

Chapter 11 Solutions, Susanna Epp Discrete Math 5th Edition

<https://github.com/spamegg1>

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Contents

1	Exercise Set 11.1	10
1.1	Exercise 1	10
1.1.1	(a)	10
1.1.2	(b)	10
1.1.3	(c)	10
1.1.4	(d)	10
1.1.5	(e)	11
1.1.6	(f)	11
1.2	Exercise 2	11
1.2.1	(a)	11
1.2.2	(b)	11
1.2.3	(c)	11
1.2.4	(d)	11
1.2.5	(e)	12
1.2.6	(f)	12
1.3	Exercise 3	12
1.4	Exercise 4	12
1.5	Exercise 5	12
1.6	Exercise 6	13
1.7	Exercise 7	13
1.8	Exercise 8	14
1.9	Exercise 9	14
1.10	Exercise 10	15
1.11	Exercise 11	15
1.12	Exercise 12	15
1.13	Exercise 13	16
1.14	Exercise 14	16
1.15	Exercise 15	16
1.16	Exercise 16	17
1.17	Exercise 17	17

1.17.1	(a)	17
1.17.2	(b)	17
1.18	Exercise 18	18
1.18.1	(a)	18
1.18.2	(b)	18
1.19	Exercise 19	18
1.20	Exercise 20	18
1.21	Exercise 21	19
1.21.1	(a)	19
1.21.2	(b)	19
1.22	Exercise 22	20
1.23	Exercise 23	20
1.24	Exercise 24	20
1.25	Exercise 25	21
1.26	Exercise 26	21
1.27	Exercise 27	22
1.28	Exercise 28	22
2	Exercise Set 11.2	22
2.1	Exercise 1	22
2.1.1	(a)	23
2.1.2	(b)	23
2.2	Exercise 2	23
2.2.1	(a)	23
2.2.2	(b)	23
2.3	Exercise 3	23
2.3.1	(a)	24
2.3.2	(b)	24
2.4	Exercise 4	24
2.5	Exercise 5	24
2.6	Exercise 6	24
2.7	Exercise 7	24
2.8	Exercise 8	25
2.9	Exercise 9	25
2.10	Exercise 10	25
2.10.1	(a)	25
2.10.2	(b)	25
2.10.3	(c)	25
2.10.4	(d)	26
2.10.5	(e)	26
2.11	Exercise 11	26
2.11.1	(a)	26
2.11.2	(b)	26
2.11.3	(c)	26
2.11.4	(d)	27
2.11.5	(e)	27

2.12	Exercise 12	27
2.12.1	(a)	27
2.12.2	(b)	27
2.12.3	(c)	28
2.12.4	(d)	28
2.12.5	(e)	28
2.13	Exercise 13	28
2.14	Exercise 14	29
2.15	Exercise 15	29
2.16	Exercise 16	29
2.17	Exercise 17	30
2.18	Exercise 18	30
2.19	Exercise 19	30
2.20	Exercise 20	31
2.21	Exercise 21	31
2.22	Exercise 22	32
2.22.1	(a)	32
2.22.2	(b)	32
2.22.3	(c)	32
2.23	Exercise 23	33
2.23.1	(a)	33
2.23.2	(b)	33
2.23.3	(c)	33
2.24	Exercise 24	33
2.24.1	(a)	33
2.24.2	(b)	34
2.24.3	(c)	34
2.25	Exercise 25	34
2.25.1	(a)	35
2.25.2	(b)	35
2.25.3	(c)	36
2.26	Exercise 26	36
2.27	Exercise 27	36
2.28	Exercise 28	36
2.29	Exercise 29	36
2.30	Exercise 30	36
2.31	Exercise 31	37
2.32	Exercise 32	37
2.33	Exercise 33	37
2.34	Exercise 34	37
2.35	Exercise 35	37
2.36	Exercise 36	38
2.37	Exercise 37	38
2.38	Exercise 38	38
2.39	Exercise 39	38
2.40	Exercise 40	39

2.40.1	(a)	39
2.40.2	(b)	39
2.41	Exercise 41	39
2.42	Exercise 42	39
2.43	Exercise 43	39
2.44	Exercise 44	40
2.45	Exercise 45	40
2.46	Exercise 46	40
2.46.1	(a)	40
2.46.2	(b)	40
2.47	Exercise 47	41
2.47.1	(a)	41
2.47.2	(b)	41
2.47.3	(c)	41
2.47.4	(d)	41
2.48	Exercise 48	42
2.49	Exercise 49	42
2.50	Exercise 50	42
2.50.1	(a)	42
2.50.2	(b)	43
2.50.3	(c)	43
2.51	Exercise 51	43
2.51.1	(a)	43
2.51.2	(b)	44
2.51.3	(c)	44
3	Exercise Set 11.3	44
3.1	Exercise 1	44
3.1.1	(a)	44
3.1.2	(b)	45
3.1.3	(c)	45
3.1.4	(d)	45
3.1.5	(e)	45
3.1.6	(f)	45
3.2	Exercise 2	45
3.2.1	(a)	45
3.2.2	(b)	46
3.2.3	(c)	46
3.3	Exercise 3	46
3.3.1	(a)	46
3.3.2	(b)	46
3.3.3	(c)	46
3.4	Exercise 4	47
3.4.1	(a)	47
3.4.2	(b)	47
3.4.3	(c)	47

3.5	Exercise 5	47
3.5.1	(a)	47
3.5.2	(b)	48
3.5.3	(c)	48
3.6	Exercise 6	48
3.6.1	(a)	48
3.6.2	(b)	48
3.7	Exercise 7	49
3.7.1	(a)	49
3.7.2	(b)	49
3.8	Exercise 8	49
3.8.1	(a)	49
3.8.2	(b)	49
3.9	Exercise 9	49
3.9.1	(a)	50
3.9.2	(b)	50
3.10	Exercise 10	50
3.10.1	(a)	50
3.10.2	(b)	50
3.11	Exercise 11	50
3.11.1	(a)	50
3.11.2	(b)	51
3.12	Exercise 12	51
3.12.1	(a)	51
3.12.2	(b)	51
3.13	Exercise 13	51
3.13.1	(a)	52
3.13.2	(b)	52
3.14	Exercise 14	52
3.14.1	(a)	52
3.14.2	(b)	53
3.15	Exercise 15	53
3.15.1	(a)	53
3.15.2	(b)	53
3.16	Exercise 16	53
3.16.1	(a)	53
3.16.2	(b)	54
3.17	Exercise 17	54
3.17.1	(a)	54
3.17.2	(b)	55
3.18	Exercise 18	55
3.18.1	(a)	55
3.18.2	(b)	56
3.19	Exercise 19	56
3.19.1	(a)	56
3.19.2	(b)	56

3.20	Exercise 20	56
3.21	Exercise 21	57
3.22	Exercise 22	57
3.23	Exercise 23	57
3.24	Exercise 24	58
3.25	Exercise 25	58
3.26	Exercise 26	58
3.27	Exercise 27	59
3.27.1	(a)	59
3.27.2	(b)	59
3.28	Exercise 28	60
3.29	Exercise 29	60
3.30	Exercise 30	61
3.31	Exercise 31	61
3.32	Exercise 32	62
3.33	Exercise 33	62
3.34	Exercise 34	62
3.35	Exercise 35	62
3.35.1	(a)	63
3.35.2	(b)	63
3.35.3	(c)	63
3.35.4	(d)	63
3.36	Exercise 36	64
3.37	Exercise 37	64
3.38	Exercise 38	65
3.39	Exercise 39	65
3.40	Exercise 40	66
3.41	Exercise 41	66
3.42	Exercise 42	67
3.43	Exercise 43	67
4	Exercise Set 11.4	67
4.1	Exercise 1	67
4.1.1	()	67
4.2	Exercise 2	67
4.2.1	()	67
4.3	Exercise 3	67
4.3.1	()	67
4.4	Exercise 4	67
4.4.1	()	67
4.5	Exercise 5	68
4.5.1	()	68
4.6	Exercise 6	68
4.6.1	()	68
4.7	Exercise 7	68
4.7.1	()	68

4.8	Exercise 8	68
4.8.1	()	68
4.9	Exercise 9	68
4.9.1	()	68
4.10	Exercise 10	68
4.10.1	()	68
4.11	Exercise 11	68
4.11.1	()	68
4.12	Exercise 12	68
4.12.1	()	68
4.13	Exercise 13	69
4.13.1	()	69
4.14	Exercise 14	69
4.14.1	()	69
4.15	Exercise 15	69
4.15.1	()	69
4.16	Exercise 16	69
4.16.1	()	69
4.17	Exercise 17	69
4.17.1	()	69
4.18	Exercise 18	69
4.18.1	()	69
4.19	Exercise 19	69
4.19.1	()	69
4.20	Exercise 20	69
4.20.1	()	69
4.21	Exercise 21	70
4.21.1	()	70
4.22	Exercise 22	70
4.22.1	()	70
4.23	Exercise 23	70
4.23.1	()	70
4.24	Exercise 24	70
4.24.1	()	70
4.25	Exercise 25	70
4.25.1	()	70
4.26	Exercise 26	70
4.26.1	()	70
4.27	Exercise 27	70
4.27.1	()	70
4.28	Exercise 28	70
4.28.1	()	70
4.29	Exercise 29	71
4.29.1	()	71
4.30	Exercise 30	71
4.30.1	()	71

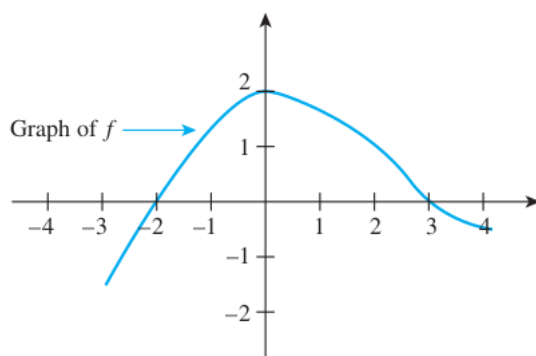
4.31	Exercise 31	71
4.31.1	()	71
4.32	Exercise 32	71
4.32.1	()	71
4.33	Exercise 33	71
4.33.1	()	71
4.34	Exercise 34	71
4.34.1	()	71
4.35	Exercise 35	71
4.35.1	()	71
4.36	Exercise 36	71
4.36.1	()	71
4.37	Exercise 37	72
4.37.1	()	72
4.38	Exercise 38	72
4.38.1	()	72
4.39	Exercise 39	72
4.39.1	()	72
4.40	Exercise 40	72
4.40.1	()	72
4.41	Exercise 41	72
4.41.1	()	72
4.42	Exercise 42	72
4.42.1	()	72
4.43	Exercise 43	72
4.43.1	()	72
4.44	Exercise 44	72
4.44.1	()	72
4.45	Exercise 45	73
4.45.1	()	73
4.46	Exercise 46	73
4.46.1	()	73
4.47	Exercise 47	73
4.47.1	()	73
4.48	Exercise 48	73
4.48.1	()	73
4.49	Exercise 49	73
4.49.1	()	73
4.50	Exercise 50	73
4.50.1	()	73
4.51	Exercise 51	73
4.51.1	()	73
5	Exercise Set 11.5	73
5.1	Exercise 1	73
5.1.1	()	73

5.2	Exercise 2	74
5.2.1	()	74
5.3	Exercise 3	74
5.3.1	()	74
5.4	Exercise 4	74
5.4.1	()	74
5.5	Exercise 5	74
5.5.1	()	74
5.6	Exercise 6	74
5.6.1	()	74
5.7	Exercise 7	74
5.7.1	()	74
5.8	Exercise 8	74
5.8.1	()	74
5.9	Exercise 9	74
5.9.1	()	74
5.10	Exercise 10	75
5.10.1	()	75
5.11	Exercise 11	75
5.11.1	()	75
5.12	Exercise 12	75
5.12.1	()	75
5.13	Exercise 13	75
5.13.1	()	75
5.14	Exercise 14	75
5.14.1	()	75
5.15	Exercise 15	75
5.15.1	()	75
5.16	Exercise 16	75
5.16.1	()	75
5.17	Exercise 17	75
5.17.1	()	75
5.18	Exercise 18	76
5.18.1	()	76
5.19	Exercise 19	76
5.19.1	()	76
5.20	Exercise 20	76
5.20.1	()	76
5.21	Exercise 21	76
5.21.1	()	76
5.22	Exercise 22	76
5.22.1	()	76
5.23	Exercise 23	76
5.23.1	()	76
5.24	Exercise 24	76
5.24.1	()	76

5.25 Exercise 25	76
5.25.1 ()	76
5.26 Exercise 26	77
5.26.1 ()	77

1 Exercise Set 11.1

1.1 Exercise 1



The graph of a function f is shown above.

1.1.1 (a)

Is $f(0)$ positive or negative?

Proof. positive

□

1.1.2 (b)

For what values of x does $f(x) = 0$?

Proof. $f(x) = 0$ when $x = -2$ and $x = 3$ (approximately)

□

1.1.3 (c)

Find approximate values for x_1 and x_2 so that $f(x_1) = f(x_2) = 1$ but $x_1 \neq x_2$.

Proof. $x_1 = -1$ and $x_2 = 2$ (approximately)

□

1.1.4 (d)

Find an approximate value for x such that $f(x) = 1.5$.

Proof. $x = 1$ or $x = -1/2$ (approximately)

□

1.1.5 (e)

As x increases from -3 to -1 , do the values of f increase or decrease?

Proof. increase

□

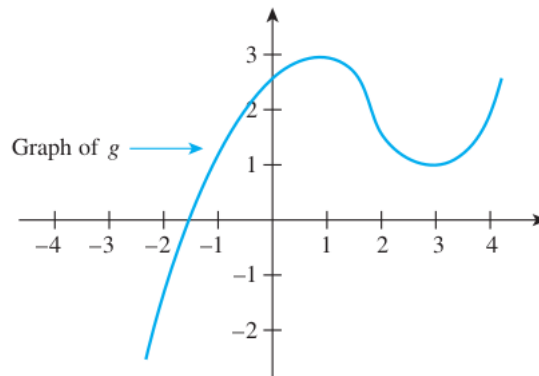
1.1.6 (f)

As x increases from 0 to 4 , do the values of f increase or decrease?

Proof. decrease

□

1.2 Exercise 2



The graph of a function g is shown above.

1.2.1 (a)

Is $g(0)$ positive or negative?

Proof. positive

□

1.2.2 (b)

Find an approximate value of x so that $g(x) = 0$.

Proof. -1.5 (approximately)

□

1.2.3 (c)

Find approximate values for x_1 and x_2 so that $g(x_1) = g(x_2) = 1$ but $x_1 \neq x_2$.

Proof. $x_1 = -1, x_2 = 3$ (approximately)

□

1.2.4 (d)

Find an approximate value for x such that $g(x) = -2$.

Proof. $x = -2.2$ (approximately)

□

1.2.5 (e)

As x increases from -2 to 1 , do the values of g increase or decrease?

Proof. increase □

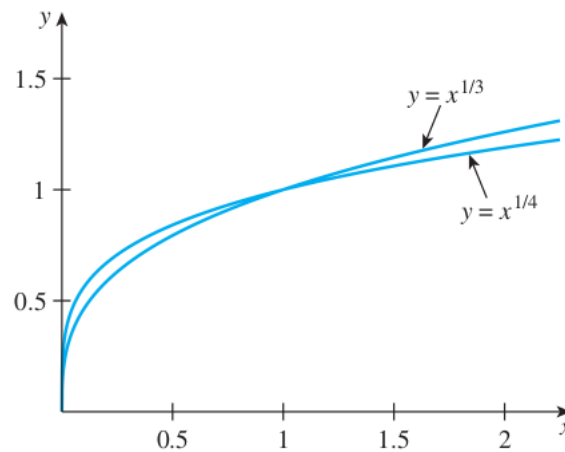
1.2.6 (f)

As x increases from 1 to 3 , do the values of g increase or decrease?

Proof. decrease □

1.3 Exercise 3

Sketch the graphs of the power functions $p_{1/3}$ and $p_{1/4}$ on the same set of axes. When $0 < x < 1$, which is greater: $x^{1/3}$ or $x^{1/4}$? When $x > 1$, which is greater: $x^{1/3}$ or $x^{1/4}$?



Proof. When $0 < x < 1$, $x^{1/3} < x^{1/4}$. When $1 < x$, $x^{1/4} < x^{1/3}$. □

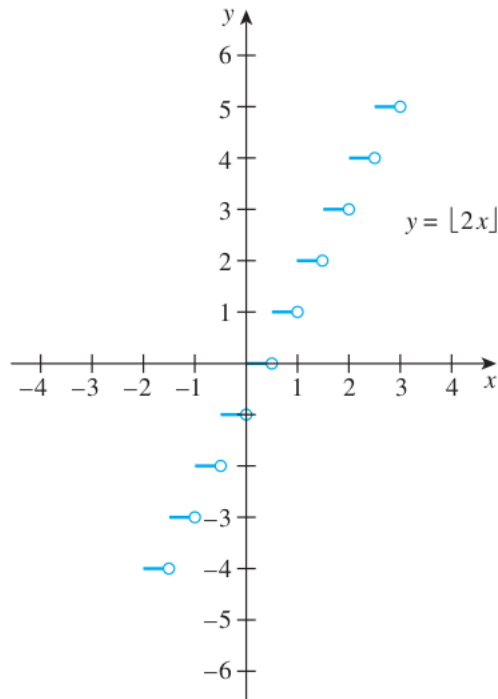
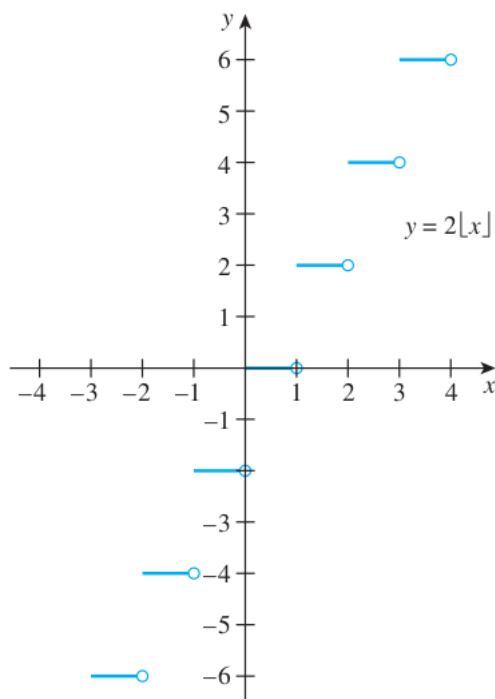
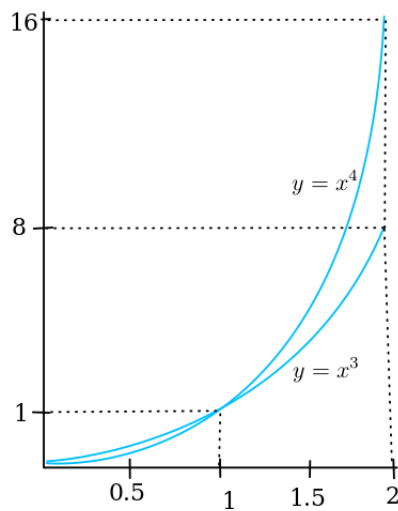
1.4 Exercise 4

Sketch the graphs of the power functions p_3 and p_4 on the same set of axes. When $0 < x < 1$, which is greater: x^3 or x^4 ? When $x > 1$, which is greater: x^3 or x^4 ?

Proof. When $0 < x < 1$, $x^4 < x^3$. When $1 < x$, $x^3 < x^4$. □

1.5 Exercise 5

Sketch the graphs of $y = 2[x]$; and $y = [2x]$ for each real number x . What can you conclude from these graphs?



Proof.

The graphs show that $2[x] \neq [2x]$ for many values of x . □

Sketch a graph for each of the functions defined in 6 – 9 below.

1.6 Exercise 6

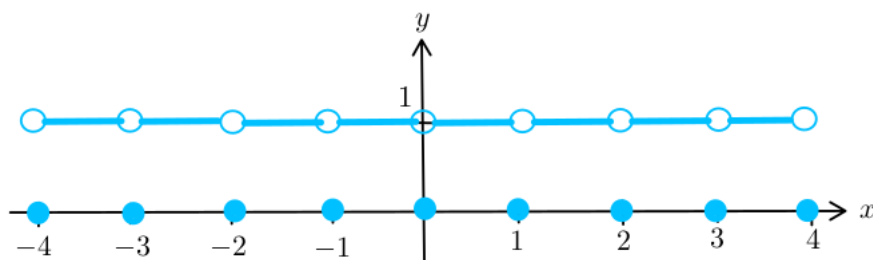
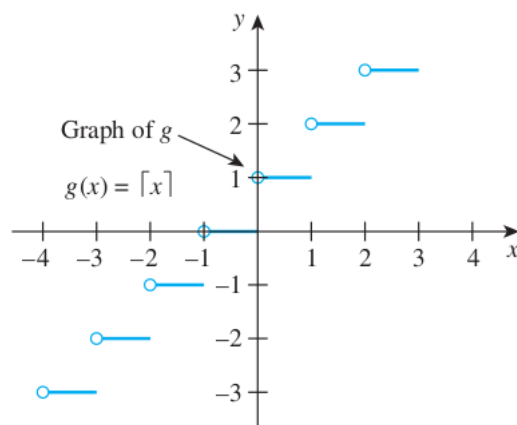
$g(x) = \lceil x \rceil$ for each real number x (Recall that the ceiling of x , $\lceil x \rceil$, is the least integer that is greater than or equal to x . That is, $\lceil x \rceil =$ the unique integer n such that $n - 1 < x \leq n$.)

Proof. □

1.7 Exercise 7

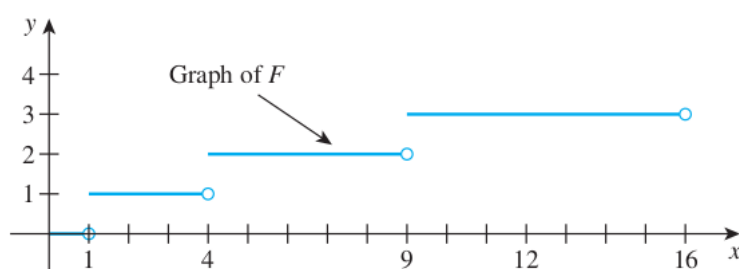
$h(x) = \lceil x \rceil - \lfloor x \rfloor$ for each real number x

Proof. □



1.8 Exercise 8

$F(x) = \lfloor x^{1/2} \rfloor$ for each real number x

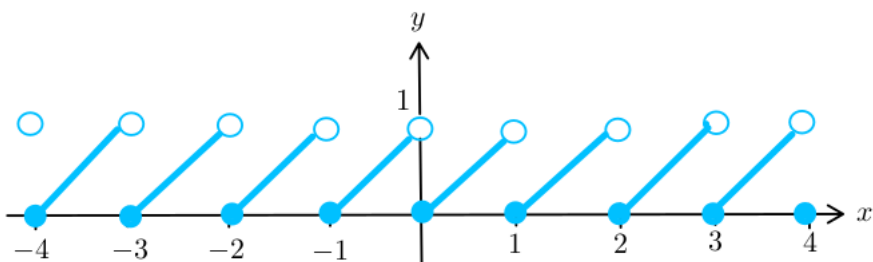


Proof.

□

1.9 Exercise 9

$G(x) = x - \lfloor x \rfloor$ for each real number x



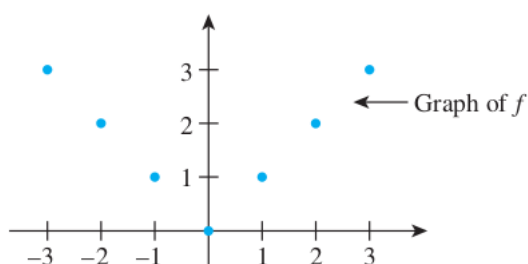
Proof.

□

In each of 10 – 13 a function is defined on a set of integers. Sketch a graph for each function.

1.10 Exercise 10

$f(n) = |n|$ for each integer n

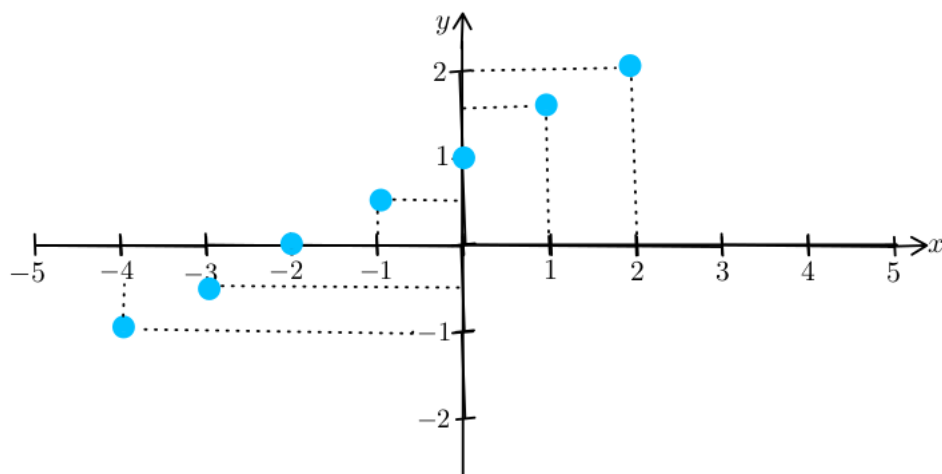


Proof.

□

1.11 Exercise 11

$g(n) = (n/2) + 1$ for each integer n

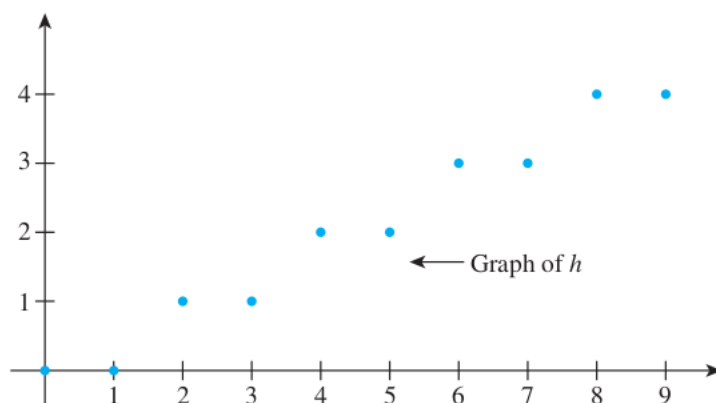


Proof.

□

1.12 Exercise 12

$h(n) = \lfloor n/2 \rfloor$ for each integer $n \geq 0$

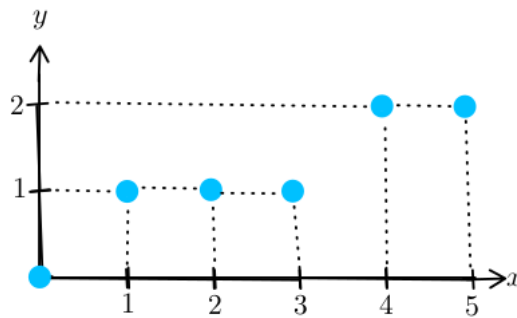


Proof.

□

1.13 Exercise 13

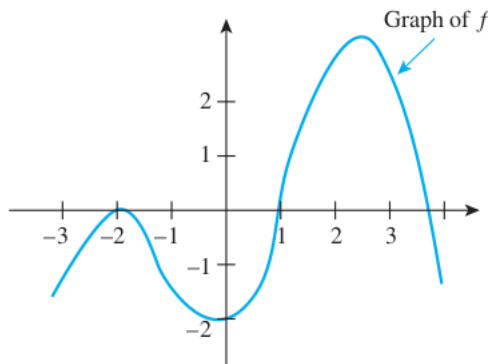
$k(n) = \lfloor n^{1/2} \rfloor$ for each integer $n \geq 0$



Proof.

□

1.14 Exercise 14



The graph of a function f is shown below. Find the intervals on which f is increasing and the intervals on which f is decreasing.

Proof. f is increasing on the intervals $\{x \in \mathbb{R} \mid -3 < x < -2\}$ and $\{x \in \mathbb{R} \mid 0 < x < 2.5\}$, and f is decreasing on $\{x \in \mathbb{R} \mid -2 < x < 0\}$ and $\{x \in \mathbb{R} \mid 2.5 < x < 4\}$ (approximately). □

1.15 Exercise 15

Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula $f(x) = 2x - 3$ is increasing on the set of real numbers.

Proof. Suppose that x_1 and x_2 are particular but arbitrarily chosen real numbers such that $x_1 < x_2$. [We must show that $f(x_1) < f(x_2)$.] Since $x_1 < x_2$ then $2x_1 < 2x_2$ and $2x_1 - 3 < 2x_2 - 3$ by basic properties of inequalities. Thus, by definition of f , $f(x_1) < f(x_2)$ [as was to be shown]. Hence f is increasing on the set of all real numbers. □

1.16 Exercise 16

Show that the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula $g(x) = -(x/3) + 1$ is decreasing on the set of real numbers.

Proof. □

1.17 Exercise 17

Let h be the function from \mathbb{R} to \mathbb{R} defined by the formula $h(x) = x^2$ for each real number x .

1.17.1 (a)

Show that h is decreasing on the set of real numbers less than zero.

Proof. Suppose that x_1 and x_2 are particular but arbitrarily chosen real numbers such that $x_1 < x_2 < 0$. [We must show that $h(x_1) > h(x_2)$.]

Since $x_1 < x_2 < 0$ then $0 < -x_2 < -x_1$ and multiplying by $-x_1$ (which is a positive number) we get $(-x_1)(-x_2) < (-x_1)(-x_1) = x_1^2$ by basic properties of inequalities.

Similarly, since $x_1 < x_2 < 0$ then $0 < -x_2 < -x_1$ and multiplying by $-x_2$ (which is a positive number) we get $(-x_2)(-x_2) = x_2^2 < (-x_1)(-x_2)$ by basic properties of inequalities.

By combining the two results we get $x_2^2 < (-x_1)(-x_2) < x_1^2$ so $x_2^2 < x_1^2$.

Thus, by definition of h , $h(x_1) > h(x_2)$ [as was to be shown]. Hence h is increasing on the set of all real numbers. □

1.17.2 (b)

Show that h is increasing on the set of real numbers greater than zero.

Proof. Suppose that x_1 and x_2 are particular but arbitrarily chosen real numbers such that $0 < x_1 < x_2$. [We must show that $h(x_1) < h(x_2)$.]

Since $0 < x_1 < x_2$ then multiplying by x_1 (which is a positive number) we get $x_1x_1 = x_1^2 < x_1x_2$ by basic properties of inequalities.

Similarly, since $0 < x_1 < x_2$ then multiplying by x_2 (which is a positive number) we get $x_1x_2 < x_2x_2 = x_2^2$ by basic properties of inequalities.

By combining the two results we get $x_1^2 < x_1x_2 < x_2^2$ so $x_1^2 < x_2^2$.

Thus, by definition of h , $h(x_1) < h(x_2)$ [as was to be shown]. Hence h is increasing on the set of all real numbers. □

1.18 Exercise 18

Let $k : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by the formula $k(x) = (x - 1)/x$ for each real number $x \neq 0$.

1.18.1 (a)

Show that k is increasing for every real number $x > 0$.

Proof. Suppose that x_1 and x_2 are positive real numbers and $x_1 < x_2$. [We must show that $k(x_1) < k(x_2)$.]

	$x_1 < x_2$	by assumption
\implies	$-x_2 < -x_1$	by multiplying by -1
\implies	$x_1x_2 - x_2 < x_1x_2 - x_1$	by adding x_1x_2 to both sides
\implies	$x_2(x_1 - 1) < x_1(x_2 - 1)$	by factoring both sides
\implies	$\frac{x_1 - 1}{x_2} < \frac{x_2 - 1}{x_1}$	by dividing both sides by $x_1x_2 > 0$
\implies	$\frac{x_1}{k(x_1)} < \frac{x_2}{k(x_2)}$	by definition of k

□

1.18.2 (b)

Is k increasing or decreasing for $x < 0$? Prove your answer.

Proof. It is increasing. The same proof as in part (a) works. Note that the only place in the proof where the signs of x_1 and x_2 matter is when we divide both sides by x_1x_2 . For the proof to work, x_1x_2 has to be positive. But if both x_1 and x_2 are negative, then x_1x_2 is positive. Therefore the proof still works. □

1.19 Exercise 19

Show that if a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, then f is one-to-one.

Proof. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing. [We must show that f is one-to-one. In other words, we must show that for all real numbers x_1 and x_2 , if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.] Suppose x_1 and x_2 are real numbers and $x_1 \neq x_2$. By the trichotomy law [Appendix A, T17] $x_1 < x_2$, or $x_1 > x_2$. In case $x_1 < x_2$, then since f is increasing, $f(x_1) < f(x_2)$ and so $f(x_1) \neq f(x_2)$. Similarly, in case $x_1 > x_2$, then $f(x_1) > f(x_2)$ and so $f(x_1) \neq f(x_2)$. Thus in either case, $f(x_1) \neq f(x_2)$ [as was to be shown]. □

1.20 Exercise 20

Given real-valued functions f and g with the same domain D , the sum of f and g , denoted $f + g$, is defined as follows: For each real number x , $(f + g)(x) = f(x) + g(x)$. Show that if f and g are both increasing on a set S , then $f + g$ is also increasing on S .

Proof. Assume $x_1, x_2 \in S$ and $x_1 < x_2$. [We want to show $(f + g)(x_1) < (f + g)(x_2)$.] Since f is increasing, $f(x_1) < f(x_2)$. Since g is increasing, $g(x_1) < g(x_2)$. By definition of $f + g$ we have $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$, [as was to be shown.] \square

1.21 Exercise 21

1.21.1 (a)

Let m be any positive integer, and define $f(x) = x^m$ for each nonnegative real number x . Use the binomial theorem to show that f is an increasing function.

Proof. Suppose u and v are nonnegative real numbers with $u < v$. [We must show that $f(u) < f(v)$.] Note that $v = u + h$ for some positive real number h . By substitution and the binomial theorem,

$$v^m = (u + h)^m = \sum_{i=0}^m \binom{m}{i} u^{m-i} h^i = u^m + \sum_{i=1}^m \binom{m}{i} u^{m-i} h^i$$

The last summation is positive because $u \geq 0$ and $h > 0$, and a sum of nonnegative terms that includes at least one positive term is positive. Hence $v^m = u^m +$ a positive number, and so $f(u) = u^m < v^m = f(v)$, [as was to be shown.] \square

1.21.2 (b)

Let m and n be any positive integers, and let $g(x) = x^{m/n}$ for each nonnegative real number x . Prove that g is an increasing function.

Note: The results of exercise 21 are used in the exercises for Sections 11.2 and 11.4.

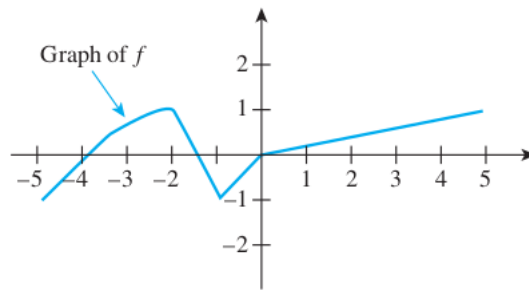
Proof. Write $f(x) = x^m$. Then $g(x) = (f(x))^{1/n}$ by the law of exponents.

Now assume $0 \leq x_1 < x_2$. In part (a) we showed that f is increasing. Therefore $f(x_1) < f(x_2)$, in other words $x_1^m < x_2^m$. So we need to show that the function $h(x) = x^{1/n}$ is an increasing function. That will imply $g(x_1) = h(x_1^m) < h(x_2^m) = g(x_2)$, in other words $x_1^{m/n} < x_2^{m/n}$, which is what we want.

To show h is increasing, assume $0 \leq z_1 < z_2$. By definition, $h(z_1) = z_1^{1/n} = y_1$ is the real number with the property that $y_1^n = z_1$. Similarly $h(z_2) = z_2^{1/n} = y_2$ is the real number with the property that $y_2^n = z_2$. [We want to show $y_1 < y_2$.]

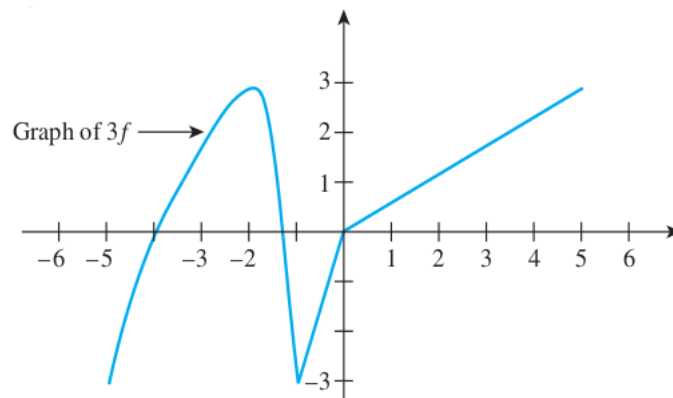
Argue by contradiction and assume $y_2 \leq y_1$. Now consider the function $e(y) = y^n$. This function is also increasing by part (a), since m and n are both any positive integers. Therefore $e(y_2) \leq e(y_1)$, in other words $z_2 \leq z_1$, which is a contradiction!

Therefore $y_1 < y_2$ and h is increasing, and thus g is increasing as a consequence. \square



1.22 Exercise 22

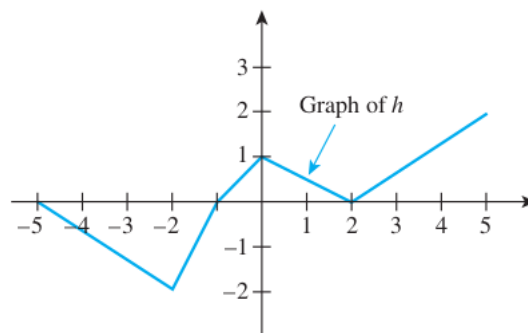
Let f be the function whose graph follows. Sketch the graph of $3f$.



Proof.

□

1.23 Exercise 23



Let h be the function whose graph is shown above. Sketch the graph of $2h$.

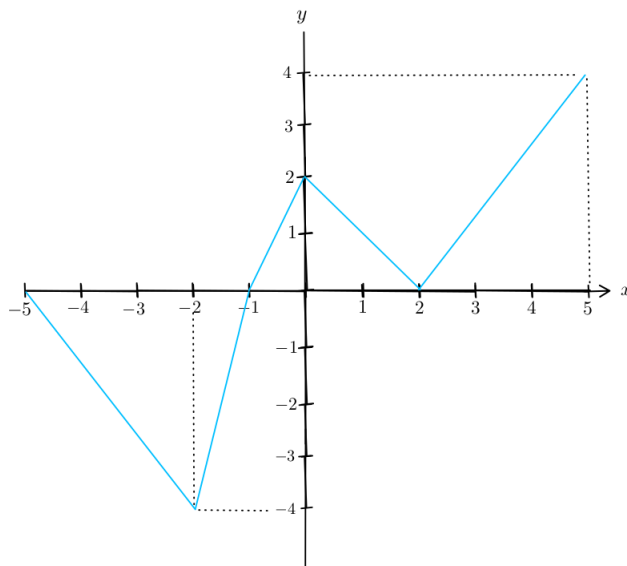
Proof.

□

1.24 Exercise 24

Let f be a real-valued function of a real variable. Show that if f is decreasing on a set S and if M is any positive real number, then Mf is decreasing on S .

Proof. Suppose that f is a real-valued function of a real variable, f is decreasing on a set S , and M is any positive real number. [We must show that Mf is decreasing



on S . In other words, we must show that for all x_1 and x_2 in S , if $x_1 < x_2$ then $(Mf)(x_1) > (Mf)(x_2)$.] Suppose x_1 and x_2 are in S and $x_1 < x_2$. Since f is decreasing on S , $f(x_1) > f(x_2)$, and since M is positive, $Mf(x_1) > Mf(x_2)$ [because when both sides of an inequality are multiplied by a positive number, the direction of the inequality is unchanged]. It follows by definition of Mf that $(Mf)(x_1) > (Mf)(x_2)$, [as was to be shown]. \square

1.25 Exercise 25

Let f be a real-valued function of a real variable. Show that if f is increasing on a set S and if M is any negative real number, then Mf is decreasing on S .

Proof. The proof is the same as in Exercise 24, except that this time we have $f(x_1) < f(x_2)$ because f is increasing, and multiplying an inequality by a negative number M reverses the direction of the equality, so $Mf(x_1) > Mf(x_2)$. \square

1.26 Exercise 26

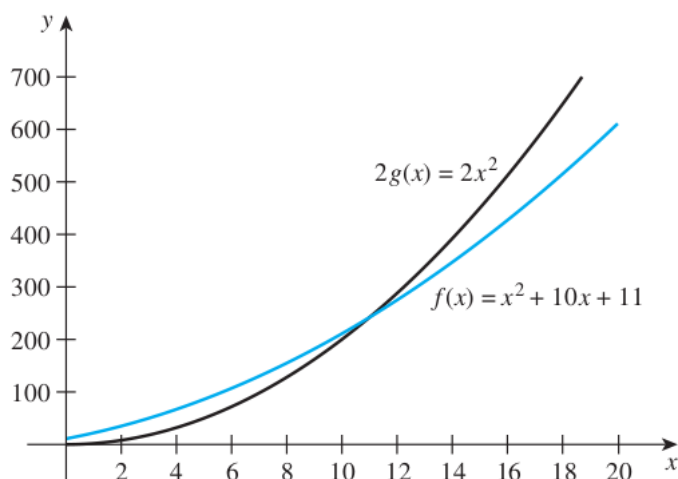
Let f be a real-valued function of a real variable. Show that if f is decreasing on a set S and if M is any negative real number, then Mf is increasing on S .

Proof. The proof is the same as in Exercise 24, except that this time multiplying an inequality by a negative number M reverses the direction of the equality, so $Mf(x_1) < Mf(x_2)$. \square

In 27 and 28, functions f and g are defined. In each case sketch the graphs of f and $2g$ on the same set of axes and find a number x_0 so that $f(x) \leq 2g(x)$ for all $x > x_0$. You can find an exact value for x_0 by solving a quadratic equation, or you can find an approximate value for x_0 by using a graphing calculator or computer.

1.27 Exercise 27

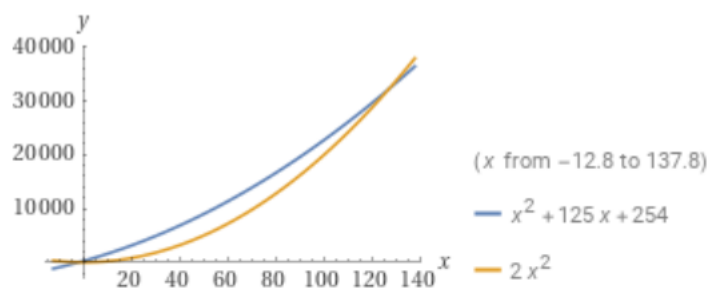
$f(x) = x^2 + 10x + 11$ and $g(x) = x^2$ for each real number $x \geq 0$



Proof. To find the answer algebraically, solve the equation $2x^2 = x^2 + 10x + 11$ for x . Subtracting x^2 from both sides gives $x^2 - 10x - 11 = 0$, and either using the quadratic formula or factoring $x^2 - 10x - 11 = (x - 11)(x + 1)$ gives $x = 11$ (since $x > 0$). To find an approximate answer with a graphing calculator, plot both $f(x) = x^2 + 10x + 11$ and $2g(x) = 2x^2$ for $x > 0$, as shown in the figure, and find that $2g(x) > f(x)$ when $x > 11$ (approximately). You can obtain only an approximate answer from a graphing calculator because the calculator computes values only to an accuracy of a finite number of decimal places. \square

1.28 Exercise 28

$f(x) = x^2 + 125x + 254$ and $g(x) = x^2$ for each real number $x \geq 0$



Proof. If we set $f(x) = 2g(x)$ and solve, we get $x^2 + 125x + 254 = 2x^2$ which gives $x^2 - 125x - 254 = 0$ which factors as $(x - 127)(x + 2) = 0$ which has solutions $x = -2, 127$. So let $x_0 = 127$, so that $f(x) < g(x)$ for all $x > x_0 = 127$. \square

2 Exercise Set 11.2

2.1 Exercise 1

The following is a formal definition for Ω -notation, written using quantifiers and variables: $f(n)$ is $\Omega(g(n))$ if, and only if, \exists positive real numbers a and A such that

$\forall n \geq a, Ag(n) \leq f(n).$

2.1.1 (a)

Write the formal negation for the definition using the symbols \forall and \exists .

Proof. Formal version of negation: $f(n)$ is not $\Omega(g(n))$ if, and only if, \forall positive real numbers a and A , \exists an integer $n \geq a$ such that $Ag(n) > f(n)$. \square

2.1.2 (b)

Restate the negation less formally without using the symbols \forall and \exists or the words “for any,” “for every,” or “there exists.”

Proof. Informal version of negation: $f(n)$ is not $\Omega(g(n))$ if, and only if, no matter what positive real numbers a and A might be chosen, it is possible to find an integer n greater than or equal to a with the property that $Ag(n) > f(n)$. \square

2.2 Exercise 2

The following is a formal definition for O -notation, written using quantifiers and variables: $f(n)$ is $O(g(n))$ if, and only if, \exists positive real numbers b and B such that $\forall n \geq b$, $0 \leq f(n) \leq Bg(n)$.

2.2.1 (a)

Write the formal negation for the definition using the symbols \forall and \exists .

Proof. $f(n)$ is not $O(g(n))$ if, and only if, \forall positive real numbers b and B , $\exists n \geq b$ such that $0 > f(n)$ or $f(n) > Bg(n)$. \square

2.2.2 (b)

Restate the negation less formally without using the symbols \forall and \exists or the words “for any,” “for every,” or “there exists.”

Proof. $f(n)$ is not $O(g(n))$ if, and only if, no matter what positive real numbers b and B are chosen, it is possible to choose an integer n greater than b with the property that either $0 > f(n)$ or $f(n) > Bg(n)$. \square

2.3 Exercise 3

The following is a formal definition for Θ -notation, written using quantifiers and variables: $f(n)$ is $\Theta(g(n))$ if, and only if, \exists positive real numbers k, A and B such that $\forall n \geq b$, $Ag(n) \leq f(n) \leq Bg(n)$.

2.3.1 (a)

Write the formal negation for the definition using the symbols \forall and \exists .

Proof. $f(n)$ is not $\Theta(g(n))$ if, and only if, \forall positive real numbers k, A and B , $\exists n \geq b$ such that $Ag(n) > f(n)$ or $f(n) > Bg(n)$. \square

2.3.2 (b)

Restate the negation less formally without using the symbols \forall and \exists or the words “for any,” “for every,” or “there exists.”

Proof. $f(n)$ is not $\Theta(g(n))$ if, and only if, no matter what positive real numbers k, A and B are chosen, it is possible to choose an integer n greater than b with the property that either $Ag(n) > f(n)$ or $f(n) > Bg(n)$. \square

In 4 – 9, express each statement using Ω -, O -, or Θ -notation.

2.4 Exercise 4

$\frac{1}{2}n \leq n - \left\lfloor \frac{n}{2} \right\rfloor + 1$ for every integer $n \geq 1$. (Use Ω -notation).

Proof. $n - \left\lfloor \frac{n}{2} \right\rfloor + 1$ is $\Omega(n)$ \square

2.5 Exercise 5

$0 \leq n - \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq n$ for every integer $n \geq 3$. (Use O -notation).

Proof. $n - \left\lfloor \frac{n}{2} \right\rfloor + 1$ is $O(n)$ \square

2.6 Exercise 6

$n^2 \leq 3n(n-2) \leq 4n^2$ for every integer $n \geq 3$. (Use Θ -notation.)

Proof. $3n(n-2)$ is $\Theta(n^2)$ \square

2.7 Exercise 7

$\frac{1}{2}n^2 \leq \frac{n(3n-2)}{2}$ for every integer $n \geq 3$. (Use Ω -notation).

Proof. $\frac{n(3n-2)}{2}$ is $\Omega(n^2)$ \square

2.8 Exercise 8

$0 \leq \frac{n(3n-2)}{2} \leq n^2$ for every integer $n \geq 1$. (Use O -notation).

Proof. $\frac{n(3n-2)}{2}$ is $O(n^2)$

□

2.9 Exercise 9

$\frac{n^3}{6} \leq n^2 \left(\left\lceil \frac{n}{3} \right\rceil - 1 \right) \leq n^3$ for every integer $n \geq 2$. (Use Θ -notation.)

Proof. $n^2 \left(\left\lceil \frac{n}{3} \right\rceil - 1 \right)$ is $\Theta(n^3)$

□

2.10 Exercise 10

2.10.1 (a)

Show that for any integer $n \geq 1$, $0 \leq 2n^2 + 15n + 4 \leq 21n^2$.

Proof. For each integer $n \geq 1$, $0 \leq 2n^2 + 15n + 4$ because all terms in $2n^2 + 15n + 4$ are positive. Moreover, $2n^2 + 15n + 4 \leq 2n^2 + 15n^2 + 4n^2$ because when $n \geq 1$, $15n \leq 15n^2$ and $4 \leq 4n^2$, which add up to $21n^2$ by combining like terms. Therefore, by transitivity of equality and order, $0 \leq 2n^2 + 15n + 4 \leq 21n^2$ for each integer $n \geq 1$. □

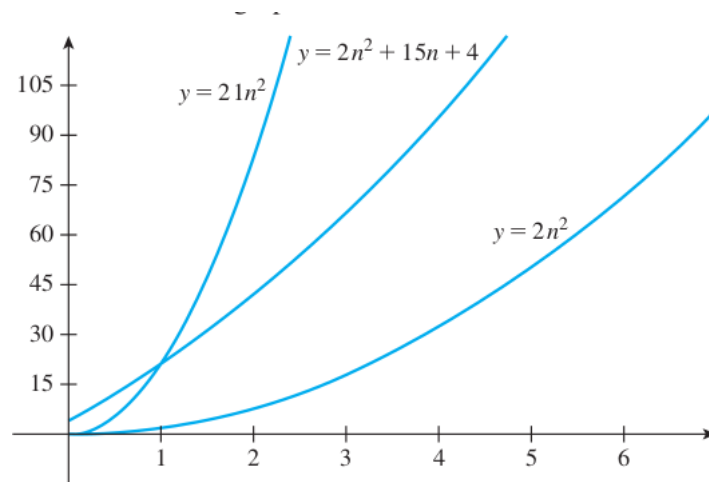
2.10.2 (b)

Show that for any integer $n \geq 1$, $2n^2 \leq 2n^2 + 15n + 4$.

Proof. For each integer $n \geq 1$, $2n^2 \leq 2n^2 + 15n + 4$ because $15n + 4 > 0$ since n is positive. □

2.10.3 (c)

Sketch a graph to illustrate the results of parts (a) and (b).



Proof.

□

2.10.4 (d)

Use the O - and Ω -notations to express the results of parts (a) and (b).

Proof. Let $A = 2$ and $a = 1$. Then, by substitution from the result of part (b), $An^2 < 2n^2 + 15n + 4$ for each integer $n \geq a$, and hence, by definition of Ω -notation, $2n^2 + 15n + 4$ is $\Omega(n^2)$. Let $B = 21$ and $b = 1$. Then, by substitution from the result of part (a), $0 < 2n^2 + 15n + 4 \leq Bn^2$ for each integer $n \geq b$, and hence by definition of O -notation, $2n^2 + 15n + 4$ is $O(n^2)$. \square

2.10.5 (e)

What can you deduce about the order of $2n^2 + 15n + 4$?

Proof. Solution 1: Let $A = 2, B = 21$, and $k = 1$. By the results of parts (a) and (b), $An^2 \leq 2n^2 + 15n + 4 \leq Bn^2$ for each integer $n \geq k$, and hence, by definition of Θ -notation, $2n^2 + 15n + 4$ is $\Theta(n^2)$.

Solution 2: By part (d), $2n^2 + 15n + 4$ is both $\Omega(n^2)$ and $O(n^2)$. Hence, by Theorem 11.2.1, $2n^2 + 15n + 4$ is $\Theta(n^2)$. \square

2.11 Exercise 11

2.11.1 (a)

Show that for any integer $n \geq 1$, $0 \leq 23n^4 + 8n^2 + 4n \leq 35n^4$.

Proof. For each integer $n \geq 1$, $0 \leq 23n^4 + 8n^2 + 4n$ because all terms in $23n^4 + 8n^2 + 4n$ are positive. Moreover, $23n^4 + 8n^2 + 4n \leq 23n^4 + 8n^4 + 4n^4$ because when $n \geq 1$, $8n^2 \leq 8n^4$ and $4n \leq 4n^4$, which add up to $35n^4$ by combining like terms. Therefore, by transitivity of equality and order, $0 \leq 23n^4 + 8n^2 + 4n \leq 35n^4$ for each integer $n \geq 1$. \square

2.11.2 (b)

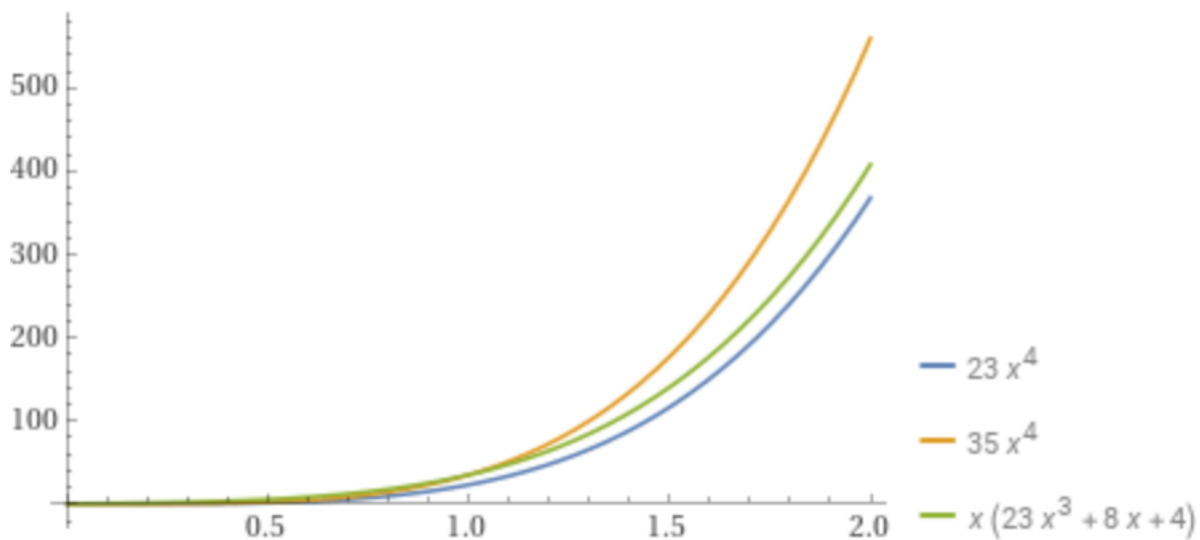
Show that for any integer $n \geq 1$, $23n^4 \leq 23n^4 + 8n^2 + 4n$.

Proof. For each integer $n \geq 1$, $23n^4 \leq 23n^4 + 8n^2 + 4n$ because $8n^2 + 4n > 0$ since n is positive. \square

2.11.3 (c)

Sketch a graph to illustrate the results of parts (a) and (b).

Proof. \square



2.11.4 (d)

Use the O - and Ω -notations to express the results of parts (a) and (b).

Proof. Let $A = 23$ and $a = 1$. Then, by substitution from the result of part (b), $An^4 < 23n^4 + 8n^2 + 4n$ for each integer $n \geq a$, and hence, by definition of Ω -notation, $23n^4 + 8n^2 + 4n$ is $\Omega(n^4)$. Let $B = 35$ and $b = 1$. Then, by substitution from the result of part (a), $0 < 23n^4 + 8n^2 + 4n \leq Bn^4$ for each integer $n \geq b$, and hence by definition of O -notation, $23n^4 + 8n^2 + 4n$ is $O(n^4)$. \square

2.11.5 (e)

What can you deduce about the order of $23n^4 + 8n^2 + 4n$?

Proof. By part (d), $23n^4 + 8n^2 + 4n$ is both $\Omega(n^4)$ and $O(n^4)$. Hence, by Theorem 11.2.1, $23n^4 + 8n^2 + 4n$ is $\Theta(n^4)$. \square

2.12 Exercise 12

2.12.1 (a)

Show that for any integer $n \geq 1$, $0 \leq 7n^3 + 10n^2 + 3 \leq 20n^3$.

Proof. For each integer $n \geq 1$, $0 \leq 7n^3 + 10n^2 + 3$ because all terms in $7n^3 + 10n^2 + 3$ are positive. Moreover, $7n^3 + 10n^2 + 3 \leq 7n^3 + 10n^3 + 3n^3$ because when $n \geq 1$, $10n^2 \leq 10n^3$ and $3 \leq 3n^3$, which add up to $20n^3$ by combining like terms. Therefore, by transitivity of equality and order, $0 \leq 7n^3 + 10n^2 + 3 \leq 20n^3$ for each integer $n \geq 1$. \square

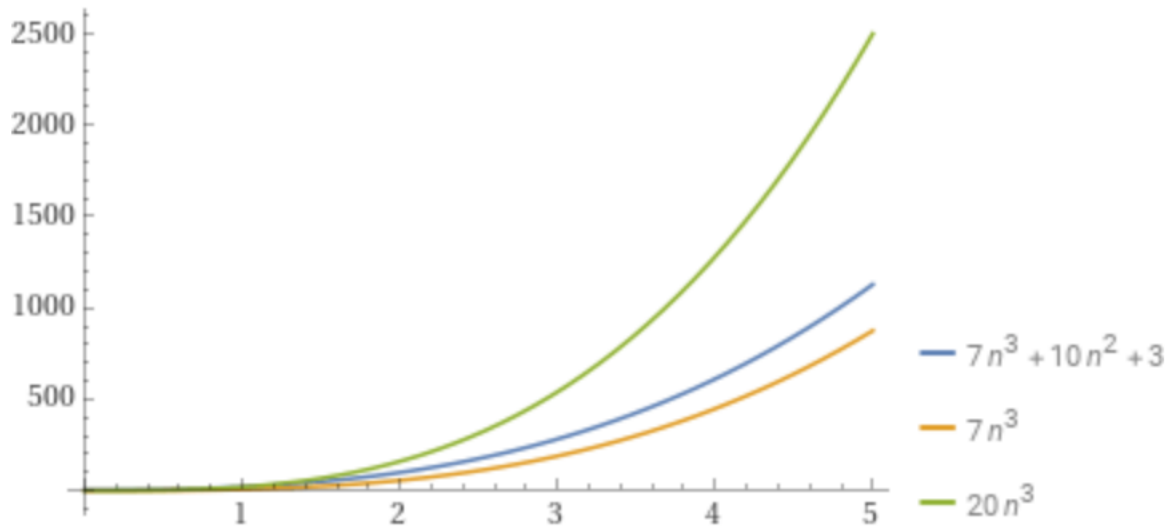
2.12.2 (b)

Show that for any integer $n \geq 1$, $7n^3 \leq 7n^3 + 10n^2 + 3$.

Proof. For each integer $n \geq 1$, $7n^3 \leq 7n^3 + 10n^2 + 3$ because $10n^2 + 3 > 0$ since n is positive. \square

2.12.3 (c)

Sketch a graph to illustrate the results of parts (a) and (b).



Proof.

□

2.12.4 (d)

Use the O - and Ω -notations to express the results of parts (a) and (b).

Proof. Let $A = 7$ and $a = 1$. Then, by substitution from the result of part (b), $An^3 < 7n^3 + 10n^2 + 3$ for each integer $n \geq a$, and hence, by definition of Ω -notation, $7n^3 + 10n^2 + 3$ is $\Omega(n^3)$. Let $B = 20$ and $b = 1$. Then, by substitution from the result of part (a), $0 < 7n^3 + 10n^2 + 3 \leq Bn^3$ for each integer $n \geq b$, and hence by definition of O -notation, $7n^3 + 10n^2 + 3$ is $O(n^3)$. □

2.12.5 (e)

What can you deduce about the order of $7n^3 + 10n^2 + 3$?

Proof. By part (d), $7n^3 + 10n^2 + 3$ is both $\Omega(n^3)$ and $O(n^3)$. Hence, by Theorem 11.2.1, $7n^3 + 10n^2 + 3$ is $\Theta(n^3)$. □

2.13 Exercise 13

Use the definition of Θ -notation to show that $5n^3 + 65n + 30$ is $\Theta(n^3)$.

Proof. For each integer $n \geq 1$, $5n^3 \leq 5n^3 + 65n + 30$ because $65n + 30 > 0$ since n is positive. Moreover, $5n^3 + 65n + 30 \leq 5n^3 + 65n^3 + 30n^3$ because when $n \geq 1$, then $65n < 65n^3$ and $30 < 30n^3$, which add up to $100n^3$ by combining like terms. Therefore, by transitivity of order and equality, $5n^3 \leq 5n^3 + 65n + 30 \leq 100n^3$. Thus, let $A = 5$, $B = 100$, and $k = 1$. Then $An^3 \leq 5n^3 + 65n + 30 \leq Bn^3$ for each integer $n \geq k$, and hence, by definition of Θ -notation, $5n^3 + 65n + 30$ is $\Theta(n^3)$. □

2.14 Exercise 14

Use the definition of Θ -notation to show that $n^2 + 100n + 88$ is $\Theta(n^2)$.

Proof. For each integer $n \geq 1$, $n^2 \leq n^2 + 100n + 88$ because $100n + 88 > 0$ since n is positive. Moreover, $n^2 + 100n + 88 \leq n^2 + 100n^2 + 88n^2$ because when $n \geq 1$, then $100n < 100n^2$ and $88 < 88n^2$, which add up to $189n^2$ by combining like terms. Therefore, by transitivity of order and equality, $n^2 \leq n^2 + 100n + 88 \leq 189n^2$. Thus, let $A = 1$, $B = 189$, and $k = 1$. Then $An^2 \leq n^2 + 100n + 88 \leq Bn^2$ for each integer $n \geq k$, and hence, by definition of Θ -notation, $n^2 + 100n + 88$ is $\Theta(n^2)$. \square

2.15 Exercise 15

Use the definition of Θ -notation to show that $\left\lfloor n + \frac{1}{2} \right\rfloor$ is $\Theta(n)$.

Proof. For each integer $n \geq 1$, $n \leq n + \frac{1}{2} < n + 1$, and so by definition of floor, $\left\lfloor n + \frac{1}{2} \right\rfloor = n$, and $\left\lfloor n + \frac{1}{2} \right\rfloor$ is nonnegative. In addition, when $n \geq 1$, then $n + 1 \leq n + n = 2n$, and thus, by transitivity of equality and order, $n \leq \left\lfloor n + \frac{1}{2} \right\rfloor \leq 2n$. Let $A = 1$, $B = 2$, and $k = 1$. Then $An \leq \left\lfloor n + \frac{1}{2} \right\rfloor \leq Bn$ for every integer $n \geq k$, and hence, by definition of Θ -notation, $\left\lfloor n + \frac{1}{2} \right\rfloor$ is $\Theta(n)$. \square

2.16 Exercise 16

Use the definition of Θ -notation to show that $\left\lceil n + \frac{1}{2} \right\rceil$ is $\Theta(n)$.

Proof. For each integer $n \geq 1$, $n < n + \frac{1}{2} \leq n + 1$, and so by definition of ceiling, $\left\lceil n + \frac{1}{2} \right\rceil = n + 1$, and $\left\lceil n + \frac{1}{2} \right\rceil$ is nonnegative. In addition, when $n \geq 1$, then $n + 1 \leq n + n = 2n$, and thus, by transitivity of equality and order, $n < \left\lceil n + \frac{1}{2} \right\rceil \leq 2n$. Let $A = 1$, $B = 2$, and $k = 1$. Then $An \leq \left\lceil n + \frac{1}{2} \right\rceil \leq Bn$ for every integer $n \geq k$, and hence, by definition of Θ -notation, $\left\lceil n + \frac{1}{2} \right\rceil$ is $\Theta(n)$. \square

2.17 Exercise 17

Use the definition of Θ -notation to show that $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$. (*Hint:* Show that if $n \geq 4$, then $\frac{n}{2} - 1 \geq \frac{1}{4}n$.)

Proof. Assume $n \geq 2$ is even.

Then $n = 2k$ for some integer k and thus $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = \frac{n}{2}$. Then notice that $\frac{1}{4}n \leq \frac{n}{2} \leq n$. So $\frac{1}{4}n \leq \left\lfloor \frac{n}{2} \right\rfloor \leq n$.

Now assume $n \geq 2$ is odd.

Then $n = 2k + 1$ for some integer k and thus $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k + 1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor = k = \frac{n - 1}{2}$.

Now $n - 1 \leq n \leq 2n$, so $\frac{n - 1}{2} \leq n$. And $2 \leq n$ implies $0 \leq n - 2$ so $n \leq 2n - 2$ then $\frac{1}{4}n \leq \frac{2n - 2}{4} = \frac{n - 1}{2}$. Thus $\frac{1}{4}n \leq \left\lfloor \frac{n}{2} \right\rfloor \leq n$.

So, in all cases, for $n \geq 2$ we have $\frac{1}{4}n \leq \left\lfloor \frac{n}{2} \right\rfloor \leq n$. Let $A = \frac{1}{4}, B = 1, k = 2$. Then for all $n \geq k, An \leq \left\lfloor \frac{n}{2} \right\rfloor \leq Bn$. So by definition of Θ notation, $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$. \square

2.18 Exercise 18

Prove Theorem 11.2.7(b): If f and g are real-valued functions defined on the same set of nonnegative integers and if $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, where r is a positive real number, then if $f(n)$ is $\Theta(g(n))$, then $g(n)$ is $\Theta(f(n))$.

Proof. Suppose f and g are real-valued functions defined on the same set of nonnegative integers, suppose $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, where r is a positive real number, and suppose $f(n)$ is $\Theta(g(n))$. [We must show that $g(n)$ is $\Theta(f(n))$.] By definition of Θ -notation, there exist positive real numbers A, B , and k with $k \geq r$ such that for each integer $n \geq k, Ag(n) \leq f(n) \leq Bg(n)$. Dividing the left-hand inequality by A and the right-hand inequality by B gives that $g(n) \leq \frac{1}{A}f(n)$ and $\frac{1}{B}f(n) \leq g(n)$, and combining the resulting inequalities produces $\frac{1}{B}f(n) \leq g(n) \leq \frac{1}{A}f(n)$ for each integer $n \geq k$. Now both $f(n) \geq 0$ and $g(n) \geq 0$ for each integer $n \geq k$. Also, since both A and B are positive real numbers, so are $1/A$ and $1/B$. Thus, by definition of Θ -notation, $g(n)$ is $\Theta(f(n))$. \square

2.19 Exercise 19

Prove Theorem 11.2.1: If f and g are real-valued functions defined on the same set of nonnegative integers and if $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, where r is a positive real number, then $f(n)$ is $\Theta(g(n))$ if, and only if, $f(n)$ is $\Omega(g(n))$ and $f(n)$ is $O(g(n))$.

Proof. Assume $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r > 0$.

(\implies) 1. Assume $f(n)$ is $\Theta(g(n))$.

2. By definition of Θ -notation, there exist positive real numbers A, B , and $k \geq r$ such that $Ag(n) \leq f(n) \leq Bg(n)$ for every integer $n \geq k$.

3. By 2 and assumption, $0 \leq f(n) \leq Bg(n)$ for all $n \geq k$, so by definition of O -notation, $f(n)$ is $O(g(n))$.

4. By 2, $Ag(n) \leq f(n)$ for all $n \geq k$, so by definition of Ω -notation, $f(n)$ is $\Omega(g(n))$.

(\impliedby) 1. Assume $f(n)$ is $\Omega(g(n))$ and $f(n)$ is $O(g(n))$.

2. By 1 and definition of Ω -notation, there exist positive real numbers A and $a \geq r$ such that $Ag(n) \leq f(n)$ for every integer $n \geq a$.

3. By 1 and definition of O -notation, there exist positive real numbers B and $b \geq r$ such that $0 \leq f(n) \leq Bg(n)$ for every integer $n \geq b$.

4. Let $c = \max(a, b)$. Then by 2 and 3, for every $n \geq c$, $Ag(n) \leq f(n) \leq Bg(n)$. So by definition of Θ -notation, $f(n)$ is $\Theta(g(n))$. \square

2.20 Exercise 20

Without using Theorem 11.2.4 prove that n^5 is not $O(n^2)$.

Proof. Suppose not. That is, suppose n^5 is $O(n^2)$. [We must show that this supposition leads to a contradiction.] By definition of O -notation, there exist positive real numbers B and b such that $0 \leq n^5 \leq Bn^2$ for each integer $n \geq b$. Dividing the inequalities by n^2 and taking the cube root of both sides gives $0 \leq n \leq \sqrt[3]{B}$ for each integer $n \geq b$. These two conditions are contradictory because on the one hand n can be any integer greater than or equal to b , but when n is greater than b , then n is less than $\sqrt[3]{B}$, which is a fixed integer. Thus the supposition leads to a contradiction, and hence the supposition is false. \square

2.21 Exercise 21

Prove Theorem 11.2.4: If f is a real-valued function defined on a set of nonnegative integers and $f(n)$ is $\Omega(n^m)$, where m is a positive integer, then $f(n)$ is not $O(n^p)$ for any positive real number $p < m$.

Proof. Assume m is a positive integer, p is a positive real number, $p < m$ and $f(n)$ is $\Omega(n^m)$.

By definition of Ω -notation there exist positive real numbers A and $a \geq 0$ such that $An^m \leq f(n)$ for every integer $n \geq a$. (We are taking $r = 0$ since $n^m \geq 0$ for all $n \geq 0$.)

Argue by contradiction and assume $f(n)$ is $O(n^p)$. By definition of O -notation, there exist positive real numbers B and $b \geq r$ such that $0 \leq f(n) \leq Bn^p$ for every integer $n \geq b$.

Let $c = \max(a, b)$. Then for all $n \geq c$ we have $An^m \leq f(n) \leq Bn^p$. In particular, $An^m \leq Bn^p$ for all $n \geq c$. Dividing by An^p we get $n^{m-p} \leq \frac{B}{A}$ for all $n \geq c$. Since $m - p > 0$, this is a contradiction: the left hand side is a function that grows without bound as n gets larger, and the right hand side is a positive constant.

So our supposition was false, and $f(n)$ is not $O(n^p)$. \square

2.22 Exercise 22

2.22.1 (a)

Use one of the methods of Example 11.2.4 to show that $2n^4 - 90n^3 + 3$ is $\Omega(n^4)$.

Proof. To use the general procedure from Example 11.2.4 to show that $2n^4 - 90n^3 + 3$ is $\Omega(n^4)$, let $A = \frac{1}{2} \cdot 2 = 1$ and $a = \frac{2}{2}(|-90| + |3|) = 93$ and note that $a \geq 1$. We will show that $n^4 \leq 2n^4 - 90n^3 + 3$ for every integer $n \geq a$. Now $n \geq a$ means that $n \geq 90 + 3$. Multiplying both sides by n^3 gives $n^4 \geq 90n^3 + 3n^3$ and subtracting first $3n^3$ and then 3 from the right-hand side gives that $n^4 \geq 90n^3 \geq 90n^3 - 3$ for every integer $n \geq a$. Subtracting the right-hand side from the left-hand side and adding n^4 to both sides gives $2n^4 - 90n^3 + 3 \geq n^4$ for every integer $n \geq a$. Thus since $A = 1$, $2n^4 - 90n^3 + 3 \geq An^4$ for every integer $n \geq a$, and so, by definition of Ω -notation, $2n^4 - 90n^3 + 3$ is $\Omega(n^4)$. \square

2.22.2 (b)

Show that $2n^4 - 90n^3 + 3$ is $O(n^4)$.

Proof. To show that $2n^4 - 90n^3 + 3$ is $O(n^4)$, observe that for every integer $n \geq 1$, $2n^4 - 90n^3 + 3 \leq 2n^4 + 90n^3 + 3$ because when $n \geq 1$, then $90n^3$ is positive,

$\leq 2n^4 + 90n^4 + 3n^4$ by Theorem 11.2.2 (since $n \geq 1$, $n^3 \leq n^4$ and $1 \leq n^4$, $90n^3 \leq 90n^4$ and $3 \leq 3n^4$),

and so $= 95n^4$ because $2 + 90 + 3 = 95$. Thus, by transitivity of order and equality, for every integer $n \geq 1$, $2n^4 - 90n^3 + 3 \leq 95n^4$.

In addition, by part (a), for every integer $n \geq 60$, $\frac{1}{2}n^4 \leq 2n^4 - 90n^3 + 3$ so since $0 \leq 12n^4$, transitivity of order gives that for every integer $n \geq 60$, $0 \leq 2n^4 - 90n^3 + 3 \leq 95n^4$.

Let $B = 95$ and $b = 60$. Then, for every integer $n \geq b$, $0 \leq 2n^4 - 90n^3 + 3 \leq Bn^4$ and hence, by definition of O -notation, $2n^4 - 90n^3 + 3$ is $O(n^4)$. \square

2.22.3 (c)

Justify the conclusion that $2n^4 - 90n^3 + 3$ is $\Theta(n^4)$.

Proof. By parts (a) and (b), $2n^4 - 90n^3 + 3$ is both $\Omega(n^4)$ and $O(n^4)$. Hence, by Theorem 11.2.1, $2n^4 - 90n^3 + 3$ is $\Theta(n^4)$. \square

2.23 Exercise 23

2.23.1 (a)

Use one of the methods of Example 11.2.4 to show that $\frac{1}{5}n^2 - 42n - 8$ is $\Omega(n^2)$.

Proof. Let $f(n) = \frac{1}{5}n^2 - 42n - 8$.

To find the lower bound, let us follow the procedure. Let $A = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$. Let $a = \frac{2}{1/5}(|-42| + |-8|) = 500$. Now we need to show that $\frac{1}{10}n^2 \leq f(n)$ for $n \geq 500$.

Assume $n \geq 500$, which means $n \geq 10(42 + 8)$. Divide by 10 and multiply by n to get $\frac{1}{10}n^2 \geq 42n + 8n$. Subtract $8n - 8$ from the right hand side to get $42n + 8n \geq 42n + 8$ (because when $n \geq 500$, $8n - 8 \geq 0$, so subtracting a positive number makes it smaller). So by transitivity of order, $\frac{1}{10}n^2 \geq 42n + 8$. Subtract right hand side from left hand side to get $\frac{1}{10}n^2 - 42n - 8 \geq 0$. Now add $\frac{1}{10}n^2$ to both sides to get $\frac{1}{5}n^2 - 42n - 8 \geq \frac{1}{10}n^2$ for all $n \geq 500$. So by definition of Ω -notation, $f(n)$ is $\Omega(n^2)$. \square

2.23.2 (b)

Show that $\frac{1}{5}n^2 - 42n - 8$ is $O(n^2)$.

Proof. Setting $f(n) = 0$ we find

$$n = \frac{42 \pm \sqrt{(-42)^2 - 4(1/5)(-8)}}{2/5} = 105 \pm \sqrt{11065} \approx 0 \text{ and } 210,$$

so $f(n) \geq 0$ for all $n \geq 211$.

To find the upper bound, we can replace $\frac{1}{5}n^2 - 42n - 8$ with the bigger $n^2 + 42n^2 + 8n^2 = 51n^2$. So $0 \leq f(n) \leq 51n^2$ for all $n \geq 211$, so $f(n)$ is $O(n^2)$. \square

2.23.3 (c)

Justify the conclusion that $\frac{1}{5}n^2 - 42n - 8$ is $\Theta(n^2)$.

Proof. By parts (a) and (b), $f(n)$ is both $\Omega(n^2)$ and $O(n^2)$. By Theorem 11.2.1, $f(n)$ is $\Theta(n^2)$. \square

2.24 Exercise 24

2.24.1 (a)

Use one of the methods of Example 11.2.4 to show that $\frac{1}{4}n^5 - 50n^3 + 3n + 12$ is $\Omega(n^5)$.

Proof. $A = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$, $a = \frac{2}{1/4}(|-50| + |3| + |12|) = 8(50 + 3 + 12) = 520$.

Assume $n \geq 520 = 8(50 + 3 + 12)$.

Divide by 8 and multiply by n^4 to get $\frac{1}{8}n^5 \geq 50n^4 + 3n^4 + 12n^4$.

From the right hand side, subtract $50n^4 - 50n^3$ to get $50n^4 + 3n^4 + 12n^4 \geq 50n^3 + 3n^4 + 12n^4$ (because when $n \geq 520$ we have $12n^4 - 12n^3 = 12n^3(n - 1) \geq 0$ so subtracting something positive makes it smaller).

From the right hand side, subtract $3n^4 + 3n$ to get $50n^3 + 3n^4 + 12n^4 \geq 50n^3 - 3n + 12n^4$ (because when $n \geq 520$ we have $3n^4 + 3n = 3n(n^3 + 1) \geq 0$ so subtracting something positive makes it smaller).

From the right hand side, subtract $12n^4 + 12$ to get $50n^3 - 3n + 12n^4 \geq 50n^3 - 3n - 12$ (because when $n \geq 520$ we have $12n^4 + 12n = 12(n^4 + 1) \geq 0$ so subtracting something positive makes it smaller).

By transitivity of order we get $\frac{1}{8}n^5 \geq 50n^3 - 3n - 12$. Moving everything to the left hand side, we get $\frac{1}{8}n^5 - 50n^3 + 3n + 12 \geq 0$. Now add $\frac{1}{8}n^5$ to both sides to finally get $\frac{1}{4}n^5 - 50n^3 + 3n + 12 \geq \frac{1}{8}n^5$ for all $n \geq 520$.

So by definition of Ω -notation, $f(n)$ is $\Omega(n^5)$. □

2.24.2 (b)

Show that $\frac{1}{4}n^5 - 50n^3 + 3n + 12$ is $O(n^5)$.

Proof. Setting $f(n) = 0$ we find $n \approx -14, 1, 14$. So $f(n) \geq 0$ for all $n \geq 15$.

$\frac{1}{4}n^5 - 50n^3 + 3n + 12 \leq n^5 + 50n^5 + 3n^5 + 12n^5 = 66n^5$ for all $n \geq 1$.

Therefore $0 \leq f(n) \leq 66n^5$ for all $n \geq 15$. By definition of O -notation, $f(n)$ is $O(n^5)$. □

2.24.3 (c)

Justify the conclusion that $\frac{1}{4}n^5 - 50n^3 + 3n + 12$ is $\Theta(n^5)$.

Proof. By parts (a) and (b), $f(n)$ is both $\Omega(n^5)$ and $O(n^5)$. By Theorem 11.2.1, $f(n)$ is $\Theta(n^5)$. □

2.25 Exercise 25

Suppose $P(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$, where all the coefficients a_0, a_1, \dots, a_m are real numbers and $a_m > 0$.

2.25.1 (a)

Prove that $P(n)$ is $\Omega(n^m)$ by using the general procedure described in Example 11.2.4.

Proof. Let $A = \frac{1}{2}a_m$, $d = \frac{2}{a_m}(|a_{m-1}| + \cdots + |a_0|)$ and $a = \max(d, 1)$. Then $n \geq a$ means that $n \geq \frac{2}{a_m}(|a_{m-1}| + \cdots + |a_0|)$. Multiplying both sides by $\frac{1}{2}a_m n^{m-1}$ gives

$$\frac{1}{2}a_m n^m \geq (|a_{m-1}| + \cdots + |a_0|)n^{m-1} = |a_{m-1}|n^{m-1} + |a_{m-2}|n^{m-1} + \cdots + |a_1|n^{m-1} + |a_0|n^{m-1}$$

which is $\geq |a_{m-1}|n^{m-1} + |a_{m-2}|n^{m-2} + \cdots + |a_1|n^1 + |a_0|n^0$. So by transitivity of order

$$\frac{1}{2}a_m n^m \geq |a_{m-1}|n^{m-1} + |a_{m-2}|n^{m-2} + \cdots + |a_1|n^1 + |a_0|n^0.$$

Subtracting the right hand side gives

$$\frac{1}{2}a_m n^m - |a_{m-1}|n^{m-1} - |a_{m-2}|n^{m-2} - \cdots - |a_1|n^1 - |a_0|n^0 \geq 0.$$

Since each $a_i \geq -|a_i|$, we have

$$\frac{1}{2}a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1n + a_0 \geq \frac{1}{2}a_m n^m - |a_{m-1}|n^{m-1} - \cdots - |a_1|n^1 - |a_0|n^0.$$

By transitivity of order $\frac{1}{2}a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1n + a_0 \geq 0$. Add $\frac{1}{2}a_m n^m$ to both sides to get $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_1n + a_0 \geq \frac{1}{2}a_m n^m$. So by definition of Ω notation, $P(n)$ is $\Omega(n^m)$. \square

2.25.2 (b)

Prove that $P(n)$ is $O(n^m)$.

Proof. For all $n \geq 1$ we have $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2n^2 + a_1n + a_0$
 $\leq |a_m|n^m + |a_{m-1}|n^m + \cdots + |a_2|n^m + |a_1|n^m + |a_0|n^m = (|a_m| + \cdots + |a_0|)n^m.$

Let $B = |a_m| + \cdots + |a_0|$. Then, by transitivity of order and equality, for each integer $n \geq 1$, $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2n^2 + a_1n + a_0 \leq Bn^m$.

In addition, by part (a), there exists a positive real number a such that for each integer $n \geq a$, $\frac{a_m}{2}n^m \leq a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2n^2 + a_1n + a_0$.

Now $\frac{a_m}{2}n^m > 0$ because $a_m > 0$, and thus, transitivity of order gives that for each integer $n \geq a$, $0 \leq a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2n^2 + a_1n + a_0$.

And hence, by definition of O -notation, $a_m n^m + a_{m-1}n^{m-1} + \cdots + a_2n^2 + a_1n + a_0$ is $O(n^m)$. \square

2.25.3 (c)

Justify the conclusion that $P(n)$ is $\Theta(n^m)$.

Proof. By parts (a) and (b), $a_m n^m + a_{m-1} n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$ is both $\Omega(n^m)$ and $O(n^m)$. Hence, by Theorem 11.2.1, it is $\Theta(n^m)$. \square

Use the theorem on polynomial orders to prove each of the statements in 26 – 31.

2.26 Exercise 26

$$\frac{(n+1)(n-2)}{4} \text{ is } \Theta(n^2)$$

Proof. $\frac{(n+1)(n-2)}{4} = \frac{n^2 - n - 2}{4} = \frac{1}{4}n^2 - \frac{1}{4}n - \frac{1}{2}$, which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

2.27 Exercise 27

$$\frac{n}{3}(4n^2 - 1) \text{ is } \Theta(n^3)$$

Proof. $\frac{n}{3}(4n^2 - 1) = \frac{4}{3}n^3 - \frac{1}{3}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders. \square

2.28 Exercise 28

$$\frac{n(n-1)}{2} + 3n \text{ is } \Theta(n^2)$$

Proof. $\frac{n(n-1)}{2} + 3n = \frac{n^2 - n}{2} + 3n = \frac{1}{2}n^2 + \frac{5}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

2.29 Exercise 29

$$\frac{n(n-1)(2n+1)}{6} \text{ is } \Theta(n^3)$$

Proof. $\frac{n(n-1)(2n+1)}{6} = \frac{(n^2 - n)(2n+1)}{6} = \frac{2n^3 - n^2 - n}{6} = \frac{1}{3}n^3 - \frac{1}{6}n^2 - \frac{1}{6}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders. \square

2.30 Exercise 30

$$\left[\frac{n(n+1)}{2} \right]^2 \text{ is } \Theta(n^4)$$

Proof. $\left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \frac{n^2(n^2+2n+1)}{4} = \frac{n^4+2n^3+n^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$, which is $\Theta(n^4)$ by the theorem on polynomial orders. \square

2.31 Exercise 31

$2(n-1) + \frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right)$ is $\Theta(n^2)$

Proof. $2(n-1) + \frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right) = 2n-2 + \frac{1}{2}n^2 + \frac{1}{2}n + 2n^2 - 2n = \frac{5}{2}n^2 + \frac{1}{2}n - 2$, which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

Prove each of the statements in 32–39. Use the theorem on polynomial orders and results from the theorems and exercises in Section 5.2 as appropriate.

2.32 Exercise 32

$1^2 + 2^2 + 3^2 + \cdots + n^2$ is $\Theta(n^3)$

Proof. By exercise 10 of Section 5.2, this sum equals $\frac{n(n-1)(2n+1)}{6}$, which is $\Theta(n^3)$ by Exercise 29 above. \square

2.33 Exercise 33

$1^3 + 2^3 + 3^3 + \cdots + n^3$ is $\Theta(n^4)$

Proof. By exercise 11 of Section 5.2, this sum equals $\left[\frac{n(n+1)}{2}\right]^2$, which is $\Theta(n^4)$ by Exercise 30 above. \square

2.34 Exercise 34

$2 + 4 + 6 + \cdots + 2n$ is $\Theta(n^2)$

Proof. $2 + 4 + 6 + \cdots + 2n = 2(1 + 2 + 3 + \cdots + n) = 2 \cdot \frac{n(n+1)}{2} = n^2 + n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

2.35 Exercise 35

$5 + 10 + 15 + 20 + 25 + \cdots + 5n$ is $\Theta(n^2)$

Proof. $5 + 10 + 15 + 20 + 25 + \cdots + 5n = 5(1 + 2 + 3 + \cdots + n) = 5 \cdot \frac{n(n+1)}{2} = \frac{5}{2}n^2 + \frac{5}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

2.36 Exercise 36

$$\sum_{i=1}^n (4i - 9) \text{ is } \Theta(n^2)$$

Proof. $\sum_{i=1}^n (4i - 9) = 4 \sum_{i=1}^n i - 9 \sum_{i=1}^n 1 = 4 \cdot \frac{n(n+1)}{2} - 9n = 2n^2 + 2n - 9n = 2n^2 - 7n$

which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

2.37 Exercise 37

$$\sum_{k=1}^n (k + 3) \text{ is } \Theta(n^2)$$

Proof. $\sum_{k=1}^n (k + 3) = \sum_{k=1}^n k + 3 \sum_{k=1}^n 1 = \frac{n(n+1)}{2} + 3n = \frac{1}{2}n^2 + \frac{7}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders. \square

2.38 Exercise 38

$$\sum_{i=1}^n i(i + 1) \text{ is } \Theta(n^3)$$

Proof. $\sum_{i=1}^n i(i + 1) = \sum_{i=1}^n i^2 + \sum_{i=1}^n i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{2n^3 + 3n^2 + n}{6} + \frac{n^2 + n}{2} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n$, which is $\Theta(n^3)$ by the theorem on polynomial orders. \square

2.39 Exercise 39

$$\sum_{k=3}^n (k^2 - 2k) \text{ is } \Theta(n^3)$$

Proof. $\sum_{k=3}^n (k^2 - 2k) = \sum_{k=1}^n (k^2 - 2k) - (1^2 - 2 \cdot 1 + 2^2 - 2 \cdot 2) = \sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k - (-1) = \frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{n(n+1)}{2} + 1 = \frac{2n^3 + 3n^2 + n}{6} + n^2 + n + 1 = \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{7}{6}n + 1$, which is $\Theta(n^3)$ by the theorem on polynomial orders. \square

2.40 Exercise 40

2.40.1 (a)

Prove: If c is a positive real number and if f is a real-valued function defined on a set of nonnegative integers with $f(n) \geq 0$ for every integer n greater than or equal to some positive real number, then $cf(n)$ is $\Theta(f(n))$.

Proof. Suppose c is a positive real number and f is a real-valued function defined on a set of nonnegative integers with $f(n) \geq 0$ for each integer n greater than or equal to a positive real number k . Now if we let $A = B = c$, we have that for each integer $n \geq k$, $Af(n) \leq cf(n) \leq Bf(n)$ and so, by definition of Θ -notation, $cf(n)$ is $\Theta(f(n))$. \square

2.40.2 (b)

Use part (a) to show that $3n$ is $\Theta(n)$.

Proof. Let $c = 3$ and $f(n) = n$. Then f is a real-valued function and $f(n) \geq 0$ for each integer $n \geq 0$. So by part (a), $cf(n)$ is $\Theta(f(n))$, or, by substitution, $3n$ is $\Theta(n)$. \square

2.41 Exercise 41

Prove: If c is a positive real number and $f(n) = c$ for every integer $n \geq 1$, then $f(n)$ is $\Theta(1)$.

Proof. Assume c is a positive real number and $f(n) = c$ for every integer $n \geq 1$. Then let $A = B = c$ and $k = 1$. Then $A \cdot 1 \leq f(n) \leq B \cdot 1$ for all $n \geq k$, so by definition, $f(n)$ is $\Theta(1)$. \square

2.42 Exercise 42

What can you say about a function f with the property that $f(n)$ is $\Theta(1)$?

Proof. If $f(n)$ is $\Theta(1)$ then by definition, there are positive reals A, B and a positive integer k such that $A \leq f(n) \leq B$ for all $n \geq k$. So the graph of f is trapped between the two horizontal lines $y = A$ and $y = B$ for $n \geq k$. \square

Use Theorems 11.2.5 – 11.2.9 and the results of exercises 15 – 17, 40, and 41 to justify the statements in 43 – 45.

2.43 Exercise 43

$\left\lfloor \frac{n+1}{2} \right\rfloor + 3n$ is $\Theta(n)$

Proof. By exercise 15 $\left\lfloor \frac{n+1}{2} \right\rfloor$ is $\Theta(n)$ and by exercise 40 (b) $3n$ is $\Theta(n)$. Thus $\left\lfloor \frac{n+1}{2} \right\rfloor + 3n$ is $\Theta(n)$ by Theorem 11.2.9(a). \square

2.44 Exercise 44

$\frac{n(n-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor + 1$ is $\Theta(n^2)$

Proof. By exercise 28 $\frac{n(n-1)}{2}$ is $\Theta(n^2)$, by exercise 17 $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$ and by exercise 41 (with $f(n) = 1$), 1 is $\Theta(1)$. So $\frac{n(n-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor + 1$ is $\Theta(n^2)$ by Theorem 11.2.9(c). \square

2.45 Exercise 45

$\left\lfloor \frac{n}{2} \right\rfloor + 4n + 3$ is $\Theta(n)$

Proof. By exercise 17 $\left\lfloor \frac{n}{2} \right\rfloor$ is $\Theta(n)$, by exercise 40 (b) $4n$ is $\Theta(n)$, and by exercise 41 (with $f(n) = 3$), 3 is $\Theta(1)$. So $\left\lfloor \frac{n}{2} \right\rfloor + 4n + 3$ is $\Theta(n)$ by Theorem 11.2.9(c). \square

2.46 Exercise 46

2.46.1 (a)

Use mathematical induction to prove that if n is any integer with $n > 1$, then for every integer $m \geq 1$, $n^m > 1$.

Proof. Let the property $P(m)$ be the sentence: “If n is any integer with $n > 1$, then $n^m > 1$ ”.

Show that $P(1)$ is true: We must show that if n is any integer with $n > 1$, then $n^1 > 1$. But this is true because $n^1 = n$. So $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be a particular but arbitrarily chosen integer with $k \geq 1$, and suppose that if n is any integer with $n > 1$, then $n^k > 1$.

We must show that if n is any integer with $n > 1$, then $n^{k+1} > 1$.

So suppose n is any integer with $n > 1$. By inductive hypothesis, $n^k > 1$, and multiplying both sides by the positive number n gives $n \cdot n^k > n \cdot 1$, or, equivalently, $n^{k+1} > n$. Thus $n^{k+1} > n$ and $n > 1$, and so, by transitivity of order, $n^{k+1} > 1$, [as was to be shown]. \square

2.46.2 (b)

Prove that if n is any integer with $n > 1$, then $n^r < n^s$ for all integers r and s with $r < s$.

Proof. Suppose n is any integer with $n > 1$ and r and s are integers with $r < s$. Then $s - r$ is an integer with $s - r \geq 1$, and so, by part (a), $n^{s-r} > 1$. Multiplying both sides by n^r gives $n^r \cdot n^{s-r} > n^r \cdot 1$, and so, by the laws of exponents, $n^s > n^r$ [as was to be shown]. \square

2.47 Exercise 47

2.47.1 (a)

Let x be any positive real number. Use mathematical induction to prove that for every integer $m \geq 1$, if $x \leq 1$ then $x^m \leq 1$.

Proof. Let the property $P(m)$ be the sentence “If $0 < x \leq 1$, then $x^m \leq 1$ ”.

Show that $P(1)$ is true: We must show that if $0 < x \leq 1$, then $x^1 \leq 1$. But $x \leq 1$ by assumption and $x^1 = x$. So $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that if $0 < x \leq 1$, then $x^k \leq 1$ (inductive hypothesis). We must show that if $0 < x \leq 1$, then $x^{k+1} \leq 1$.

So let x be any number with $0 < x \leq 1$. By inductive hypothesis, $x^k \leq 1$, and multiplying both sides of this inequality by the nonnegative number x gives $x \cdot x^k \leq x^1$. Thus, by the laws of exponents, $x^{k+1} \leq x$. Then $x^{k+1} \leq x$ and $x \leq 1$, and hence, by the transitive property of order (T18 in Appendix A), $x^{k+1} \leq 1$. \square

2.47.2 (b)

Explain how it follows from part (b) that if x is any positive real number, then for every integer $m \geq 1$, if $x^m > 1$ then $x > 1$.

Proof. This is the contrapositive of the statement in part (a), therefore it's true. \square

2.47.3 (c)

Explain how it follows from part (b) that if x is any positive real number, then for every integer $m \geq 1$, if $x > 1$ then $x^{1/m} > 1$.

Proof. Let $y = x^{1/m}$. Then by part (b), with y replacing x , we have: if $y^m > 1$ then $y > 1$. Now substitute $y = x^{1/m}$ to get: if $(x^{1/m})^m > 1$ then $x^{1/m} > 1$. In other words: if $x > 1$ then $x^{1/m} > 1$. \square

2.47.4 (d)

Let p, q, r , and s be positive integers, and suppose $p/q > r/s$. Use part (c) and the result of exercise 46 to prove Theorem 11.2.2. In other words, show that for any integer n , if $n > 1$ then $n^{p/q} > n^{r/s}$.

Proof. 1. Assume n is any integer with $n > 1$, p, q, r , and s are positive integers with $p/q > r/s$.

2. Notice $ps > qr$, so by part exercise 46 (b), $n^{ps} > n^{qr}$. By algebra, $\frac{n^{ps}}{n^{qr}} > 1$.

3. Let $x = \frac{n^{ps}}{n^{qr}}$. By 2, $x > 1$. Therefore by part (c), $x^{1/s} > 1$.

4. Rewriting 3, $\left(\frac{n^{ps}}{n^{qr}}\right)^{1/s} > 1$. So by law of exponents $\frac{n^p}{n^{qr/s}} > 1$.
5. Let $y = \frac{n^p}{n^{qr/s}}$. By 4, $y > 1$. So by part (c) $y^{1/q} > 1$.
6. Rewriting 5, $\left(\frac{n^p}{n^{qr/s}}\right)^{1/q} > 1$. So by law of exponents $\frac{n^{p/q}}{n^{r/s}} > 1$.
7. By 6 and algebra, $n^{p/q} > n^{r/s}$. □

2.48 Exercise 48

Prove Theorem 11.2.6(b): If f and g are real-valued functions defined on the same set of nonnegative integers, and if there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$, and if $g(n)$ is $O(f(n))$, then $f(n)$ is $\Omega(g(n))$.

Proof. Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for each integer $n \geq r$. Suppose also that $g(n)$ is $O(f(n))$. We will show that $f(n)$ is $\Omega(g(n))$. By definition of O -notation, there are positive real numbers B and b such that $b \geq r$, and, for each integer $n \geq b$, $0 \leq g(n) \leq Bf(n)$. Divide the right-hand inequality by B to obtain $\frac{1}{B}g(n) \leq f(n)$, for each integer $n \geq b$. Let $A = 1/B$ and $a = b$. Then for each integer $n \geq a$, $Ag(n) \leq f(n)$ and so $f(n)$ is $\Omega(g(n))$ by definition of Ω -notation. □

2.49 Exercise 49

Prove Theorem 11.2.7(a): If f is a real-valued function defined on a set of nonnegative integers and there is a real number r such that $f(n) \geq 0$ for every integer $n \geq r$, then $f(n)$ is $\Theta(f(n))$.

Proof. Since $f(n) \geq 0$ for all $n \geq r$, and since $f(n) \leq f(n) \leq f(n)$, we can let $g(n) = f(n)$, $A = B = 1$ and $k = r$ in the definition of Θ -notation to obtain that $f(n)$ is $\Theta(f(n))$. □

2.50 Exercise 50

Prove Theorem 11.2.8:

2.50.1 (a)

Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) > 0$ for every integer $n \geq r$. If $f(n)$ is $\Omega(g(n))$ and c is any positive real number, then $cf(n)$ is $\Omega(g(n))$.

Proof. Assume r is a positive real number such that $f(n) \geq 0$ and $g(n) > 0$ for every integer $n \geq r$, and $f(n)$ is $\Omega(g(n))$. Assume c is any positive real number.

By definition of Ω -notation, there exist positive real numbers A and $a \geq r$ such that $Ag(n) \leq f(n)$ for every integer $n \geq a$.

Let $A' = cA$ and $a' = a$. Then A' and $a' \geq r$ are positive real numbers, and by 2 $A'g(n) = cAg(n) \leq cf(n)$ for every integer $n \geq a'$. So by definition of Ω -notation, $cf(n)$ is $\Omega(g(n))$. \square

2.50.2 (b)

Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$. If $f(n)$ is $O(g(n))$ and c is any positive real number, then $cf(n)$ is $O(g(n))$.

Proof. The proof is almost identical to part (a), except start with $0 \leq f(n) \leq Bg(n)$ for every integer $n \geq b$, let $B' = cB$, $b' = b$ and end with $0 \leq cf(n) \leq cBg(n) = B'g(n)$ for every integer $n \geq b'$. \square

2.50.3 (c)

Let f and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$. If $f(n)$ is $\Theta(g(n))$ and c is any positive real number, then $cf(n)$ is $\Theta(g(n))$.

Proof. This follows from parts (a) and (b) and Theorem 11.2.1. \square

2.51 Exercise 51

Prove Theorem 11.2.9:

2.51.1 (a)

Let f_1, f_2 and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0$, $f_2(n) \geq 0$ and $g(n) \geq 0$ for every integer $n \geq r$. If $f_1(n)$ is $\Theta(g(n))$ and $f_2(n)$ is $\Theta(g(n))$, then $(f_1(n) + f_2(n))$ is $\Theta(g(n))$.

Proof. Let f_1, f_2 , and g be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0$, $f_2(n) \geq 0$, and $g(n) \geq 0$ for each integer $n \geq r$. Suppose also that $f_1(n)$ is $\Theta(g(n))$ and $f_2(n)$ is $\Theta(g(n))$. [We will show that $(f_1(n) + f_2(n))$ is $\Theta(g(n))$.] By definition of Θ -notation, there are positive real numbers A, B, A', B', k , and k' such that $k \geq r, k' \geq r$ and, for each integer n such that $n \geq k$ and $n \geq k'$, $Ag(n) \leq f_1(n) \leq Bg(n)$ and $A'g(n) \leq f_2(n) \leq B'g(n)$.

Notice that $Ag(n) + A'g(n) \leq f_1(n) + f_2(n) \leq Bg(n) + B'g(n)$ for every integer $n \geq \max(k, k')$. Let $k'' = \max(k, k')$, $A'' = A + A'$ and $B'' = B + B'$. So $A''g(n) \leq f_1(n) + f_2(n) \leq B''g(n)$ for every integer $n \geq k''$. Then by definition of Θ -notation, $(f_1(n) + f_2(n))$ is $\Theta(g(n))$. \square

2.51.2 (b)

Let f_1, f_2, g_1 , and g_2 be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0, f_2(n) \geq 0, g_1(n) \geq 0$, and $g_2(n) \geq 0$ for every integer $n \geq r$. If $f_1(n)$ is $\Theta(g_1(n))$ and $f_2(n)$ is $\Theta(g_2(n))$, then $(f_1(n)f_2(n))$ is $\Theta(g_1(n)g_2(n))$.

Proof. The proof is almost identical to part (a), except in the crucial step we have $AA'g_1(n)g_2(n) \leq f_1(n)f_2(n) \leq BB'g_1(n)g_2(n)$ for every integer $n \geq \max(k, k')$. \square

2.51.3 (c)

Let f_1, f_2, g_1 , and g_2 be real-valued functions defined on the same set of nonnegative integers, and suppose there is a positive real number r such that $f_1(n) \geq 0, f_2(n) \geq 0, g_1(n) \geq 0$, and $g_2(n) \geq 0$ for every integer $n \geq r$. If $f_1(n)$ is $\Theta(g_1(n))$ and $f_2(n)$ is $\Theta(g_2(n))$ and if there is a real number s so that $g_1(n) \leq g_2(n)$ for every integer $n \geq s$, then $(f_1(n) + f_2(n))$ is $\Theta(g_2(n))$.

Proof. The proof is almost identical to part (a), except in the crucial step we have:

$$A'g_2(n) \leq Ag_1(n) + A'g_2(n) \leq f_1(n) + f_2(n) \leq Bg_1(n) + B'g_2(n) \leq Bg_2(n) + B'g_2(n)$$

for all $n \geq \max(s, k, k')$. So we can let $A'' = A', B'' = B + B'$ and $k'' = \max(s, k, k')$. Then for every integer $n \geq k''$, we have $A''g_2(n) \leq f_1(n) + f_2(n) \leq B''g_2(n)$. \square

3 Exercise Set 11.3

3.1 Exercise 1

Suppose a computer takes 1 nanosecond ($= 10^{-9}$ second) to execute each operation. Approximately how long will it take the computer to execute the following numbers of operations? Convert your answers into seconds, minutes, hours, days, weeks, or years, as appropriate. For example, instead of 2^{50} nanoseconds, write 13 days.

3.1.1 (a)

$\log_2 200$

Proof. $\log_2(200) = \frac{\ln 200}{\ln 2} \approx 7.6$ nanoseconds $= 0.0000000076$ second \square

3.1.2 (b)

200

Proof. $200 \cdot 10^{-9} = 2 \cdot 10^{-7} = 0.0000002$ seconds □

3.1.3 (c)

$200 \log_2 200$

Proof. $\approx 200 \cdot 7.6 \cdot 10^{-9} = 0.000001520$ seconds □

3.1.4 (d)

200^2

Proof. $200^2 = 40,000$ nanoseconds = 0.00004 second □

3.1.5 (e)

200^8

Proof. $200^8 = 2.56 \times 10^{18}$ nanoseconds $\frac{2.56 \times 10^9}{10^9 \cdot 60 \cdot 60 \cdot 24 \cdot (365.25)} \approx 81.1215$ years [because there are 10^9 nanoseconds in a second, 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day, and approximately 365.25 days in a year on average]. □

3.1.6 (f)

2^{200}

Proof. $2^{200} = (2^{50})^4 = 13^4$ days (since 2^{50} nanoseconds = 13 days) = 28561 days ≈ 78.2 years □

3.2 Exercise 2

Suppose an algorithm requires cn^2 operations when performed with an input of size n (where c is a constant).

3.2.1 (a)

How many operations will be required when the input size is increased from m to $2m$ (where m is a positive integer)?

Proof. When the input size is increased from m to $2m$, the number of operations increases from cm^2 to $c(2m)^2 = 4cm^2$. □

3.2.2 (b)

By what factor will the number of operations increase when the input size is doubled?

Proof. By part (a), the number of operations increases by a factor of $(4cm^2)/(cm^2) = 4$. \square

3.2.3 (c)

By what factor will the number of operations increase when the input size is increased by a factor of ten?

Proof. When the input size is increased by a factor of 10 (from m to $10m$), the number of operations increases by a factor of $(c(10m)^2)/(cm^2) = (100cm^2)/(cm^2) = 100$. \square

3.3 Exercise 3

Suppose an algorithm requires cn^3 operations when performed with an input of size n (where c is a constant).

3.3.1 (a)

How many operations will be required when the input size is increased from m to $2m$ (where m is a positive integer)?

Proof. When the input size is increased from m to $2m$, the number of operations increases from cm^3 to $c(2m)^3 = 8cm^3$. \square

3.3.2 (b)

By what factor will the number of operations increase when the input size is doubled?

Proof. By part (a), the number of operations increases by a factor of $(8cm^3)/(cm^3) = 8$. \square

3.3.3 (c)

By what factor will the number of operations increase when the input size is increased by a factor of ten?

Proof. When the input size is increased by a factor of 10 (from m to $10m$), the number of operations increases by a factor of $(c(10m)^3)/(cm^3) = (1000cm^3)/(cm^3) = 1000$. \square

Exercises 4–5 explore the fact that for relatively small values of n , algorithms with larger orders can be more efficient than algorithms with smaller orders.

3.4 Exercise 4

Suppose that when run with an input of size n , algorithm A requires $2n^2$ operations and algorithm B requires $80n^{3/2}$ operations.

3.4.1 (a)

What are orders for algorithms A and B from among the set of power functions?

Proof. Algorithm A has order n^2 and algorithm B has order $n^{3/2}$. □

3.4.2 (b)

For what values of n is algorithm A more efficient than algorithm B ?

Proof. Algorithm A is more efficient than algorithm B when $2n^2 < 80n^{3/2}$. This occurs exactly when

$$n^2 < 40n^{3/2} \iff \frac{n^2}{40n^{3/2}} < 40 \iff n^{1/2} < 40 \iff n < 40^2.$$

Thus, algorithm A is more efficient than algorithm B when $n < 40^2 = 1,600$. □

3.4.3 (c)

For what values of n is algorithm B at least 100 times more efficient than algorithm A ?

Proof. Algorithm B is at least 100 times more efficient than algorithm A for values of n with $100(80n^{3/2}) \leq 2n^2$. This occurs exactly when

$$8,000n^{3/2} \leq 2n^2 \iff 4,000 \leq \frac{n^2}{n^{3/2}} \iff 4,000 \leq \sqrt{n} \iff 16,000,000 \leq n.$$

Thus, algorithm B is at least 100 times more efficient than algorithm A when $n \geq 16,000,000$. □

3.5 Exercise 5

Suppose that when run with an input of size n , algorithm A requires 10^6n^2 operations and algorithm B requires n^3 operations.

3.5.1 (a)

What are orders for algorithms A and B from among the set of power functions?

Proof. Algorithm A has order n^2 and algorithm B has order n^3 . □

3.5.2 (b)

For what values of n is algorithm A more efficient than algorithm B ?

Proof. Algorithm A is more efficient than algorithm B when $10^6 n^2 < n^3$. This occurs exactly when

$$10^6 n^2 < n^3 \iff 10^6 < \frac{n^3}{n^2} \iff 10^6 < n.$$

Thus, algorithm A is more efficient than algorithm B when $1,000,000 < n$. \square

3.5.3 (c)

For what values of n is algorithm B at least 100 times more efficient than algorithm A ?

Proof. Algorithm B is at least 100 times more efficient than algorithm A for values of n with $100(n^3) \leq 10^6 n^2$. This occurs exactly when

$$100(n^3) \leq 10^6 n^2 \iff \frac{n^3}{n^2} \leq \frac{10^6}{100} \iff n \leq 10^4.$$

Thus, algorithm B is at least 100 times more efficient than algorithm A when $n \leq 10,000$. \square

For each of the algorithm segments in 6 – 19, assume that n is a positive integer. (a) Compute the actual number of elementary operations (additions, subtractions, multiplications, divisions, and comparisons) that are performed when the algorithm segment is executed. For simplicity, however, count only comparisons that occur within if-then statements; ignore those implied by for-next loops. (b) Use the theorem on polynomial orders to find an order for the algorithm segment.

3.6 Exercise 6

```
for  $i := 3$  to  $n - 1$ 
     $a := 3 \cdot n + 2 \cdot i - 1$ 
next  $i$ 
```

3.6.1 (a)

Proof. There are two multiplications, one addition, and one subtraction for each iteration of the loop, so there are four times as many operations as there are iterations of the loop. The loop is iterated $(n - 1) - 3 + 1 = n - 3$ times (since the number of iterations equals the top minus the bottom index plus 1). Thus the total number of operations is $4(n - 3) = 4n - 12$. \square

3.6.2 (b)

Proof. By the theorem on polynomial orders, $4n - 12$ is $\Theta(n)$, so the algorithm segment has order n . \square

3.7 Exercise 7

```
max := a[1]
for i := 2 to n
    if max < a[i] then max := a[i]
next i
```

3.7.1 (a)

Proof. There is 1 comparison for every iteration of the loop. The loop is iterated $n - 2 + 1 = n - 1$ times. So the total number of operations is $n - 1$. \square

3.7.2 (b)

Proof. By the theorem on polynomial orders, $n - 1$ is $\Theta(n)$, so the algorithm segment has order n . \square

3.8 Exercise 8

```
a := 0
for i := 1 to  $\lfloor n/2 \rfloor$ 
    a := a + 3
next i
```

3.8.1 (a)

Proof. There is one addition for each iteration of the loop, and there are $\lfloor n/2 \rfloor$ iterations of the loop. \square

3.8.2 (b)

Proof. Because

$$\lfloor n/2 \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

$\lfloor n/2 \rfloor$ is $\Theta(n)$ by theorem on polynomial orders. So the algorithm segment has order n . \square

3.9 Exercise 9

```
s := 0
for i := 1 to n
    for j := 1 to 2n
        s := s + i · j
    next j
next i
```

3.9.1 (a)

Proof. For each iteration of the inner loop, there is one multiplication and one addition. There are $2n$ iterations of the inner loop for each iteration of the outer loop, and there are n iterations of the outer loop. Therefore, the number of iterations of the inner loop is $2n \cdot n = 2n^2$. It follows that the total number of elementary operations that must be performed when the algorithm is executed is $2 \cdot 2n^2 = 4n^2$. \square

3.9.2 (b)

Proof. Since $4n^2$ is $\Theta(n^2)$ (by the theorem on polynomial orders), the algorithm segment has order n^2 . \square

3.10 Exercise 10

```
for  $k := 2$  to  $n$ 
  for  $j := 1$  to  $3n$ 
     $x := a[k] - b[j]$ 
  next  $j$ 
next  $k$ 
```

3.10.1 (a)

Proof. There is 1 subtraction for each iteration of the inner loop. The inner loop iterates $3n - 1 + 1 = 3n$ times for each time the outer loop iterates. The outer loop iterates $n - 2 + 1 = n - 1$ times. So the total number of operations is $1 \cdot (n - 1) \cdot 3n = 3n^2 - 3n$. \square

3.10.2 (b)

Proof. $3n^2 - 3n$ is $\Theta(n^2)$ by theorem on polynomial orders, so the order of the algorithm is n^2 . \square

3.11 Exercise 11

```
for  $k := 1$  to  $n - 1$ 
  for  $j := 1$  to  $k + 1$ 
     $x := a[k] + b[j]$ 
  next  $j$ 
next  $k$ 
```

3.11.1 (a)

Proof. There is one addition for each iteration of the inner loop. The number of iterations in the inner loop equals the number of columns in the table below, which shows the values of k and j for which the inner loop is executed.

k	1		2			3				...	$n-1$				
j	1	2	1	2	3	1	2	3	4	...	1	2	3	...	n
	$\underbrace{\hspace{1.5cm}}$		$\underbrace{\hspace{1.5cm}}$			$\underbrace{\hspace{1.5cm}}$				$\underbrace{\hspace{2.5cm}}$					
	2		3			4				n					

Hence the total number of iterations of the inner loop is $2+3+\cdots+n = (1+2+3+\cdots+n)-1 = \frac{n(n+1)}{2}-1 = \frac{1}{2}n^2+\frac{1}{2}n-1$ (by Theorem 5.2.1). Because one operation is performed for each iteration of the inner loop, the total number of operations is $\frac{1}{2}n^2 + \frac{1}{2}n - 1$. \square

3.11.2 (b)

Proof. By the theorem on polynomial orders, $\frac{1}{2}n^2+\frac{1}{2}n-1$ is $\Theta(n^2)$, and so the algorithm segment has order n^2 . \square

3.12 Exercise 12

```

for  $k := 1$  to  $n - 1$ 
     $max := a[k]$ 
    for  $i := k + 1$  to  $n$ 
        if  $max < a[i]$  then  $max := a[i]$ 
    next  $i$ 
 $a[k] := max$ 
next  $k$ 

```

3.12.1 (a)

Proof. There is 1 comparison operation in the inner loop. The outer loop iterates $n-1-1+1 = n-1$ times. But the inner loop iterates a variable number of times. When $k=1$ the inner loop iterates $n-2+1 = n-1$ times, when $k=2$ the inner loop iterates $n-3+1 = n-2$ times, and so on. Finally when $k=n-1$ the inner loop iterates $n-(n-1+1)+1 = 1$ times. The total number of iterations of the inner loop is $(n-1)+(n-2)+\cdots+2+1 = (1+2+\cdots+n)-n = \frac{n(n+1)}{2}-n = \frac{1}{2}n^2 - \frac{1}{2}n$, which is also the total number of operations. \square

3.12.2 (b)

Proof. $\frac{1}{2}n^2 - \frac{1}{2}n$ is $\Theta(n^2)$ by theorem on polynomial orders, so the algorithm has order n^2 . \square

3.13 Exercise 13

```

for  $i := 1$  to  $n - 1$ 
    for  $j := i$  to  $n$ 
        if  $a[j] > a[i]$  then do
             $temp := a[i]$ 
             $a[i] := a[j]$ 

```

```

        a[j] := temp
    end do
next j
next i

```

3.13.1 (a)

Proof. The inner loop contains 1 comparison operation. When $i = 1$ the inner loop iterates $n - 1 + 1 = n$ times, when $i = 2$ the inner loop iterates $n - 2 + 1 = n - 1$ times, and so on. Finally when $i = n - 1$ the inner loop iterates $n - (n - 1) + 1 = 2$ times. The total number of operations is then $n + (n - 1) + \cdots + 3 + 2 = (1 + 2 + 3 + \cdots + n) - 1 = \frac{n(n+1)}{2} - 1 = \frac{1}{2}n^2 + \frac{1}{2}n - 1$. \square

3.13.2 (b)

Proof. $\frac{1}{2}n^2 + \frac{1}{2}n - 1$ is $\Theta(n^2)$ by theorem on polynomial orders, so the algorithm has order n^2 . \square

3.14 Exercise 14

```

t := 0
for i := 1 to n
    s := 0
    for j := 1 to i
        s := s + a[j]
    next j
    t := t + s^2
next i

```

3.14.1 (a)

Proof. There is one addition for each iteration of the inner loop, and there is one additional addition and one multiplication for each iteration of the outer loop. The number of iterations in the inner loop equals the number of columns in the following table, which shows the values of i and j for which the inner loop is executed.

i	1	2		3			...	n				
j	1	1	2	1	2	3	...	1	2	3	...	n
	1		2		3			n				

Hence the total number of iterations of the inner loop is $= 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$ (by Theorem 5.2.1). Because one addition is performed for each iteration of the inner loop, the number of operations performed when the inner loop is executed is $\frac{1}{2}n^2 + \frac{1}{2}n$. Now an additional two operations are performed each time the outer loop is executed, and because the outer loop is executed n times, this gives an additional $2n$ operations. Therefore, the total number of operations is $\frac{1}{2}n^2 + \frac{1}{2}n + 2n = \frac{1}{2}n^2 + \frac{5}{2}n$. \square

3.14.2 (b)

Proof. By the theorem on polynomial orders, $\frac{1}{2}n^2 + \frac{5}{2}n$ is $\Theta(n^2)$, and so the algorithm segment has order n^2 . \square

3.15 Exercise 15

```

for  $i := 1$  to  $n - 1$ 
     $p := 1$ 
     $q := 1$ 
    for  $j := i + 1$  to  $n$ 
         $p := p \cdot c[j]$ 
         $q := q \cdot c[j]^2$ 
    next  $j$ 
     $r := p + q$ 
next  $i$ 

```

3.15.1 (a)

Proof. The inner loop has 3 multiplications, the outer loop has 1 addition.

The inner loop iterates: $i = 1$: $n - (1 + 1) + 1 = n - 1$ times, $i = 2$: $n - (2 + 1) + 1 = n - 2$ times, \dots , $i = n - 1$: $n - (n - 1 + 1) + 1 = 1$ time. Total: $1 + 2 + \dots + (n - 2) + (n - 1) = \frac{n(n+1)}{2} - n = \frac{1}{2}n^2 - \frac{1}{2}n$ times.

The outer loop iterates: $n - 1 - 1 + 1 = n - 1$ times. So the total number of operations is $3 \left(\frac{1}{2}n^2 - \frac{1}{2}n \right) + 1 \cdot (n - 1) = \frac{3}{2}n^2 - \frac{1}{2}n - 1$. \square

3.15.2 (b)

Proof. $\frac{3}{2}n^2 - \frac{1}{2}n - 1$ is $\Theta(n^2)$ by theorem on polynomial orders, so the algorithm has order n^2 . \square

3.16 Exercise 16

```

for  $i := 1$  to  $n$ 
     $s := 0$ 
    for  $j := 1$  to  $i - 1$ 
         $s := s + j \cdot (i - j + 1)$ 
    next  $j$ 
     $r := s^2$ 
next  $i$ 

```

3.16.1 (a)

Proof. The inner loop has 4 operations: 2 additions, 1 subtraction, 1 multiplication. The outer loop has 1 multiplication. The outer loop iterates $n - 1 + 1 = n$ times.

When $i = 1$, the inner loop runs from $j = 1$ to $j = i - 1 = 1 - 1 = 0$, so it cannot run (from 1 to 0). I think this might be a typo in the book.

When $i = 2$, the inner loop runs from $j = 1$ to $j = i - 1 = 2 - 1 = 1$, so it iterates $1 - 1 + 1 = 1$ time.

When $i = 3$, the inner loop runs from $j = 1$ to $j = i - 1 = 3 - 1 = 2$, so it iterates $2 - 1 + 1 = 2$ times. And so on.

Finally, when $i = n$ the inner loop iterates $n - 1 - 1 + 1 = n - 1$ times. So the total number of iterations of the inner loop is $1 + 2 + \dots + (n - 2) + (n - 1) = \frac{(n-1)(n-1+1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$.

The total number of operations is: 4 times the inner loop, plus 1 times the outer loop, $= 4 \left(\frac{1}{2}n^2 - \frac{1}{2}n \right) + 1 \cdot n = 2n^2 - 2n + n = 2n^2 - n$. \square

3.16.2 (b)

Proof. $2n^2 - n$ is $\Theta(n^2)$ by theorem on polynomial orders, so the algorithm has order n^2 . \square

3.17 Exercise 17

```

for  $i := 1$  to  $n$ 
  for  $j := 1$  to  $\lfloor (i + 1)/2 \rfloor$ 
     $a := (n - i) \cdot (n - j)$ 
  next  $j$ 
next  $i$ 

```

3.17.1 (a)

Proof. There are two subtractions and one multiplication for each iteration of the inner loop. If n is odd, the number of iterations of the inner loop equals the number of columns in the following table, which shows the values of i and j for which the inner loop is executed.

i	1	2	3		4		5			6			...	$n - 1$...		n		...	
$\lfloor \frac{i+1}{2} \rfloor$	1	1	2		2		3			3			...	$\frac{n-1}{2}$...		$\frac{n+1}{2}$...	
j	1	1	1	2	1	2	1	2	3	1	2	3	...	1	2	...	$\frac{n-1}{2}$	1	2	...	$\frac{n+1}{2}$
	1		1		2		2		3			3			$\frac{n-1}{2}$			$\frac{n+1}{2}$			

Thus the number of iterations of the inner loop is

$$\begin{aligned}
&= 1 + 1 + 2 + 2 + \cdots + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n+1}{2} \\
&= 2 \left(1 + 2 + 3 + \cdots + \frac{n-1}{2} \right) + \frac{n+1}{2} \\
&= 2 \cdot \frac{\frac{n-1}{2} \left(\frac{n-1}{2} + 1 \right)}{2} + \frac{n+1}{2} && \text{by Theorem 5.2.1} \\
&= \frac{n^2 - 2n + 1}{4} + \frac{n-1}{2} + \frac{n+1}{2} \\
&= \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}
\end{aligned}$$

By similar reasoning, if n is even, then the number of iterations of the inner loop is

$$\begin{aligned}
&= 1 + 1 + 2 + 2 + \cdots + \frac{n}{2} + \frac{n}{2} \\
&= 2 \left(1 + 2 + 3 + \cdots + \frac{n}{2} \right) \\
&= 2 \cdot \frac{\frac{n}{2} \left(\frac{n}{2} + 1 \right)}{2} && \text{by Theorem 5.2.1} \\
&= \frac{1}{4}n^2 + \frac{1}{2}n
\end{aligned}$$

Because three operations are performed for each iteration of the inner loop, the answer is $3 \left(\frac{1}{4}n^2 + \frac{1}{2}n \right)$ when n is even, and $3 \left(\frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4} \right)$ when n is odd. \square

3.17.2 (b)

Proof. Both $3 \left(\frac{1}{4}n^2 + \frac{1}{2}n \right)$ and $3 \left(\frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4} \right)$ are $\Theta(n^2)$ by theorem on polynomial orders, so the algorithm has order n^2 . \square

3.18 Exercise 18

```

for  $i := 1$  to  $n$ 
  for  $j := \lfloor (i+1)/2 \rfloor$  to  $n$ 
     $x := i \cdot j$ 
  next  $j$ 
next  $i$ 

```

3.18.1 (a)

Proof. When $i = 1$ the inner loop runs from $j = \lfloor (1+1)/2 \rfloor = 1$ to n , so it iterates $n - 1 + 1 = n$ times.

When $i = 2$ the inner loop runs from $j = \lfloor (2+1)/2 \rfloor = 1$ to n , so it iterates $n - 1 + 1 = n$ times.

When $i = 3$ the inner loop runs from $j = \lfloor (3+1)/2 \rfloor = 2$ to n , so it iterates $n - 2 + 1 = n - 1$ times.

When $i = 4$ the inner loop runs from $j = \lfloor (4 + 1)/2 \rfloor = 2$ to n , so it iterates $n - 2 + 1 = n - 1$ times.

And so on. Finally when $i = n$ the inner loop runs from $j = \lfloor (n + 1)/2 \rfloor$ to n . If n is even, this is from $n/2$ to n ; and if n is odd this is from $(n + 1)/2$ to n . So it iterates roughly $n/2$ times.

So the total iterations of the inner loop is $n + n + (n - 1) + (n - 1) + \cdots + (n/2 + n/2) = 2(\frac{n}{2} + \cdots + (n - 1) + n) = 2(1 + 2 + \cdots + n - (1 + 2 + \cdots \frac{n}{2})) = 2\left(\frac{n(n+1)}{2} - \frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\right) = n(n + 1) - \frac{n}{2}(\frac{n}{2} + 1) = n^2 + n - \frac{1}{4}n^2 - \frac{1}{2}n = \frac{3}{4}n^2 - \frac{1}{2}n$.

This is the same as the number of operations, as there is only one multiplication inside the inner loop. \square

3.18.2 (b)

Proof. $\frac{3}{4}n^2 - \frac{1}{2}n$ is $\Theta(n^2)$ by theorem on polynomial orders. So the algorithm has order n^2 . \square

3.19 Exercise 19

```

for i := 1 to n
  for j := 1 to i
    for k := 1 to j
      x := i · j · k
    next k
  next j
next i

```

Hint: See Section 9.6 for a discussion of how to count the number of iterations of the innermost loop.

3.19.1 (a)

Proof. By Example 9.6.4 there are $[n(n + 1)(n + 2)]/6$ iterations of the innermost loop. There are two multiplications, so the total number of operations is $[n(n + 1)(n + 2)]/3 = \frac{1}{3}(n^3 + 3n^2 + 2n)$. \square

3.19.2 (b)

Proof. $\frac{1}{3}(n^3 + 3n^2 + 2n)$ is $\Theta(n^3)$ by theorem on polynomial orders. So the algorithm has order n^3 . \square

3.20 Exercise 20

Construct a table showing the result of each step when insertion sort is applied to the array $a[1] = 6$, $a[2] = 2$, $a[3] = 1$, $a[4] = 8$, and $a[5] = 4$.

Proof.

	$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
Initial order	6	2	1	8	4
Step 1 result	2	6	1	8	4
Step 2 result	1	2	6	8	4
Step 3 result	1	2	6	8	4
Final order	1	2	4	6	8

□

3.21 Exercise 21

Construct a table showing the result of each step when insertion sort is applied to the array $a[1] = 7$, $a[2] = 3$, $a[3] = 6$, $a[4] = 9$, and $a[5] = 5$.

Proof.

	$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
Initial order	7	3	6	9	5
Step 1 result	3	7	6	9	5
Step 2 result	3	6	7	9	5
Step 3 result	3	6	7	9	5
Final order	3	5	6	7	9

□

3.22 Exercise 22

Construct a trace table showing the action of insertion sort on the array of exercise 20.

Proof.

n	5										
$a[1]$	6	2			1						
$a[2]$	2	6		1	2						
$a[3]$	1			6						4	
$a[4]$	8								4	6	
$a[5]$	4								8		
k	2		3			4		5			
x	2		1			8		4			
j	1	0	2	1	0	3	0	4	3	2	0

□

3.23 Exercise 23

Construct a trace table showing the action of insertion sort on the array of exercise 21.

<i>n</i>	5											
<i>a</i> [1]	7	3										
<i>a</i> [2]	3	7		6							5	
<i>a</i> [3]	6			7						5	6	
<i>a</i> [4]	9								5	7		
<i>a</i> [5]	5								9			
<i>k</i>	2		3			4		5				
<i>x</i>	3		6			9		5				
<i>j</i>	1	0	2	1	0	3	0	4	3	2	1	0

Proof.

□

3.24 Exercise 24

How many comparisons between values of $a[j]$ and x actually occur when insertion sort is applied to the array of exercise 20?

Proof. There are seven comparisons between values of x and values of $a[j]$: one when $k = 2$, two when $k = 3$, one when $k = 4$, and three when $k = 5$. □

3.25 Exercise 25

How many comparisons between values of $a[j]$ and x actually occur when insertion sort is applied to the array of exercise 21?

Proof. There are eight comparisons between values of x and values of $a[j]$: one when $k = 2$, two when $k = 3$, one when $k = 4$, and four when $k = 5$. □

3.26 Exercise 26

According to Example 11.3.6, the maximum number of comparisons needed to perform insertion sort on an array of length five is $\frac{1}{2}(5^2 + 5 - 2) = 14$. Find an array of length five that requires the maximum number of comparisons when insertion sort is applied to it.

Proof. Example 11.3.6 has a mistake. It says that the maximum number of comparisons for any k is $k - 1$, then it says that k goes from 2 to n , so the sum should be: $(2 - 1) + (3 - 1) + \cdots + (n - 1 - 1) + (n - 1) = 1 + 2 + \cdots + (n - 2) + (n - 1)$, which is equal to $\frac{(n - 1)(n - 1 + 1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$.

Let $a = [5, 4, 3, 2, 1]$. When $k = 2$ 4 will be compared to 5 and swapped. When $k = 3$, 3 will be compared to 5 first, and swapped with it, then it will be compared to 4 and swapped with it. Similarly there will be 3 comparisons when $k = 4$ and finally 4 comparisons when $k = 5$, for a total of $1 + 2 + 3 + 4 = 10$. This matches the formula above, since $\frac{1}{2}5^2 - \frac{1}{2}5 = 25/2 - 5/2 = 20/2 = 10$. □

3.27 Exercise 27

Consider the recurrence relation that arose in Example 11.3.7: $E_1 = 0$ and $E_k = E_{k-1} + \frac{k+1}{2}$, for each integer $k \geq 2$.

3.27.1 (a)

Use iteration to find an explicit formula for the sequence.

Hint: $E_n = \frac{1}{2}[3 + 4 + \cdots + (n + 1)]$, which equals $\frac{1}{2}[(1 + 2 + 3 + \cdots + (n + 1)) - (1 + 2)]$.

Proof. $E_1 = 0$, $E_2 = E_1 + \frac{2+1}{2} = 3/2$, $E_3 = E_2 + \frac{3+1}{2} = 3/2 + 4/2 = (3 + 4)/2$,
 $E_4 = E_3 + \frac{4+1}{2} = (3 + 4)/2 + 5/2 = (3 + 4 + 5)/2$,
 $E_5 = E_4 + \frac{5+1}{2} = (3 + 4 + 5)/2 + 6/2 = (3 + 4 + 5 + 6)/2$.

We guess $E_n = (3 + 4 + \cdots + n)/2$. □

3.27.2 (b)

Use mathematical induction to verify the correctness of the formula.

Proof. Let $P(n)$ be the equation $E_n = (3 + 4 + \cdots + n)/2$.

Show that $P(2)$ is true: $E_2 = 3/2$ which agrees with the formula above, so $P(2)$ is true.

Show that for any integer $k \geq 2$ if $P(k)$ is true then $P(k + 1)$ is true: Assume $k \geq 2$ is any integer and $E_k = (3 + 4 + \cdots + k)/2$. [We want to show $E_{k+1} = (3 + 4 + \cdots + k + (k + 1))/2$.]

$E_{k+1} = E_k + \frac{k+1}{2} = (3 + 4 + \cdots + k)/2 + \frac{k+1}{2} = (3 + 4 + \cdots + k + (k + 1))/2$, so $P(k + 1)$ holds. □

Exercises 28–35 refer to selection sort, which is another algorithm to arrange the items in an array in ascending order.

Algorithm 11.3.2 Selection Sort

[Given an array $a[1], a[2], a[3], \dots, a[n]$, this algorithm selects the smallest element and places it in the first position, then selects the second smallest element and places it in the second position, and so forth, until the entire array is sorted. In general, for each $k = 1$ to $n - 1$, the k th step of the algorithm selects the index of the array item with minimum value from among $a[k + 1], a[k + 2], a[k + 3], \dots, a[n]$. Once this index is found, the value of the corresponding array item is interchanged with the value of $a[k]$ unless the index already equals k . At the end of execution the array elements are in order.]

Input: n [a positive integer], $a[1], a[2], a[3], \dots, a[n]$ [an array of data items capable of being ordered]

Algorithm Body:

```
for  $k := 1$  to  $n - 1$ 
     $indexMin := k$ 
    for  $i := k + 1$  to  $n$ 
        if  $a[i] < a[indexMin]$ 
            then  $indexMin := i$ 
    next  $i$ 
    if  $indexMin \neq k$  then
         $Temp := a[k]$ 
         $a[k] := a[indexMin]$ 
         $a[indexMin] := Temp$ 
next  $k$ 
```

Output: $a[1], a[2], a[3], \dots, a[n]$ [in ascending order]

The action of selection sort can be represented pictorially as follows:

$a[1] \quad a[2] \quad \cdots \quad \boxed{a[k]} \quad a[k + 1] \quad \cdots \quad a[n]$



k th step: Find the index of the array element with minimum value from among $a[k + 1], \dots, a[n]$. If the value of this array element is less than the value of $a[k]$, then its value and the value of $a[k]$ are interchanged.

3.28 Exercise 28

Construct a table showing the interchanges that occur when selection sort is applied to the array $a[1] = 7, a[2] = 3, a[3] = 8, a[4] = 4$, and $a[5] = 2$.

Proof. The top row of the table shows the initial values of the array, and the bottom row shows the final values. The results for executing each step in the for-next loop are shown in separate rows.

□

3.29 Exercise 29

Construct a table showing the interchanges that occur when selection sort is applied to the array $a[1] = 6, a[2] = 4, a[3] = 5, a[4] = 8$, and $a[5] = 1$.

k	$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
Initial	7	3	8	4	2
1	2	3	8	4	7
2	2	3	8	4	7
3	2	3	4	8	7
4	2	3	4	7	8
5	2	3	4	7	8

k	$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
Initial	6	4	5	8	1
1	1	4	5	8	6
2	1	4	5	8	6
3	1	4	5	8	6
4	1	4	5	6	8
Final	1	4	5	6	8

Proof.

□

3.30 Exercise 30

Construct a trace table showing the action of selection sort on the array of exercise 28.

n	5													
$a[1]$	7	3			2									
$a[2]$	3	7												
$a[3]$	8									4				
$a[4]$	4									8			7	
$a[5]$	2				7								8	
k	1					2			3			4		
$IndexOfMin$	1	2		5		2		3	4		4	5		
i	2		3	4	5		3	4	5	4	5		5	
$temp$						7					8			7

Proof.

□

3.31 Exercise 31

Construct a trace table showing the action of selection sort on the array of exercise 29.

Proof.

<i>n</i>	5												
<i>a</i>[1]	6					1							
<i>a</i>[2]	4												
<i>a</i>[3]	5												
<i>a</i>[4]	8												6
<i>a</i>[5]	1					6							8
<i>k</i>	1						2			3		4	
<i>indexMin</i>	1	2			5		2			3		4	5
<i>i</i>	2		3	4	5		3	4	5	4	5	5	
<i>temp</i>						6							8

□

3.32 Exercise 32

When selection sort is applied to the array of exercise 28, how many times is the comparison in the if-then statement performed?

Note: The book is not clear about this, but it is referring to the if-then comparison inside the inner for-loop, not to the “**if** *IndexOfMin* \neq *k* **then**” after it.

Proof. There is one comparison for each combination of values of *k* and *i*: namely, $4 + 3 + 2 + 1 = 10$. □

3.33 Exercise 33

When selection sort is applied to the array of exercise 29, how many times is the comparison in the if-then statement performed?

Proof. There is one comparison for each combination of values of *k* and *i*: namely, $4 + 3 + 2 + 1 = 10$. □

3.34 Exercise 34

When selection sort is applied to an array $a[1], a[2], a[3], a[4]$, how many times is the comparison in the if-then statement performed?

Proof. *k* ranges from 1 to $4 - 1 = 3$, and for each value of *k*, *i* ranges from *k* + 1 to 4. So *i* ranges from: 2 to 4, then 3 to 4, then 4 to 4. So $(4 - 2 + 1) + (4 - 3 + 1) + (4 - 4 + 1) = 3 + 2 + 1 = 6$ comparisons. □

3.35 Exercise 35

Consider applying selection sort to an array $a[1], a[2], a[3], \dots, a[n]$.

3.35.1 (a)

How many times is the comparison in the **if-then** statement performed when $a[1]$ is compared to each of $a[2], a[3], \dots, a[n]$?

Proof. Since $k = 1$, i ranges from $k = 2$ to n , so $n - 2 + 1 = n - 1$ times. \square

3.35.2 (b)

How many times is the comparison in the **if-then** statement performed when $a[2]$ is compared to each of $a[3], a[4], \dots, a[n]$?

Proof. Since $k = 2$, i ranges from $k = 3$ to n , so $n - 3 + 1 = n - 2$ times. \square

3.35.3 (c)

How many times is the comparison in the **if-then** statement performed when $a[k]$ is compared to each of $a[k + 1], a[k + 2], \dots, a[n]$?

Proof. i ranges from $k + 1$ to n , so $n - (k + 1) + 1 = n - k$ times. \square

3.35.4 (d)

Using the number of times the comparison in the if-then statement is performed as a measure of the time efficiency of selection sort, find a worst-case order for selection sort. Use the theorem on polynomial orders.

Hint: The answer is n^2 .

Proof. From parts (a), (b), (c) the total number of comparisons is: $(n - 1) + (n - 2) + \dots + (n - (n - 1)) = 1 + 2 + \dots + (n - 1) = \frac{(n-1)(n-1+1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$. This is $\Theta(n^2)$ by theorem on polynomial orders, so the algorithm has order n^2 . \square

Exercises 36 – 39 refer to the following algorithm to compute the value of a real polynomial.

Algorithm 11.3.3 Term-by-Term Polynomial Evaluation

[This algorithm computes the value of a polynomial $a[n]x^n + a[n-1]x^{n-1} + \dots + a[2]x^2 + a[1]x + a[0]$ by computing each term separately, starting with $a[0]$, and adding it to an accumulating sum.]

Input: n [a nonnegative integer], $a[0], a[1], a[2], a[3], \dots, a[n]$ [an array of real numbers], x [a real number]

Algorithm Body:

$polyval := a[0]$

for $i := 1$ **to** n

$term := a[i]$

for $j := 1$ **to** i

$term := term \cdot x$

next j

$polyval := polyval + term$

next i

[At this point $polyval = a[n]x^n + a[n-1]x^{n-1} + \dots + a[2]x^2 + a[1]x + a[0]$.]

Output: $polyval$ [a real number]

3.36 Exercise 36

Trace Algorithm 11.3.3 for the input $n = 3, a[0] = 2, a[1] = 1, a[2] = -1, a[3] = 3$, and $x = 2$.

n	3								
$a[0]$	2								
$a[1]$	1								
$a[2]$	-1								
$a[3]$	3								
x	2								
$polyval$	2	4			0			24	
i	1	2			3				
$term$	1	2	-1	-2	-4	3	6	12	24
j	1		1	2		1	2	3	

Proof.

□

3.37 Exercise 37

Trace Algorithm 11.3.3 for the input $n = 2, a[0] = 5, a[1] = -1, a[2] = 2$, and $x = 3$.

<i>n</i>	2					
<i>a</i> [0]	5					
<i>a</i> [1]	−1					
<i>a</i> [2]	2					
<i>Proof.</i> <i>x</i>	3					
<i>polyval</i>	5		2			20
<i>i</i>	1		2			
<i>term</i>	−1	−3	2	6	18	
<i>j</i>	1		1	2		

□

3.38 Exercise 38

Let s_n = the number of additions and multiplications that are performed when Algorithm 11.3.3 is executed for a polynomial of degree n . Express s_n as a function of n .

Proof. Number of multiplications = number of iterations of the inner loop = $1 + 2 + 3 + \cdots + n = n(n+1)/2$ by Theorem 5.2.1. Number of additions = number of iterations of the outer loop = n . Hence the total number of multiplications and additions is $n(n+1)/2 + n = n^2/2 + 3n/2$. □

3.39 Exercise 39

Use the theorem on polynomial orders to find an order for Algorithm 11.3.3.

Proof. It's n^2 . □

Exercises 40 – 42 refer to another algorithm, known as Horner's rule, for finding the value of a polynomial.

Algorithm 11.3.4 Horner's Rule

[This algorithm computes the value of a polynomial $a[n]x^n + a[n-1]x^{n-1} + \dots + a[2]x^2 + a[1]x + a[0]$ by nesting successive additions and multiplications as indicated in the following parenthesization: $((\dots((a[n]x + a[n-1])x + a[n-2])x + \dots + a[2])x + a[1])x + a[0]$. At each stage, starting with $a[n]$, the current value of *polyval* is multiplied by x and the next lower coefficient of the polynomial is added to it.]

Input: n [a nonnegative integer], $a[0], a[1], a[2], a[3], \dots, a[n]$ [an array of real numbers], x [a real number]

Algorithm Body:

$\text{polyval} := a[n]$

for $i := 1$ **to** n

$\text{polyval} := \text{polyval} \cdot x + a[n-i]$

next i

[At this point $\text{polyval} = a[n]x^n + a[n-1]x^{n-1} + \dots + a[2]x^2 + a[1]x + a[0]$.]

Output: *polyval* [a real number]

3.40 Exercise 40

Trace Algorithm 11.3.4 for the input $n = 3, a[0] = 2, a[1] = 1, a[2] = -1, a[3] = 3$, and $x = 2$.

<i>n</i>	3			
<i>a</i>[0]	2			
<i>a</i>[1]	1			
<i>a</i>[2]	-1			
<i>a</i>[3]	3			
<i>x</i>	2			
<i>polyval</i>	3	5	11	24
<i>i</i>	1	2	3	

Proof.

□

3.41 Exercise 41

Trace Algorithm 11.3.4 for the input $n = 2, a[0] = 5, a[1] = -1, a[2] = 2$, and $x = 3$.

Proof.

<i>n</i>	2		
<i>a</i>[0]	5		
<i>a</i>[1]	-1		
<i>a</i>[2]	2		
<i>x</i>	3		
<i>polyval</i>	2	5	20
<i>i</i>	1	2	

□

3.42 Exercise 42

Let t_n = the number of additions and multiplications that are performed when Algorithm 11.3.4 is executed for a polynomial of degree n . Express t_n as a function of n .

Hint: $t_n = 2n$.

Proof. There is one multiplication and one subtraction for each iteration of the loop. The loop iterates n times. Therefore $t_n = 2n$. □

3.43 Exercise 43

Use the theorem on polynomial orders to find an order for Algorithm 11.3.4. How does this order compare with that of Algorithm 11.3.3?

Proof. The order is n which is one degree less than the order of Algorithm 11.3.3. So this algorithm runs in the square root of the time the other one takes to run. □

4 Exercise Set 11.4

4.1 Exercise 1

4.1.1 ()

Proof.

□

4.2 Exercise 2

4.2.1 ()

Proof.

□

4.3 Exercise 3

4.3.1 ()

Proof.

□

4.4 Exercise 4

4.4.1 ()

Proof.

□

4.5 Exercise 5

4.5.1 ()

Proof.



4.6 Exercise 6

4.6.1 ()

Proof.



4.7 Exercise 7

4.7.1 ()

Proof.



4.8 Exercise 8

4.8.1 ()

Proof.



4.9 Exercise 9

4.9.1 ()

Proof.



4.10 Exercise 10

4.10.1 ()

Proof.



4.11 Exercise 11

4.11.1 ()

Proof.



4.12 Exercise 12

4.12.1 ()

Proof.



4.13 Exercise 13

4.13.1 $()$

Proof. ☐

4.14 Exercise 14

4.14.1 $()$

Proof. ☐

4.15 Exercise 15

4.15.1 $()$

Proof. ☐

4.16 Exercise 16

4.16.1 $()$

Proof. ☐

4.17 Exercise 17

4.17.1 $()$

Proof. ☐

4.18 Exercise 18

4.18.1 $()$

Proof. ☐

4.19 Exercise 19

4.19.1 $()$

Proof. ☐

4.20 Exercise 20

4.20.1 $()$

Proof. ☐

4.21 Exercise 21

4.21.1 ()

Proof.



4.22 Exercise 22

4.22.1 ()

Proof.



4.23 Exercise 23

4.23.1 ()

Proof.



4.24 Exercise 24

4.24.1 ()

Proof.



4.25 Exercise 25

4.25.1 ()

Proof.



4.26 Exercise 26

4.26.1 ()

Proof.



4.27 Exercise 27

4.27.1 ()

Proof.



4.28 Exercise 28

4.28.1 ()

Proof.



4.29 Exercise 29

4.29.1 ()

Proof.

□

4.30 Exercise 30

4.30.1 ()

Proof.

□

4.31 Exercise 31

4.31.1 ()

Proof.

□

4.32 Exercise 32

4.32.1 ()

Proof.

□

4.33 Exercise 33

4.33.1 ()

Proof.

□

4.34 Exercise 34

4.34.1 ()

Proof.

□

4.35 Exercise 35

4.35.1 ()

Proof.

□

4.36 Exercise 36

4.36.1 ()

Proof.

□

4.37 Exercise 37

4.37.1 ()

Proof.



4.38 Exercise 38

4.38.1 ()

Proof.



4.39 Exercise 39

4.39.1 ()

Proof.



4.40 Exercise 40

4.40.1 ()

Proof.



4.41 Exercise 41

4.41.1 ()

Proof.



4.42 Exercise 42

4.42.1 ()

Proof.



4.43 Exercise 43

4.43.1 ()

Proof.



4.44 Exercise 44

4.44.1 ()

Proof.



4.45 Exercise 45

4.45.1 ()

Proof.



4.46 Exercise 46

4.46.1 ()

Proof.



4.47 Exercise 47

4.47.1 ()

Proof.



4.48 Exercise 48

4.48.1 ()

Proof.



4.49 Exercise 49

4.49.1 ()

Proof.



4.50 Exercise 50

4.50.1 ()

Proof.



4.51 Exercise 51

4.51.1 ()

Proof.



5 Exercise Set 11.5

5.1 Exercise 1

5.1.1 ()

Proof.



5.2 Exercise 2

5.2.1 ()

Proof.



5.3 Exercise 3

5.3.1 ()

Proof.



5.4 Exercise 4

5.4.1 ()

Proof.



5.5 Exercise 5

5.5.1 ()

Proof.



5.6 Exercise 6

5.6.1 ()

Proof.



5.7 Exercise 7

5.7.1 ()

Proof.



5.8 Exercise 8

5.8.1 ()

Proof.



5.9 Exercise 9

5.9.1 ()

Proof.



5.10 Exercise 10

5.10.1 ()

Proof.



5.11 Exercise 11

5.11.1 ()

Proof.



5.12 Exercise 12

5.12.1 ()

Proof.



5.13 Exercise 13

5.13.1 ()

Proof.



5.14 Exercise 14

5.14.1 ()

Proof.



5.15 Exercise 15

5.15.1 ()

Proof.



5.16 Exercise 16

5.16.1 ()

Proof.



5.17 Exercise 17

5.17.1 ()

Proof.



5.18 Exercise 18

5.18.1 ()

Proof.



5.19 Exercise 19

5.19.1 ()

Proof.



5.20 Exercise 20

5.20.1 ()

Proof.



5.21 Exercise 21

5.21.1 ()

Proof.



5.22 Exercise 22

5.22.1 ()

Proof.



5.23 Exercise 23

5.23.1 ()

Proof.



5.24 Exercise 24

5.24.1 ()

Proof.



5.25 Exercise 25

5.25.1 ()

Proof.



5.26 Exercise 26

5.26.1 ()

Proof.

□