|\, 2 < x < 5\}$. Prove that S and V have the same cardinality.

Proof. Define $h : S \to V$ as follows: $h(x) = 3x + 2$, for every $x \in S$. h is 1-1 because if $3x_1 + 2 = 3x_2 + 2$ then $3x_1 = 3x_2$ and $x_1 = x_2$. h is onto because, given any $y \in V$, let $x = (y-2)/3$. Since $2 < y < 5$ we have $(2-2)/3 < x < (5-2)/3$, or $0 < x < 1$ so $x \in S$, and $h(x) = h((y-2)/3) = 3(y-2)/3 + 2 = (y-2) + 2 = y$. Hence h is a one-to-one correspondence, and so S and V have the same cardinality. □

4.12 Exercise 12

Let a and b be real numbers with $a < b$, and suppose that $W = \{x \in \mathbb{R} \,|\, a < x < b\}$. Prove that S and W have the same cardinality.

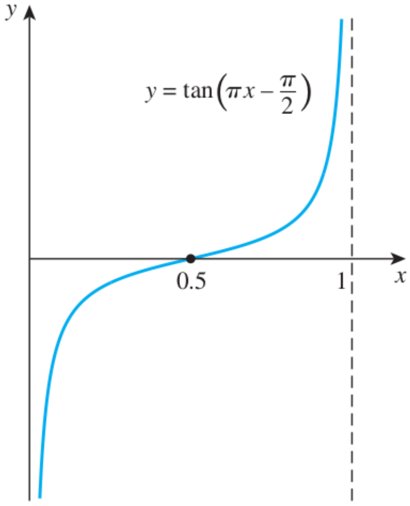
Proof. Define $f : S \to W$ as follows: $f(x) = (b-a)x + a$, for every $x \in S$. f is 1-1 because if $(b-a)x_1 + a = (b-a)x_2 + a$ then $(b-a)x_1 = (b-a)x_2$ and since $b - a > 0$ we can divide by $b - a$ to get $x_1 = x_2$. f is onto because, given any $y \in W$, let $x = (y-a)/(b-a)$. Since $a < y < b$ we have $(a-a)/(b-a) < x < (b-a)/(b-a)$, or $0 < x < 1$; so $x \in S$, and $f(x) = f((y-a)/(b-a)) = (b-a) \cdot \frac{y-a}{b-a} + a = (y-a) + a = y$. Hence f is a one-to-one correspondence, and so S and W have the same cardinality. □

4.13 Exercise 13

Draw the graph of the function f defined by the following formula: For each real number x with $0 < x < 1$, $f(x) = \tan(\pi x - \frac{\pi}{2})$. Use the graph to explain why S and R have the same cardinality.

Proof. It is clear from the graph that f is one-to-one (since it is increasing) and that the image of f is all of $\mathbb{R}$ (since the lines $x = 0$ and $x = 1$ are vertical asymptotes). Thus S and $\mathbb{R}$ have the same cardinality.

□



4.14 Exercise 14

Define a function g from the set of real numbers to S by the following formula: For each real number x,

$$g(x) = \frac{1}{2} \cdot \frac{x}{1+|x|} + \frac{1}{2}.$$

Prove that g is a one-to-one correspondence. (It is possible to prove this statement either with calculus or without it.) What conclusion can you draw from this fact?

Proof. g **is onto:** Given $y \in S$ (so $0 < y < 1$) let

$$x = \begin{cases} \frac{1}{2} \cdot \frac{1}{-y} + 1 & \text{if } 0 < y \le 1/2 \\ \frac{1}{2} \cdot \frac{1}{1-y} - 1 & \text{if } 1/2 < y < 1 \end{cases}$$

We claim that $g(x) = y$.

Case 1 ($0 < y \le 1/2$): Notice that $-y \ge -1/2$, so $\frac{1}{-y} \le -2$, so $\frac{1}{2} \cdot \frac{1}{-y} \le -1$, so $\frac{1}{2} \cdot \frac{1}{-y} + 1 \le 0$. Thus $\left|\frac{1}{2} \cdot \frac{1}{-y} + 1\right| = \frac{1}{2} \cdot \frac{1}{y} - 1$. Then $g(x)$

$$\begin{aligned} &= g\left(\frac{1}{2} \cdot \frac{1}{-y} + 1\right) &&= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \frac{1}{-y} + 1}{1 + \left|\frac{1}{2} \cdot \frac{1}{-y} + 1\right|} + \frac{1}{2} &&= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \frac{1}{-y} + 1}{1 + \frac{1}{2} \cdot \frac{1}{y} - 1} + \frac{1}{2} \\ &= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \frac{1}{-y} + 1}{\frac{1}{2} \cdot \frac{1}{y}} + \frac{1}{2} &&= y \cdot \left(\frac{1}{2} \cdot \frac{1}{-y} + 1\right) + \frac{1}{2} &&= -\frac{1}{2} + y + \frac{1}{2} = y \end{aligned}$$

Case 2 ($1/2 < y < 1$): Notice that $-1/2 > -y$, so $1/2 > 1 - y$, so $2 < \frac{1}{1-y}$, so $1 < \frac{1}{2} \cdot \frac{1}{1-y}$, so $0 < \frac{1}{2} \cdot \frac{1}{1-y} - 1$. Thus $\left|\frac{1}{2} \cdot \frac{1}{1-y} - 1\right| = \frac{1}{2} \cdot \frac{1}{1-y} - 1$. Then $g(x)$

$$\begin{aligned} &= g\left(\frac{1}{2} \cdot \frac{1}{1-y} - 1\right) &&= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \frac{1}{1-y} - 1}{1 + \left|\frac{1}{2} \cdot \frac{1}{1-y} - 1\right|} + \frac{1}{2} &&= \frac{1}{2} \cdot \frac{\frac{1}{2} \cdot \frac{1}{1-y} - 1}{1 + \frac{1}{2} \cdot \frac{1}{1-y} - 1} + \frac{1}{2} \\ &= \frac{1}{2} \cdot \frac{\frac{2y-1}{2-2y}}{\frac{1}{2-2y}} + \frac{1}{2} &&= \frac{2y-1}{2} + \frac{1}{2} &&= y - \frac{1}{2} + \frac{1}{2} = y \end{aligned}$$

g is 1-1: Assume $x_1, x_2 \in \mathbb{R}$ and $g(x_1) = g(x_2)$. *[We want to show $x_1 = x_2$].* So

$$\frac{1}{2} \cdot \frac{x_1}{1+|x_1|} + \frac{1}{2} = \frac{1}{2} \cdot \frac{x_2}{1+|x_2|} + \frac{1}{2}$$

Subtracting 1/2 and multiplying by 2 we get $\dfrac{x_1}{1+|x_1|} = \dfrac{x_2}{1+|x_2|}$. Cross multiplying we get $x_1(1+|x_2|) = x_2(1+|x_1|)$, so (*) $x_1 + x_1|x_2| = x_2 + x_2|x_1|$. There are 4 cases:

Case 1 ($\boldsymbol{x_1 \geq 0, x_2 \geq 0}$): In this case $|x_1| = x_1$ and $|x_2| = x_2$. So (*) becomes $x_1 + x_1x_2 = x_2 + x_2x_1$, after canceling we get $x_1 = x_2$.

Case 2 ($\boldsymbol{x_1 \geq 0, x_2 \leq 0}$): In this case $|x_1| = x_1$ and $|x_2| = -x_2$. So (*) becomes $x_1 - x_1x_2 = x_2 + x_2x_1$, solving for x_1 we get $x_1 - 2x_1x_2 = x_2 \implies x_1(1-2x_2) = x_2$ so $x_1 = \dfrac{x_2}{1-2x_2}$. The left hand side is ≥ 0 while the right hand side is ≤ 0, which forces $x_1 = x_2 = 0$. So $x_1 = x_2$.

Case 3 ($\boldsymbol{x_1 \leq 0, x_2 \geq 0}$): In this case $|x_1| = -x_1$ and $|x_2| = x_2$. So (*) becomes $x_1 + x_1x_2 = x_2 - x_2x_1$, solving for x_2 we get $x_1 = x_2 - 2x_1x_2 \implies x_1 = x_2(1-2x_1)$ so $x_2 = \dfrac{x_1}{1-2x_1}$. The left hand side is ≥ 0 while the right hand side is ≤ 0, which forces $x_1 = x_2 = 0$. So $x_1 = x_2$.

Case 4 ($\boldsymbol{x_1 \leq 0, x_2 \leq 0}$): In this case $|x_1| = -x_1$ and $|x_2| = -x_2$. So (*) becomes $x_1 - x_1x_2 = x_2 - x_2x_1$, after canceling we get $x_1 = x_2$.

Conclusion: S has the same cardinality as the set of real numbers. □

4.15 Exercise 15

Show that the set of all bit strings (strings of 0's and 1's) is countable.

Proof. We can describe a one-to-one correspondence between this set and $\mathbb{Z}^+$ as follows: first consider the bit strings of length 0, namely the empty string. Map it to 1. Then consider the bit strings of length 1, namely 0 and 1. Map them to 2 and 3. Then consider the bit strings of length 2, namely 00, 01, 10 and 11. Map them to 4,5,6,7. And so on. Generally, for each integer $n \geq 0$ there are 2^n bit strings of length n, and we map them to the positive integers between 2^n (inclusive) and $2^{n+1} - 1$ (inclusive). This mapping is a one-to-one correspondence by definition, since we are never mapping two different bit strings to the same positive integer (so it's 1-1) and we are not skipping over any positive integer in the range (so it's onto). □

4.16 Exercise 16

Show that $\mathbb{Q}$, the set of all rational numbers, is countable.

Proof. In Example 7.4.4 it was shown that there is a one-to-one correspondence from $\mathbb{Z}^+$ to $\mathbb{Q}^+$. This implies that the positive rational numbers can be written as an infinite sequence: $r_1, r_2, r_3, r_4, \ldots$. Now the set $\mathbb{Q}$ of all rational numbers consists of the numbers in this sequence together with 0 and the negative rational numbers: $-r_1, -r_2, -r_3, -r_4, \ldots$. Let $r_0 = 0$. Then the elements of the set of all rational numbers can be "counted" as follows: $r_0, r_1, -r_1, r_2, -r_2, r_3, -r_3, r_4, -r_4, \ldots$. In other words, we can define a one-to-one correspondence as follows: for each integer $n \geq 1$,

$$G(x) = \begin{cases} r_{n/2} & \text{if } n \text{ is even} \\ -r_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore, $\mathbb{Q}$ is countably infinite and hence countable. □

4.17 Exercise 17

Show that $\mathbb{Q}$, the set of all rational numbers, is dense along the number line by showing that given any two rational numbers r_1 and r_2 with $r_1 < r_2$, there exists a rational number x such that $r_1 < x < r_2$.

Proof. Assume r_1 and r_2 are any two rational numbers with $r_1 < r_2$. Let $x = (r_1 + r_2)/2$. Then x is a rational number, because $r_1 + r_2$ is rational (being the sum of two rationals) and thus x is the ratio of two rational numbers. Now we want to show $r_1 < x < r_2$.

$\boldsymbol{r_1 < x}$: Since $r_1 < r_2$, we have $2r_1 < r_1 + r_2$. Dividing by 2 we get $r_1 < (r_1 + r_2)/2 = x$.

$\boldsymbol{x < r_2}$: Since $r_1 < r_2$, we have $r_1 + r_2 < 2r_2$. Dividing by 2 we get $x = (r_1 + r_2)/2 < r_2$. □

4.18 Exercise 18

Must the average of two irrational numbers always be irrational? Prove or give a counterexample.

Proof. No. Counterexample: Both π and $-\pi$ are irrational, but their average is 0, which is rational. □

4.19 Exercise 19

Show that the set of all irrational numbers is dense along the number line by showing that given any two real numbers, there is an irrational number in between.

Hint: Suppose r and s are real numbers with $0 < r < s$. Let n be an integer such that $\dfrac{\sqrt{2}}{s-r} < n$, and let $m = \left\lfloor \dfrac{nr}{\sqrt{2}} \right\rfloor + 1$. Show that $m - 1 \leq \dfrac{nr}{\sqrt{2}} < m$, and use the fact that $s = r + (s - r)$ to conclude that $r < \dfrac{\sqrt{2}m}{n} < s$.

Proof. (following the Hint)

1. Suppose r and s are real numbers with $0 < r < s$.

2. By 1, $0 < s - r$, so $\dfrac{\sqrt{2}}{s-r}$ is a positive real number.

3. By 2 and Archimedean property, there is a positive integer n such that $\dfrac{\sqrt{2}}{s-r} < n$.

4. Define $m = \left\lfloor \dfrac{nr}{\sqrt{2}} \right\rfloor + 1$. Then m is an integer by definition of floor and $+$.

5. By definition of floor, $\left\lfloor \dfrac{nr}{\sqrt{2}} \right\rfloor \leq \dfrac{nr}{\sqrt{2}} < \left\lfloor \dfrac{nr}{\sqrt{2}} \right\rfloor + 1$. So $m - 1 \leq \dfrac{nr}{\sqrt{2}} < m$.

6. By 5, $\dfrac{nr}{\sqrt{2}} < m$, so $r < \dfrac{\sqrt{2}m}{n}$.

7. By 3, $\dfrac{\sqrt{2}}{n} < s - r$. By 5, $\dfrac{\sqrt{2}(m-1)}{n} \leq r$.

8. Since $s = r + (s - r)$, by 7, $\dfrac{\sqrt{2}(m-1)}{n} + \dfrac{\sqrt{2}}{n} < r + (s - r) = s$. So $\dfrac{\sqrt{2}m}{n} < s$.

9. By 6 and 8, $r < \dfrac{\sqrt{2}m}{n} < s$.

10. $\dfrac{\sqrt{2}m}{n}$ is irrational, since m/n is rational and $\sqrt{2}$ is irrational, and the product of a rational and an irrational is irrational.

11. By 9 and 10, there is an irrational number between r and s, as needed.

Generalizing to any real numbers $r < s$:

So far we have assumed $0 < r < s$. Now we need to generalize the argument to any two real numbers $r < s$.

Case 1 ($r = 0$): Then $0 < s$, so let $y = (r + s)/2$ and notice $r = 0 < y < s$. So we can use the above argument with y instead of r to obtain an irrational number x between y and s: $y < x < s$, which implies $r < x < s$, as needed.

Case 2 ($s = 0$): Then $r < 0$ so $0 < -r$, so we can use Case 1 with $s = 0$ and $-r$ instead of r and s to get an irrational x with $0 = s < x < -r$, which means that $r < -x < s$ (and $-x$ is also irrational), as needed.

Now for the rest ($r \neq 0$ and $s \neq 0$), there are 3 cases:

Case 3 ($0 < r < s$): This is handled by the main argument above.

Case 4 ($r < s < 0$): We have $0 < -s < -r$, so we can use Case 1 to get an irrational x such that $-s < x < -r$, which implies $r < -x < s$ (where $-x$ is also irrational), as needed.

Case 5 ($r < 0 < s$): By Case 1 there is an irrational x with $0 < x < s$, so $r < 0 < x < s$, thus $r < x < s$ as needed. $\square$

4.20 Exercise 20

Give two examples of functions from $\mathbb{Z}$ to $\mathbb{Z}$ that are one-to-one but not onto.

Proof. $f(n) = 2n$ is 1-1: if $2n_1 = 2n_2$ then $n_1 = n_2$; but not onto: there is no $n \in \mathbb{Z}$ such that $f(n) = 2n = 5$. Similarly $g(n) = 3n$ is 1-1 but not onto. □

4.21 Exercise 21

Give two examples of functions from $\mathbb{Z}$ to $\mathbb{Z}$ that are onto but not one-to-one.

Proof. Define $f : \mathbb{Z} \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} n & \text{if } n \leq 0 \\ 0 & \text{if } n = 1 \\ n-1 & \text{if } 2 \leq n \end{cases}$$

Then f is not 1-1 because $f(0) = f(1) = 0$ but $0 \neq 1$. But f is onto, because the first case covers the range of all negative integers and 0, the second case covers 0, and the last case covers the range of all positive integers. Similarly

$$g(n) = \begin{cases} n & \text{if } n \leq 0 \\ 0 & \text{if } n = 1 \\ 0 & \text{if } n = 2 \\ n-2 & \text{if } 3 \leq n \end{cases}$$

is also onto but not 1-1. □

4.22 Exercise 22

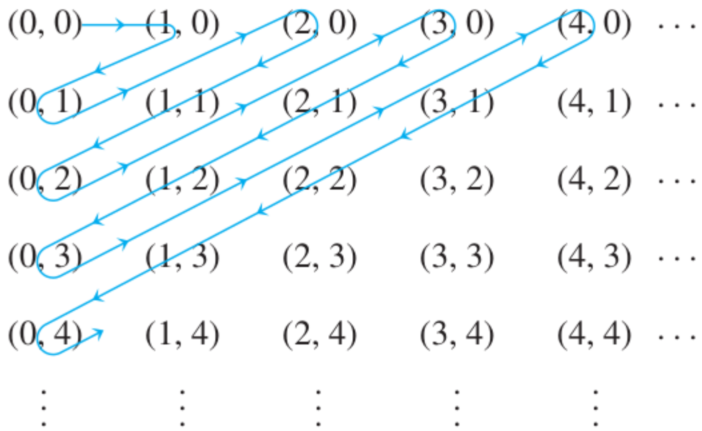
Define a function $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ by the formula $g(m, n) = 2^m 3^n$ for all $(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Show that g is one-to-one and use this result to prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Proof. Assume $g(m_1, n_1) = g(m_2, n_2)$. Then $2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2}$. So we have two prime factorizations of the same positive integer. By the uniqueness of prime factorizations, we have $m_1 = m_2$ and $n_1 = n_2$. Thus g is 1-1. Then g is a one-to-one correspondence between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $g(\mathbb{Z}^+ \times \mathbb{Z}^+)$. Since $g(\mathbb{Z}^+ \times \mathbb{Z}^+) \subseteq \mathbb{Z}^+$ and $\mathbb{Z}^+$ is countable, by Theorem 7.4.3 $g(\mathbb{Z}^+ \times \mathbb{Z}^+)$ is countable. Since g is a one-to-one correspondence, $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. □

4.23 Exercise 23

4.23.1 (a)

Explain how to use the following diagram to show that $\mathbb{Z}^{nonneg} \times \mathbb{Z}^{nonneg}$ and $\mathbb{Z}^{nonneg}$ have the same cardinality.



Proof. Define a function $G : \mathbb{Z}^{nonneg} \to \mathbb{Z}^{nonneg} \times \mathbb{Z}^{nonneg}$ as follows: Let $G(0) = (0, 0)$, and then follow the arrows in the diagram, letting each successive ordered pair of integers be the value of G for the next successive integer. Thus, for instance,

$G(1) = (1, 0), G(2) = (0, 1), G(3) = (2, 0), G(4) = (1, 1), G(5) = (0, 2), G(6) = (3, 0),$

$G(7) = (2, 1), G(8) = (1, 2)$, and so forth. □

4.23.2 (b)

Define a function $H : \mathbb{Z}^{nonneg} \times \mathbb{Z}^{nonneg} \to \mathbb{Z}^{nonneg}$ by the formula

$$H(m, n) = n + \frac{(m+n)(m+n-1)}{2}$$

for all nonnegative integers m and n. Interpret the action of H geometrically using the diagram of part (a).

Proof. Observe that if the top ordered pair of any given diagonal is $(k, 0)$, the entire diagonal (moving from top to bottom) consists of $(k, 0), (k-1, 1), (k-2, 2), \ldots, (2, k-2), (1, k-1), (0, k)$. Thus for every ordered pair (m, n) within any given diagonal, the value of $m + n$ is constant, and as you move down the ordered pairs in the diagonal, starting at the top, the value of the second element of the pair keeps increasing by 1. □

4.24 Exercise 24

Prove that the function H defined analytically in exercise 23b is a one-to-one correspondence.

Proof. ***H* is 1-1:** Assume $H(m_1, n_1) = H(m_2, n_2)$. So

$$n_1 + \frac{(m_1+n_1)(m_1+n_1-1)}{2} = n_2 + \frac{(m_2+n_2)(m_2+n_2-1)}{2}.$$

Multiplying by 2 and moving everything to the left, we get

$$2n_1 - 2n_2 + (m_1+n_1)(m_1+n_1-1) - (m_2+n_2)(m_2+n_2-1) = 0.$$

???

***H* is onto:** Assume $z \in \mathbb{Z}^{nonneg}$. The set

$$S = \{w \in \mathbb{Z}^{nonneg} \,|\, z \leq \frac{(w+1)w}{2}\}$$

is nonempty, since $z \in S$. By the well-ordering principle S has a least element, say y. Therefore

$$\frac{y(y-1)}{2} < z \leq \frac{(y+1)y}{2}$$

So there exists $n \in \mathbb{Z}^{nonneg}$ such that $z = \dfrac{y(y-1)}{2} + n$. Let $m = y - n$. Then $m \in \mathbb{Z}^{nonneg}$ and

$$z = n + \frac{y(y-1)}{2} = n + \frac{(m+n)(m+n-1)}{2}$$

Therefore $H(m, n) = z$, *[as was to be shown.]* □

4.25 Exercise 25

Prove that $0.1999\ldots = 0.2$.

Proof. Let $x = 0.1999\ldots$. Then $10x = 1.9999\ldots$ and $100x = 19.9999\ldots$. So $100x - 10x = 18$, or $90x = 18$. Thus $x = 18/90 = 1/5 = 0.2$. □

4.26 Exercise 26

Prove that any infinite set contains a countably infinite subset.

Proof. Let A be an infinite set. Construct a countably infinite subset $a_1, a_2, a_3, \ldots$ of A, by letting a_1 be any element of A, letting a_2 be any element of A other than a_1, letting a_3 be any element of A other than a_1 or a_2, and so forth. This process never stops (and hence $a_1, a_2, a_3, \ldots$ is an infinite sequence) because A is an infinite set. More formally,

1. Let a_1 be any element of A.

2. For each integer $n \neq 2$, let an be any element of $A = \{a_1, a_2, a_3, \ldots, a_{n-1}\}$. Such an element exists, for otherwise $A - \{a_1, a_2, a_3, \ldots, a_{n-1}\}$ would be empty and A would be finite. □

4.27 Exercise 27

Prove that if A is any countably infinite set, B is any set, and $g : A \to B$ is onto, then B is countable.

Proof. Suppose A is any countably infinite set, B is any set, and $g : A \to B$ is onto. Since A is countably infinite, there is a one-to-one correspondence $f : \mathbb{Z}^+ \to A$. Then, in particular, f is onto, and so by Theorem 7.3.4, $g \circ f$ is an onto function from $\mathbb{Z}^+$ to B. Define a function $h : B \to \mathbb{Z}^+$ as follows: Suppose x is any element of B. Since $g \circ f$ is onto, $\{m \in \mathbb{Z}^+ \,|\, (g \circ f)(m) = x\} \neq \varnothing$. Thus, by the well-ordering principle for the integers, this set has a least element. In other words, there is a least positive integer n

with $(g \circ f)(n) = x$. Let $h(x)$ be this integer. We claim that h is one-to-one. Suppose $h(x_1) = h(x_2) = n$. By definition of h, n is the least positive integer with $(g \circ f)(n) = x_1$. Moreover, by definition of h, n is the least positive integer with $(g \circ f)(n) = x_2$. Hence $x_1 = (g \circ f)(n) = x_2$. Thus h is a one-to-one correspondence between B and a subset S of positive integers (the range of h). Since any subset of a countable set is countable (Theorem 7.4.3), S is countable, and so there is a one-to-one correspondence between B and a countable set. It follows from the transitive property of cardinality that B is countable. □

4.28 Exercise 28

Prove that a disjoint union of any finite set and any countably infinite set is countably infinite.

Proof. 1. Assume $A = \{a_1, \ldots, a_n\}$ is a finite set, B is a countably infinite set, and A and B are disjoint.

2. By 1 and definition of countably infinite, there is a one-to-one correspondence $f : \mathbb{Z}^+ \to B$.

3. Using f from 2, define $g : \mathbb{Z}^+ \to (A \cup B)$ as follows: for each $i \in \mathbb{Z}^+$, let

$$g(i) = \begin{cases} a_i & \text{if } 1 \leq i \leq n \\ f(i-n) & \text{if } n+1 \leq i. \end{cases}$$

4. Then g is 1-1:

Assume $g(i) = g(j)$. Since A and B are disjoint by 1, either both $g(i)$ and $g(j)$ are in A, or they are both in B.

4.1. If they are both in A then $a_i = a_j$, which implies $i = j$.

4.2. If they are both in B, then $f(i-n) = f(j-n)$. Since f is 1-1 by 2, $i - n = j - n$, thus $i = j$.

So g is 1-1.

5. And g is onto: assume $x \in A \cup B$.

5.1. If $x \in A$ then $x = a_i$ for some $i \in \{1, \ldots, n\}$, so $g(i) = a_i = x$, *[as needed]*.

5.2. If $x \in B$ then since f is onto by 2, there exists $m \in \mathbb{Z}^+$ such that $f(m) = x$. Let $i = m + n$. Then $g(i) = g(m+n) = f(m+n-n) = f(m) = x$, *[as needed]*.

6. So by 4 and 5 g is a one-to-one correspondence between $\mathbb{Z}^+$ and $A \cup B$. □

4.29 Exercise 29

Prove that a union of any two countably infinite sets is countably infinite.

Proof. Suppose A and B are any two countably infinite sets. Then there are one-to-one correspondences $f : \mathbb{Z}^+ \to A$ and $g : \mathbb{Z}^+ \to B$.

Case 1 ($\boldsymbol{A \cap B = \varnothing}$): In this case define $h : \mathbb{Z}^+ \to A \cup B$ as follows: For every integer $n \geq 1$,

$$h(n) = \begin{cases} f(n/2) & \text{if } n \text{ is even} \\ g((n+1)/2) & \text{if } n \text{ is odd.} \end{cases}$$

h **is one-to-one:** Assume $h(n_1) = h(n_2)$.

Since $A \cap B = \varnothing$, n_1 and n_2 are either both odd or both even.

If n_1 and n_2 are both even, then $f(n_1/2) = f(n_2/2)$. Since f is 1-1, $n_1/2 = n_2/2$ so $n_1 = n_2$.

If n_1 and n_2 are both odd, then $g((n_1+1)/2) = g((n_2+1)/2)$. Since g is 1-1, $(n_1+1)/2 = (n_2+1)/2$ so $n_1 = n_2$.

h **is onto:** Assume $x \in A \cup B$.

Since $A \cap B = \varnothing$, either $x \in A$ or $x \in B$.

If $x \in A$, since f is onto, there is $n \in \mathbb{Z}^+$ such that $f(n) = x$. Then $h(2n) = f(2n/2) = f(n) = x$.

If $x \in B$, since g is onto, there is $n \in \mathbb{Z}^+$ such that $g(n) = x$. Then $h(2n-1) = g((2n-1+1)/2) = g(n) = x$.

So h is a one-to-one correspondence between $\mathbb{Z}^+$ and $A \cup B$. So $A \cup B$ is countably infinite.

Case 2 ($\boldsymbol{A \cap B \neq \varnothing}$): In this case let $C = B - A$. Then $A \cup B = A \cup C$ and $A \cap C = \varnothing$.

If C is countably infinite, then by Case 1 $A \cup C$ is countably infinite. If C is finite, then by exercise 28 $A \cup C$ is countably infinite.

Since $A \cup B = A \cup C$, $A \cup B$ is also countably infinite. □

4.30 Exercise 30

Use the result of exercise 29 to prove that the set of all irrational numbers is uncountable.

Hint: Use proof by contradiction and the fact that the set of all real numbers is uncountable.

Proof. Argue by contradiction and assume the set of all irrational numbers, let's call it IR, is countable.

We know that the set of all rational numbers, $\mathbb{Q}$, is countable.

By exercise 29, $IR \cup \mathbb{Q}$ is countable. But $IR \cup \mathbb{Q} = \mathbb{R}$, the set of all real numbers, which is uncountable, contradiction.

Thus our supposition was false and IR is uncountable. □

4.31 Exercise 31

Use the results of exercises 28 and 29 to prove that a union of any two countable sets is countable.

Proof. Consider the following cases:

(1) A and B are both finite: then $A \cup B$ is also finite, therefore countable.

(2) One of A, B is finite and the other is countably infinite, and $A \cap B = \varnothing$: then by exercise 28 $A \cup B$ is countably infinite.

(3) One of A, B is finite and the other is countably infinite, and $A \cap B \neq \varnothing$:

First assume B is finite. Let $C = B - A$. Notice $A \cap C = \varnothing$ and $A \cup B = A \cup C$. Then C is finite, so by exercise 28 $A \cup C$ is countably infinite. Hence $A \cup B$ is countably infinite.

Now assume A is finite. Let $C = A - B$. Notice $B \cap C = \varnothing$ and $A \cup B = B \cup C$. Then C is finite, so by exercise 28 $B \cup C$ is countably infinite. Hence $A \cup B$ is countably infinite.

(4) both A and B are countably infinite: then by exercise 29 $A \cup B$ is countably infinite. □

4.32 Exercise 32

Prove that $\mathbb{Z} \times \mathbb{Z}$, the Cartesian product of the set of integers with itself, is countably infinite.

Proof. By Example 7.4.2, the function $F : \mathbb{Z}^+ \to \mathbb{Z}$ defined by

$$F(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is a positive even integer} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is a positive odd integer} \end{cases}$$

is a one-to-one correspondence. Define $G : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z} \times \mathbb{Z}$ by $G(m, n) = (F(m), F(n))$.

G **is 1-1:** Assume $G(m_1, n_1) = G(m_2, n_2)$. Then $(F(m_1), F(n_1)) = (F(m_2), F(n_2))$. By definition of an ordered pair, $F(m_1) = F(m_2)$ and $F(n_1) = F(n_2)$. Since F is 1-1, $m_1 = m_2$ and $n_1 = n_2$. So by definition of ordered pair, $(m_1, n_1) = (m_2, n_2)$.

G **is onto:** Assume $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Since F is onto, there exist $m, n \in \mathbb{Z}^+$ such that $F(m) = x$ and $F(n) = y$. Thus $G(m, n) = (F(m), F(n)) = (x, y)$.

So G is a one-to-one correspondence, thus $\mathbb{Z}^+ \times \mathbb{Z}^+$ has the same cardinality as $\mathbb{Z} \times \mathbb{Z}$. By exercise 22, $\mathbb{Z}^+ \times \mathbb{Z}^+$ has the same cardinality as $\mathbb{Z}^+$. By the transitivity of cardinality, $\mathbb{Z} \times \mathbb{Z}$ has the same cardinality as $\mathbb{Z}^+$, *[as was to be shown.]* □

4.33 Exercise 33

Use the results of exercises 27, 31, and 32 to prove the following: If R is the set of all solutions to all equations of the form $x^2 + bx + c = 0$, where b and c are integers, then R is countable.

Proof. 1. By exercise 32, $\mathbb{Z} \times \mathbb{Z}$ is countable.

2. There is a one-to-one correspondence F between $\mathbb{Z} \times \mathbb{Z}$ and the set of quadratic functions $S = \{f \mid f(x) = x^2 + bx + c \text{ for some integers } b, c\}$, given by $F(b, c) = f$ where $f(x) = x^2 + bx + c$. Thus S is countable.

3. Consider the subset $S_{1,2}$ of S consisting of only those quadratic functions that have 1 or 2 solutions (because some quadratics like $f(x) = x^2 + 1$ have no real solutions). Then $S_{1,2}$ is countable by Theorem 7.4.3.

4. We can split $S_{1,2}$ into two disjoint subsets S_1 and S_2, consisting of those quadratic functions that have exactly 1 or exactly 2 real solutions, respectively. Then both S_1 and S_2 are countable by Theorem 7.4.3.

5. Let R_1 and R_2 be sets of real numbers defined as the sets of solutions to all the quadratic functions in S_1 and S_2, respectively.

6. There is a function $G : S_1 \to R_1$ given by, for each $f \in S_1$, letting $G(f) =$ the (one) real solution to $f(x) = 0$. G is a one-to-one correspondence, because for any $r \in R_1$ there is only one $f \in S_1$ which has r as its only solution, namely $f(x) = (x - r)^2$. So R_1 is countable.

7. Each $f \in S_2$ has two distinct roots, so one root is smaller, the other is bigger. We can split R_2 into two (possibly overlapping) subsets R_2^{small} and R_2^{big}, by defining R_2^{small} to be the set of the *smaller* solutions to quadratic functions in S_2, and R_2^{big} to be the set of the *bigger* solutions to quadratic functions in S_2. Then there are onto functions $H : S_2 \to R_2^{small}$ and $J : S_2 \to R_2^{big}$ defined by, for each $f \in S_2$, $H(f) =$ the smaller root of f and $J(f) =$ the bigger root of f, respectively. Therefore by exercise 27, both R_2^{small} and R_2^{big} are countable.

8. By exercise 31 $R_2^{small} \cup R_2^{big}$ is countable. Notice that $R_2 \subseteq R_2^{small} \cup R_2^{big}$, so by Theorem 7.4.3 R_2 is countable.

9. Finally, by exercise 31 $R_1 \cup R_2$ is countable. Notice that $R \subseteq R_1 \cup R_2$, so by Theorem 7.4.3 R is countable. □

4.34 Exercise 34

Let $\mathcal{P}(S)$ be the set of all subsets of S, and let T be the set of all functions from S to $\{0, 1\}$. Show that $\mathcal{P}(S)$ and T have the same cardinality.

Proof. Define a function $f : \mathcal{P}(S) \to T$ as follows: For each subset A of S, let $f(A) = \chi_A$, the characteristic function of A, where $\chi_A : S \to \{0, 1\}$ is defined by the rule: for every $x \in S$,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Show that f is one-to-one: Assume A_1 and A_2 are subsets of S and $\chi_{A_1} = \chi_{A_2}$. So for all $x \in S$, $\chi_{A_1}(x) = \chi_{A_2}(x)$. This means that if $\chi_{A_1}(x) = \chi_{A_2}(x) = 1$ then $x \in A_1$

and $x \in A_2$, and if $\chi_{A_1}(x) = \chi_{A_2}(x) = 0$ then $x \notin A_1$ and $x \notin A_2$. So $x \in A_1$ if and only if $x \in A_2$ for all $x \in S$, therefore $A_1 = A_2$.

Show that f is onto: Assume $g : S \to \{0, 1\}$ is any function. Define $A = \{x \in S \mid g(x) = 1\}$. Notice that $A \subseteq S$. Moreover, notice that $g(x) = 1$ for all $x \in A$ and $g(x) = 0$ for all $x \notin A$. Therefore $g(x) = \chi_A(x)$ for all $x \in S$. Thus $g = \chi_A$.

So f is a one-to-one correspondence between $\mathscr{P}(S)$ and T. □

4.35 Exercise 35

Let S be a set and let $\mathscr{P}(S)$ be the set of all subsets of S. Show that S is "smaller than" $\mathscr{P}(S)$ in the sense that there is a one-to-one function from S to $\mathscr{P}(S)$ but there is no onto function from S to $\mathscr{P}(S)$.

Proof. Let $g : S \to \mathscr{P}(S)$ be defined by: $g(s) = \{s\}$ for all $s \in S$. Then g is one-to-one: if $g(r) = g(s)$ then $\{r\} = \{s\}$, which implies $r = s$.

Suppose there is an onto function $f : S \to \mathscr{P}(S)$. Let $A = \{x \in S \mid x \notin f(x)\}$. Then $A \in \mathscr{P}(S)$ and since f is onto, there exists $z \in S$ such that $A = f(z)$. Now if $z \in A$, then $z \in f(z)$ because $A = f(z)$, but also $z \notin f(z)$ by the definition of A, a contradiction. So $z \notin A$. Then $z \notin f(z)$ because $A = f(z)$, but since now $z \notin f(z)$, z satisfies the definition of being an element of A, so $z \in A$, a contradiction again. Thus our initial supposition was false, and there is no onto function $f : S \to \mathscr{P}(S)$. □

4.36 Exercise 36

The Schroeder–Bernstein theorem states the following: If A and B are any sets with the property that there is a one-to-one function from A to B and a one-to-one function from B to A, then A and B have the same cardinality. Use this theorem to prove that there are as many functions from $\mathbb{Z}^+$ to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as there are functions from $\mathbb{Z}^+$ to $\{0, 1\}$.

Proof. Let R be the set of functions from $\mathbb{Z}^+$ to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Let S be the set of functions from $\mathbb{Z}^+$ to $\{0, 1, \}$.

Define $F : S \to R$ as follows: given $g \in S$, define $f \in R$ by $f(z) = g(z)$ for all $z \in \mathbb{Z}^+$.

F is 1-1: assume $f_1 = F(g_1), f_2 = F(g_2)$ and assume $f_1 = f_2$. So by definition of f_1, f_2, $g_1(z) = g_2(z)$ for all $z \in \mathbb{Z}^+$, which implies $g_1 = g_2$, *[as needed.]*

Given $f \in R$, we can think of f as an infinite sequence that represents the decimal expansion of a real number between 0 and 1. For example, if $f(1) = 5, f(2) = 9, f(3) = 3, \ldots$ then f corresponds to the real number $f_{dec} = 0.593...$ which is an infinite series written in terms of negative powers of 10:

$$f_{dec} = 5 \cdot 10^{-1} + 9 \cdot 10^{-2} + 3 \cdot 10^{-3} + \cdots$$

For every such number between 0 and 1 with infinite decimal expansion, there is a corresponding infinite *binary* expansion, which converges to the same real number but uses 0's and 1's in its digits. For the example of f_{dec} above, the corresponding binary expansion would be: $f_{bin} = 0.10010111110011101111...$:

$$f_{bin} = 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 0 \cdot 2^{-3} + 1 \cdot 2^{-4} + \cdots$$

This infinite binary sequence can be viewed as a function $g : \mathbb{Z}^+ \to \{0, 1\}$.

Define $G : R \to S$ as $G(f) = f_{bin}$ (viewed as a function) for each $f \in S$. G is 1-1 by the uniqueness of binary expansions, because both the decimal expansion and the binary expansion converge to the same real number.

So there is a 1-1 function $R \to S$ and a 1-1 function $S \to R$. Then by the Schroeder-Bernstein theorem, R and S have the same cardinality. □

4.37 Exercise 37

Prove that if A and B are any countably infinite sets, then $A \times B$ is countably infinite.

Proof. Since A and B are countable, their elements can be listed as $A : a_1, a_2, a_3, \ldots$ and $B : b_1, b_2, b_3, \ldots$.

We can represent the elements of $A \times B$ in a grid:

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \ldots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \ldots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

Define $F : A \times B \to \mathbb{Z}^+ \times \mathbb{Z}^+$ by $F(a_i, b_j) = (i, j)$. It is easy to see that F is a one-to-one correspondence. Since $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by exercise 22, $A \times B$ is also countable. □

4.38 Exercise 38

Suppose $A_1, A_2, A_3, \ldots$ is an infinite sequence of countable sets. Recall that

$$\bigcup_{i=1}^{\infty} A_i = \{x \,|\, x \in A_i \text{ for some positive integer } i\}.$$

Prove that A is countable. (In other words, prove that a countably infinite union of countable sets is countable.)

Proof. First suppose $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Since $A_1, A_2, A_3, \ldots$ are all countable, their elements can be listed as:

$$\begin{array}{ccccc} A_1 : & a_1^1 & a_1^2 & a_1^3 & \ldots \\ A_2 : & a_2^1 & a_2^2 & a_2^3 & \ldots \\ A_3 : & a_3^1 & a_3^2 & a_3^3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

This listing also lists the elements of their union, $\bigcup_{i=1}^{\infty} A_i$.

Define $F : \bigcup_{i=1}^{\infty} A_i \to \mathbb{Z}^+ \times \mathbb{Z}^+$ by $F(a_i^j) = (i, j)$. It is easy to see that F is a one-to-one correspondence (since the union is disjoint). Therefore $\bigcup_{i=1}^{\infty} A_i$ is countable.

Now consider the general case, where the sets are not necessarily disjoint. We can replace them with sets $B_1, B_2, B_3, \ldots$ that are disjoint, and such that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ by discarding the duplicate elements. Then by the above argument, $\bigcup_{i=1}^{\infty} B_i$ is countable, and then so is $\bigcup_{i=1}^{\infty} A_i$. $\square$