Solutions to Chapter 8, Susanna Epp Discrete Math 5th Edition

https://github.com/spamegg1

September 27, 2023

Contents

1 Exercise Set 8.1

1.1 Exercise 1

As in Example 8.1.2, the **congruence modulo 2** relation E is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For every ordered pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $m\ E\ n \iff m - n$ is even.

1.1.1 (a)

Is $0\ E\ 0$? Is $5\ E\ 2$? Is $(6, 6) \in E$? Is $(21, 7) \in E$?

Proof. $0\ E\ 0$ because $0 - 0 = 0 = 2 \cdot 0$, so $2 \mid (0 - 0)$. $5 \not E\ 2$ because $5 - 2 = 3$ and $3 \neq 2k$ for any integer k, so $2 \nmid (5 - 2)$. $(6, 6) \in E$ because $6 - 6 = 0 = 2 \cdot 0$, so $2 \mid (6 - 6)$.$(-1, 7) \in E$ because $-1 - 7 = -8 = 2 \cdot (-4)$, so $2 \mid (-1 - 7)$. □

1.1.2 (b)

Prove that for any even integer n, $n\ E\ 0$.

Proof. Assume n is even. By definition of even, $n = 2k$ for some integer k. Then $n - 0 = 2k - 0 = 2k$ is also even. Therefore by definition of E, $n\ E\ 0$. □

1.2 Exercise 2

Prove that for all integers m and n, $m - n$ is even if, and only if, both m and n are even or both m and n are odd.

Proof. $\implies$: Assume $m - n$ is even. *[We want to prove that both m and n are even or both m and n are odd.]* By definition of even, $m - n = 2k$ for some integer k. There are 4 cases:

Case 1: both m and n are even: Nothing to prove.

Case 2: both m and n are odd: Nothing to prove.

Case 3: m is even, n is odd: By definitions of even and odd, $m = 2k, n = 2l + 1$ for some integers k, l. So $m - n = 2k - 2l - 1 = 2(k - l - 1) + 1$ where $k - l - 1$ is an integer. So by definition of odd, $m - n$ is odd, a contradiction. So this case is impossible.

Case 4: m is odd, n is even: By definitions of even and odd, $m = 2k+1, n = 2l$ for some integers k, l. So $m - n = 2k + 1 - 2l = 2(k-l) + 1$ where $k - l$ is an integer. So by definition of odd, $m - n$ is odd, a contradiction. So this case is impossible.

$\Longleftarrow$: Assume both m and n are even or both m and n are odd. *[We want to prove that $m - n$ is even.]* There are 2 cases:

Case 1: both m and n are even: By definition of even, $m = 2k, n = 2l$ for some integers k, l. Then $m - n = 2k - 2l = 2(k - l)$ where $k - l$ is an integer. So by definition, $m - n$ is even.

Case 2: both m and n are odd: By definition of even, $m = 2k + 1, n = 2l + 1$ for some integers k, l. Then $m - n = 2k + 1 - 2l - 1 = 2(k - l)$ where $k - l$ is an integer. So by definition, $m - n$ is even. □

1.3 Exercise 3

The congruence modulo 3 relation, T, is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all integers m and n, $m \; T \; n \iff 3 \mid (m - n)$.

1.3.1 (a)

Is $10 \; T \; 1$? Is $1 \; T \; 10$? Is $(2, 2) \in T$? Is $(8, 1) \in T$?

Proof. $10 \; T \; 1$ because $10 - 1 = 9 = 3 \cdot 3$, and so $3 \mid (10 - 1)$.

$1 \; T \; 10$ because $1 - 10 = -9 = 3 \cdot (-3)$, and so $3 \mid (1 - 10)$.

$2 \; T \; 2$ because $2 - 2 = 0 = 3 \cdot 0$, and so $3 \mid (2 - 2)$.

$8 \not{T} \; 1$ because $8 - 1 = 7 \neq 3k$, for any integer k. So $3 \nmid (8 - 1)$. □

1.3.2 (b)

List five integers n such that $n \; T \; 0$.

Proof. One possible answer: 3, 6, 9, -3, -6 □

1.3.3 (c)

List five integers n such that $n \; T \; 1$.

Proof. One possible answer: 4, 7, 10, -2, -5 □

1.3.4 (d)

List five integers n such that $n \; T \; 2$.

Proof. One possible answer: 5, 8, 11, -1, -4 □

1.3.5 (e)

Make and prove a conjecture about which integers are related by T to 0, which integers are related by T to 1, and which integers are related by T to 2.

All integers of the form $3k + 1$, for some integer k, are related by T to 1.

Proof. All integers of the form $3k$, for some integer k, are related by T to 0.

All integers of the form $3k + 1$, for some integer k, are related by T to 1.

All integers of the form $3k + 2$, for some integer k, are related by T to 2. □

1.4 Exercise 4

Define a relation P on $\mathbb{Z}$ as follows: For every ordered pair $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, $m \; P \; n \iff m$ and n have a common prime factor.

1.4.1 (a)

Is $15 \; P \; 25$?

Proof. Yes, because 15 and 25 are both divisible by 5, which is prime. □

1.4.2 (b)

Is $22 \; P \; 27$?

Proof. No, because 22 and 27 have no common prime factor. □

1.4.3 (c)

Is $0 \; P \; 5$?

Proof. Yes, because 0 and 5 are both divisible by 5, which is prime. □

1.4.4 (d)

Is $8 \; P \; 8$?

Proof. Yes, because 8 and 8 are both divisible by 2, which is prime. □

1.5 Exercise 5

Let $X = \{a, b, c\}$. Recall that $\mathscr{P}(X)$ is the power set of X. Define a relation $\mathbf{S}$ on $\mathscr{P}(X)$ as follows: For all sets A and B in $\mathscr{P}(X)$, $A \; \mathbf{S} \; B \iff A$ has the same number of elements as B.

1.5.1 (a)

Is $\{a, b\}$ $\mathbf{S}$ $\{b, c\}$?

Proof. Yes, because both $\{a, b\}$ and $\{b, c\}$ have two elements. $\square$

1.5.2 (b)

Is $\{a\}$ $\mathbf{S}$ $\{a, b\}$?

Proof. No, one has 1 element, the other has 2 elements. $\square$

1.5.3 (c)

Is $\{c\}$ $\mathbf{S}$ $\{b\}$?

Proof. Yes, because both $\{c\}$ and $\{b\}$ have one element. $\square$

1.6 Exercise 6

Let $X = \{a, b, c\}$. Define a relation $\mathbf{J}$ on $\mathscr{P}(X)$ as follows: For all sets A and B in $\mathscr{P}(X)$, $A \mathbf{J} B \iff A \cap B \neq \varnothing$.

1.6.1 (a)

Is $\{a\}$ $\mathbf{J}$ $\{c\}$?

Proof. No, because $\{a\} \cap \{c\} = \varnothing$. $\square$

1.6.2 (b)

Is $\{a, b\}$ $\mathbf{J}$ $\{b, c\}$?

Proof. Yes, because $\{a, b\} \cap \{b, c\} = \{b\} \neq \varnothing$. $\square$

1.6.3 (c)

Is $\{a, b\}$ $\mathbf{J}$ $\{a, b, c\}$?

Proof. Yes, because $\{a, b\} \cap \{a, b, c\} = \{a, b\} \neq \varnothing$. $\square$

1.7 Exercise 7

Define a relation R on $\mathbb{Z}$ as follows: For all integers m and n, $m \, R \, n \iff 5 \mid (m^2 - n^2)$.

1.7.1 (a)

Is $1 \ R \ (-9)$?

Proof. Yes. $1 \ R \ (-9) \iff 5 \mid (1^2 - (-9)^2)$. But $1^2 - (-9)^2 = 1 - 81 = -80$, and $5 \mid (-80)$ because $-80 = 5 \cdot (-16)$. $\square$

1.7.2 (b)

Is $2 \ R \ 13$?

Proof. Yes, $2^2 - (13)^2 = 4 - 169 = -165 = 5 \cdot (-33)$. So $5 \mid 2^2 - (13)^2$. $\square$

1.7.3 (c)

Is $2 \ R \ (-8)$?

Proof. Yes, $2^2 - (-8)^2 = 4 - 64 = -60 = 5 \cdot (-12)$. So $5 \mid 2^2 - (-8)^2$. $\square$

1.7.4 (d)

Is $(-8) \ R \ 2$?

Proof. Yes, $(-8)^2 - 2^2 = 64 - 4 = 60 = 5 \cdot 12$. So $5 \mid (-8)^2 - 2^2$. $\square$

1.8 Exercise 8

Let A be the set of all strings of a's and b's of length 4. Define a relation R on A as follows: For every $s, t \in A$, $s \ R \ t \iff s$ has the same first two characters as t.

1.8.1 (a)

Is $abaa \ R \ abba$?

Proof. Yes, because both $abaa$ and $abba$ have the same first two characters ab. $\square$

1.8.2 (b)

Is $aabb \ R \ bbaa$?

Proof. No, because the first two characters of $aabb$ are different from the first two characters of $bbaa$. $\square$

1.8.3 (c)

Is $aaaa \ R \ aaab$?

Proof. Yes, because both $aaaa$ and $aaab$ have the same first two characters aa. $\square$

1.8.4 (d)

Is $baaa\ R\ abaa$?

Proof. No, because the first two characters of $baaa$ are different from the first two characters of $abaa$. □

1.9 Exercise 9

Let A be the set of all strings of 0's, 1's, and 2's of length 4. Define a relation R on A as follows: For every $s, t \in A$, $s\ R\ t \iff$ the sum of the characters in s equals the sum of the characters in t.

1.9.1 (a)

Is $0121\ R\ 2200$?

Proof. Yes, because the sum of the characters in 0121 is 4 and the sum of the characters in 2200 is also 4. □

1.9.2 (b)

Is $1011\ R\ 2101$?

Proof. No, because the sum of the characters in 1011 is 3, whereas the sum of the characters in 2101 is 4. □

1.9.3 (c)

Is $2212\ R\ 2121$?

Proof. No, because the sum of the characters in 2212 is 7, whereas the sum of the characters in 2121 is 6. □

1.9.4 (d)

Is $1220\ R\ 2111$?

Proof. Yes, because the sum of the characters in 1220 is 5 and the sum of the characters in 2111 is also 5. □

1.10 Exercise 10

Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is, for every ordered pair $(x, y) \in A \times B, x\,R\,y \iff x < y$. State explicitly which ordered pairs are in R and R^{-1}.

Proof. $R = \{(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

$R^{-1} = \{(4,3),(5,3),(6,3),(5,4),(6,4),(6,5)\}$ □

1.11 Exercise 11

Let $A = \{3,4,5\}$ and $B = \{4,5,6\}$ and let S be the "divides" relation. That is, for every ordered pair $(x,y) \in A \times B, x\,S\,y \iff x \mid y$. State explicitly which ordered pairs are in S and S^{-1}.

Proof. $S = \{(3,6),(4,4),(5,5)\}, S^{-1} = \{(6,3),(4,4),(5,5)\}$ □

1.12 Exercise 12

1.12.1 (a)

Suppose a function $F : X \to Y$ is one-to-one but not onto. Is F^{-1} (the inverse relation for F) a function? Explain your answer.

Proof. No. If $F : X \to Y$ is not onto, then F fails to be defined on all of Y. In other words, there is an element y in Y such that $(y,x) \notin F^{-1}$ for any $x \in X$. Consequently, F^{-1} does not satisfy property (1) of the definition of function. □

1.12.2 (b)

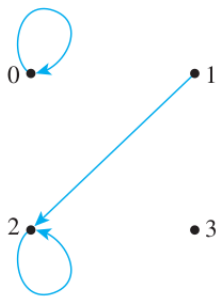
Suppose a function $F : X \to Y$ is onto but not one-to-one. Is F^{-1} (the inverse relation for F) a function? Explain your answer.

Proof. No. If $F : X \to Y$ is not one-to-one, then F for some y in Y, there will be multiple potential values for $F^{-1}(y)$. In other words, there is an element y in Y and elements $x_1, x_2 \in X$ such that $(y,x_1) \in F^{-1}$ and $(y,x_2) \in F^{-1}$. Consequently, F^{-1} does not satisfy property (2) of the definition of function. □

Draw the directed graphs of the relations defined in $13-18$.

1.13 Exercise 13

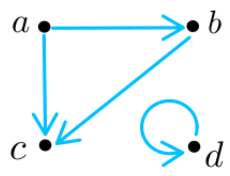
Define a relation R on $A = \{0,1,2,3\}$ by $R = \{(0,0),(1,2),(2,2)\}$.



Proof. □

1.14 Exercise 14

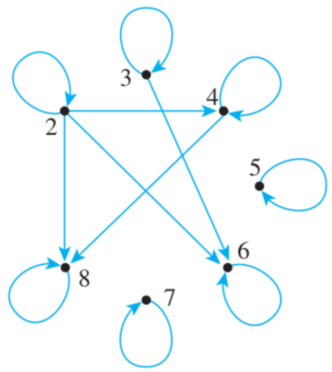
Define a relation S on $B = \{a, b, c, d\}$ by $S = \{(a, b), (a, c), (b, c), (d, d)\}$.



Proof. □

1.15 Exercise 15

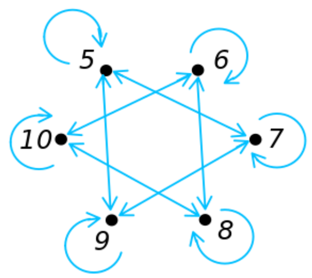
Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For every $x, y \in A, x\, R\, y \iff x \mid y$.



Proof. □

1.16 Exercise 16

Let $A = \{5, 6, 7, 8, 9, 10\}$ and define a relation S on A as follows: For every $x, y \in A, x\, S\, y \iff 2 \mid (x - y)$.

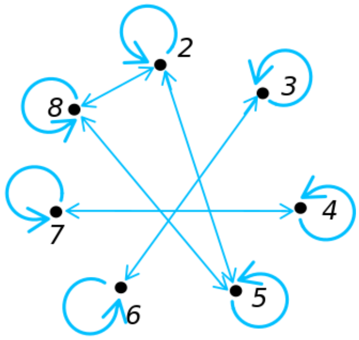


Proof. □

1.17 Exercise 17

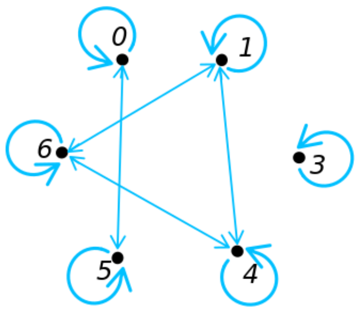
Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation T on A as follows: For every $x, y \in A, x\, T\, y \iff 3 \mid (x - y)$.

Proof. □



1.18 Exercise 18

Let $A = \{0, 1, 3, 4, 5, 6\}$ and define a relation V on A as follows: For every $x, y \in A, x\,V\,y \iff 5 \mid (x^2 - y^2)$.



Proof. □

1.19 Exercise 19

Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define relations R and S from A to B as follows: For every $(x, y) \in A \times B, x\,R\,y \iff x \mid y$ and $x\,S\,y \iff y - 4 = x$. State explicitly which ordered pairs are in $A \times B, R, S, R \cup S$, and $R \cap S$.

Proof. $A \times B = \{(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)\}$

$R = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$, $S = \{(2, 6), (4, 8)\}$, $R \cup S = R, R \cap S = S$ □

1.20 Exercise 20

Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For every $(x, y) \in A \times B, x\,R\,y \iff |x| \mid |y|$ and $x\,S\,y \iff x - y$ is even. State explicitly which ordered pairs are in $A \times B, R, S, R \cup S$, and $R \cap S$.

Proof. $A \times B = \{(-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2), (4, 1), (4, 2)\}$

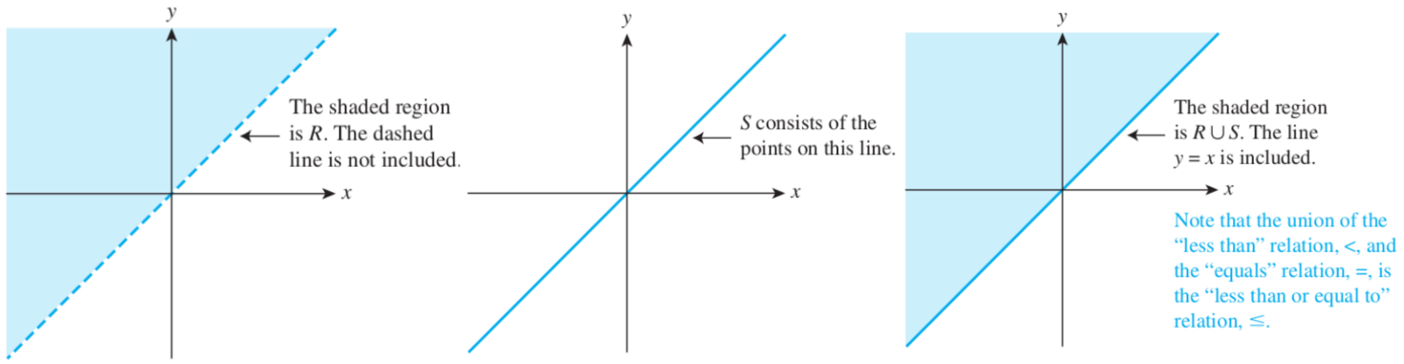
$R = \{(-1, 1), (1, 1), (2, 2)\}$, $S = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$, $R \cup S = S, R \cap S = R$ □

1.21 Exercise 21

Define relations R and S on R as follows: $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$ and

$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y\}$. That is, R is the "less than" relation and S is the "equals" relation on $\mathbb{R}$. Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.

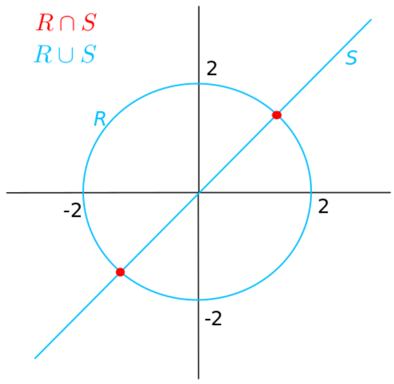
Proof. The graph of the intersection of R and S is obtained by finding the set of all points common to both graphs. But there are no points for which both $x < y$ and $x = y$. Hence $R \cap S = \varnothing$ and the graph consists of no points at all.



□

1.22 Exercise 22

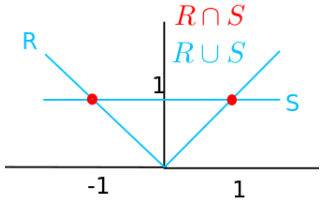
Define relations R and S on R as follows: $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 4\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y\}$. Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.



Proof. □

1.23 Exercise 23

Define relations R and S on R as follows: $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = |x|\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 1\}$. Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.



Proof. □

1.24 Exercise 24

In Example 8.1.7 consider the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = X. The response to the query is the projection onto the first two coordinates of the intersection of the database with the set $A_1 \times A_2 \times A_3 \times \{X\}$.

1.24.1 (a)

Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = pneumonia.

Proof. 574329 Tak Kurosawa, 011985 John Schmidt □

1.24.2 (b)

Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = appendicitis.

Proof. 466581 Mary Lazars, 778400 Jamal Baskers □

2 Exercise Set 8.2

In $1-8$, a number of relations are defined on the set $A = \{0, 1, 2, 3\}$. For each relation:

a. Draw the directed graph.

b. Determine whether the relation is reflexive.

c. Determine whether the relation is symmetric.

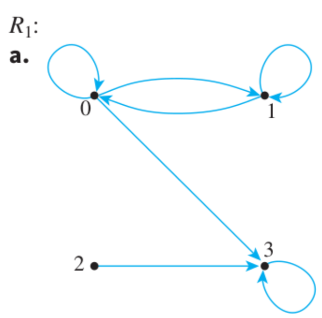
d. Determine whether the relation is transitive.

Give a counterexample in each case in which the relation does not satisfy one of the properties.

2.1 Exercise 1

$R_1 = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$

2.1.1 (a)



Proof. ∎

2.1.2 (b)

Proof. R_1 is not reflexive: $2 \not R_1 2$. ∎

2.1.3 (c)

Proof. R_1 is not symmetric: $2 \, R_1 \, 3$ but $3 \not R_1 2$. ∎

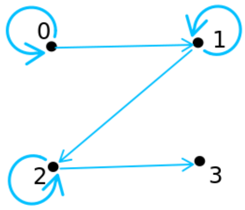
2.1.4 (d)

Proof. R_1 is not transitive: $1 \, R_1 \, 0$ and $0 \, R_1 \, 3$ but $1 \not R_1 3$. ∎

2.2 Exercise 2

$R_2 = \{(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)\}$

2.2.1 (a)



Proof. ∎

2.2.2 (b)

Proof. R_2 is not reflexive: $3 \not R_2 3$. ∎

2.2.3 (c)

Proof. R_2 is not symmetric: $2 \, R_2 \, 3$ but $3 \not R_2 2$. ∎

2.2.4 (d)

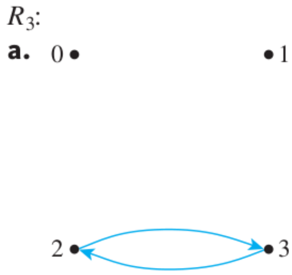
Proof. R_2 is not transitive: $0 \, R_2 \, 1$ and $1 \, R_2 \, 2$ but $0 \not R_2 2$. ∎

2.3 Exercise 3

$R_3 = \{(2,3),(3,2)\}$

2.3.1 (a)

Proof. ∎



2.3.2 (b)

Proof. R_3 is not reflexive: $0 \not{R}_3 0$. ☐

2.3.3 (c)

Proof. R_3 is symmetric. ☐

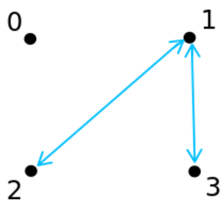
2.3.4 (d)

Proof. R_3 is not transitive: $2 \, R_3 \, 3$ and $3 \, R_3 \, 2$ but $2 \not{R}_3 2$. ☐

2.4 Exercise 4

$R_4 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

2.4.1 (a)



Proof. ☐

2.4.2 (b)

Proof. R_4 is not reflexive: $0 \not{R}_4 0$. ☐

2.4.3 (c)

Proof. R_4 is symmetric. ☐

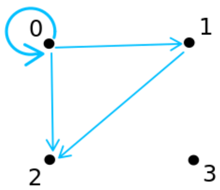
2.4.4 (d)

Proof. R_4 is not transitive: $2 \, R_4 \, 1$ and $1 \, R_4 \, 3$ but $2 \not{R}_4 3$. ☐

2.5 Exercise 5

$R_5 = \{(0, 0), (0, 1), (0, 2), (1, 2)\}$

2.5.1 (a)



Proof. □

2.5.2 (b)

Proof. R_5 is not reflexive: $3 \not R_5 3$. □

2.5.3 (c)

Proof. R_5 is not symmetric: $1 R_5 2$ but $2 \not R_5 1$. □

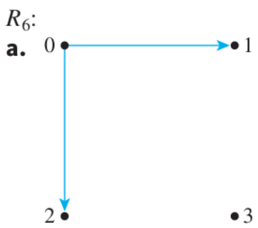
2.5.4 (d)

Proof. R_5 is transitive. □

2.6 Exercise 6

$R_6 = \{(0,1), (0,2)\}$

2.6.1 (a)



Proof. □

2.6.2 (b)

Proof. R_6 is not reflexive: $3 \not R_6 3$. □

2.6.3 (c)

Proof. R_6 is not symmetric: $0 R_6 1$ but $1 \not R_6 0$. □

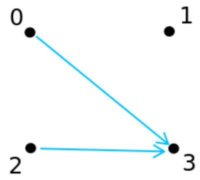
2.6.4 (d)

Proof. R_6 is transitive. □

2.7 Exercise 7

$R_7 = \{(0, 3), (2, 3)\}$

2.7.1 (a)



Proof. □

2.7.2 (b)

Proof. R_7 is not reflexive: $3 \not R_7 3$. □

2.7.3 (c)

Proof. R_7 is not symmetric: $0 \, R_7 \, 3$ but $3 \not R_7 0$. □

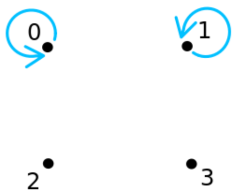
2.7.4 (d)

Proof. R_7 is transitive. □

2.8 Exercise 8

$R_8 = \{(0, 0), (1, 1)\}$

2.8.1 (a)



Proof. □

2.8.2 (b)

Proof. R_8 is not reflexive: $3 \not R_8 3$. □

2.8.3 (c)

Proof. R_8 is symmetric. □

2.8.4 (d)

Proof. R_8 is transitive. $\square$

In 9–33, determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

2.9 Exercise 9

R is the "greater than or equal to" relation on the set of real numbers: For every $x, y \in \mathbb{R}, x\,R\,y \iff x \geq y$.

Proof. ***R* is reflexive:** R is reflexive iff for every real number $x, x\,R\,x$. By definition of R, this means that for every real number $x, x \geq x$. In other words, for every real number $x, x > x$ or $x = x$, which is true.

***R* is not symmetric:** R is symmetric iff for all real numbers x and y, if $x\,R\,y$ then $y\,R\,x$. By definition of R, this means that for all real numbers x and y, if $x \geq y$ then $y \geq x$. The following counterexample shows that this is false. $x = 1$ and $y = 0$. Then $x \geq y$, but $y \not\geq x$ because $1 \geq 0$ and $0 \not\geq 1$.

***R* is transitive:** R is transitive iff for all real numbers x, y, and z, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$. By definition of R, this means that for all real numbers x, y, and z, if $x \geq y$ and $y \geq z$ then $x \geq z$. This is true by definition of $\geq$ and the transitive property of order for the real numbers. (See Appendix A, T18.) $\square$

2.10 Exercise 10

C is the circle relation on the set of real numbers: For every $x, y \in \mathbb{R}, x\,C\,y \iff x^2 + y^2 = 1$.

Proof. ***C* is not reflexive:** Let $x = 0$. Then $0^2 + 0^2 = 0 \neq 1$, therefore $0 \not{C} 0$.

***C* is symmetric:** Assume $x\,C\,y$. Then $x^2 + y^2 = 1$. So $y^2 + x^2 = 1$. So $y\,C\,x$.

***C* is not transitive:** Let $x = 1, y = 0, z = 1$. Then $x\,C\,y$ because $1^2 + 0^2 = 1$, and $y\,C\,z$ because $0^2 + 1^2 = 1$. However $x \not{C} z$ because $1^2 + 1^2 = 2 \neq 1$. $\square$

2.11 Exercise 11

D is the relation defined on $\mathbb{R}$ as follows: For every $x, y \in \mathbb{R}, x\,D\,y \iff xy \geq 0$.

Proof. ***D* is reflexive:** For all real numbers $x, x \cdot x = x^2 \geq 0$ so $x\,D\,x$.

***D* is symmetric:** Assume $x\,D\,y$. Then $xy \geq 0$. So $yx \geq 0$. So $y\,D\,x$.

***D* is not transitive:** Let $x = 1, y = 0, z = -1$. Then $xy = 0 \geq 0$ so $x\,D\,y$, and $yz = 0 \geq 0$ so $y\,D\,z$, but $xz = -1 \not\geq 0$ so $x \not{D} z$. $\square$

2.12 Exercise 12

E is the congruence modulo 4 relation on $\mathbb{Z}$: For every $m, n \in \mathbb{Z}, m\,E\,n \iff 4 \mid (m-n)$.

Proof. ***E* is reflexive:** For all $m \in \mathbb{Z}, (m-m) = 0 = 4 \cdot 0$ so $4 \mid (m-m)$ thus $m\,E\,m$.

***E* is symmetric:** Assume $m\,E\,n$. Then $4 \mid (m-n)$. So $m - n = 4 \cdot k$ for some integer k. So $n - m = 4 \cdot (-k)$ where $-k$ is an integer. So $4 \mid (n-m)$ and $n\,E\,m$.

***E* is transitive:** Assume $m\,E\,n$ and $n\,E\,o$. Then $4 \mid (m-n)$ and $4 \mid (n-o)$. So $m - n = 4k$ and $n - o = 4l$ for some integers k, l. So $m - o = (m-n) + (n-o) = 4k + 4l = 4(k+l)$ where $k + l$ is an integer. Thus $4 \mid (m-o)$ and $m\,E\,o$. □

2.13 Exercise 13

F is the congruence modulo 5 relation on $\mathbb{Z}$: For every $m, n \in \mathbb{Z}, m\,F\,n \iff 5 \mid (m-n)$.

Proof. ***F* is reflexive:** The proof is the same as in exercise 12.

***F* is symmetric:** The proof is the same as in exercise 12.

***F* is transitive:** The proof is the same as in exercise 12. □

2.14 Exercise 14

O is the relation defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,O\,n \iff m - n$ is odd.

Proof. ***O* is not reflexive:** $0 - 0 = 0$ is even, therefore $0 \not{O} 0$.

***O* is symmetric:** Assume $m\,O\,n$. So $m - n$ is odd. So $m - n = 2k + 1$ for some integer k. So $n - m = -2k - 1 = 2(-k-1) + 1$ where $-k - 1$ is an integer. So $n - m$ is odd and $n\,O\,m$.

***O* is not transitive:** $2 - 1 = 1$ is odd so $2\,O\,1$, and $1 - 0 = 1$ is odd so $1\,O\,0$, but $2 - 0 = 2$ is even so $2 \not{O} 0$. □

2.15 Exercise 15

D is the "divides" relation on Z^+: For all positive integers m and $n, m\,D\,n \iff m \mid n$.

Proof. ***D* is reflexive:** For all $m \in \mathbb{Z}^+$ $m = m \cdot 1$ therefore $m \mid m$, so $m\,D\,m$.

***D* is not symmetric:** $3\,D\,6$ because $3 \mid 6$ because $6 = 3 \cdot 2$, but $6 \not{D} 3$ because $6 \nmid 3$ since $3/6 = 1/2$ is not an integer.

***D* is transitive:** Assume $m\,D\,n$ and $n\,D\,o$. Then $m \mid n$ and $n \mid o$. So $n = mk$ and $o = nl$ for some integers k, l. So $o = nl = (mk)l = m(kl)$ where kl is an integer. So $m \mid o$ and $m\,D\,o$. □

2.16 Exercise 16

A is the "absolute value" relation on $\mathbb{R}$: For all real numbers x and y, $x\,A\,y \iff |x| = |y|$.

Proof. **A is reflexive:** For all real numbers x, $|x| = |x|$ so $x\,A\,x$.

A is symmetric: Assume $x\,A\,y$ so $|x| = |y|$. Then $|y| = |x|$ so $y\,A\,x$.

A is transitive: Assume $x\,A\,y$ and $y\,A\,z$, so $|x| = |y|$ and $|y| = |z|$. Then $|x| = |y| = |z|$ so $x\,A\,z$. □

2.17 Exercise 17

Recall that a prime number is an integer that is greater than 1 and has no positive integer divisors other than 1 and itself. (In particular, 1 is not prime.) A relation P is defined on $\mathbb{Z}$ as follows: For every $m, n \in \mathbb{Z}, m\,P\,n \iff \exists$ a prime number p such that $p \mid m$ and $p \mid n$.

Proof. **P is not reflexive:** There is no prime number p such that $p \mid 1$ and $p \mid 1$. Thus $1 \not P 1$.

P is symmetric: Assume $m\,P\,n$. So there is a prime number p such that $p \mid m$ and $p \mid n$. So $p \mid n$ and $p \mid m$, and thus $n\,P\,m$.

P is not transitive: Let $m = 6, n = 15, o = 35$. Then the prime $p = 3$ divides both m and n, so $m\,P\,n$, and the prime $q = 5$ divides both n and o, so $n\,P\,o$, but there is no prime that divides both $m = 2 \cdot 3$ and $o = 5 \cdot 7$, so $m \not P o$. □

2.18 Exercise 18

Define a relation Q on $\mathbb{R}$ as follows: For all real numbers x and y, $x\,Q\,y \iff x - y$ is rational.

Proof. **Q is reflexive:** For all reals $x \in \mathbb{R}, x - x = 0$ and 0 is rational, so $x\,Q\,x$.

Q is symmetric: Assume $x\,Q\,y$. Then $x - y$ is rational. Then $y - x = -(x - y)$ is rational (being the negative of a rational). So $y\,Q\,x$.

Q is transitive: Assume $x\,Q\,y$ and $y\,Q\,z$. Then $x - y$ and $y - z$ are rational. So $x - z = (x - y) + (y - z)$ is also rational (being the sum of two rationals). Thus $x\,Q\,z$. □

2.19 Exercise 19

Define a relation I on $\mathbb{R}$ as follows: For all real numbers x and y, $x\,I\,y \iff x - y$ is irrational.

Proof. **I is not reflexive:** For all reals $x \in \mathbb{R}, x - x = 0$ and 0 is not irrational, so $x \not I x$.

***I* is symmetric:** Assume $x\,I\,y$. Then $x-y$ is irrational. So $y-x=-(x-y)$ is irrational (being the negative of an irrational). So $y\,I\,x$.

***I* is not transitive:** Let $x=\sqrt{2}, y=0, z=\sqrt{2}$. Then $x\,I\,y$ because $x-y=\sqrt{2}$ is irrational. Also $y\,I\,z$ because $y-z=-\sqrt{2}$ is irrational. But $x-z=0$ is not irrational, thus $x \not{I} z$. $\square$

2.20 Exercise 20

Let $X=\{a,b,c\}$ and $\mathscr{P}(X)$ be the power set of X (the set of all subsets of X). A relation $\mathbf{E}$ is defined on $\mathscr{P}(X)$ as follows: For every $A,B \in \mathscr{P}(X), A\,\mathbf{E}\,B \iff$ the number of elements in A equals the number of elements in B.

Proof. **E is reflexive:** For every $A \in \mathscr{P}(X)$, the number of elements in A equals the number of elements in A. So $A\,\mathbf{E}\,A$.

E is symmetric: Assume $A\,\mathbf{E}\,B$. Then the number of elements in A equals the number of elements in B. So, the number of elements in B equals the number of elements in A. So $B\,\mathbf{E}\,A$.

E is transitive: Assume $A\,\mathbf{E}\,B$ and $B\,\mathbf{E}\,C$. Then the number of elements in A equals the number of elements in B, and the number of elements in B equals the number of elements in C. So the number of elements in A equals the number of elements in C. So $A\,\mathbf{E}\,C$. $\square$

2.21 Exercise 21

Let $X=\{a,b,c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation $\mathbf{L}$ is defined on $\mathscr{P}(X)$ as follows: For every $A,B \in \mathscr{P}(X), A\,\mathbf{L}\,B \iff$ the number of elements in A is less than the number of elements in B.

Proof. **L is not reflexive:** For all $A \in \mathscr{P}(X)$, the number of elements in A is not less than the number of elements in A. So $A \not{\mathbf{L}} A$.

L is not symmetric: Let $A=\varnothing, B=\{a\}$. Then the number of elements in A (which is 0) is less than the number of elements in B (which is 1). So $A\,\mathbf{L}\,B$. But the number of elements in B (which is 1) is not less than the number of elements in A (which is 0). So $B \not{\mathbf{L}} A$.

L is transitive: Assume $A\,\mathbf{L}\,B$ and $B\,\mathbf{L}\,C$. Then the number of elements in A is less than the number of elements in B, and the number of elements in B is less than the number of elements in C. Then the number of elements in A is less than the number of elements in C. So $A\,\mathbf{L}\,C$. $\square$

2.22 Exercise 22

Let $X=\{a,b,c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation $\mathbf{N}$ is defined on $\mathscr{P}(X)$ as follows: For every $A,B \in \mathscr{P}(X), A\mathbf{N}B \iff$ the number of elements in A is not equal to the number of elements in B.

Proof. **N is not reflexive:** Let $A = \{a\}$ which has 1 element. Then the number of elements in A is equal to the number of elements in A. So $A \,\not\!\mathbf{N}\, A$.

N is symmetric: Assume $A \,\mathbf{N}\, B$. Then the number of elements in A is not equal to the number of elements in B. So the number of elements in B is not equal to the number of elements in A, and $B \,\mathbf{N}\, A$.

N is not transitive: Let $A = \{a\}, B = \varnothing, C = \{c\}$. Then $A \,\mathbf{N}\, B$ because A has 1 element and B has 0 elements, and $0 \neq 1$. Similarly $B \,\mathbf{N}\, C$. But $A \,\not\!\mathbf{N}\, C$ because both A and C have 1 element, and $1 = 1$. □

2.23 Exercise 23

Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. Define the "subset" relation $\mathbf{S}$ on $\mathscr{P}(X)$ as follows: For every $A, B \in \mathscr{P}(X), A \,\mathbf{S}\, B \iff A \subseteq B$.

Proof. **S is reflexive:** For all $A \in \mathscr{P}(X), A \subseteq A$ therefore $A \,\mathbf{S}\, A$.

S is not symmetric: Let $A = \{a\}, B = \{a, b\}$. Then $A \subseteq B$, so $A \,\mathbf{S}\, B$. But $B \not\subseteq A$ therefore $B \,\not\!\mathbf{S}\, A$.

S is transitive: Assume $A \,\mathbf{S}\, B$ and $B \,\mathbf{S}\, C$. So $A \subseteq B$ and $B \subseteq C$. Then by transitivity of subsets, $A \subseteq C$, and $A \,\mathbf{S}\, C$. □

2.24 Exercise 24

Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. Define the "not equal to" relation $\mathbf{U}$ on $\mathscr{P}(X)$ as follows: For every $A, B \in \mathscr{P}(X), A\mathbf{U}B \iff A \neq B$.

Proof. **U is not reflexive:** For every $A \in \mathscr{P}(X), A = A$ therefore $A \,\not\!\mathbf{U}\, A$.

U is symmetric: Assume $A \,\mathbf{U}\, B$. Then $A \neq B$. So $B \neq A$, and $B \,\mathbf{U}\, A$.

U is not transitive: Let $X = \{x\}, A = \{x\}, B = \varnothing, C = \{x\}$. Then $A \,\mathbf{U}\, B$ because $A \neq B$, and $B \,\mathbf{U}\, C$ because $B \neq C$, but $A = C$ so $A \,\not\!\mathbf{U}\, C$. □

2.25 Exercise 25

Let A be the set of all strings of a's and b's of length 4. Define a relation R on A as follows: For every $s, t \in A, s \,R\, t \iff s$ has the same first two characters as t.

Proof. ***R* is reflexive:** For every string $s \in A$, s has the same first two characters as s. Thus $s \,R\, s$.

***R* is symmetric:** Assume $s \,R\, t$. Then s has the same first two characters as t. Then t has the same first two characters as s, so $t \,R\, s$.

***R* is transitive:** Assume $s \,R\, t$ and $t \,R\, r$. Then s has the same first two characters as t, and t has the same first two characters as r. So s has the same first two characters as r, and $s \,R\, r$. □

2.26 Exercise 26

Let A be the set of all strings of 0's, 1's, and 2's that have length 4 and for which the sum of the characters in the string is less than or equal to 2. Define a relation R on A as follows: For every $s, t \in A, s\,R\,t \iff$ the sum of the characters of s equals the sum of the characters of t.

Proof. ***R* is reflexive:** For every $s, \in A$, the sum of the characters of s equals the sum of the characters of s. So $s\,R\,s$.

***R* is symmetric:** Assume $s\,R\,t$. Then the sum of the characters of s equals the sum of the characters of t. So the sum of the characters of t equals the sum of the characters of s, and $s\,R\,t$.

***R* is transitive:** Assume $s\,R\,t$ and $t\,R\,r$. Then the sum of the characters of s equals the sum of the characters of t, and the sum of the characters of t equals the sum of the characters of r. So the sum of the characters of s equals the sum of the characters of r, and $s\,R\,r$. □

2.27 Exercise 27

Let A be the set of all English statements. A relation $\mathbf{I}$ is defined on A as follows: For every $p, q \in A, p\,\mathbf{I}\,q \iff p \implies q$ is true.

Proof. **I is reflexive:** For every $p \in A, p \implies p$ is true, therefore $p\,\mathbf{I}\,p$.

I is not symmetric: Let p be "$1 is greater than 2$" and let q be "2 is greater than 1". So p is false and q is true. Therefore $p \implies q$ is true and $q \implies p$ is false. So $p\,\mathbf{I}\,q$ but $q\,\not{\mathbf{I}}\,p$.

I is transitive: Assume $p\,\mathbf{I}\,q$ and $q\,\mathbf{I}\,r$. So $p \implies q$ is true and $q \implies r$ is true. By transitivity of implication, $p \implies r$ is true, and $p\,\mathbf{I}\,r$. □

2.28 Exercise 28

Let $A = \mathbb{R} \times \mathbb{R}$. A relation $\mathbf{F}$ is defined on A as follows: For every (x_1, y_1) and (x_2, y_2) in A, $(x_1, y_1)\mathbf{F}(x_2, y_2) \iff x_1 = x_2$.

Proof. **F is reflexive:** For every $(x, y) \in A, x = x$, therefore $(x, y)\mathbf{F}(x, y)$.

F is symmetric: Assume $(x_1, y_1)\mathbf{F}(x_2, y_2)$. Then $x_1 = x_2$. Then $x_2 = x_1$. So $(x_2, y_2)\mathbf{F}(x_1, y_1)$.

F is transitive: Assume $(x_1, y_1)\mathbf{F}(x_2, y_2)$ and $(x_2, y_2)\mathbf{F}(x_3, y_3)$. Then $x_1 = x_2$ and $x_2 = x_3$. Thus $x_1 = x_3$ and so $(x_1, y_1)\mathbf{F}(x_3, y_3)$. □

2.29 Exercise 29

Let $A = \mathbb{R} \times \mathbb{R}$. A relation $\mathbf{S}$ is defined on A as follows: For every (x_1, y_1) and (x_2, y_2) in $A, (x_1, y_1)\mathbf{S}(x_2, y_2) \iff y_1 = y_2$.

Proof. **S is reflexive:**

S is symmetric:

S is transitive: □

2.30 Exercise 30

Let A be the "punctured plane"; that is, A is the set of all points in the Cartesian plane except the origin (0, 0). A relation R is defined on A as follows: For every p_1 and p_2 in $A, p_1 \, R \, p_2 \iff p_1$ and p_2 lie on the same half line emanating from the origin.

Proof. ***R* is reflexive:** For all $p \in A, p$ and p lie on the same half line emanating from the origin. So $p \, R \, p$.

***R* is symmetric:** Assume $p_1 \, R \, p_2$. Then p_1 and p_2 lie on the same half line emanating from the origin. Then p_2 and p_1 lie on the same half line emanating from the origin. So $p_2 \, R \, p_1$.

***R* is transitive:** First notice that for any $p \in A$ there is exactly one half line emanating from the origin on which p lies.

Assume $p_1 \, R \, p_2$ and $p_2 \, R \, p_3$. Then p_1 and p_2 lie on the same half line emanating from the origin, say l_1. And p_2 and p_3 lie on the same half line emanating from the origin, say l_2. Since p_2 lies on both l_1 and l_2, by the previous paragraph $l_1 = l_2$. Then p_1 and p_3 lie on the same half line emanating from the origin. So $p_1 \, R \, p_3$. □

2.31 Exercise 31

Let A be the set of people living in the world today. A relation R is defined on A as follows: For all people p and q in $A, p \, R \, q \iff p$ lives within 100 miles of q.

Proof. ***R* is reflexive:** For every person p, p lives within 0 miles of p, so in particular p lives within 100 miles of p. Therefore $p \, R \, p$.

***R* is symmetric:** Assume $p \, R \, q$. So p lives within 100 miles of q. Then q lives within 100 miles of p. Thus $q \, R \, p$.

***R* is not transitive:** As a counterexample, take p to be an inhabitant of Chicago, Illinois, q an inhabitant of Kankakee, Illinois, and r an inhabitant of Champaign, Illinois. Then $p \, R \, q$ because Chicago is less than 100 miles from Kankakee, and $q \, R \, r$ because Kankakee is less than 100 miles from Champaign, but $p \not R \, r$ because Chicago is not less than 100 miles from Champaign. □

2.32 Exercise 32

Let A be the set of all lines in the plane. A relation R is defined on A as follows: For every l_1 and l_2 in $A, l_1 \, R \, l_2 \iff l_1$ is parallel to l_2. (Assume that a line is parallel to itself.)

Proof. **R is reflexive:** For every line $l \in A, l$ is parallel to itself, therefore $l \, R \, l$.

R is symmetric: Assume $l_1 \, R \, l_2$. Then l_1 is parallel to l_2. Then l_2 is parallel to l_1, so $l_2 \, R \, l_1$.

R is transitive: Assume $l_1 \, R \, l_2$ and $l_2 \, R \, l_3$. Then l_1 is parallel to l_2 and l_2 is parallel to l_3. By transitivity of parallelism l_1 is parallel to l_3 so $l_1 \, R \, l_3$. $\square$

2.33 Exercise 33

Let A be the set of all lines in the plane. A relation R is defined on A as follows: For every l_1 and l_2 in $A, l_1 \, R \, l_2 \iff l_1$ is perpendicular to l_2.

Proof. **R is not reflexive:** For evert line l in A, l is not perpendicular to itself (l is parallel to itself). Therefore $l \not{R} \, l$.

R is symmetric: Assume $l_1 \, R \, l_2$. Then l_1 is perpendicular to l_2. Then l_2 is perpendicular to l_1. So $l_2 \, R \, l_1$.

R is not transitive: Let l_1 be the line $y = 0$, let l_2 be the line $x = 0$ and l_3 be the line $y = 1$. Then l_2 is perpendicular to both l_1 and l_3 so $l_1 \, R \, l_2$ and $l_2 \, R \, l_3$. But l_1 is parallel to l_3 so $l_1 \not{R} \, l_3$. $\square$

In $34 - 36$, assume that R is a relation on a set A. Prove or disprove each statement.

2.34 Exercise 34

If R is reflexive, then R^{-1} is reflexive.

Proof. Suppose R is any reflexive relation on a set A. *[We must show that R^{-1} is reflexive. To show this, we must show that for every x in $A, x \, R^{-1} x$.]* Given any element x in A, since R is reflexive, $x \, R \, x$, and by definition of relation, this means that $(x, x) \in R$. It follows, by definition of the inverse of a relation, that $(x, x) \in R^{-1}$, and so, by definition of relation, $x \, R^{-1} \, x$ *[as was to be shown]*. $\square$

2.35 Exercise 35

If R is symmetric, then R^{-1} is symmetric.

Proof. Assume R is symmetric. *[We want to show R^{-1} is symmetric.]* Assume $x \, R^{-1} \, y$. We need to show $y \, R^{-1} \, x$. By definition of $R^{-1}, y \, R \, x$. Since R is symmetric, $x \, R \, y$. By definition of R^{-1} again, $y \, R^{-1} \, x$. $\square$

2.36 Exercise 36

If R is transitive, then R^{-1} is transitive.

Proof. Assume R is transitive. *[We want to show R^{-1} is transitive.]* Assume $x\,R^{-1}\,y$ and $y\,R^{-1}\,z$. We need to show $x\,R^{-1}\,z$. By definition of R^{-1}, $y\,R\,x$ and $z\,R\,y$. Since R is transitive, $z\,R\,x$. By definition of R^{-1} again, $x\,R^{-1}\,z$. □

In 37 – 42, assume that R and S are relations on a set A. Prove or disprove each statement.

2.37 Exercise 37

If R and S are reflexive, is $R \cap S$ reflexive? Why?

Proof. Yes. Suppose R and S are reflexive. *[To show that $R \cap S$ is reflexive, we must show that $\forall x \in A, (x,x) \in R \cap S$.]* So suppose $x \in A$. Since R is reflexive, $(x,x) \in R$, and since S is reflexive, $(x,x) \in S$. Thus, by definition of intersection, $(x,x) \in R \cap S$ *[as was to be shown]*. □

2.38 Exercise 38

If R and S are symmetric, is $R \cap S$ symmetric? Why?

Proof. Yes. Suppose R and S are symmetric. *[To show that $R \cap S$ is symmetric, we must show that $\forall x, y \in A$, if $(x,y) \in R \cap S$ then $(y,x) \in R \cap S$.]* So suppose $x, y \in A$ and $(x,y) \in R \cap S$. By definition of intersection $(x,y) \in R$ and $(x,y) \in S$. Since R is symmetric, $(y,x) \in R$, and since S is symmetric, $(y,x) \in S$. Thus, by definition of intersection, $(y,x) \in R \cap S$ *[as was to be shown]*. □

2.39 Exercise 39

If R and S are transitive, is $R \cap S$ transitive? Why?

Proof. Yes. Suppose R and S are transitive. *[To show that $R \cap S$ is transitive, we must show that $\forall x, y, z \in A$, if $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$ then $(x,z) \in R \cap S$.]* So suppose $x, y, z \in A$ and $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$. By definition of intersection $(x,y) \in R$ and $(x,y) \in S$ and $(y,z) \in R$ and $(y,z) \in S$. Since R is transitive, $(x,z) \in R$, and since S is transitive, $(x,z) \in S$. Thus, by definition of intersection, $(x,z) \in R \cap S$ *[as was to be shown]*. □

2.40 Exercise 40

If R and S are reflexive, is $R \cup S$ reflexive? Why?

Proof. Yes. To prove this we must show that for all x in A, $(x,x) \in R \cup S$. So suppose x is a particular but arbitrarily chosen element in A. *[We must show that $(x,x) \in R \cup S$.]* Then $(x,x) \in R$ because R is reflexive, and hence $(x,x) \in R \cup S$ by definition of union, *[as was to be shown]*. □

2.41 Exercise 41

If R and S are symmetric, is $R \cup S$ symmetric? Why?

Proof. Yes. To prove this we must show that for all x and y in A, if $(x, y) \in R \cup S$ then $(y, x) \in R \cup S$. So suppose (x, y) is a particular but arbitrarily chosen element in $R \cup S$. *[We must show that $(y, x) \in R \cup S$.]* By definition of union, $(x, y) \in R$ or $(x, y) \in S$. In case $(x, y) \in R$, then $(y, x) \in R$ because R is symmetric, and hence $(y, x) \in R \cup S$ by definition of union. In case $(x, y) \in S$ then $(y, x) \in S$ because S is symmetric, and hence $(y, x) \in R \cup S$ by definition of union. Thus, in both cases, $(y, x) \in R \cup S$ *[as was to be shown]*. $\square$

2.42 Exercise 42

If R and S are transitive, is $R \cup S$ transitive? Why?

Proof. No. Let $A = \{a, b, c, d\}, R = \{(a, b), (b, c), (a, c)\}, S = \{(c, a), (a, d), (c, d)\}$. Then R and S are transitive but $R \cup S$ is not: $(a, c) \in R \cup S$ and $(c, a) \in R \cup S$ but $(a, a) \notin R \cup S$. $\square$

In $43 - 50$, the following definitions are used: a relation on a set A is defined to be irreflexive if, and only if, for every $x \in A, x \not{R} x$; asymmetric if, and only if, for every $x, y \in A$ if $x \, R \, y$ then $y \not{R} x$; intransitive if, and only if, for every $x, y, z \in A$, if xRy and yRz then $x \not{R} z$. For each of the relations in the referenced exercise, determine whether the relation is irreflexive, asymmetric, intransitive, or none of these.

2.43 Exercise 43

Exercise 1

Proof. R_1 is not irreflexive because $(0, 0) \in R_1$. R_1 is not asymmetric because $(0, 1) \in R_1$ and $(1, 0) \in R_1$. R_1 is not intransitive because $(0, 1) \in R_1$ and $(1, 0) \in R_1$ and $(0, 0) \in R_1$. $\square$

2.44 Exercise 44

Exercise 2

Proof. Recall $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\}$.

R_2 is not irreflexive because $(0, 0) \in R_2$.

R_2 is not asymmetric because $(0, 0) \in R_2$ and $(0, 0) \in R_2$.

R_2 is not intransitive because $(0, 0) \in R_2$ and $(0, 1) \in R_2$ and $(0, 1) \in R_2$. $\square$

2.45 Exercise 45

Exercise 3

Proof. R_3 is irreflexive because no element of A is related by R_3 to itself. R_3 is not asymmetric because $(2,3) \in R_3$ and $(3,2) \in R_3$. R_3 is intransitive. To see why, observe that R_3 consists only of $(2,3)$ and $(3,2)$. Now $(2,3) \in R_3$ and $(3,2) \in R_3$ but $(2,2) \notin R_3$. Also $(3,2) \in R_3$ and $(2,3) \in R_3$ but $(3,3) \notin R_3$. □

2.46 Exercise 46

Exercise 4

Proof. Recall $R_4 = \{(1,2),(2,1),(1,3),(3,1)\}$.

R_4 is irreflexive.

R_4 is not asymmetric because $(1,2) \in R_4$ and $(2,1) \in R_4$.

R_4 is intransitive. □

2.47 Exercise 47

Exercise 5

Proof. Recall $R_5 = \{(0,0),(0,1),(0,2),(1,2)\}$.

R_5 is not irreflexive because $(0,0) \in R_5$.

R_5 is not asymmetric because $(0,0) \in R_5$ and $(0,0) \in R_5$.

R_5 is not intransitive because $(0,1) \in R_5$ and $(1,2) \in R_5$ and $(0,2) \in R_5$. □

2.48 Exercise 48

Exercise 6

Proof. Recall $R_6 = \{(0,1),(0,2)\}$. R_6 is irreflexive because no element of A is related by R_6 to itself. R_6 is asymmetric because R_6 consists only of $(0,1)$ and $(0,2)$ and neither $(1,0)$ nor $(2,0)$ is in R_6. R_6 is not intransitive. Let $x = y = z = 0$. Then $(x,y) \in R_6$ and $(y,z) \in R_6$ and $(x,z) \in R_6$. □

2.49 Exercise 49

Exercise 7

Proof. Recall $R_7 = \{(0,3),(2,3)\}$.

R_7 is irreflexive, asymmetric and intransitive. □

2.50 Exercise 50

Exercise 8

Proof. Recall $R_8 = \{(0,0),(1,1)\}$. R_8 is not irreflexive because $(0,0) \in R_8$. R_8 is not asymmetric because $(0,0) \in R_8$ and $(0,0) \in R_8$. R_8 is intransitive. $\square$

In $51 - 53, R, S$, and T are relations defined on $A = \{0,1,2,3\}$.

2.51 Exercise 51

Let $R = \{(0,1),(0,2),(1,1),(1,3),(2,2),(3,0)\}$. Find R^t, the transitive closure of R.

Proof. $R^t = R \cup \{(0,0),(0,3),(1,0),(3,1),(3,2),(3,3),(0,2),(1,2)\} =$

$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,2),(3,0),(3,1),(3,2),(3,3)\}$. $\square$

2.52 Exercise 52

Let $S = \{(0,0),(0,3),(1,0),(1,2),(2,0),(3,2)\}$. Find S^t, the transitive closure of S.

Proof. $S^t = S \cup \{(0,2),(1,3),(2,2),(2,3),(3,3)\}$

$= \{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,2),(2,3),(3,2),(3,3)\}$ $\square$

2.53 Exercise 53

Let $T = \{(0,2),(1,0),(2,3),(3,1)\}$. Find T^t, the transitive closure of T.

Proof. $T^t = T \cup \{(0,3),(0,1),(0,0),(1,2),(1,3),(1,1),(2,1),(2,0),(2,2),(3,0),(3,2),$
$(3,3)\} = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3),$
$(3,0),(3,1),(3,2),(3,1)\}$ $\square$

2.54 Exercise 54

Write a computer algorithm to test whether a relation R defined on a finite set A is reflexive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Algorithm: Test for Reflexivity
[The input for this algorithm is a binary relation R defined on a set A, that is represented as the one- dimensional array $a[1], a[2], \ldots, a[n]$. To test whether R is reflexive, a variable called answer is initially set equal to "yes," and each element $a[i]$ of A is examined in turn to see whether it is related by R to itself. If any element is not related to itself by R, then answer is set equal to "no," the while loop is not repeated, and processing terminates.]

Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of $A \times A$]*
Algorithm Body:
$i := 1$, answer $:=$ "yes"
while (answer $=$ "yes" and $i \leq n$)
if $(a[i], a[i]) \notin R$ **then** answer $:=$ "no"
$i := i + 1$
end while
Output: answer *[a string]*

2.55 Exercise 55

Write a computer algorithm to test whether a relation R defined on a finite set A is symmetric, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Algorithm: Test for Symmetry
Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of $A \times A$]*
Algorithm Body:
$i := 1, j := 1$, answer $:=$ "yes"
while (answer $=$ "yes" and $i \leq n$)
while (answer $=$ "yes" and $j \leq n$)
if $(a[i], a[j]) \in R$ and $(a[j], a[i]) \notin R$ **then** answer $:=$ "no"
$j := j + 1$
end while
$i := i + 1$
end while
Output: answer *[a string]*

2.56 Exercise 56

Write a computer algorithm to test whether a relation R defined on a finite set A is transitive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Algorithm: Test for Transitivity
Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing a set A]*, R *[a subset of $A \times A$]*
Algorithm Body:
$i := 1, j := 1, k := 1$, answer $:=$ "yes"

```
while (answer = "yes" and i ≤ n)
    while (answer = "yes" and j ≤ n)
        while (answer = "yes" and k ≤ n)
            if (a[i], a[j]) ∈ R and (a[j], a[k]) ∈ R and (a[i], a[k]) ∉ R
            then answer := "no"
            k := k + 1
        end while
        j := j + 1
    end while
    i := i + 1
end while
Output: answer [a string]
```

3 Exercise Set 8.3

3.1 Exercise 1

3.1.1 (a)

Proof. □

3.1.2 (b)

Proof. □

3.1.3 (c)

Proof. □

3.1.4 (d)

Proof. □

3.2 Exercise 2

3.2.1 (a)

Proof. □

3.2.2 (b)

Proof. □

3.2.3 (c)

Proof. □

3.3 Exercise 3

Proof. □

3.4 Exercise 4

Proof. □

3.5 Exercise 5

Proof. □

3.6 Exercise 6

Proof. □

3.7 Exercise 7

Proof. □

3.8 Exercise 8

Proof. □

3.9 Exercise 9

Proof. □

3.10 Exercise 10

Proof. □

3.11 Exercise 11

Proof. □

3.12 Exercise 12

Proof. □

3.13 Exercise 13

Proof. □

3.14 Exercise 14

Proof. □

3.15 Exercise 15

3.15.1 (a)

Proof. □

3.15.2 (b)

Proof. □

3.15.3 (c)

Proof. □

3.15.4 (d)

Proof. □

3.16 Exercise 16

3.16.1 (a)

Proof. □

3.16.2 (b)

Proof. □

3.17 Exercise 17

3.17.1 (a)

Proof. □

3.17.2 (b)

Proof. □

3.18 Exercise 18

3.18.1 (a)

Proof. □

3.18.2 (b)

Proof. □

3.19 Exercise 19

3.19.1 (a)

Proof. □

3.19.2 (b)

Proof. □

3.20 Exercise 20

Proof. □

3.21 Exercise 21

Proof. □

3.22 Exercise 22

Proof. □

3.23 Exercise 23

Proof. □

3.24 Exercise 24

Proof. □

3.25 Exercise 25

Proof. □

3.26 Exercise 26

Proof. □

3.27 Exercise 27

Proof. □

3.28 Exercise 28

Proof. □

3.29 Exercise 29

Proof. □

3.30 Exercise 30

Proof. □

3.31 Exercise 31

Proof. □

3.32 Exercise 32

Proof. □

3.33 Exercise 33

Proof. □

3.34 Exercise 34

Proof. □

3.35 Exercise 35

Proof. □

3.36 Exercise 36

Proof. □

3.37 Exercise 37

Proof. □

3.38 Exercise 38

Proof. □

3.39 Exercise 39

Proof. □

3.40 Exercise 40

Proof. □

3.41 Exercise 41

Proof. □

3.42 Exercise 42

3.42.1 (a)

Proof. □

3.42.2 (b)

Proof. □

3.42.3 (c)

Proof. □

3.42.4 (d)

Proof. □

3.43 Exercise 43

3.43.1 (a)

Proof. □

3.43.2 (b)

Proof. □

3.43.3 (c)

Proof. □

3.43.4 (d)

Proof. □

3.43.5 (e)

Proof. □

3.43.6 (f)

Proof. □

3.44 Exercise 44

3.44.1 (a)

Proof. □

3.44.2 (b)

Proof. □

3.44.3 (c)

Proof. □

3.44.4 (d)

Proof. □

3.44.5 (e)

Proof. □

3.44.6 (f)

Proof. □

3.44.7 (g)

Proof. □

3.45 Exercise 45

Proof. □

3.46 Exercise 46

Proof. □

3.47 Exercise 47

3.47.1 (a)

Proof. □

3.47.2 (b)

Proof. □

3.47.3 (c)

Proof. □

3.47.4 (d)

Proof. □

3.47.5 (e)

Proof. □

3.47.6 (f)

Proof. □

3.47.7 (g)

Proof. □

4 Exercise Set 8.4

4.1 Exercise 1

4.1.1 (a)

Proof. □

4.1.2 (b)

Proof. □

4.2 Exercise 2

4.2.1 (a)

Proof. □

4.2.2 (b)

Proof. □

4.3 Exercise 3

4.3.1 (a)

Proof. □

4.3.2 (b)

Proof. □

4.3.3 (c)

Proof. □

4.3.4 (d)

Proof. □

4.3.5 (e)

Proof. □

4.4 Exercise 4

4.4.1 (a)

Proof. □

4.4.2 (b)

Proof. □

4.4.3 (c)

Proof. □

4.4.4 (d)

Proof. □

4.4.5 (e)

Proof. □

4.5 Exercise 5

Proof. □

4.6 Exercise 6

Proof. □

4.7 Exercise 7

4.7.1 (a)

Proof. □

4.7.2 (b)

Proof. □

4.7.3 (c)

Proof. □

4.7.4 (d)

Proof. □

4.7.5 (e)

Proof. □

4.8 Exercise 8

4.8.1 (a)

Proof. □

4.8.2 (b)

Proof. □

4.8.3 (c)

Proof. □

4.8.4 (d)

Proof. □

4.8.5 (e)

Proof. □

4.9 Exercise 9

4.9.1 (a)

Proof. □

4.9.2 (b)

Proof. ◻

4.10 Exercise 10

Proof. ◻

4.11 Exercise 11

Proof. ◻

4.12 Exercise 12

4.12.1 (a)

Proof. ◻

4.12.2 (b)

Proof. ◻

4.13 Exercise 13

4.13.1 (a)

Proof. ◻

4.13.2 (b)

Proof. ◻

4.14 Exercise 14

Proof. ◻

4.15 Exercise 15

Proof. ◻

4.16 Exercise 16

Proof. ◻

4.17 Exercise 17

Proof. ◻

4.18 Exercise 18

Proof. □

4.19 Exercise 19

Proof. □

4.20 Exercise 20

Proof. □

4.21 Exercise 21

Proof. □

4.22 Exercise 22

Proof. □

4.23 Exercise 23

Proof. □

4.24 Exercise 24

Proof. □

4.25 Exercise 25

Proof. □

4.26 Exercise 26

Proof. □

4.27 Exercise 27

Proof. □

4.28 Exercise 28

Proof. □

4.29 Exercise 29

Proof. □

4.30 Exercise 30

Proof. □

4.31 Exercise 31

4.31.1 (a)

Proof. □

4.31.2 (b)

Proof. □

4.31.3 (c)

Proof. □

4.32 Exercise 32

4.32.1 (a)

Proof. □

4.32.2 (b)

Proof. □

4.33 Exercise 33

Proof. □

4.34 Exercise 34

Proof. □

4.35 Exercise 35

Proof. □

4.36 Exercise 36

Proof. □

4.37 Exercise 37

Proof. □

4.38 Exercise 38

Proof. □

4.39 Exercise 39

Proof. □

4.40 Exercise 40

Proof. □

4.41 Exercise 41

4.41.1 (a)

Proof. □

4.41.2 (b)

Proof. □

4.42 Exercise 42

Proof. □

4.43 Exercise 43

Proof. □

5 Exercise Set 8.5

5.1 Exercise 1

5.1.1 (a)

Proof. □

5.1.2 (b)

Proof. □

5.1.3 (c)

Proof. □

5.1.4 (d)

Proof. □

5.2 Exercise 2

Proof. □

5.3 Exercise 3

Proof. □

5.4 Exercise 4

Proof. □

5.5 Exercise 5

Proof. □

5.6 Exercise 6

Proof. □

5.7 Exercise 7

Proof. □

5.8 Exercise 8

Proof. □

5.9 Exercise 9

Proof. □

5.10 Exercise 10

Proof. □

5.11 Exercise 11

5.11.1 (a)

Proof. □

5.11.2 (b)

Proof. □

5.11.3 (c)

Proof. □

5.11.4 (d)

Proof. □

5.11.5 (e)

Proof. □

5.11.6 (f)

Proof. □

5.11.7 (g)

Proof. □

5.12 Exercise 12

Proof. □

5.13 Exercise 13

Proof. □

5.14 Exercise 14

5.14.1 (a)

Proof. □

5.14.2 (b)

Proof. □

5.15 Exercise 15

Proof. □

5.16 Exercise 16

5.16.1 (a)

Proof. □

5.16.2 (b)

Proof. □

5.17 Exercise 17

Proof. □

5.18 Exercise 18

Proof. □

5.19 Exercise 19

Proof. □

5.20 Exercise 20

Proof. □

5.21 Exercise 21

5.21.1 (a)

Proof. □

5.21.2 (b)

Proof. □

5.22 Exercise 22

Proof. □

5.23 Exercise 23

Proof. □

5.24 Exercise 24

Proof. □

5.25 Exercise 25

Proof. ∎

5.26 Exercise 26

Proof. ∎

5.27 Exercise 27

Proof. ∎

5.28 Exercise 28

Proof. ∎

5.29 Exercise 29

Proof. ∎

5.30 Exercise 30

5.30.1 (a)

Proof. ∎

5.30.2 (b)

Proof. ∎

5.30.3 (c)

Proof. ∎

5.30.4 (d)

Proof. ∎

5.31 Exercise 31

Proof. ∎

5.32 Exercise 32

Proof. ∎

5.33 Exercise 33

Proof. □

5.34 Exercise 34

Proof. □

5.35 Exercise 35

Proof. □

5.36 Exercise 36

Proof. □

5.37 Exercise 37

Proof. □

5.38 Exercise 38

Proof. □

5.39 Exercise 39

Proof. □

5.40 Exercise 40

5.40.1 (a)

Proof. □

5.40.2 (b)

Proof. □

5.41 Exercise 41

5.41.1 (a)

Proof. □

5.41.2 (b)

Proof. □

5.42 Exercise 42

Proof. □

5.43 Exercise 43

Proof. □

5.44 Exercise 44

Proof. □

5.45 Exercise 45

Proof. □

5.46 Exercise 46

Proof. □

5.47 Exercise 47

Proof. □

5.48 Exercise 48

Proof. □

5.49 Exercise 49

5.49.1 (a)

Proof. □

5.49.2 (b)

Proof. □

5.50 Exercise 50

5.50.1 (a)

Proof. □

5.50.2 (b)

Proof. □

5.51 Exercise 51

5.51.1 (a)

Proof. $\square$

5.51.2 (b)

Proof. $\square$