Solutions to Chapter 8, Susanna Epp Discrete Math 5th Edition

https://github.com/spamegg1

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Contents

1 Exercise Set 8.1

1.1 Exercise 1

As in Example 8.1.2, the **congruence modulo 2** relation E is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For every ordered pair $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, $m\ E\ n \iff m - n$ is even.

1.1.1 (a)

Is $0\ E\ 0$? Is $5\ E\ 2$? Is $(6,6) \in E$? Is $(21,7) \in E$?

Proof. $0\ E\ 0$ because $0 - 0 = 0 = 2 \cdot 0$, so $2 \mid (0-0)$. $5 \not E\ 2$ because $5 - 2 = 3$ and $3 \neq 2k$ for any integer k, so $2 \nmid (5-2)$. $(6,6) \in E$ because $6 - 6 = 0 = 2 \cdot 0$, so $2 \mid (6-6)$.$(-1,7) \in E$ because $-1-7 = -8 = 2 \cdot (-4)$, so $2 \mid (-1-7)$. □

1.1.2 (b)

Prove that for any even integer n, $n\ E\ 0$.

Proof. Assume n is even. By definition of even, $n = 2k$ for some integer k. Then $n - 0 = 2k - 0 = 2k$ is also even. Therefore by definition of E, $n\ E\ 0$. □

1.2 Exercise 2

Prove that for all integers m and n, $m - n$ is even if, and only if, both m and n are even or both m and n are odd.

Proof. $\implies$: Assume $m - n$ is even. *[We want to prove that both m and n are even or both m and n are odd.]* By definition of even, $m - n = 2k$ for some integer k. There are 4 cases:

Case 1: both m and n are even: Nothing to prove.

Case 2: both m and n are odd: Nothing to prove.

Case 3: m is even, n is odd: By definitions of even and odd, $m = 2k, n = 2l + 1$ for some integers k, l. So $m - n = 2k - 2l - 1 = 2(k - l - 1) + 1$ where $k - l - 1$ is an integer. So by definition of odd, $m - n$ is odd, a contradiction. So this case is impossible.

Case 4: m is odd, n is even: By definitions of even and odd, $m = 2k+1, n = 2l$ for some integers k, l. So $m - n = 2k + 1 - 2l = 2(k-l) + 1$ where $k - l$ is an integer. So by definition of odd, $m - n$ is odd, a contradiction. So this case is impossible.

$\Longleftarrow$: Assume both m and n are even or both m and n are odd. *[We want to prove that $m - n$ is even.]* There are 2 cases:

Case 1: both m and n are even: By definition of even, $m = 2k, n = 2l$ for some integers k, l. Then $m - n = 2k - 2l = 2(k - l)$ where $k - l$ is an integer. So by definition, $m - n$ is even.

Case 2: both m and n are odd: By definition of even, $m = 2k+1, n = 2l+1$ for some integers k, l. Then $m - n = 2k + 1 - 2l - 1 = 2(k - l)$ where $k - l$ is an integer. So by definition, $m - n$ is even. □

1.3 Exercise 3

The congruence modulo 3 relation, T, is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all integers m and n, $m \; T \; n \iff 3 \mid (m - n)$.

1.3.1 (a)

Is $10 \; T \; 1$? Is $1 \; T \; 10$? Is $(2, 2) \in T$? Is $(8, 1) \in T$?

Proof. $10 \; T \; 1$ because $10 - 1 = 9 = 3 \cdot 3$, and so $3 \mid (10 - 1)$.

$1 \; T \; 10$ because $1 - 10 = -9 = 3 \cdot (-3)$, and so $3 \mid (1 - 10)$.

$2 \; T \; 2$ because $2 - 2 = 0 = 3 \cdot 0$, and so $3 \mid (2 - 2)$.

$8 \not{T} \; 1$ because $8 - 1 = 7 \neq 3k$, for any integer k. So $3 \nmid (8 - 1)$. □

1.3.2 (b)

List five integers n such that $n \; T \; 0$.

Proof. One possible answer: 3, 6, 9, −3, −6 □

1.3.3 (c)

List five integers n such that $n \; T \; 1$.

Proof. One possible answer: 4, 7, 10, −2, −5 □

1.3.4 (d)

List five integers n such that $n \; T \; 2$.

Proof. One possible answer: 5, 8, 11, −1, −4 □

1.3.5 (e)

Make and prove a conjecture about which integers are related by T to 0, which integers are related by T to 1, and which integers are related by T to 2.

All integers of the form $3k + 1$, for some integer k, are related by T to 1.

Proof. All integers of the form $3k$, for some integer k, are related by T to 0.

All integers of the form $3k + 1$, for some integer k, are related by T to 1.

All integers of the form $3k + 2$, for some integer k, are related by T to 2. □

1.4 Exercise 4

Define a relation P on $\mathbb{Z}$ as follows: For every ordered pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $m \; P \; n \iff m$ and n have a common prime factor.

1.4.1 (a)

Is $15 \; P \; 25$?

Proof. Yes, because 15 and 25 are both divisible by 5, which is prime. □

1.4.2 (b)

Is $22 \; P \; 27$?

Proof. No, because 22 and 27 have no common prime factor. □

1.4.3 (c)

Is $0 \; P \; 5$?

Proof. Yes, because 0 and 5 are both divisible by 5, which is prime. □

1.4.4 (d)

Is $8 \; P \; 8$?

Proof. Yes, because 8 and 8 are both divisible by 2, which is prime. □

1.5 Exercise 5

Let $X = \{a, b, c\}$. Recall that $\mathscr{P}(X)$ is the power set of X. Define a relation $\mathbf{S}$ on $\mathscr{P}(X)$ as follows: For all sets A and B in $\mathscr{P}(X)$, $A \; \mathbf{S} \; B \iff A$ has the same number of elements as B.

1.5.1 (a)

Is $\{a, b\}$ **S** $\{b, c\}$?

Proof. Yes, because both $\{a, b\}$ and $\{b, c\}$ have two elements. $\square$

1.5.2 (b)

Is $\{a\}$ **S** $\{a, b\}$?

Proof. No, one has 1 element, the other has 2 elements. $\square$

1.5.3 (c)

Is $\{c\}$ **S** $\{b\}$?

Proof. Yes, because both $\{c\}$ and $\{b\}$ have one element. $\square$

1.6 Exercise 6

Let $X = \{a, b, c\}$. Define a relation **J** on $\mathscr{P}(X)$ as follows: For all sets A and B in $\mathscr{P}(X)$, $A \mathbf{J} B \iff A \cap B \neq \varnothing$.

1.6.1 (a)

Is $\{a\}$ **J** $\{c\}$?

Proof. No, because $\{a\} \cap \{c\} = \varnothing$. $\square$

1.6.2 (b)

Is $\{a, b\}$ **J** $\{b, c\}$?

Proof. Yes, because $\{a, b\} \cap \{b, c\} = \{b\} \neq \varnothing$. $\square$

1.6.3 (c)

Is $\{a, b\}$ **J** $\{a, b, c\}$?

Proof. Yes, because $\{a, b\} \cap \{a, b, c\} = \{a, b\} \neq \varnothing$. $\square$

1.7 Exercise 7

Define a relation R on $\mathbb{Z}$ as follows: For all integers m and n, $m \, R \, n \iff 5 \mid (m^2 - n^2)$.

1.7.1 (a)

Is $1\ R\ (-9)$?

Proof. Yes. $1\ R\ (-9) \iff 5 \mid (1^2 - (-9)^2)$. But $1^2 - (-9)^2 = 1 - 81 = -80$, and $5 \mid (-80)$ because $-80 = 5 \cdot (-16)$. $\square$

1.7.2 (b)

Is $2\ R\ 13$?

Proof. Yes, $2^2 - (13)^2 = 4 - 169 = -165 = 5 \cdot (-33)$. So $5 \mid 2^2 - (13)^2$. $\square$

1.7.3 (c)

Is $2\ R\ (-8)$?

Proof. Yes, $2^2 - (-8)^2 = 4 - 64 = -60 = 5 \cdot (-12)$. So $5 \mid 2^2 - (-8)^2$. $\square$

1.7.4 (d)

Is $(-8)\ R\ 2$?

Proof. Yes, $(-8)^2 - 2^2 = 64 - 4 = 60 = 5 \cdot 12$. So $5 \mid (-8)^2 - 2^2$. $\square$

1.8 Exercise 8

Let A be the set of all strings of a's and b's of length 4. Define a relation R on A as follows: For every $s, t \in A$, $s\ R\ t \iff s$ has the same first two characters as t.

1.8.1 (a)

Is $abaa\ R\ abba$?

Proof. Yes, because both *abaa* and *abba* have the same first two characters *ab*. $\square$

1.8.2 (b)

Is $aabb\ R\ bbaa$?

Proof. No, because the first two characters of *aabb* are different from the first two characters of *bbaa*. $\square$

1.8.3 (c)

Is $aaaa\ R\ aaab$?

Proof. Yes, because both *aaaa* and *aaab* have the same first two characters *aa*. $\square$

1.8.4 (d)

Is $baaa\ R\ abaa$?

Proof. No, because the first two characters of $baaa$ are different from the first two characters of $abaa$. ☐

1.9 Exercise 9

Let A be the set of all strings of 0's, 1's, and 2's of length 4. Define a relation R on A as follows: For every $s, t \in A$, $s\ R\ t \iff$ the sum of the characters in s equals the sum of the characters in t.

1.9.1 (a)

Is $0121\ R\ 2200$?

Proof. Yes, because the sum of the characters in 0121 is 4 and the sum of the characters in 2200 is also 4. ☐

1.9.2 (b)

Is $1011\ R\ 2101$?

Proof. No, because the sum of the characters in 1011 is 3, whereas the sum of the characters in 2101 is 4. ☐

1.9.3 (c)

Is $2212\ R\ 2121$?

Proof. No, because the sum of the characters in 2212 is 7, whereas the sum of the characters in 2121 is 6. ☐

1.9.4 (d)

Is $1220\ R\ 2111$?

Proof. Yes, because the sum of the characters in 1220 is 5 and the sum of the characters in 2111 is also 5. ☐

1.10 Exercise 10

Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is, for every ordered pair $(x, y) \in A \times B, x\,R\,y \iff x < y$. State explicitly which ordered pairs are in R and R^{-1}.

Proof. $R = \{(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

$R^{-1} = \{(4,3),(5,3),(6,3),(5,4),(6,4),(6,5)\}$ □

1.11 Exercise 11

Let $A = \{3,4,5\}$ and $B = \{4,5,6\}$ and let S be the "divides" relation. That is, for every ordered pair $(x,y) \in A \times B, x\,S\,y \iff x \mid y$. State explicitly which ordered pairs are in S and S^{-1}.

Proof. $S = \{(3,6),(4,4),(5,5)\}, S^{-1} = \{(6,3),(4,4),(5,5)\}$ □

1.12 Exercise 12

1.12.1 (a)

Suppose a function $F : X \to Y$ is one-to-one but not onto. Is F^{-1} (the inverse relation for F) a function? Explain your answer.

Proof. No. If $F : X \to Y$ is not onto, then F fails to be defined on all of Y. In other words, there is an element y in Y such that $(y,x) \notin F^{-1}$ for any $x \in X$. Consequently, F^{-1} does not satisfy property (1) of the definition of function. □

1.12.2 (b)

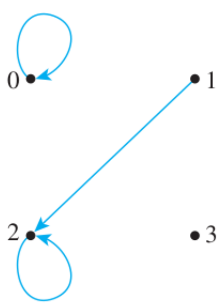
Suppose a function $F : X \to Y$ is onto but not one-to-one. Is F^{-1} (the inverse relation for F) a function? Explain your answer.

Proof. No. If $F : X \to Y$ is not one-to-one, then F for some y in Y, there will be multiple potential values for $F^{-1}(y)$. In other words, there is an element y in Y and elements $x_1, x_2 \in X$ such that $(y,x_1) \in F^{-1}$ and $(y,x_2) \in F^{-1}$. Consequently, F^{-1} does not satisfy property (2) of the definition of function. □

Draw the directed graphs of the relations defined in 13 – 18.

1.13 Exercise 13

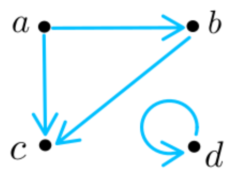
Define a relation R on $A = \{0,1,2,3\}$ by $R = \{(0,0),(1,2),(2,2)\}$.



Proof. □

1.14 Exercise 14

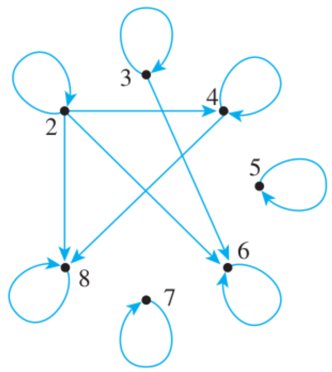
Define a relation S on $B = \{a, b, c, d\}$ by $S = \{(a, b), (a, c), (b, c), (d, d)\}$.



Proof. □

1.15 Exercise 15

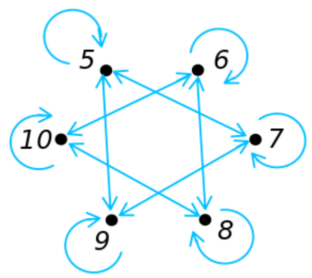
Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For every $x, y \in A, x\,R\,y \iff x \mid y$.



Proof. □

1.16 Exercise 16

Let $A = \{5, 6, 7, 8, 9, 10\}$ and define a relation S on A as follows: For every $x, y \in A, x\,S\,y \iff 2 \mid (x - y)$.

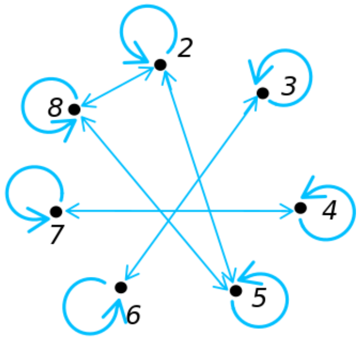


Proof. □

1.17 Exercise 17

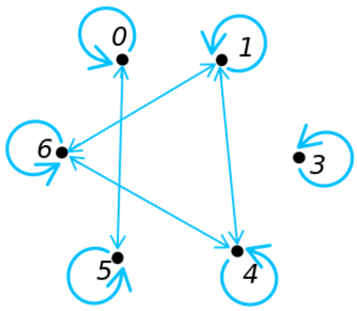
Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation T on A as follows: For every $x, y \in A, x\,T\,y \iff 3 \mid (x - y)$.

Proof. □



1.18 Exercise 18

Let $A = \{0, 1, 3, 4, 5, 6\}$ and define a relation V on A as follows: For every $x, y \in A, x\,V\,y \iff 5 \mid (x^2 - y^2)$.



Proof. □

1.19 Exercise 19

Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define relations R and S from A to B as follows: For every $(x, y) \in A \times B, x\,R\,y \iff x \mid y$ and $x\,S\,y \iff y - 4 = x$. State explicitly which ordered pairs are in $A \times B, R, S, R \cup S$, and $R \cap S$.

Proof. $A \times B = \{(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)\}$

$R = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$, $S = \{(2, 6), (4, 8)\}$, $R \cup S = R, R \cap S = S$ □

1.20 Exercise 20

Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For every $(x, y) \in A \times B, x\,R\,y \iff |x| \mid |y|$ and $x\,S\,y \iff x - y$ is even. State explicitly which ordered pairs are in $A \times B, R, S, R \cup S$, and $R \cap S$.

Proof. $A \times B = \{(-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2), (4, 1), (4, 2)\}$

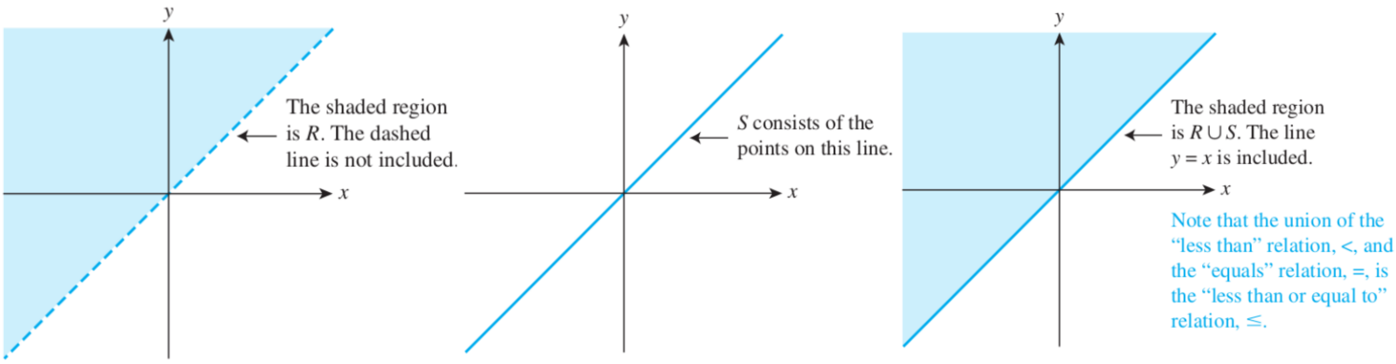
$R = \{(-1, 1), (1, 1), (2, 2)\}$, $S = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$, $R \cup S = S, R \cap S = R$ □

1.21 Exercise 21

Define relations R and S on R as follows: $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$ and

$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x = y\}$. That is, R is the "less than" relation and S is the "equals" relation on $\mathbb{R}$. Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.

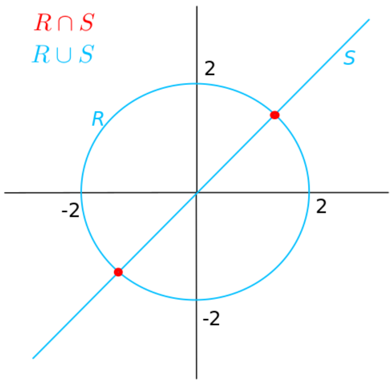
Proof. The graph of the intersection of R and S is obtained by finding the set of all points common to both graphs. But there are no points for which both $x < y$ and $x = y$. Hence $R \cap S = \varnothing$ and the graph consists of no points at all.



□

1.22 Exercise 22

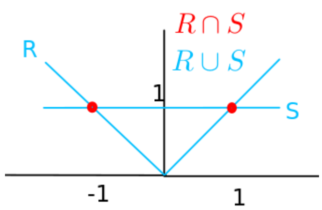
Define relations R and S on R as follows: $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x^2 + y^2 = 4\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x = y\}$. Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.



Proof. □

1.23 Exercise 23

Define relations R and S on R as follows: $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, y = |x|\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, y = 1\}$. Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.



Proof. □

1.24 Exercise 24

In Example 8.1.7 consider the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = X. The response to the query is the projection onto the first two coordinates of the intersection of the database with the set $A_1 \times A_2 \times A_3 \times \{X\}$.

1.24.1 (a)

Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = pneumonia.

Proof. 574329 Tak Kurosawa, 011985 John Schmidt □

1.24.2 (b)

Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = appendicitis.

Proof. 466581 Mary Lazars, 778400 Jamal Baskers □

2 Exercise Set 8.2

In $1-8$, a number of relations are defined on the set $A = \{0, 1, 2, 3\}$. For each relation:

a. Draw the directed graph.

b. Determine whether the relation is reflexive.

c. Determine whether the relation is symmetric.

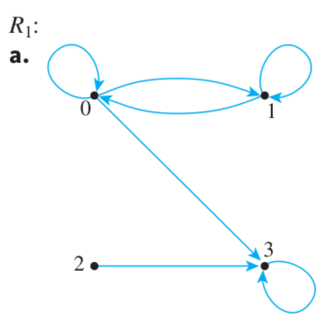
d. Determine whether the relation is transitive.

Give a counterexample in each case in which the relation does not satisfy one of the properties.

2.1 Exercise 1

$R_1 = \{(0, 0), (0, 1), (0, 3), (1, 1), (1, 0), (2, 3), (3, 3)\}$

2.1.1 (a)



Proof. ☐

2.1.2 (b)

Proof. R_1 is not reflexive: $2 \not{R_1} 2$. ☐

2.1.3 (c)

Proof. R_1 is not symmetric: $2 \, R_1 \, 3$ but $3 \not{R_1} 2$. ☐

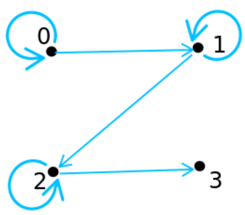
2.1.4 (d)

Proof. R_1 is not transitive: $1 \, R_1 \, 0$ and $0 \, R_1 \, 3$ but $1 \not{R_1} 3$. ☐

2.2 Exercise 2

$R_2 = \{(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)\}$

2.2.1 (a)



Proof. ☐

2.2.2 (b)

Proof. R_2 is not reflexive: $3 \not{R_2} 3$. ☐

2.2.3 (c)

Proof. R_2 is not symmetric: $2 \, R_2 \, 3$ but $3 \not{R_2} 2$. ☐

2.2.4 (d)

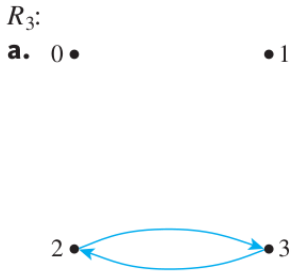
Proof. R_2 is not transitive: $0 \, R_2 \, 1$ and $1 \, R_2 \, 2$ but $0 \not{R_2} 2$. ☐

2.3 Exercise 3

$R_3 = \{(2,3),(3,2)\}$

2.3.1 (a)

Proof. ☐



2.3.2 (b)

Proof. R_3 is not reflexive: $0 \not R_3 0$. □

2.3.3 (c)

Proof. R_3 is symmetric. □

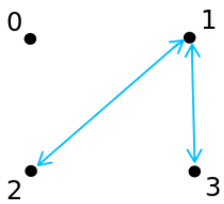
2.3.4 (d)

Proof. R_3 is not transitive: $2 R_3 3$ and $3 R_3 2$ but $2 \not R_3 2$. □

2.4 Exercise 4

$R_4 = \{(1,2),(2,1),(1,3),(3,1)\}$

2.4.1 (a)



Proof. □

2.4.2 (b)

Proof. R_4 is not reflexive: $0 \not R_4 0$. □

2.4.3 (c)

Proof. R_4 is symmetric. □

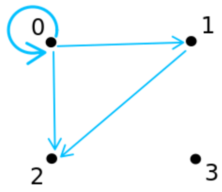
2.4.4 (d)

Proof. R_4 is not transitive: $2 R_4 1$ and $1 R_4 3$ but $2 \not R_4 3$. □

2.5 Exercise 5

$R_5 = \{(0,0),(0,1),(0,2),(1,2)\}$

2.5.1 (a)



Proof. □

2.5.2 (b)

Proof. R_5 is not reflexive: $3 \not R_5 3$. □

2.5.3 (c)

Proof. R_5 is not symmetric: $1\, R_5\, 2$ but $2 \not R_5 1$. □

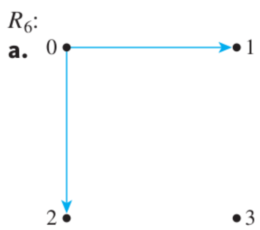
2.5.4 (d)

Proof. R_5 is transitive. □

2.6 Exercise 6

$R_6 = \{(0, 1), (0, 2)\}$

2.6.1 (a)



Proof. □

2.6.2 (b)

Proof. R_6 is not reflexive: $3 \not R_6 3$. □

2.6.3 (c)

Proof. R_6 is not symmetric: $0\, R_6\, 1$ but $1 \not R_6 0$. □

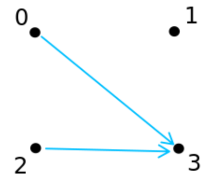
2.6.4 (d)

Proof. R_6 is transitive. □

2.7 Exercise 7

$R_7 = \{(0,3),(2,3)\}$

2.7.1 (a)



Proof. □

2.7.2 (b)

Proof. R_7 is not reflexive: $3 \not R_7 3$. □

2.7.3 (c)

Proof. R_7 is not symmetric: $0\, R_7\, 3$ but $3 \not R_7 0$. □

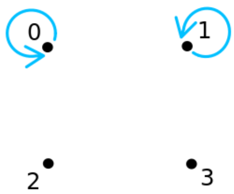
2.7.4 (d)

Proof. R_7 is transitive. □

2.8 Exercise 8

$R_8 = \{(0,0),(1,1)\}$

2.8.1 (a)



Proof. □

2.8.2 (b)

Proof. R_8 is not reflexive: $3 \not R_8 3$. □

2.8.3 (c)

Proof. R_8 is symmetric. □

2.8.4 (d)

Proof. R_8 is transitive. □

In 9–33, determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

2.9 Exercise 9

R is the "greater than or equal to" relation on the set of real numbers: For every $x, y \in \mathbb{R}, x\,R\,y \iff x \geq y$.

Proof. ***R*** **is reflexive:** R is reflexive iff for every real number $x, x\,R\,x$. By definition of R, this means that for every real number $x, x \geq x$. In other words, for every real number $x, x > x$ or $x = x$, which is true.

R **is not symmetric:** R is symmetric iff for all real numbers x and y, if $x\,R\,y$ then $y\,R\,x$. By definition of R, this means that for all real numbers x and y, if $x \geq y$ then $y \geq x$. The following counterexample shows that this is false. $x = 1$ and $y = 0$. Then $x \geq y$, but $y \not\geq x$ because $1 \geq 0$ and $0 \not\geq 1$.

R **is transitive:** R is transitive iff for all real numbers x, y, and z, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$. By definition of R, this means that for all real numbers x, y, and z, if $x \geq y$ and $y \geq z$ then $x \geq z$. This is true by definition of $\geq$ and the transitive property of order for the real numbers. (See Appendix A, T18.) □

2.10 Exercise 10

C is the circle relation on the set of real numbers: For every $x, y \in \mathbb{R}, x\,C\,y \iff x^2 + y^2 = 1$.

Proof. ***C*** **is not reflexive:** Let $x = 0$. Then $0^2 + 0^2 = 0 \neq 1$, therefore $0 \not{C} 0$.

C **is symmetric:** Assume $x\,C\,y$. Then $x^2 + y^2 = 1$. So $y^2 + x^2 = 1$. So $y\,C\,x$.

C **is not transitive:** Let $x = 1, y = 0, z = 1$. Then $x\,C\,y$ because $1^2 + 0^2 = 1$, and $y\,C\,z$ because $0^2 + 1^2 = 1$. However $x \not{C} z$ because $1^2 + 1^2 = 2 \neq 1$. □

2.11 Exercise 11

D is the relation defined on $\mathbb{R}$ as follows: For every $x, y \in \mathbb{R}, x\,D\,y \iff xy \geq 0$.

Proof. ***D*** **is reflexive:** For all real numbers $x, x \cdot x = x^2 \geq 0$ so $x\,D\,x$.

D **is symmetric:** Assume $x\,D\,y$. Then $xy \geq 0$. So $yx \geq 0$. So $y\,D\,x$.

D **is not transitive:** Let $x = 1, y = 0, z = -1$. Then $xy = 0 \geq 0$ so $x\,D\,y$, and $yz = 0 \geq 0$ so $y\,D\,z$, but $xz = -1 \not\geq 0$ so $x \not{D} z$. □

2.12 Exercise 12

E is the congruence modulo 4 relation on $\mathbb{Z}$: For every $m, n \in \mathbb{Z}, m\,E\,n \iff 4 \mid (m-n)$.

Proof. **E is reflexive:** For all $m \in \mathbb{Z}, (m-m) = 0 = 4 \cdot 0$ so $4 \mid (m-m)$ thus $m\,E\,m$.

E is symmetric: Assume $m\,E\,n$. Then $4 \mid (m-n)$. So $m-n = 4 \cdot k$ for some integer k. So $n-m = 4 \cdot (-k)$ where $-k$ is an integer. So $4 \mid (n-m)$ and $n\,E\,m$.

E is transitive: Assume $m\,E\,n$ and $n\,E\,o$. Then $4 \mid (m-n)$ and $4 \mid (n-o)$. So $m-n = 4k$ and $n-o = 4l$ for some integers k, l. So $m-o = (m-n)+(n-o) = 4k+4l = 4(k+l)$ where $k+l$ is an integer. Thus $4 \mid (m-o)$ and $m\,E\,o$. $\square$

2.13 Exercise 13

F is the congruence modulo 5 relation on $\mathbb{Z}$: For every $m, n \in \mathbb{Z}, m\,F\,n \iff 5 \mid (m-n)$.

Proof. **F is reflexive:** The proof is the same as in exercise 12.

F is symmetric: The proof is the same as in exercise 12.

F is transitive: The proof is the same as in exercise 12. $\square$

2.14 Exercise 14

O is the relation defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,O\,n \iff m-n$ is odd.

Proof. **O is not reflexive:** $0-0=0$ is even, therefore $0 \not{O} 0$.

O is symmetric: Assume $m\,O\,n$. So $m-n$ is odd. So $m-n = 2k+1$ for some integer k. So $n-m = -2k-1 = 2(-k-1)+1$ where $-k-1$ is an integer. So $n-m$ is odd and $n\,O\,m$.

O is not transitive: $2-1=1$ is odd so $2\,O\,1$, and $1-0=1$ is odd so $1\,O\,0$, but $2-0=2$ is even so $2 \not{O} 0$. $\square$

2.15 Exercise 15

D is the "divides" relation on Z^+: For all positive integers m and $n, m\,D\,n \iff m \mid n$.

Proof. **D is reflexive:** For all $m \in \mathbb{Z}^+$ $m = m \cdot 1$ therefore $m \mid m$, so $m\,D\,m$.

D is not symmetric: $3\,D\,6$ because $3 \mid 6$ because $6 = 3 \cdot 2$, but $6 \not{D} 3$ because $6 \nmid 3$ since $3/6 = 1/2$ is not an integer.

D is transitive: Assume $m\,D\,n$ and $n\,D\,o$. Then $m \mid n$ and $n \mid o$. So $n = mk$ and $o = nl$ for some integers k, l. So $o = nl = (mk)l = m(kl)$ where kl is an integer. So $m \mid o$ and $m\,D\,o$. $\square$

2.16 Exercise 16

A is the "absolute value" relation on $\mathbb{R}$: For all real numbers x and y, $x\,A\,y \iff |x| = |y|$.

Proof. **$\boldsymbol{A}$ is reflexive:** For all real numbers x, $|x| = |x|$ so $x\,A\,x$.

$\boldsymbol{A}$ is symmetric: Assume $x\,A\,y$ so $|x| = |y|$. Then $|y| = |x|$ so $y\,A\,x$.

$\boldsymbol{A}$ is transitive: Assume $x\,A\,y$ and $y\,A\,z$, so $|x| = |y|$ and $|y| = |z|$. Then $|x| = |y| = |z|$ so $x\,A\,z$. $\square$

2.17 Exercise 17

Recall that a prime number is an integer that is greater than 1 and has no positive integer divisors other than 1 and itself. (In particular, 1 is not prime.) A relation P is defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,P\,n \iff \exists$ a prime number p such that $p \mid m$ and $p \mid n$.

Proof. **$\boldsymbol{P}$ is not reflexive:** There is no prime number p such that $p \mid 1$ and $p \mid 1$. Thus $1 \not{P} 1$.

$\boldsymbol{P}$ is symmetric: Assume $m\,P\,n$. So there is a prime number p such that $p \mid m$ and $p \mid n$. So $p \mid n$ and $p \mid m$, and thus $n\,P\,m$.

$\boldsymbol{P}$ is not transitive: Let $m = 6, n = 15, o = 35$. Then the prime $p = 3$ divides both m and n, so $m\,P\,n$, and the prime $q = 5$ divides both n and o, so $n\,P\,o$, but there is no prime that divides both $m = 2 \cdot 3$ and $o = 5 \cdot 7$, so $m \not{P} o$. $\square$

2.18 Exercise 18

Define a relation Q on $\mathbb{R}$ as follows: For all real numbers x and y, $x\,Q\,y \iff x - y$ is rational.

Proof. **$\boldsymbol{Q}$ is reflexive:** For all reals $x \in \mathbb{R}, x - x = 0$ and 0 is rational, so $x\,Q\,x$.

$\boldsymbol{Q}$ is symmetric: Assume $x\,Q\,y$. Then $x - y$ is rational. Then $y - x = -(x - y)$ is rational (being the negative of a rational). So $y\,Q\,x$.

$\boldsymbol{Q}$ is transitive: Assume $x\,Q\,y$ and $y\,Q\,z$. Then $x - y$ and $y - z$ are rational. So $x - z = (x - y) + (y - z)$ is also rational (being the sum of two rationals). Thus $x\,Q\,z$. $\square$

2.19 Exercise 19

Define a relation I on $\mathbb{R}$ as follows: For all real numbers x and y, $x\,I\,y \iff x - y$ is irrational.

Proof. **$\boldsymbol{I}$ is not reflexive:** For all reals $x \in \mathbb{R}, x - x = 0$ and 0 is not irrational, so $x \not{I} x$.

***I* is symmetric:** Assume $x\,I\,y$. Then $x-y$ is irrational. So $y-x=-(x-y)$ is irrational (being the negative of an irrational). So $y\,I\,x$.

***I* is not transitive:** Let $x=\sqrt{2}, y=0, z=\sqrt{2}$. Then $x\,I\,y$ because $x-y=\sqrt{2}$ is irrational. Also $y\,I\,z$ because $y-z=-\sqrt{2}$ is irrational. But $x-z=0$ is not irrational, thus $x \not I z$. $\square$

2.20 Exercise 20

Let $X=\{a,b,c\}$ and $\mathscr{P}(X)$ be the power set of X (the set of all subsets of X). A relation $\mathbf{E}$ is defined on $\mathscr{P}(X)$ as follows: For every $A,B\in\mathscr{P}(X), A\,\mathbf{E}\,B \iff$ the number of elements in A equals the number of elements in B.

Proof. **E is reflexive:** For every $A\in\mathscr{P}(X)$, the number of elements in A equals the number of elements in A. So $A\,\mathbf{E}\,A$.

E is symmetric: Assume $A\,\mathbf{E}\,B$. Then the number of elements in A equals the number of elements in B. So, the number of elements in B equals the number of elements in A. So $B\,\mathbf{E}\,A$.

E is transitive: Assume $A\,\mathbf{E}\,B$ and $B\,\mathbf{E}\,C$. Then the number of elements in A equals the number of elements in B, and the number of elements in B equals the number of elements in C. So the number of elements in A equals the number of elements in C. So $A\,\mathbf{E}\,C$. $\square$

2.21 Exercise 21

Let $X=\{a,b,c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation $\mathbf{L}$ is defined on $\mathscr{P}(X)$ as follows: For every $A,B\in\mathscr{P}(X), A\,\mathbf{L}\,B \iff$ the number of elements in A is less than the number of elements in B.

Proof. **L is not reflexive:** For all $A\in\mathscr{P}(X)$, the number of elements in A is not less than the number of elements in A. So $A \not\mathbf{L} A$.

L is not symmetric: Let $A=\varnothing, B=\{a\}$. Then the number of elements in A (which is 0) is less than the number of elements in B (which is 1). So $A\,\mathbf{L}\,B$. But the number of elements in B (which is 1) is not less than the number of elements in A (which is 0). So $B \not\mathbf{L} A$.

L is transitive: Assume $A\,\mathbf{L}\,B$ and $B\,\mathbf{L}\,C$. Then the number of elements in A is less than the number of elements in B, and the number of elements in B is less than the number of elements in C. Then the number of elements in A is less than the number of elements in C. So $A\,\mathbf{L}\,C$. $\square$

2.22 Exercise 22

Let $X=\{a,b,c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation $\mathbf{N}$ is defined on $\mathscr{P}(X)$ as follows: For every $A,B\in\mathscr{P}(X), A\mathbf{N}B \iff$ the number of elements in A is not equal to the number of elements in B.

Proof. **N is not reflexive:** Let $A = \{a\}$ which has 1 element. Then the number of elements in A is equal to the number of elements in A. So $A \not\mathbf{N} A$.

N is symmetric: Assume $A \mathbf{N} B$. Then the number of elements in A is not equal to the number of elements in B. So the number of elements in B is not equal to the number of elements in A, and $B \mathbf{N} A$.

N is not transitive: Let $A = \{a\}, B = \varnothing, C = \{c\}$. Then $A \mathbf{N} B$ because A has 1 element and B has 0 elements, and $0 \neq 1$. Similarly $B \mathbf{N} C$. But $A \not\mathbf{N} C$ because both A and C have 1 element, and $1 = 1$. □

2.23 Exercise 23

Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. Define the "subset" relation $\mathbf{S}$ on $\mathscr{P}(X)$ as follows: For every $A, B \in \mathscr{P}(X), A \mathbf{S} B \iff A \subseteq B$.

Proof. **S is reflexive:** For all $A \in \mathscr{P}(X), A \subseteq A$ therefore $A \mathbf{S} A$.

S is not symmetric: Let $A = \{a\}, B = \{a, b\}$. Then $A \subseteq B$, so $A \mathbf{S} B$. But $B \nsubseteq A$ therefore $B \not\mathbf{S} A$.

S is transitive: Assume $A \mathbf{S} B$ and $B \mathbf{S} C$. So $A \subseteq B$ and $B \subseteq C$. Then by transitivity of subsets, $A \subseteq C$, and $A \mathbf{S} C$. □

2.24 Exercise 24

Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. Define the "not equal to" relation $\mathbf{U}$ on $\mathscr{P}(X)$ as follows: For every $A, B \in \mathscr{P}(X), A\mathbf{U}B \iff A \neq B$.

Proof. **U is not reflexive:** For every $A \in \mathscr{P}(X), A = A$ therefore $A \not\mathbf{U} A$.

U is symmetric: Assume $A \mathbf{U} B$. Then $A \neq B$. So $B \neq A$, and $B \mathbf{U} A$.

U is not transitive: Let $X = \{x\}, A = \{x\}, B = \varnothing, C = \{x\}$. Then $A \mathbf{U} B$ because $A \neq B$, and $B \mathbf{U} C$ because $B \neq C$, but $A = C$ so $A \not\mathbf{U} C$. □

2.25 Exercise 25

Let A be the set of all strings of a's and b's of length 4. Define a relation R on A as follows: For every $s, t \in A, s\, R\, t \iff s$ has the same first two characters as t.

Proof. ***R* is reflexive:** For every string $s \in A$, s has the same first two characters as s. Thus $s\, R\, s$.

***R* is symmetric:** Assume $s\, R t$. Then s has the same first two characters as t. Then t has the same first two characters as s, so $t\, R\, s$.

***R* is transitive:** Assume $s\, R\, t$ and $t\, R\, r$. Then s has the same first two characters as t, and t has the same first two characters as r. So s has the same first two characters as r, and $s\, R\, r$. □

2.26 Exercise 26

Let A be the set of all strings of 0's, 1's, and 2's that have length 4 and for which the sum of the characters in the string is less than or equal to 2. Define a relation R on A as follows: For every $s, t \in A, s\, R\, t \iff$ the sum of the characters of s equals the sum of the characters of t.

Proof. ***R* is reflexive:** For every $s, \in A$, the sum of the characters of s equals the sum of the characters of s. So $s\, R\, s$.

***R* is symmetric:** Assume $s\, R\, t$. Then the sum of the characters of s equals the sum of the characters of t. So the sum of the characters of t equals the sum of the characters of s, and $s\, R\, t$.

***R* is transitive:** Assume $s\, R\, t$ and $t\, R\, r$. Then the sum of the characters of s equals the sum of the characters of t, and the sum of the characters of t equals the sum of the characters of r. So the sum of the characters of s equals the sum of the characters of r, and $s\, R\, r$. □

2.27 Exercise 27

Let A be the set of all English statements. A relation **I** is defined on A as follows: For every $p, q \in A, p\, \mathbf{I}\, q \iff p \implies q$ is true.

Proof. **I is reflexive:** For every $p \in A, p \implies p$ is true, therefore $p\, \mathbf{I}\, p$.

I is not symmetric: Let p be "$1 is greater than 2$" and let q be "2 is greater than 1". So p is false and q is true. Therefore $p \implies q$ is true and $q \implies p$ is false. So $p\, \mathbf{I}\, q$ but $q \not\mathbf{I}\, p$.

I is transitive: Assume $p\, \mathbf{I}\, q$ and $q\, \mathbf{I}\, r$. So $p \implies q$ is true and $q \implies r$ is true. By transitivity of implication, $p \implies r$ is true, and $p\, \mathbf{I}\, r$. □

2.28 Exercise 28

Let $A = \mathbb{R} \times \mathbb{R}$. A relation **F** is defined on A as follows: For every (x_1, y_1) and (x_2, y_2) in A, $(x_1, y_1)\mathbf{F}(x_2, y_2) \iff x_1 = x_2$.

Proof. **F is reflexive:** For every $(x, y) \in A, x = x$, therefore $(x, y)\mathbf{F}(x, y)$.

F is symmetric: Assume $(x_1, y_1)\mathbf{F}(x_2, y_2)$. Then $x_1 = x_2$. Then $x_2 = x_1$. So $(x_2, y_2)\mathbf{F}(x_1, y_1)$.

F is transitive: Assume $(x_1, y_1)\mathbf{F}(x_2, y_2)$ and $(x_2, y_2)\mathbf{F}(x_3, y_3)$. Then $x_1 = x_2$ and $x_2 = x_3$. Thus $x_1 = x_3$ and so $(x_1, y_1)\mathbf{F}(x_3, y_3)$. □

2.29 Exercise 29

Let $A = \mathbb{R} \times \mathbb{R}$. A relation **S** is defined on A as follows: For every (x_1, y_1) and (x_2, y_2) in $A, (x_1, y_1)\mathbf{S}(x_2, y_2) \iff y_1 = y_2$.

Proof. **S is reflexive:**

S is symmetric:

S is transitive: $\square$

2.30 Exercise 30

Let A be the "punctured plane"; that is, A is the set of all points in the Cartesian plane except the origin (0, 0). A relation R is defined on A as follows: For every p_1 and p_2 in $A, p_1 \, R \, p_2 \iff p_1$ and p_2 lie on the same half line emanating from the origin.

Proof. ***R* is reflexive:** For all $p \in A, p$ and p lie on the same half line emanating from the origin. So $p \, R \, p$.

***R* is symmetric:** Assume $p_1 \, R \, p_2$. Then p_1 and p_2 lie on the same half line emanating from the origin. Then p_2 and p_1 lie on the same half line emanating from the origin. So $p_2 \, R \, p_1$.

***R* is transitive:** First notice that for any $p \in A$ there is exactly one half line emanating from the origin on which p lies.

Assume $p_1 \, R \, p_2$ and $p_2 \, R \, p_3$. Then p_1 and p_2 lie on the same half line emanating from the origin, say l_1. And p_2 and p_3 lie on the same half line emanating from the origin, say l_2. Since p_2 lies on both l_1 and l_2, by the previous paragraph $l_1 = l_2$. Then p_1 and p_3 lie on the same half line emanating from the origin. So $p_1 \, R \, p_3$. $\square$

2.31 Exercise 31

Let A be the set of people living in the world today. A relation R is defined on A as follows: For all people p and q in $A, p \, R \, q \iff p$ lives within 100 miles of q.

Proof. ***R* is reflexive:** For every person p, p lives within 0 miles of p, so in particular p lives within 100 miles of p. Therefore $p \, R \, p$.

***R* is symmetric:** Assume $p \, R \, q$. So p lives within 100 miles of q. Then q lives within 100 miles of p. Thus $q \, R \, p$.

***R* is not transitive:** As a counterexample, take p to be an inhabitant of Chicago, Illinois, q an inhabitant of Kankakee, Illinois, and r an inhabitant of Champaign, Illinois. Then $p \, R \, q$ because Chicago is less than 100 miles from Kankakee, and $q \, R \, r$ because Kankakee is less than 100 miles from Champaign, but $p \not R \, r$ because Chicago is not less than 100 miles from Champaign. $\square$

2.32 Exercise 32

Let A be the set of all lines in the plane. A relation R is defined on A as follows: For every l_1 and l_2 in $A, l_1 \, R \, l_2 \iff l_1$ is parallel to l_2. (Assume that a line is parallel to itself.)

Proof. **R is reflexive:** For every line $l \in A, l$ is parallel to itself, therefore $l\,R\,l$.

R is symmetric: Assume $l_1\,R\,l_2$. Then l_1 is parallel to l_2. Then l_2 is parallel to l_1, so $l_2\,R\,l_1$.

R is transitive: Assume $l_1\,R\,l_2$ and $l_2\,R\,l_3$. Then l_1 is parallel to l_2 and l_2 is parallel to l_3. By transitivity of parallelism l_1 is parallel to l_3 so $l_1\,R\,l_3$. □

2.33 Exercise 33

Let A be the set of all lines in the plane. A relation R is defined on A as follows: For every l_1 and l_2 in $A, l_1\,R\,l_2 \iff l_1$ is perpendicular to l_2.

Proof. **R is not reflexive:** For evert line l in A, l is not perpendicular to itself (l is parallel to itself). Therefore $l \not R\, l$.

R is symmetric: Assume $l_1\,R\,l_2$. Then l_1 is perpendicular to l_2. Then l_2 is perpendicular to l_1. So $l_2\,R\,l_1$.

R is not transitive: Let l_1 be the line $y = 0$, let l_2 be the line $x = 0$ and l_3 be the line $y = 1$. Then l_2 is perpendicular to both l_1 and l_3 so $l_1\,R\,l_2$ and $l_2\,R\,l_3$. But l_1 is parallel to l_3 so $l_1 \not R\, l_3$. □

In $34 - 36$, assume that R is a relation on a set A. Prove or disprove each statement.

2.34 Exercise 34

If R is reflexive, then R^{-1} is reflexive.

Proof. Suppose R is any reflexive relation on a set A. *[We must show that R^{-1} is reflexive. To show this, we must show that for every x in $A, x\,R^{-1}x$.]* Given any element x in A, since R is reflexive, $x\,R\,x$, and by definition of relation, this means that $(x, x) \in R$. It follows, by definition of the inverse of a relation, that $(x, x) \in R^{-1}$, and so, by definition of relation, $x\,R^{-1}\,x$ *[as was to be shown]*. □

2.35 Exercise 35

If R is symmetric, then R^{-1} is symmetric.

Proof. Assume R is symmetric. *[We want to show R^{-1} is symmetric.]* Assume $x\,R^{-1}\,y$. We need to show $y\,R^{-1}\,x$. By definition of $R^{-1}, y\,R\,x$. Since R is symmetric, $x\,R\,y$. By definition of R^{-1} again, $y\,R^{-1}\,x$. □

2.36 Exercise 36

If R is transitive, then R^{-1} is transitive.

Proof. Assume R is transitive. *[We want to show R^{-1} is transitive.]* Assume $x\,R^{-1}\,y$ and $y\,R^{-1}\,z$. We need to show $x\,R^{-1}\,z$. By definition of R^{-1}, $y\,R\,x$ and $z\,R\,y$. Since R is transitive, $z\,R\,x$. By definition of R^{-1} again, $x\,R^{-1}\,z$. □

In $37-42$, assume that R and S are relations on a set A. Prove or disprove each statement.

2.37 Exercise 37

If R and S are reflexive, is $R \cap S$ reflexive? Why?

Proof. Yes. Suppose R and S are reflexive. *[To show that $R \cap S$ is reflexive, we must show that $\forall x \in A, (x, x) \in R \cap S$.]* So suppose $x \in A$. Since R is reflexive, $(x, x) \in R$, and since S is reflexive, $(x, x) \in S$. Thus, by definition of intersection, $(x, x) \in R \cap S$ *[as was to be shown]*. □

2.38 Exercise 38

If R and S are symmetric, is $R \cap S$ symmetric? Why?

Proof. Yes. Suppose R and S are symmetric. *[To show that $R \cap S$ is symmetric, we must show that $\forall x, y \in A$, if $(x, y) \in R \cap S$ then $(y, x) \in R \cap S$.]* So suppose $x, y \in A$ and $(x, y) \in R \cap S$. By definition of intersection $(x, y) \in R$ and $(x, y) \in S$. Since R is symmetric, $(y, x) \in R$, and since S is symmetric, $(y, x) \in S$. Thus, by definition of intersection, $(y, x) \in R \cap S$ *[as was to be shown]*. □

2.39 Exercise 39

If R and S are transitive, is $R \cap S$ transitive? Why?

Proof. Yes. Suppose R and S are transitive. *[To show that $R \cap S$ is transitive, we must show that $\forall x, y, z \in A$, if $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$ then $(x, z) \in R \cap S$.]* So suppose $x, y, z \in A$ and $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. By definition of intersection $(x, y) \in R$ and $(x, y) \in S$ and $(y, z) \in R$ and $(y, z) \in S$. Since R is transitive, $(x, z) \in R$, and since S is transitive, $(x, z) \in S$. Thus, by definition of intersection, $(x, z) \in R \cap S$ *[as was to be shown]*. □

2.40 Exercise 40

If R and S are reflexive, is $R \cup S$ reflexive? Why?

Proof. Yes. To prove this we must show that for all x in A, $(x, x) \in R \cup S$. So suppose x is a particular but arbitrarily chosen element in A. *[We must show that $(x, x) \in R \cup S$.]* Then $(x, x) \in R$ because R is reflexive, and hence $(x, x) \in R \cup S$ by definition of union, *[as was to be shown]*. □

2.41 Exercise 41

If R and S are symmetric, is $R \cup S$ symmetric? Why?

Proof. Yes. To prove this we must show that for all x and y in A, if $(x, y) \in R \cup S$ then $(y, x) \in R \cup S$. So suppose (x, y) is a particular but arbitrarily chosen element in $R \cup S$. *[We must show that* $(y, x) \in R \cup S$*.]* By definition of union, $(x, y) \in R$ or $(x, y) \in S$. In case $(x, y) \in R$, then $(y, x) \in R$ because R is symmetric, and hence $(y, x) \in R \cup S$ by definition of union. In case $(x, y) \in S$ then $(y, x) \in S$ because S is symmetric, and hence $(y, x) \in R \cup S$ by definition of union. Thus, in both cases, $(y, x) \in R \cup S$ *[as was to be shown]*. □

2.42 Exercise 42

If R and S are transitive, is $R \cup S$ transitive? Why?

Proof. No. Let $A = \{a, b, c, d\}, R = \{(a, b), (b, c), (a, c)\}, S = \{(c, a), (a, d), (c, d)\}$. Then R and S are transitive but $R \cup S$ is not: $(a, c) \in R \cup S$ and $(c, a) \in R \cup S$ but $(a, a) \notin R \cup S$. □

In $43 - 50$, the following definitions are used: a relation on a set A is defined to be irreflexive if, and only if, for every $x \in A, x \not R x$; asymmetric if, and only if, for every $x, y \in A$ if $x \, R \, y$ then $y \not R x$; intransitive if, and only if, for every $x, y, z \in A$, if xRy and yRz then $x \not R z$. For each of the relations in the referenced exercise, determine whether the relation is irreflexive, asymmetric, intransitive, or none of these.

2.43 Exercise 43

Exercise 1

Proof. R_1 is not irreflexive because $(0, 0) \in R_1$. R_1 is not asymmetric because $(0, 1) \in R_1$ and $(1, 0) \in R_1$. R_1 is not intransitive because $(0, 1) \in R_1$ and $(1, 0) \in R_1$ and $(0, 0) \in R_1$. □

2.44 Exercise 44

Exercise 2

Proof. Recall $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\}$.

R_2 is not irreflexive because $(0, 0) \in R_2$.

R_2 is not asymmetric because $(0, 0) \in R_2$ and $(0, 0) \in R_2$.

R_2 is not intransitive because $(0, 0) \in R_2$ and $(0, 1) \in R_2$ and $(0, 1) \in R_2$. □

2.45 Exercise 45

Exercise 3

Proof. R_3 is irreflexive because no element of A is related by R_3 to itself. R_3 is not asymmetric because $(2,3) \in R_3$ and $(3,2) \in R_3$. R_3 is intransitive. To see why, observe that R_3 consists only of $(2,3)$ and $(3,2)$. Now $(2,3) \in R_3$ and $(3,2) \in R_3$ but $(2,2) \notin R_3$. Also $(3,2) \in R_3$ and $(2,3) \in R_3$ but $(3,3) \notin R_3$. $\square$

2.46 Exercise 46

Exercise 4

Proof. Recall $R_4 = \{(1,2),(2,1),(1,3),(3,1)\}$.

R_4 is irreflexive.

R_4 is not asymmetric because $(1,2) \in R_4$ and $(2,1) \in R_4$.

R_4 is intransitive. $\square$

2.47 Exercise 47

Exercise 5

Proof. Recall $R_5 = \{(0,0),(0,1),(0,2),(1,2)\}$.

R_5 is not irreflexive because $(0,0) \in R_5$.

R_5 is not asymmetric because $(0,0) \in R_5$ and $(0,0) \in R_5$.

R_5 is not intransitive because $(0,1) \in R_5$ and $(1,2) \in R_5$ and $(0,2) \in R_5$. $\square$

2.48 Exercise 48

Exercise 6

Proof. Recall $R_6 = \{(0,1),(0,2)\}$. R_6 is irreflexive because no element of A is related by R_6 to itself. R_6 is asymmetric because R_6 consists only of $(0,1)$ and $(0,2)$ and neither $(1,0)$ nor $(2,0)$ is in R_6. R_6 is intransitive. $\square$

2.49 Exercise 49

Exercise 7

Proof. Recall $R_7 = \{(0,3),(2,3)\}$.

R_7 is irreflexive, asymmetric and intransitive. $\square$

2.50 Exercise 50

Exercise 8

Proof. Recall $R_8 = \{(0,0),(1,1)\}$. R_8 is not irreflexive because $(0,0) \in R_8$. R_8 is not asymmetric because $(0,0) \in R_8$ and $(0,0) \in R_8$. R_8 is intransitive. □

In $51-53, R, S$, and T are relations defined on $A = \{0,1,2,3\}$.

2.51 Exercise 51

Let $R = \{(0,1),(0,2),(1,1),(1,3),(2,2),(3,0)\}$. Find R^t, the transitive closure of R.

Proof. $R^t = R \cup \{(0,0),(0,3),(1,0),(3,1),(3,2),(3,3),(0,2),(1,2)\} =$

$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,2),(3,0),(3,1),(3,2),(3,3)\}$. □

2.52 Exercise 52

Let $S = \{(0,0),(0,3),(1,0),(1,2),(2,0),(3,2)\}$. Find S^t, the transitive closure of S.

Proof. $S^t = S \cup \{(0,2),(1,3),(2,2),(2,3),(3,3)\}$

$= \{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,2),(2,3),(3,2),(3,3)\}$ □

2.53 Exercise 53

Let $T = \{(0,2),(1,0),(2,3),(3,1)\}$. Find T^t, the transitive closure of T.

Proof. $T^t = T \cup \{(0,3),(0,1),(0,0),(1,2),(1,3),(1,1),(2,1),(2,0),(2,2),(3,0),(3,2),$
$(3,3)\} = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3),$
$(3,0),(3,1),(3,2),(3,1)\}$ □

2.54 Exercise 54

Write a computer algorithm to test whether a relation R defined on a finite set A is reflexive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Algorithm: Test for Reflexivity
[The input for this algorithm is a binary relation R defined on a set A, that is represented as the one- dimensional array $a[1], a[2], \ldots, a[n]$. To test whether R is reflexive, a variable called answer is initially set equal to "yes," and each element $a[i]$ of A is examined in turn to see whether it is related by R to itself. If any element is not related to itself by R, then answer is set equal to "no," the while loop is not repeated, and processing terminates.]

Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of $A \times A$]*
Algorithm Body:
$i := 1$, answer := "yes"
while (answer = "yes" and $i \leq n$)
if $(a[i], a[i]) \notin R$ **then** answer := "no"
$i := i + 1$
end while
Output: answer *[a string]*

2.55 Exercise 55

Write a computer algorithm to test whether a relation R defined on a finite set A is symmetric, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Algorithm: Test for Symmetry
Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of $A \times A$]*
Algorithm Body:
$i := 1, j := 1$, answer := "yes"
while (answer = "yes" and $i \leq n$)
while (answer = "yes" and $j \leq n$)
if $(a[i], a[j]) \in R$ and $(a[j], a[i]) \notin R$ **then** answer := "no"
$j := j + 1$
end while
$i := i + 1$
end while
Output: answer *[a string]*

2.56 Exercise 56

Write a computer algorithm to test whether a relation R defined on a finite set A is transitive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Algorithm: Test for Transitivity
Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of $A \times A$]*
Algorithm Body:
$i := 1, j := 1, k := 1$, answer := "yes"

while (answer = "yes" and $i \leq n$)
  while (answer = "yes" and $j \leq n$)
    while (answer = "yes" and $k \leq n$)
      if $(a[i], a[j]) \in R$ and $(a[j], a[k]) \in R$ and $(a[i], a[k]) \notin R$
      then answer := "no"
      $k := k + 1$
    end while
    $j := j + 1$
  end while
  $i := i + 1$
end while
Output: answer *[a string]*

3 Exercise Set 8.3

3.1 Exercise 1

Suppose that $S = \{a, b, c, d, e\}$ and R is a relation on S such that $a \, R \, b$, $b \, R \, c$, and $d \, R \, e$. List all of the following:

$$c \, R \, b, c \, R \, c, a \, R \, c, b \, R \, a, a \, R \, d, e \, R \, a, e \, R \, d, c \, R \, a$$

that must be true if R is:

3.1.1 (a)

reflexive (but not symmetric or transitive)

Proof. $c \, R \, c$

□

3.1.2 (b)

symmetric (but not reflexive or transitive)

Proof. $b \, R \, a, c \, R \, b, e \, R \, d$ □

3.1.3 (c)

transitive (but not reflexive or symmetric)

Proof. $a \, R \, c$ □

3.1.4 (d)

an equivalence relation.

Proof. $c\,R\,c, b\,R\,a, c\,R\,b, e\,R\,d, a\,R\,c, c\,R\,a$ □

3.2 Exercise 2

Each of the following partitions of $\{0, 1, 2, 3, 4\}$ induces a relation R on $\{0, 1, 2, 3, 4\}$. In each case, find the ordered pairs in R.

3.2.1 (a)

$\{0, 2\}, \{1\}, \{3, 4\}$

Proof. $R = \{(0, 0), (0, 2), (2, 0), (2, 2), (1, 1), (3, 3), (3, 4), (4, 3), (4, 4)\}$ □

3.2.2 (b)

$\{0\}, \{1, 3, 4\}, \{2\}$

Proof. $R = \{(0, 0), (1, 1), (3, 3), (4, 4), (1, 3), (3, 1), (1, 4), (4, 1), (3, 4), (4, 3), (2, 2)\}$ □

3.2.3 (c)

$\{0\}, \{1, 2, 3, 4\}$

Proof. $R = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$ □

In each of $3-6$, the relation R is an equivalence relation on A. As in example 8.3.5, first find the specified equivalence classes. then state the number of distinct equivalence classes for R and list them.

3.3 Exercise 3

$A = \{0, 1, 2, 3, 4\}, R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$, equivalence classes: $[0]$, $[1]$, $[2]$, $[3]$

Proof. $[0] = \{0, 4\}, [1] = \{1, 3\}, [2] = \{2\}, [3] = \{1, 3\}$. So there are three distinct equivalence classes: $[0] = \{0, 4\} = [4], [1] = \{1, 3\} = [3], [2] = \{2\}$ □

3.4 Exercise 4

$A = \{a, b, c, d\}, R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$, classes: $[a], [b], [c], [d]$

Proof. $[a] = \{a\}, [b] = \{b, d\}, [c] = \{c\}, [d] = \{b, d\}$. So there are four distinct equivalence classes: $[a] = \{a\}, [b] = \{b, d\} = [d], [c] = \{c\}$. □

3.5 Exercise 5

$A = \{1, 2, 3, 4, \ldots, 20\}$. R is defined on A as follows: for all $x, y \in A, x\, R\, y \iff 4 \mid (x - y)$. Equivalence classes: [1], [2], [3], [4], [5]

Proof. $[1] = \{1, 5, 9, 13, 17\}, [2] = \{2, 6, 10, 14, 18\}, [3] = \{3, 7, 11, 15, 19\}, [4] = \{4, 8, 12, 16, 20\}, [5] = \{5, 9, 13, 17, 1\} = [1]$, four distinct equivalence classes: [1], [2], [3], [4] □

3.6 Exercise 6

$A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows: for all $x, y \in A, x\, R\, y \iff 3 \mid (x - y)$. Equivalence classes: [0], [1], [2], [3]

Proof. $[0] = \{-3, 0, 3\}, [1] = \{-2, 1, 4\}, [2] = \{-4, -1, 2, 5\}, [3] = \{-3, 0, 3\}$

There are 3 distinct equivalence classes: [0], [1], [2] □

In each of $7 - 14$, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

3.7 Exercise 7

$A = \{(1, 3), (2, 4), (-4, -8), (3, 9), (1, 5), (3, 6)\}$. R is defined on A as follows: For every $(a, b), (c, d) \in A, (a, b)\, R\, (c, d) \iff ad = bc$.

Proof. $\{(1, 3), (3, 9)\}, \{(2, 4), (24, 28), (3, 6)\}, \{(1, 5)\}$ □

3.8 Exercise 8

$X = \{a, b, c\}$ and $A = \mathscr{P}(X)$. R is defined on A as follows: For all sets u, v in $\mathscr{P}(X)$, $u\, R\, v \iff N(u) = N(v)$. (That is, the number of elements in u equals the number of elements in v.)

Proof. $\{\varnothing\}, \{\{a\}, \{b\}, \{c\}\}, \{\{a, b\}, \{a, c\}, \{b, c\}\}, \{\{a, b, c\}\}$ □

3.9 Exercise 9

$X = \{-1, 0, 1\}$ and $A = \mathscr{P}(X)$. R is defined on $\mathscr{P}(X)$ as follows: For all sets s and t in $\mathscr{P}(X), s\, R\, t \iff$ the sum of the elements in s equals the sum of the elements in t.

Proof. $\{\varnothing, \{0\}, \{-1, 1\}, \{-1, 0, 1\}\}, \{\{-1\}, \{-1, 0\}\}, \{\{1\}, \{0, 1\}\}$ □

3.10 Exercise 10

$A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows: For all $m, n \in \mathbb{Z}, m\, R\, n \iff 3 \mid (m^2 - n^2)$.

Proof. $\{-5, -4, -2, -1, 1, 2, 4, 5\}, \{-3, 0, 3\}$ □

3.11 Exercise 11

$A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. R is defined on A as follows: For every $(m, n) \in A, m\,R\,n \iff 4 \mid (m^2 - n^2)$.

Proof. $\{-4, -2, 0, 2, 4\}, \{-3, -1, 1, 3\}$ □

3.12 Exercise 12

$A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. R is defined on A as follows: For all $(m, n) \in A, m\,R\,n \iff 5 \mid (m^2 - n^2)$.

Proof. $\{-4, -1, 1, 4\}, \{-3, -2, 2, 3\}, \{0\}$ □

3.13 Exercise 13

A is the set of all strings of length 4 in a's and b's. R is defined on A as follows: For all strings s and t in $A, s\,R\,t \iff s$ has the same first two characters as t.

Proof. $\{aaaa, aaab, aaba, aabb\}$, $\{abaa, abab, abba, abbb\}$, $\{baaa, baab, baba, babb\}$,

$\{bbaa, bbab, bbba, bbbb\}$ □

3.14 Exercise 14

A is the set of all strings of 0's, 1's, and 2's that have length 4 and for which the sum of the characters in the string is less than or equal to 2. R is defined on A as follows: For every $s, t \in A, s\,R\,t \iff$ the sum of the characters of s equals the sum of the characters of t.

Proof. $\{0000\}, \{0001, 0010, 0100, 1000\}, \{0011, 0101, 1001, 0110, 1010, 1100, 0002, 0020,$

$0200, 2000\}$ □

3.15 Exercise 15

Determine which of the following congruence relations are true and which are false.

3.15.1 (a)

$17 \equiv 2 \pmod{5}$

Proof. True. $17 - 2 = 15 = 5 \cdot 3$ □

3.15.2 (b)

$4 \equiv -5 \pmod{7}$

Proof. False. $4 - (-5) = 9$ is not divisible by 7. □

3.15.3 (c)

$-2 \equiv -8 \pmod 3$

Proof. True. $-2-(-8) = 6 = 3 \cdot 2$ □

3.15.4 (d)

$-6 \equiv -2 \pmod 2$

Proof. True. $-6-(-2) = -4 = 2 \cdot (-2)$ □

3.16 Exercise 16

3.16.1 (a)

Let R be the relation of congruence modulo 3. Which of the following equivalence classes are equal? $[7], [-4], [-6], [17], [4], [27], [19]$

Proof. $[7] = [4] = [19], [-4] = [17], [-6] = [27]$ □

3.16.2 (b)

Let R be the relation of congruence modulo 7. Which of the following equivalence classes are equal? $[35], [3], [-7], [12], [0], [-2], [17]$

Proof. $[35] = [0] = [-7], [3] = [17], [12] = [-2]$ □

3.17 Exercise 17

3.17.1 (a)

Prove that for all integers m and n, $m \equiv n \pmod 3$ iff $m \mod 3 = n \mod 3$.

Proof. ($\implies$) Suppose m, n are integers such that $m \equiv n \pmod 3$. *[We want to show that $m \mod 3 = n \mod 3$].* By definition of congruence, $3 \mid (m-n)$, and so, by definition of divisibility, $m - n = 3a$ for some integer a. Let $r = m \mod 3$. Then $m = 3b+r$ for some integer b. Since $m-n = 3a$, it follows that $m-n = (3b+r)-n = 3a$, or, equivalently, $n = 3(b-a)+r$. Now $b-a$ is an integer and $0 \leq r < 3$. So, by definition of mod, $n \mod 3 = r$, which equals $m \mod 3$.

($\impliedby$) Suppose m, n are integers such that $m \mod 3 = n \mod 3$. *[We want to show that $m \equiv n \pmod 3$].* Let $r = m \mod 3 = n \mod 3$. Then, by definition of mod, $m = 3p + r$ and $n = 3q + r$ for some integers p and q. By substitution, $m - n = (3p + r) - (3q + r) = 3(p - q)$. Since $p - q$ is an integer, it follows that $3 \mid (m - n)$, and so, by definition of congruence, $m \equiv n \pmod 3$. □

3.17.2 (b)

Prove for all integers $d > 0$ and m, n, $m \equiv n \pmod{d}$ iff $m \mod d = n \mod d$.

Proof. Assume m, n, d are integers with $d > 0$.

$(\implies)$ 1. Assume $m \equiv n \pmod{d}$.

2. By 1 and definition of congruence, $d \mid (m - n)$.

3. By 2 and definition of divisibility, $m - n = da$ for some integer a.

4. Let $r = m \mod d$. Then by definition of mod, $m = db + r$ for some integer b and $0 \leq r < d$.

5. By 3 and 4, $m - n = (db + r) - n = da$, so $n = (db + r) - da = d(b - a) + r$ where $b - a$ is an integer and $0 \leq r < d$.

6. By 5 and definition of mod, $r = n \mod d$. Therefore $n \mod d = r = m \mod d$.

$(\impliedby)$ 1. Assume $m \mod d = n \mod d$. Let $r = m \mod d = n \mod d$.

2. By 1 and definition of mod, $m = da + r$ and $n = db + r$ for some integers a, b.

3. By 2, $m - n = da + r - (db + r) = d(a - b)$ where $a - b$ is an integer. Thus $d \mid (m - n)$.

4. By 3 and definition of congruence, $m \equiv n \pmod{d}$. □

3.18 Exercise 18

3.18.1 (a)

Give an example of two sets that are distinct but not disjoint.

Proof. One possible answer: Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $A \neq B$, so A and B are distinct. But A and B are not disjoint since $2 \in A \cap B$. □

3.18.2 (b)

Find sets A_1 and A_2 and elements x, y, and z such that x and y are in A_1 and y and z are in A_2 but x and z are not both in either of the sets A_1 or A_2.

Proof. Let $A_1 = \{x, y\}, A_2 = \{y, z\}$. □

In 19 – 31, (1) prove that the relation is an equivalence relation, and (2) describe the distinct equivalence classes of each relation.

3.19 Exercise 19

A is the set of all students at your college.

3.19.1 (a)

R is the relation defined on A as follows: For every x and y in A, $x\,R\,y \iff x$ has the same major (or double major) as y. (Assume "undeclared" is a major.)

Proof. (1) R is reflexive because it is true that for each student x at a college, x has the same major (or double major) as x.

R is symmetric because it is true that for all students x and y at a college, if x has the same major (or double major) as y, then y has the same major (or double major) as x.

R is transitive because it is true that for all students x, y, and z at a college, if x has the same major (or double major) as y and y has the same major (or double major) as z, then x has the same major (or double major) as z.

R is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one equivalence class for each major and double major at the college. Each class consists of all students with that major (or double major). □

3.19.2 (b)

S is the relation defined on A as follows: For every $x, y \in A, x\,S\,y \iff x$ is the same age as y.

Proof. (1) S is reflexive because for each student x at a college, x is the same age as x.

S is symmetric because it is true that for all students x and y at a college, if x is the same age as y, then y is the same age as x.

S is transitive because it is true that for all students x, y, and z at a college, if x is the same age as y and y is the same age as z, then x is the same age as z.

S is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one equivalence class for each age at the college. Each class consists of all students with that age. □

3.20 Exercise 20

E is the relation defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,E\,n \iff 4 \mid (m-n)$.

Proof. (1) The solution to exercise 12 in Section 8.2 proved that E is reflexive, symmetric, and transitive. Thus E is an equivalence relation.

(2) Observe that for any integer a, the equivalence class of a is

$$\begin{aligned}
[a] &= \{x \in \mathbb{Z} \mid x\,E\,a\} && \text{by definition of equivalence class} \\
&= \{x \in \mathbb{Z} \mid x - a \text{ is divisible by } 4\} && \text{by definition of } E \\
&= \{x \in \mathbb{Z} \mid x - a = 4k \text{ for some integer } k\} && \text{by definition of divisibility} \\
&= \{x \in \mathbb{Z} \mid x = 4k + a \text{ for some integer } k\} && \text{by algebra.}
\end{aligned}$$

Now when any integer a is divided by 4, the only possible remainders are 0, 1, 2, and 3 and no integer has two distinct remainders when it is divided by 4. Thus every integer is contained in exactly one of the following four equivalence classes:

$\{x \in \mathbb{Z} \,|\, x = 4k \text{ for some integer } k\}$, $\{x \in \mathbb{Z} \,|\, x = 4k+1 \text{ for some integer } k\}$,

$\{x \in \mathbb{Z} \,|\, x = 4k+2 \text{ for some integer } k\}$, $\{x \in \mathbb{Z} \,|\, x = 4k+3 \text{ for some integer } k\}$ □

3.21 Exercise 21

R is the relation defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,R\,n \iff 7m - 5n$ is even.

Proof. (1) R is reflexive because for all $m \in \mathbb{Z}, 7m - 5m = 2m$ is even, therefore $m\,R\,m$.

R is symmetric: assume $m\,R\,n$. Then $7m - 5n$ is even. So $7m - 5n = 2k$ for some integer k. Then $7n - 5m = (-5n + 12n) + (7m - 12m) = (12n - 12m) + (7m - 5n) = 2(6n - 6m) + 2k = 2(6n - 6m + k)$ where $6n - 6m + k$ is an integer. Thus $7n - 5m$ is even and $n\,R\,m$.

R is transitive: assume $m\,R\,n$ and $n\,R\,o$. Then $7m - 5n = 2k$ and $7n - 5o = 2l$ for some integers k, l. So $7m - 5o = (7m - 5n + 5n) + (-7n + 7n - 5o) = (7m - 5n) + (5n - 7n) + (7n - 5o) = 2k - 2n + 2l = 2(k - n + l)$, where $k - n + l$ is an integer. So $7m - 5o$ is even and $m\,R\,o$.

(2) Assume $m\,R\,n$. So $7m - 5n$ is even. By properties of even and odd integers, either $7m$ and $5n$ are both even, or they are both odd. Since 7 and 5 are odd, and odd times odd is odd, this means that either m and n are both even, or they are both odd. Thus R has two equivalence classes: the set of all even integers and the set of all odd integers. □

3.22 Exercise 22

Let A be the set of all statement forms in three variables p, q, and r. $\mathbf{R}$ is the relation defined on A as follows: For all P and Q in A, $P\,\mathbf{R}\,Q \iff P$ and Q have the same truth table.

Proof. (1) $\mathbf{R}$ is reflexive because for all $P \in A$, P and P have the same truth table, so $P\,\mathbf{R}\,P$.

$\mathbf{R}$ is symmetric: assume $P\,\mathbf{R}\,Q$. Then P and Q have the same truth table. Then Q and P have the same truth table, so $Q\,\mathbf{R}\,P$.

$\mathbf{R}$ is transitive: assume $P\,\mathbf{R}\,Q$ and $Q\,\mathbf{R}\,S$. Then P and Q, and Q and S have the same truth tables. So P and S have the same truth table, and $P\,\mathbf{R}\,S$.

(2) There is an equivalence class corresponding to every possible truth table in 3 variables p, q, r. There are 8 lines in every truth table, and each line has 2 options (true or false), so there are 2^8 equivalence classes. □

3.23 Exercise 23

Let P be a set of parts shipped to a company from various suppliers. S is the relation defined on P as follows: For every $x, y \in P, x\, S\, y \iff x$ has the same part number and is shipped from the same supplier as y.

Proof. (1) S is reflexive because for all $x \in P$, x has the same part number and is shipped from the same supplier as x, so $x\, S\, x$.

S is symmetric: assume $x\, S\, y$. Then x has the same part number and is shipped from the same supplier as y. So y has the same part number and is shipped from the same supplier as x. Thus $y\, S\, x$.

S is transitive: assume $x\, S\, y$ and $y\, S\, z$. So x has the same part number and is shipped from the same supplier as y and y has the same part number and is shipped from the same supplier as z, so x has the same part number and is shipped from the same supplier as z. Therefore $x\, S\, z$.

(2) For each distinct part number shipped from each distinct supplier, there is a distinct equivalence class corresponding to that part number. □

3.24 Exercise 24

Let A be the set of identifiers in a computer program. It is common for identifiers to be used for only a short part of the execution time of a program and not to be used again to execute other parts of the program. In such cases, arranging for identifiers to share memory locations makes efficient use of a computer's memory capacity. Define a relation R on A as follows: For all identifiers x and y, $x\, R\, y \iff$ the values of x and y are stored in the same memory location during execution of the program.

Proof. (1) R is reflexive because for all identifiers x, the values of x and x are stored in the same memory location during execution of the program.

R is symmetric: assume $x\, R\, y$. Then the values of x and y are stored in the same memory location during execution of the program. So the values of y and x are stored in the same memory location during execution of the program. So $y\, R\, x$.

R is transitive: assume $x\, R\, y$ and $y\, R\, z$. So the values of x and y are stored in the same memory location during execution of the program, and the values of y and z are stored in the same memory location during execution of the program. Then the values of x and z are stored in the same memory location during execution of the program. So $x\, R\, z$.

(2) There is a distinct equivalence class corresponding to each distinct memory location during execution of the program. □

3.25 Exercise 25

A is the "absolute value" relation defined on $\mathbb{R}$ as follows: For every $x, y \in \mathbb{R}, x\, A\, y \iff |x| = |y|$.

Proof. (1) A is reflexive because for all $x \in \mathbb{R}, |x| = |x|$, so $x\,A\,x$.

A is symmetric: assume $x\,A\,y$. Then $|x| = |y|$. So $|y| = |x|$, and $y\,A\,x$.

A is transitive: assume $x\,A\,y$ and $y\,A\,z$. Then $|x| = |y|$ and $|y| = |z|$, so $|x| = |z|$ and therefore $x\,A\,z$.

(2) There is a distinct equivalence class for each nonnegative real number. Each class is a set of the form $\{x, -x\}$ where x is a nonnegative real number. □

3.26 Exercise 26

D is the relation defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,D\,n \iff 3 \mid (m^2 - n^2)$.

Proof. (1) D is reflexive because for all $m \in \mathbb{Z}, m^2 - m^2 = 0 = 3 \cdot 0$, so $3 \mid (m^2 - m^2)$ and thus $m\,D\,m$.

D is symmetric: assume $m\,D\,n$. Then $3 \mid (m^2 - n^2)$. Then $m^2 - n^2 = 3r$ for some integer r. Then $n^2 - m^2 = 3 \cdot (-r)$ where $-r$ is an integer. So $3 \mid (n^2 - m^2)$ and thus $n\,D\,m$.

D is transitive: assume $m\,D\,n$ and $n\,D\,o$. So $3 \mid (m^2 - n^2)$ and $3 \mid (n^2 - o^2)$. So $m^2 - n^2 = 3r$ and $n^2 - o^2 = 3s$ for some integers r, s. Then $m^2 - o^2 = (m^2 - n^2) + (n^2 - o^2) = 3(r + s)$ where $r + s$ is an integer. Therefore $3 \mid (m^2 - n^2)$ and $m\,D\,o$.

(2) There are two distinct equivalence classes: $[0] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$ and $[1] = \{\ldots, -5, -4, -2, -1, 1, 2, 4, 5, \ldots\}$. □

3.27 Exercise 27

R is the relation defined on $\mathbb{Z}$ as follows: For every $(m, n) \in \mathbb{Z}, m\,R\,n \iff 4 \mid (m^2 - n^2)$.

Proof. (1) R is an equivalence relation, the proofs are exactly the same as above in Exercise 26 (replace 3 with 4).

(2) There are 2 distinct equivalence classes: $[0]$ = the set of all even integers, $[1]$ = the set of all odd integers. This is because, if we want $4 \mid (m - n)(m + n)$, it is sufficient that both $m - n$ and $m + n$ are even; this is the case when either both m, n are odd or both m, n are even. □

3.28 Exercise 28

I is the relation defined on $\mathbb{R}$ as follows: For every $x, y \in \mathbb{R}, m\,I\,n \iff x - y$ is an integer.

Proof. (1) I is reflexive because the difference between each real number and itself is 0, which is an integer.

I is symmetric because for all real numbers x and y, if $x - y$ is an integer, then $y - x = (-1)(x - y)$, which is also an integer.

I is transitive because for all real numbers x, y, and z, if $x - y$ is an integer and $y - z$ is an integer, then $x - z = (x - y) + (y - z)$ is the sum of two integers and thus is an integer.

I is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one class for each real number x with $0 \le x < 1$. The distinct classes are all sets of the form $\{y \in \mathbb{R} \,|\, y = n + x, \text{ for some integer } n\}$, where x is a real number such that $0 \le x < 1$. $\square$

3.29 Exercise 29

Define P on the set $\mathbb{R} \times \mathbb{R}$ of ordered pairs of real numbers as follows: For every $(w, x), (y, z) \in \mathbb{R} \times \mathbb{R}, (w, x)\, P\, (y, z) \iff w = y$.

Proof. (1) P is reflexive because each ordered pair of real numbers has the same first element as itself.

P is symmetric for the following reason: Suppose (w, x) and (y, z) are ordered pairs of real numbers such that $(w, x)\, P\, (y, z)$. Then, by definition of P, $w = y$. Now by the symmetric property of equality, this implies that $y = w$, and so, by definition of P, $(y, z)\, P\, (w, x)$.

P is transitive for the following reason: Suppose $(u, v), (w, x)$, and (y, z) are ordered pairs of real numbers such that $(u, v)\, P\, (w, x)$ and $(w, x)\, P\, (y, z)$. Then, by definition of P, $u = w$ and $w = y$. It follows from the transitive property of equality that $u = y$. Hence, by definition of P, $(u, v)\, P\, (y, z)$.

P is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one equivalence class for each real number. The distinct equivalence classes are all sets of ordered pairs $\{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x = a\}$, for each real number a. Equivalently, the equivalence classes consist of all vertical lines in the Cartesian plane. $\square$

3.30 Exercise 30

Define Q on the set $\mathbb{R} \times \mathbb{R}$ as follows: For every $(w, x), (y, z) \in \mathbb{R} \times \mathbb{R}, (w, x)\, Q\, (y, z) \iff x = z$.

Proof. (1) Q is reflexive because for all $(w, x) \in \mathbb{R} \times \mathbb{R}, x = x$ so $(w, x)\, Q\, (w, x)$.

Q is symmetric: assume $(w, x)\, Q\, (y, z)$. Then $x = z$. So $z = x$ and thus $(y, z)\, Q\, (w, x)$.

Q is transitive: assume $(w, x)\, Q\, (y, z)$ and $(y, z)\, Q\, (s, t)$. Then $x = z$ and $z = t$. So $x = t$ and thus $(w, x)\, Q\, (s, t)$.

(2) Q has a distinct equivalence class, for each real number a, of the form $\{(w, x) \in \mathbb{R} \times \mathbb{R} \,|\, x = a\}$. $\square$

3.31 Exercise 31

Let P be the set of all points in the Cartesian plane except the origin. R is the relation defined on P as follows: For every p_1 and p_2 in $P, p_1 \, R \, p_2 \iff p_1$ and p_2 lie on the same half-line emanating from the origin.

Proof. (1) R is reflexive: for every $p \in P$, p and p lie on the same half-line emanating from the origin (namely the half-line that connects p to the origin). Thus $p \, R \, p$.

R is symmetric: assume $p \, R \, q$. Then p and q lie on the same half-line l emanating from the origin. Then q and p lie on the same half-line l emanating from the origin. Thus $q \, R \, p$.

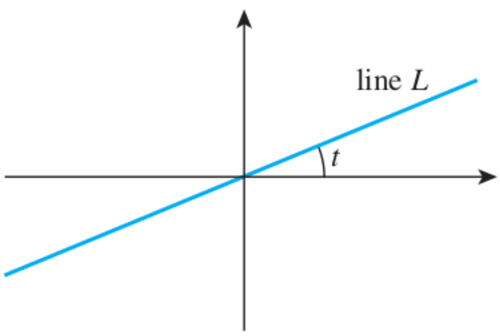
R is transitive: assume $p \, R \, q$ and $q \, R \, r$. Then p and q lie on the same half-line l_1 emanating from the origin and q and r lie on the same half-line l_2 emanating from the origin. Then it must be that $l_1 = l_2$ since q lies on both half-lines. Thus p and r lie on the same half-line emanating from the origin, and $p \, R \, r$.

(2) Each equivalence class is a half-line l emanating from the origin, containing all the points p that lie on l. □

3.32 Exercise 32

Let A be the set of all straight lines in the Cartesian plane. Define a relation $\parallel$ on A as follows: For every l_1 and l_2 in A, $l_1 \parallel l_2 \iff l_1$ is parallel to l_2. Then $\parallel$ is an equivalence relation on A. Describe the equivalence classes of this relation.

Proof. There is one equivalence class for each real number t such that $0 \leq t < \pi$. One line in each class goes through the origin, and that line makes an angle of t with the positive horizontal axis.



Alternatively, there is one equivalence class for every possible slope: all real numbers plus "undefined." □

3.33 Exercise 33

Let A be the set of points in the rectangle with x and y coordinates between 0 and 1. That is,

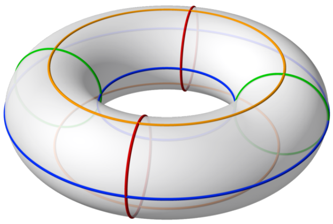
$$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}.$$

Define a relation R on A as follows: For all (x_1, y_1) and (x_2, y_2) in A,

$$
\begin{array}{rcccccccc}
(x_1, y_1)\, R\, (x_2, y_2) & \iff & (x_1, y_1) & = & (x_2, y_2); & \text{or} & & & \\
& & x_1 = 0 & \text{and} & x_2 = 1 & \text{and} & y_1 = y_2; & \text{or} \\
& & x_1 = 1 & \text{and} & x_2 = 0 & \text{and} & y_1 = y_2; & \text{or} \\
& & y_1 = 0 & \text{and} & y_2 = 1 & \text{and} & x_1 = x_2; & \text{or} \\
& & y_1 = 1 & \text{and} & y_2 = 0 & \text{and} & x_1 = x_2. &
\end{array}
$$

In other words, all points along the top edge of the rectangle are related to the points along the bottom edge directly beneath them, and all points directly opposite each other along the left and right edges are related to each other. The points in the interior of the rectangle are not related to anything other than themselves. Then R is an equivalence relation on A. Imagine gluing together all the points that are in the same equivalence class. Describe the resulting figure.

Proof. Gluing the top and bottom edges of the rectangle results in a horizontal cylinder. Then, if we also glue the left and right circular ends of this cylinder, we get a doughnut shaped figure:



□

3.34 Exercise 34

The documentation for the computer language Java recommends that when an "equals method" is defined for an object, it be an equivalence relation. That is, if R is defined as follows: $x\,R\,y \iff x.equals(y)$ for all objects in the class, then R should be an equivalence relation. Suppose that in trying to optimize some of the mathematics of a graphics application, a programmer creates an object called a point, consisting of two coordinates in the plane. The programmer defines an equals method as follows: If p and q are any points, then $p.equals(q)$ iff the distance from p to q is less than or equal to c where c is a small positive number that depends on the resolution of the computer display. Is the programmer's equals method an equivalence relation? Justify your answer.

Proof. No. If points p, q, and r all lie on a straight line with q in the middle, and if p is c units from q and q is c units from r, then p is more than c units from r. □

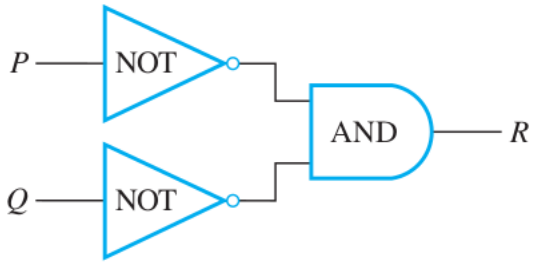
3.35 Exercise 35

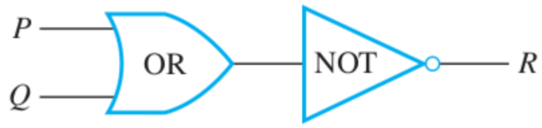
Find an additional representative circuit for the input/output table of Example 8.3.9.

Proof. Recall that the output table is:

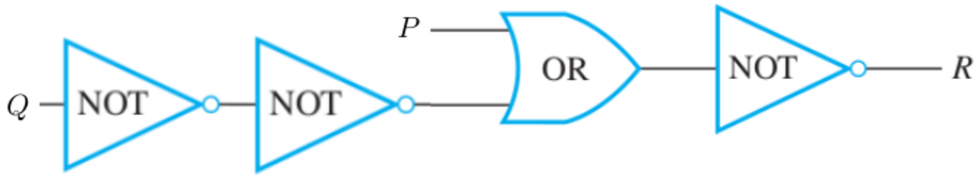
Input		Output
P	Q	R
1	1	0
1	0	0
0	1	0
0	0	1

And two representative circuits for this table were given:





Here is another:



□

Let R be an equivalence relation on a set A. Prove each of the statements in $36 - 41$ directly from the definitions of equivalence relation and equivalence class without using the results of Lemma 8.3.2, Lemma 8.3.3, or Theorem 8.3.4.

3.36 Exercise 36

For every a in A, $a \in [a]$.

Proof. Suppose R is an equivalence relation on a set A and $a \in A$. Because R is an equivalence relation, R is reflexive, and because R is reflexive, each element of A is related to itself by R. In particular $a\,R\,a$. Hence, by definition of equivalence class, $a \in [a]$. □

3.37 Exercise 37

For every a and b in A, if $b \in [a]$ then $a\,R\,b$.

Proof. Yes, by definition of $[a], b \in [a] \iff b\,R\,a$. So $b\,R\,a$. By symmetry, $a\,R\,b$. □

3.38 Exercise 38

For every a, b, and c in A, if $b\,R\,c$ and $c \in [a]$ then $b \in [a]$.

Proof. Suppose R is an equivalence relation on a set A and a, b, and c are elements of A with $b\,R\,c$ and $c \in [a]$. Since $c \in [a]$, then $c\,R\,a$ by definition of equivalence class. Now R is transitive because R is an equivalence relation. Thus, since $b\,R\,c$ and $c\,R\,a$, then $b\,R\,a$. It follows that $b \in [a]$ by definition of equivalence class. $\square$

3.39 Exercise 39

For every a and b in A, if $[a] = [b]$ then $a\,R\,b$.

Proof. Assume $[a] = [b]$. By Exercise 36 $b \in [b]$, and since $[a] = [b]$ we have $b \in [a]$. So by Exercise 37 $a\,R\,b$. $\square$

3.40 Exercise 40

For every a, b, and x in A, if $a\,R\,b$ and $x \in [a]$ then $x \in [b]$.

Proof. Suppose a, b, and x are in $A, a\,R\,b$, and $x \in [a]$. By definition of equivalence class, $x\,R\,a$. So $x\,R\,a$ and $a\,R\,b$, and thus, by transitivity, $x\,R\,b$. Hence $x \in [b]$. $\square$

3.41 Exercise 41

For every a and b in A, if $a \in [b]$ then $[a] = [b]$.

Proof. Assume $x \in [a]$. Then $x\,R\,a$ by definition of $[a]$. Since $a \in [b]$, by Exercise 37 $b\,R\,a$. By symmetry, $a\,R\,b$. Then by transitivity $x\,R\,b$. So $x \in [b]$ by definition of $[b]$. Thus $[a] \subseteq [b]$.

Assume $x \in [b]$. Then $x\,R\,b$ by definition of $[b]$. Since $a \in [b]$, by Exercise 37 $b\,R\,a$. Then by transitivity $x\,R\,a$. So $x \in [a]$ by definition of $[a]$. Thus $[b] \subseteq [a]$.

So by definition of set equality $[a] = [b]$. $\square$

3.42 Exercise 42

Let R be the relation defined in Example 8.3.12: $(a, b)\,R\,(c, d) \iff ad = bc$.

3.42.1 (a)

Prove that R is reflexive.

Proof. For all $a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\}, ab = ba$, therefore $(a, b)\,R\,(a, b)$. So R is reflexive. $\square$

3.42.2 (b)

Prove that R is symmetric.

Proof. Assume $(a, b)\, R\, (c, d)$. Then $ad = bc$. So $bc = ad$ and thus $(c, d)\, R\, (a, b)$. So R is symmetric. $\square$

3.42.3 (c)

List four distinct elements in [(1, 3)].

Proof. One possible answer: $(2, 6), (-2, -6), (3, 9), (-3, -9)$. $\square$

3.42.4 (d)

List four distinct elements in [(2, 5)].

Proof. One possible answer: $(4, 10), (6, 15), (8, 20), (10, 25)$. $\square$

3.43 Exercise 43

In Example 8.3.12, define operations of addition (+) and multiplication ($\cdot$) as follows: For every $(a, b), (c, d) \in A$, $[(a, b)]+[(c, d)] = [(ad+bc, bd)]$ and $[(a, b)]\cdot[(c, d)] = [(ac, bd)]$.

3.43.1 (a)

Prove that this addition is well defined. That is, show that if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(ad + bc, bd)] = [(a'd' + b'c', b'd')]$.

Proof. Suppose that $(a, b), (a', b'), (c, d)$, and (c', d') are any elements of A such that $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$. By definition of R, $ab' = ba'$ (*) and $cd' = dc'$ (**). We must show that $[(a, b)] + [(c, d)] = [(a', b')] + [(c', d')]$. By definition of the addition on A, this equation is true if, and only if, $[(ad + bc, bd)] = [(a'd' + b'c', b'd')]$. And, by definition of the relation, this equation is true if, and only if, $(ad + bc)b'd' = bd(a'd' + b'c')$. After multiplying out, this becomes $adb'd' + bcb'd' = bda'd' + bdb'c'$, and regrouping, turns it into $(ab')(dd') + (cd')(bb') = (ba')(dd') + (dc')(bb')$. Substituting the values from (*) and (**) shows that this last equation is true. $\square$

3.43.2 (b)

Prove that this multiplication is well defined. That is, show that if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(ac, bd)] = [(a'c', b'd')]$.

Proof. 1. Assume $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$.

2. By 1 and the definition of equivalence classes of R, $(a, b)\, R\, (a', b')$ and $(c, d)\, R\, (c', d')$.

3. By 2 and definition of R, $ab' = ba'$ and $cd' = dc'$.

4. By 3, multiplying the left hand sides together and the right hand sides together, $ab'cd' = ba'dc'$.

5. By 4 and commutativity, reorganizing, $(ac)(b'd') = (bd)(a'c')$.

6. By 5 and definition of R, $(ad, bc)\, R\, (a'c', b'd')$.

7. By 6 and definition of equivalence classes of R, $[(ad, bc)] = [(a'c', b'd')]$. $\square$

3.43.3 (c)

Show that $[(0, 1)]$ is an identity element for addition. That is, show that for any $(a, b) \in A$, $[(a, b)] + [(0, 1)] = [(0, 1)] + [(a, b)] = [(a, b)]$.

Proof. Suppose that (a, b) is any element of A. We must show that $[(a, b)] + [(0, 1)] = [(a, b)]$. By definition of the addition on A, this equation is true if, and only if, $[(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)]$. And this last equation is true because $a \cdot 1 + b \cdot 0 = a$ and $b \cdot 1 = b$. $\square$

3.43.4 (d)

Find an identity element for multiplication. That is, find (i, j) in A so that for every (a, b) in A, $[(a, b)] \cdot [(i, j)] = [(i, j)] \cdot [(a, b)] = [(a, b)]$.

Proof. The multiplicative identity is $(1, 1)$. Indeed, by definition of multiplication on A, $[(a, b)] \cdot [(1, 1)] = [(a \cdot 1, b \cdot 1)] = [(a, b)]$ and $[(1, 1)] \cdot [(a, b)] = [(1 \cdot a, 1 \cdot b)] = [(a, b)]$. $\square$

3.43.5 (e)

For any $(a, b) \in A$, show that $[(-a, b)]$ is an inverse for $[(a, b)]$ for addition. That is, show that $[(-a, b)] + [(a, b)] = [(a, b)] + [(-a, b)] = [(0, 1)]$.

Proof. Suppose that (a, b) is any element of A. We must show that $[(a, b)] + [(-a, b)] = [(-a, b)] + [(a, b)] = [(0, 1)]$. By definition of the addition on A, this equation is true if, and only if, $[(ab + b(-a), bb)] = [(0, 1)]$, or, equivalently, $[(0, bb)] = [(0, 1)]$. By definition of the relation, this last equation is true if, and only if, $0 \cdot 1 = bb \cdot 0$, which is true. $\square$

3.43.6 (f)

Given any $(a, b) \in A$ with $a \neq 0$, find an inverse for $[(a, b)]$ for multiplication. That is, find (c, d) in A so that $[(a, b)] \cdot [(c, d)] = [(c, d)] \cdot [(a, b)] = [(i, j)]$, where $[(i, j)]$ is the identity element you found in part (d).

Proof. Given $[(a, b)]$ we want to find $[(c, d)]$ such that $[(a, b)] \cdot [(c, d)] = [(ac, bd)] = [(1, 1)]$.

So by the definition of equivalence classes of R on A, (ac, bd) is related to $(1, 1)$ by R, in other words $ac \cdot 1 = bd \cdot 1$, or $ac = bd$. Then let $(c, d) = (b, a)$. So $ac = ab = ba = bd$, therefore $(ac, bd)\, R\, (1, 1)$ and thus $[(ac, bd)] = [(1, 1)]$, in other words $[(a, b)] \cdot [(c, d)] = [(1, 1)]$.

Similarly we can prove that $[(c, d)] \cdot [(a, b)] = [(1, 1)]$. $\square$

3.44 Exercise 44

Let $A = Z^+ \times Z^+$. Define a relation R on A as follows: For every (a, b) and (c, d) in $A, (a, b)\, R\, (c, d) \iff a + d = c + b$.

3.44.1 (a)

Prove that R is reflexive.

Proof. Let (a, b) be any element of $Z^+ \times Z^+$. We must show that $(a, b)\, R\, (a, b)$. By definition of R, this relationship holds if, and only if, $a + b = b + a$. But this equation is true by the commutative law of addition for real numbers. Hence R is reflexive. □

3.44.2 (b)

Prove that R is symmetric.

Proof. Assume $(a, b)\, R\, (c, d)$. Then by definition of R, $a + d = c + b$. So $c + b = a + d$. Thus $(c, d)\, R\, (a, b)$ by definition of R. So R is symmetric. □

3.44.3 (c)

Prove that R is transitive.

Proof. Assume $(a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f)$. By definition of R, $a + d = c + b$ and $c + f = e + d$. Adding the two equations we get $a + d + c + f = c + b + e + d$. Canceling $c + d$ on both sides we get $a + f = e + b$, thus $(a, b)\, R\, (e, f)$ so R is transitive. □

3.44.4 (d)

List five elements in [(1, 1)].

Proof. One possible answer: (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) □

3.44.5 (e)

List five elements in [(3, 1)].

Proof. One possible answer: (4, 2), (5, 3), (6, 4), (7, 5), (8, 6) □

3.44.6 (f)

List five elements in [(1, 2)].

Proof. One possible answer: (1, 2), (2, 3), (3, 4), (4, 5), (5, 6) □

3.44.7 (g)

Describe the distinct equivalence classes of R.

Proof. Observe that for any positive integers a and b, the equivalence class of (a, b) consists of all ordered pairs in $Z^+ \times Z^+$ for which the difference between the first and second coordinates equals $a - b$. Thus there is one equivalence class for each integer: positive, negative, and zero. Each positive integer n corresponds to the class of $(n+1, 1)$; each negative integer $-n$ corresponds to the class of $(1, n+1)$; and zero corresponds to the class $(1, 1)$. □

3.45 Exercise 45

The following argument claims to prove that the requirement that an equivalence relation be reflexive is redundant. In other words, it claims to show that if a relation is symmetric and transitive, then it is reflexive. Find the mistake in the argument.

"Proof: Let R be a relation on a set A and suppose R is symmetric and transitive. For any two elements x and y in A, if $x\,R\,y$ then $y\,R\,x$ since R is symmetric. Thus it follows by transitivity that $x\,R\,x$, and hence R is reflexive."

Proof. The conclusion $x\,R\,x$ only follows under the assumption that $x\,R\,y$, which has not been discharged from the proof. (See next exercise.) □

3.46 Exercise 46

Let R be a relation on a set A and suppose R is symmetric and transitive. Prove the following: If for every x in A there is a y in A such that $x\,R\,y$, then R is an equivalence relation.

Proof. Let R be a relation on a set A and suppose R is symmetric and transitive. Assume x is any element in A. By the assumption, there exists y in A such that $x\,R\,y$. Then $y\,R\,x$ since R is symmetric. Thus it follows by transitivity that $x\,R\,x$, and hence R is reflexive. Hence R is an equivalence relation. □

3.47 Exercise 47

Refer to the quote at the beginning of this section to answer the following questions.

3.47.1 (a)

What is the name of the Knight's song called?

Proof. ... The name of the song is called 'Haddocks' Eyes.' □

3.47.2 (b)

What is the name of the Knight's song?

Proof. The name really is 'The Aged Aged Man.' □

3.47.3 (c)

What is the Knight's song called?

Proof. "Ways and Means" □

3.47.4 (d)

What is the Knight's song?

Proof. The song really is 'A-sitting on a Gate' □

3.47.5 (e)

What is your (full, legal) name?

Proof. Spam, Egg □

3.47.6 (f)

What are you called?

Proof. Spam □

3.47.7 (g)

What are you? (Do not answer this on paper; just think about it.)

Proof. *???* □

4 Exercise Set 8.4

4.1 Exercise 1

4.1.1 (a)

Proof. □

4.1.2 (b)

Proof. □

4.2 Exercise 2

4.2.1 (a)

Proof. □

4.2.2 (b)

Proof. □

4.3 Exercise 3

4.3.1 (a)

Proof. □

4.3.2 (b)

Proof. □

4.3.3 (c)

Proof. □

4.3.4 (d)

Proof. □

4.3.5 (e)

Proof. □

4.4 Exercise 4

4.4.1 (a)

Proof. □

4.4.2 (b)

Proof. □

4.4.3 (c)

Proof. □

4.4.4 (d)

Proof. □

4.4.5 (e)

Proof. □

4.5 Exercise 5

Proof. □

4.6 Exercise 6

Proof. □

4.7 Exercise 7

4.7.1 (a)

Proof. □

4.7.2 (b)

Proof. □

4.7.3 (c)

Proof. □

4.7.4 (d)

Proof. □

4.7.5 (e)

Proof. □

4.8 Exercise 8

4.8.1 (a)

Proof. □

4.8.2 (b)

Proof. □

4.8.3 (c)

Proof. □

4.8.4 (d)

Proof. □

4.8.5 (e)

Proof. □

4.9 Exercise 9

4.9.1 (a)

Proof. □

4.9.2 (b)

Proof. □

4.10 Exercise 10

Proof. □

4.11 Exercise 11

Proof. □

4.12 Exercise 12

4.12.1 (a)

Proof. □

4.12.2 (b)

Proof. □

4.13 Exercise 13

4.13.1 (a)

Proof. □

4.13.2 (b)

Proof. □

4.14 Exercise 14

Proof. □

4.15 Exercise 15

Proof. □

4.16 Exercise 16

Proof. □

4.17 Exercise 17

Proof. □

4.18 Exercise 18

Proof. □

4.19 Exercise 19

Proof. □

4.20 Exercise 20

Proof. □

4.21 Exercise 21

Proof. □

4.22 Exercise 22

Proof. □

4.23 Exercise 23

Proof. □

4.24 Exercise 24

Proof. □

4.25 Exercise 25

Proof. □

4.26 Exercise 26

Proof. □

4.27 Exercise 27

Proof. □

4.28 Exercise 28

Proof. □

4.29 Exercise 29

Proof. □

4.30 Exercise 30

Proof. □

4.31 Exercise 31

4.31.1 (a)

Proof. □

4.31.2 (b)

Proof. □

4.31.3 (c)

Proof. □

4.32 Exercise 32

4.32.1 (a)

Proof. □

4.32.2 (b)

Proof. □

4.33 Exercise 33

Proof. □

4.34 Exercise 34

Proof. □

4.35 Exercise 35

Proof. ☐

4.36 Exercise 36

Proof. ☐

4.37 Exercise 37

Proof. ☐

4.38 Exercise 38

Proof. ☐

4.39 Exercise 39

Proof. ☐

4.40 Exercise 40

Proof. ☐

4.41 Exercise 41

4.41.1 (a)

Proof. ☐

4.41.2 (b)

Proof. ☐

4.42 Exercise 42

Proof. ☐

4.43 Exercise 43

Proof. ☐

5 Exercise Set 8.5

5.1 Exercise 1

5.1.1 (a)

Proof. □

5.1.2 (b)

Proof. □

5.1.3 (c)

Proof. □

5.1.4 (d)

Proof. □

5.2 Exercise 2

Proof. □

5.3 Exercise 3

Proof. □

5.4 Exercise 4

Proof. □

5.5 Exercise 5

Proof. □

5.6 Exercise 6

Proof. □

5.7 Exercise 7

Proof. □

5.8 Exercise 8

Proof. □

5.9 Exercise 9

Proof. □

5.10 Exercise 10

Proof. □

5.11 Exercise 11

5.11.1 (a)

Proof. □

5.11.2 (b)

Proof. □

5.11.3 (c)

Proof. □

5.11.4 (d)

Proof. □

5.11.5 (e)

Proof. □

5.11.6 (f)

Proof. □

5.11.7 (g)

Proof. □

5.12 Exercise 12

Proof. □

5.13 Exercise 13

Proof. □

5.14 Exercise 14

5.14.1 (a)

Proof. □

5.14.2 (b)

Proof. □

5.15 Exercise 15

Proof. □

5.16 Exercise 16

5.16.1 (a)

Proof. □

5.16.2 (b)

Proof. □

5.17 Exercise 17

Proof. □

5.18 Exercise 18

Proof. □

5.19 Exercise 19

Proof. □

5.20 Exercise 20

Proof. □

5.21 Exercise 21

5.21.1 (a)

Proof. □

5.21.2 (b)

Proof. □

5.22 Exercise 22

Proof. □

5.23 Exercise 23

Proof. □

5.24 Exercise 24

Proof. □

5.25 Exercise 25

Proof. □

5.26 Exercise 26

Proof. □

5.27 Exercise 27

Proof. □

5.28 Exercise 28

Proof. □

5.29 Exercise 29

Proof. □

5.30 Exercise 30

5.30.1 (a)

Proof. □

5.30.2 (b)

Proof. □

5.30.3 (c)

Proof. □

5.30.4 (d)

Proof. □

5.31 Exercise 31

Proof. □

5.32 Exercise 32

Proof. □

5.33 Exercise 33

Proof. □

5.34 Exercise 34

Proof. □

5.35 Exercise 35

Proof. □

5.36 Exercise 36

Proof. □

5.37 Exercise 37

Proof. □

5.38 Exercise 38

Proof. □

5.39 Exercise 39

Proof. □

5.40 Exercise 40

5.40.1 (a)

Proof. □

5.40.2 (b)

Proof. □

5.41 Exercise 41

5.41.1 (a)

Proof. □

5.41.2 (b)

Proof. □

5.42 Exercise 42

Proof. □

5.43 Exercise 43

Proof. □

5.44 Exercise 44

Proof. □

5.45 Exercise 45

Proof. □

5.46 Exercise 46

Proof. □

5.47 Exercise 47

Proof. □

5.48 Exercise 48

Proof. □

5.49 Exercise 49

5.49.1 (a)

Proof. □

5.49.2 (b)

Proof. □

5.50 Exercise 50

5.50.1 (a)

Proof. □

5.50.2 (b)

Proof. □

5.51 Exercise 51

5.51.1 (a)

Proof. □

5.51.2 (b)

Proof. □