Zbirka riješenih zadataka iz matematike

Baraković Elvis

Sadržaj

1 Elementi matematičke logike

Svaka izjavna rečenica nekog jezika koja ima smisla i koja ima osobinu da je tačna ili netačna naziva se iskaz (sud). Svaki iskaz možemo obilježiti nekim slovom (najčešće slovom latinice), brojem ili nekim drugim simbolom. Takve simbole nazivamo iskazna slova.
Činjenicu da je iskaz p tačan označavaćemo sa $\tau(p) = \top$ ili $\tau(p) = 1$, a činjenicu da je netačan sa $\tau(p) = \perp$ ili $\tau(p) = 0$.
Od dva ili više iskaza možemo sastaviti novi, složeni iskaz. Sa iskazima, kao elementima skupa iskaza, uvodimo sljedeće operacije:

1. **NEGACIJA** iskaza p je novi iskaz $\rceil p$ koji ima suprotnu istinitosnu vrijednost od istinitosne vrijednosti iskaza p.

2. **KONJUNKCIJA** iskaza p i q je novi iskaz $p \wedge q$ koji je tačan ako su iskazi p i q istovremeno tačni a netačan u ostalim slučajevima.

3. **DISJUNKCIJA** iskaza p i q je novi iskaz $p \vee q$ koji je netačan ako su iskazi p i q istvremeno netačni a tačan u ostalim slučajevima.

4. **IMPLIKACIJA** iskaza p i q je novi iskaz $p \implies q$ koji je netačan ako je iskaz p tačan a iskaz q netačan a tačan u ostalim slučajevima.

5. **EKVIVALENCIJA** iskaza p i q je novi iskaz $p \iff q$ koji je tačan ako su iskazi p i q istvremeno iste istinitosne vrijednosti, a netačan u ostalim slučajevima.

Operacije sa iskazima možemo predstaviti u obliku tabele.

$\tau(p)$	$\tau(q)$	$\tau(\rceil p)$	$\tau(p \wedge q)$	$\tau(p \vee q)$	$\tau(p \implies q)$	$\tau(p \iff q)$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

Zadatak 1.1 *Provjeriti da li je sljedeća iskazna formula tautologija*

$$\mathcal{F}: \qquad \underbrace{\rceil (p \iff q)}_{L} \iff \underbrace{(p \iff \rceil q)}_{D}.$$

Rješenje:

Posmatrajmo tablicu

$\tau(p)$	$\tau(q)$	$\tau(p \iff q)$	$\tau(L)$	$\tau(\rceil q)$	$\tau(D)$	$\tau(\mathcal{F})$
1	1	1	0	0	0	**1**
1	0	0	1	1	1	**1**
0	1	0	1	0	1	**1**
0	0	1	0	1	0	**1**

Formula je tautologija.

Zadatak 1.2 *Provjeriti da li je sljedeća iskazna formula tautologija*

$$\mathcal{F}: \qquad \underbrace{\rceil (p \iff q)}_{L} \iff \underbrace{(p \wedge \rceil q) \vee (\rceil p \wedge q)}_{D}.$$

Rješenje:
Posmatrajmo tablicu

$\tau(p)$	$\tau(q)$	$\tau(p \iff q)$	$\tau(L)$	$\tau(\rceil p)$	$\tau(\rceil q)$	$\tau(p \wedge \rceil q)$	$\tau(\rceil p \wedge q)$	$\tau(D)$	$\tau(\mathcal{F})$
1	1	1	0	0	0	0	0	0	**1**
1	0	0	1	0	1	1	0	1	**1**
0	1	0	1	1	0	0	1	1	**1**
0	0	1	0	1	1	0	0	0	**1**

Formula je tautologija.

Zadatak 1.3 *Provjeriti da li je sljedeća iskazna formula tautologija*

$$\mathcal{F}: \qquad \underbrace{p \wedge (q \vee r)}_{L} \iff \underbrace{(p \wedge q) \vee (p \wedge r)}_{D}.$$

Rješenje:
Posmatrajmo tablicu

$\tau(p)$	$\tau(q)$	$\tau(r)$	$\tau(q \vee r)$	$\tau(L)$	$\tau(p \wedge q)$	$\tau(p \wedge r)$	$\tau(D)$	$\tau(\mathcal{F})$
1	1	1	1	1	1	1	1	**1**
1	1	0	1	1	1	0	1	**1**
1	0	1	1	1	0	1	1	**1**
1	0	0	0	0	0	0	0	**1**
0	1	1	1	0	0	0	0	**1**
0	1	0	1	0	0	0	0	**1**
0	0	1	1	0	0	0	0	**1**
0	0	0	0	0	0	0	0	**1**

Formula je tautologija.

Zadatak 1.4 *Provjeriti da li je sljedeća iskazna formula tautologija*

$$\mathcal{F}: \qquad \underbrace{(p \implies (q \vee r))}_{L} \implies \underbrace{(p \wedge q) \vee (p \wedge r)}_{D}.$$

Rješenje:
Posmatrajmo tablicu

$\tau(p)$	$\tau(q)$	$\tau(r)$	$\tau(q \vee r)$	$\tau(L)$	$\tau(p \wedge q)$	$\tau(p \wedge r)$	$\tau(D)$	$\tau(\mathcal{F})$
1	1	1	1	1	1	1	1	**1**
1	1	0	1	1	1	0	1	**1**
1	0	1	1	1	0	1	1	**1**
1	0	0	0	0	0	0	0	**1**
0	1	1	1	1	0	0	0	**0**
0	1	0	1	1	0	0	0	**0**
0	0	1	1	1	0	0	0	**0**
0	0	0	0	1	0	0	0	**0**

Formula nije tautologija

Zadatak 1.5 *Bez upotrebe tablice dokazati da je sljedeća formula tautologija*

$$\mathcal{F}: \qquad \underbrace{((p \implies q) \wedge \rceil q)}_{L} \implies \underbrace{\rceil p}_{D}.$$

Rješenje:

Pretpostavimo suprotno, tj da je formula nije tautologija. Tada postoje istinitosne vrijednosti iskaza p i q za koje vrijedi $\tau(L \implies D) = 0$. To će se desiti samo onda ako je iskaz L tačan a iskaz D netačan. Međutim, da bi iskaz D bio netačan onda iskaz p mora biti tačan. Da bi iskaz P bio tačan, onda moraju biti tačni iskazi $p \implies q$ i $\rceil q$ a odavdje vidimo da iskaz q mora biti netačan. Ali kako je iskaz p tačan a iskaz q netačan tada je iskaz $p \implies q$ netačan a nama se zahtijeva da bude tačan.

Kontradikcija sa pretpostavkom.

Dakle, iskaz je tautologija.

Zadaci za samostalan rad

Ispitati da li su sljedeće iskazne formule tautologije

1. $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
2. $(p \implies q) \iff \rceil p \vee q$

3. $(p \iff q) \iff (p \wedge q) \vee (\rceil p \wedge \rceil q)$
4. $((p \implies q) \wedge (q \implies r)) \implies (p \implies r)$
5. $((p \implies q) \wedge (q \wedge \rceil r)) \implies (\rceil p \implies r)$

2 Skupovi i relacije

Pojam skupa je osnovni pojam u matematici pa se zato i ne definiše. Ovaj pojam objašnjavamo navodeći primjer nekog skupa i ukazujući na pravila njegove upotrebe u matematici.
Skupove označavamo velikim slovima latinice $A, B, C, ..., X, Y, ...$ a elemente nekog skupa malim slovima latinice $a, b, c, ..., x, y, ...$.
Za skupove A i B kažemo da su jednaki ako su sastavljeni od istih elemenata.
Za skup A kažemo da je podskup skupa b ako svaki element iz skupa A istovremeno pripada i skupu B. To označavamo sa $A \subseteq B$.
Operacije sa skupovima definišemo sa:

$$\begin{aligned} A \cap B &= \{x : x \in A \wedge x \in B\} \\ A \cup B &= \{x : x \in A \vee x \in B\} \\ A \backslash B &= \{x : x \in A \wedge x \notin B\} \\ A \Delta B &= (A \backslash B) \cup (B \backslash A) \\ A \times B &= \{(a,b) : a \in A \wedge b \in B\} \end{aligned}$$

Zadatak 2.1 *Dokazati:*

$$(A \cup B)^C = A^C \cap B^C.$$

Rješenje:

$$\begin{aligned} x &\in (A \cup B)^C \\ &\Longleftrightarrow x \notin A \cup B \\ &\Longleftrightarrow \rceil (x \in A \cup B) \\ &\Longleftrightarrow \rceil (x \in A \vee x \in B) \\ &\Longleftrightarrow x \notin A \wedge x \notin B \\ &\Longleftrightarrow x \in A^C \wedge x \in B^C \\ &\Longleftrightarrow x \in A^C \cap B^C. \end{aligned}$$

Zadatak 2.2 *Dokazati:*

$$A \backslash B = A \cap B^C.$$

Rješenje:

$$\begin{aligned} x &\in A \backslash B \\ &\Longleftrightarrow x \in A \wedge x \notin B \\ &\Longleftrightarrow x \in A \wedge x \in B^C \\ &\Longleftrightarrow x \in A \cap B^C. \end{aligned}$$

Zadatak 2.3 *Dokazati*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Rješenje:

$$\begin{aligned} x &\in (A \cap B) \cup C \\ &\iff x \in A \cap B \vee x \in C \\ &\iff x \in A \wedge x \in B \vee x \in C \\ &\iff x \in A \vee x \in C \wedge x \in B \vee x \in C \\ &\iff (x \in A \vee x \in C) \wedge (x \in B \vee x \in C) \\ &\iff x \in A \cup C \wedge x \in B \cup C \\ &\iff x \in (A \cup C) \cap (B \cup C). \end{aligned}$$

Zadatak 2.4 *Dokazati:*

$$(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C).$$

Rješenje:

$$\begin{aligned} x &\in (A \cap B) \backslash C \\ &\iff x \in A \cap B \wedge x \notin C \\ &\iff x \in A \wedge x \in B \wedge x \notin C \\ &\iff x \in A \wedge x \notin C \wedge x \in B \wedge x \notin C \\ &\iff (x \in A \wedge x \notin C) \wedge (x \in B \wedge x \notin C) \\ &\iff x \in A \backslash C \wedge x \in B \backslash C \\ &\iff x \in (A \backslash C) \cap (B \backslash C). \end{aligned}$$

Zadatak 2.5 *Dokazati:*

$$(A \backslash B) \cap (C \backslash D) = (A \cap C) \backslash (B \cup C).$$

Rješenje:

$$\begin{aligned} x &\in (A \backslash B) \cap (C \backslash D) \\ &\iff x \in A \backslash B \wedge x \in C \backslash D \\ &\iff x \in A \wedge x \notin B \wedge x \in C \wedge x \notin D \\ &\iff x \in A \wedge x \in C \wedge x \notin B \wedge x \notin D \\ &\iff (x \in A \wedge x \in C) \wedge (x \notin B \wedge x \notin D) \\ &\iff x \in A \cap C \wedge x \notin B \cup D \\ &\iff x \in (A \cap C) \backslash (B \cup C). \end{aligned}$$

Zadatak 2.6 *Dokazati:*

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

Rješenje:

$$\begin{aligned}
(x,y) &\in (A \cap B) \times (C \cap D) \\
&\Longleftrightarrow x \in A \cap B \wedge y \in C \cap D \\
&\Longleftrightarrow x \in A \wedge x \in B \wedge y \in C \wedge y \in D \\
&\Longleftrightarrow x \in A \wedge y \in C \wedge x \in B \wedge y \in D \\
&\Longleftrightarrow (x \in A \wedge y \in C) \wedge (x \in B \wedge y \in D) \\
&\Longleftrightarrow (x,y) \in A \times C \wedge (x,y) \in B \times D \\
&\Longleftrightarrow (x,y) \in (A \times C) \cap (B \times D).
\end{aligned}$$

Zadatak 2.7 *Dat je skup* $X = \{1,2,3,4\}$. *Napisati relacije definisane sa*

$$\begin{aligned}
\rho_1 &= \left\{(x,y) \in X^2 : x < y\right\} \\
\rho_2 &= \left\{(x,y) \in X^2 : x \leq y\right\} \\
\rho_3 &= \left\{(x,y) \in X^2 : x > y\right\}.
\end{aligned}$$

Rješenje:
Kako je

$$\begin{aligned}
X^2 &= X \times X = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4), \\
&\quad (3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}
\end{aligned}$$

tada je

$$\begin{aligned}
\rho_1 &= \left\{(x,y) \in X^2 : x < y\right\} = \\
&= \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}. \\
\rho_2 &= \left\{(x,y) \in X^2 : x \leq y\right\} = \\
&= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}. \\
\rho_3 &= \left\{(x,y) \in X^2 : x > y\right\} = \\
&= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}.
\end{aligned}$$

Zadatak 2.8 *Neka je*$X = \{1,2,3\}$. *Napisati elemente relacije definisane sa*

$$\rho = \left\{(x,y,z) \in X^3 : x + y \leq z\right\}.$$

Rješenje:
Kako je

$$\begin{aligned} X^3 &= X\times X\times X=\{(1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,2,2),\\ &\quad (1,2,3),(1,3,1),(1,3,2),(1,3,3)\,(2,1,1),(2,1,2),\\ &\quad (2,1,3),(2,2,1),(2,2,2),(2,2,3),(2,3,1),(2,3,2),\\ &\quad (2,3,3)\,(3,1,1),(3,1,2),(3,1,3),(3,2,1),(3,2,2),\\ &\quad (3,2,3),(3,3,1),(3,3,2),(3,3,3)\}\end{aligned}$$

pa je

$$\begin{aligned}\rho &= \left\{(x,y,z)\in X^3 : x+y\le z\right\}\\ &= \{(1,1,2),(1,1,3),(1,2,3)\}.\end{aligned}$$

Zadatak 2.9 *U skupu* $\mathbb{N}$ *definisana je relacija*

$$\rho=\left\{(x,y)\in\mathbb{N}^2 : x+5y=25\right\}.$$

Napisati elemente relacije ρ.

Rješenje:
Iz

$$\begin{aligned} x+5y &= 25\\ x &= 25-5y\\ x &= 5\,(5-y)\end{aligned}$$

vidimo da $y\in\{1,2,3,4\}$, jer u protivnom x ne bi bio prirodan broj. Sada

$$\begin{aligned} y &= 1 \Longrightarrow x=20\\ y &= 2 \Longrightarrow x=15\\ y &= 3 \Longrightarrow x=10\\ y &= 4 \Longrightarrow x=5\end{aligned}$$

pa je

$$\begin{aligned}\rho &= \left\{(x,y)\in\mathbb{N}^2 : x+5y=25\right\}=\\ &= \{(20,1),(15,2),(10,5),(5,4)\}.\end{aligned}$$

3 Jednačine i nejednačine sa apsolutnim vrijednostima

Zadatak 3.1 *Riješiti jednačinu*

$$|x-1|-|x+2|=1.$$

Rješenje:

$$\begin{aligned} x-1 &= 0 \Longrightarrow x=1 \\ x+2 &= 0 \Longrightarrow x=-2 \end{aligned}$$

Posmatrajmo tablicu

x : $-\infty$	-2	1	$+\infty$
$x-1$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$

Imamo tri slučaja:

1) $x\in(-\infty,-2]$

Sada je

$$\begin{aligned} |x-1| &= -(x-1)=1-x \\ |x+2| &= -(x+2)=-x-2 \end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned} -(x-1)-(-(x+2)) &= 1 \\ 1-x+x+2 &= 1 \\ 3 &= 1 \end{aligned}$$

pa jednačina nema rješenja.

2) $x\in(-2,1]$

Sada je

$$\begin{aligned} |x-1| &= -(x-1)=1-x \\ |x+2| &= x+2 \end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned} -(x-1)-(x+2) &= 1 \\ 1-x-x-2 &= 1 \\ -2x-1 &= 1 \\ -2x &= 2 \\ x &= -1\in(-2,1] \end{aligned}$$

pa je rješenje jednačine $x = -1$.

3) $x \in (1, +\infty)$

Sada je

$$\begin{aligned}|x-1| &= x-1\\ |x+2| &= x+2\end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned}x-1-(x+2) &= 1\\ x-1-x-2 &= 1\\ -3 &= 1\end{aligned}$$

pa jednačina nema rješenja.

Dakle, jednačina ima samo jedno rješenje $x = -1$.

Zadatak 3.2 *Riješiti jednačinu*

$$\left|x^2+x-2\right| + \left|x^2-x-2\right| = 2.$$

Rješenje:

$$\begin{aligned}x^2+x-2 &= 0 \implies x_1=-2, x_2=1\\ x^2-x-2 &= 0 \implies x_1=-1, x_2=2\end{aligned}$$

Posmatrajmo tablicu

$x:$ $-\infty$	-2	-1	1	2	$+\infty$
x^2+x-2	$+$	$-$	$-$	$+$	$+$
x^2-x-2	$+$	$+$	$-$	$-$	$+$

Imamo pet slučajeva:

1) $x \in (-\infty, -2]$

Sada je

$$\begin{aligned}\left|x^2+x-2\right| &= x^2+x-2\\ \left|x^2-x-2\right| &= x^2-x-2\end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned}\left|x^2+x-2\right| + \left|x^2-x-2\right| &= 2\\ x^2+x-2+x^2-x-2 &= 2\\ 2x^2 &= 6\\ x^2 &= 3 \implies \begin{array}{l} x_1 = -\sqrt{3} \notin (-\infty,-2] \\ x_2 = \sqrt{3} \notin (-\infty,-2]\,. \end{array}\end{aligned}$$

Jednačina u ovom slučaju nema rješenja.

2) $x \in (-2,-1]$

Sada je

$$\begin{aligned}
\left|x^2+x-2\right| &= -\left(x^2+x-2\right) = -x^2-x+2 \\
\left|x^2-x-2\right| &= x^2-x-2
\end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned}
\left|x^2+x-2\right| + \left|x^2-x-2\right| &= 2 \\
-x^2-x+2+x^2-x-2 &= 2 \\
-2x &= 2 \\
x &= -1 \in (-2,-1]
\end{aligned}$$

U ovom slučaju rješenje jednačine je $x=-1$.

3) $x \in (-1,1]$

Sada je

$$\begin{aligned}
\left|x^2+x-2\right| &= -\left(x^2+x-2\right) = -x^2-x+2 \\
\left|x^2-x-2\right| &= -\left(x^2-x-2\right) = -x^2+x+2
\end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned}
\left|x^2+x-2\right| + \left|x^2-x-2\right| &= 2 \\
-x^2-x+2-x^2+x+2 &= 2 \\
-2x^2 &= -2 \\
x^2 &= 1 \Longrightarrow \begin{array}{l} x_1 = -1 \notin (-1,1] \\ x_2 = 1 \in (-1,1]\,. \end{array}
\end{aligned}$$

U ovom slučaju rješenje jednačine je $x=1$.

4) $x \in (1,2]$

Sada je

$$\begin{aligned}
\left|x^2+x-2\right| &= x^2+x-2 \\
\left|x^2-x-2\right| &= -\left(x^2-x-2\right) = -x^2+x+2
\end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned}
\left|x^2+x-2\right| + \left|x^2-x-2\right| &= 2 \\
x^2+x-2-x^2+x+2 &= 2 \\
2x &= 2 \\
x &= 1 \notin (1,2]\,.
\end{aligned}$$

Jednačina u ovom slučaju nema rješenja.

5) $x \in (2, +\infty]$

Sada je

$$\begin{aligned} \left|x^2 + x - 2\right| &= x^2 + x - 2 \\ \left|x^2 - x - 2\right| &= x^2 - x - 2 \end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned} \left|x^2 + x - 2\right| + \left|x^2 - x - 2\right| &= 2 \\ x^2 + x - 2 + x^2 - x - 2 &= 2 \\ 2x^2 &= 6 \\ x^2 &= 3 \Longrightarrow \begin{array}{l} x_1 = -\sqrt{3} \notin (2, +\infty] \\ x_2 = \sqrt{3} \notin (2, +\infty]\,. \end{array} \end{aligned}$$

Jednačina u ovom slučaju nema rješenja.

Dakle, rješenja jednačine su $x = -1$ i $x = 1$.

Zadatak 3.3 *Riješiti jednačinu*

$$|x - 1| + \left|x^2 + 3x - 4\right| = 5.$$

Rješenje:

$$\begin{aligned} x - 1 &= 0 \Longrightarrow x = 1 \\ x^2 + 3x - 4 &= 0 \Longrightarrow x = 1 \vee x = -4. \end{aligned}$$

Posmatrajmo tablicu

$x :$ $-\infty$	-4	1	$+\infty$
$x - 1$	$-$	$-$	$+$
$x^2 + 3x - 4$	$+$	$-$	$+$

Imamo tri slučaja:

1) $x \in (-\infty, -4]$

Sada je

$$\begin{aligned} |x - 1| &= -(x - 1) = 1 - x \\ \left|x^2 + 3x - 4\right| &= x^2 + 3x - 4 \end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned} 1 - x + x^2 + 3x - 4 &= 5 \\ x^2 + 2x - 8 &= 0 \Longrightarrow \begin{array}{l} x_1 = -4 \in (-\infty, -4] \\ x_2 = 2 \notin (-\infty, -4]\,. \end{array} \end{aligned}$$

U ovom slučaju rješenje jednačine je $x = -4$.

2) $x \in (-4, 1]$

Sada je

$$\begin{aligned} |x-1| &= -(x-1) = 1-x \\ \left|x^2+3x-4\right| &= -\left(x^2+3x-4\right) = -x^2-3x+4 \end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned} 1-x-x^2-3x+4 &= 5 \\ -x^2-4x &= 0 \implies \begin{array}{l} x_1 = -4 \notin (-4,1] \\ x_2 = 0 \in (-4,1]. \end{array} \end{aligned}$$

U ovom slučaju rješenje jednačine je $x = 0$.

3) $x \in (1, +\infty)$

Sada je

$$\begin{aligned} |x-1| &= x-1 \\ \left|x^2+3x-4\right| &= x^2+3x-4 \end{aligned}$$

pa jednačina ima oblik

$$\begin{aligned} x-1+x^2+3x-4 &= 5 \\ x^2+4x-10 &= 0 \implies \begin{array}{l} x_1 = -2-\sqrt{11} \notin (1,+\infty) \\ x_2 = -2+\sqrt{11} \in (1,+\infty). \end{array} \end{aligned}$$

U ovom slučaju rješenje jednačine je $x = -2 + \sqrt{11}$.

Dakle, rješenja jednačine su: $x_1 = -4, x_2 = 0, x_3 = -2 + \sqrt{11}$.

Zadatak 3.4 *Riješiti nejednačinu*

$$\left|x^2+2x-3\right| < 3x+3.$$

Rješenje:

$$x^2+2x-3 = 0 \implies x_1 = -3, x_2 = 1$$

Posmatrajmo tablicu

$x:$ $-\infty$	-3	1	$+\infty$
x^2+2x-3	$+$	$-$	$+$

Imamo tri slučaja

1) $x \in (-\infty, -3]$

Sada je

$$\left|x^2+2x-3\right| = x^2+2x-3$$

pa nejednačina ima oblik

$$\begin{aligned} x^2+2x-3 &< 3x+3 \\ x^2+x-6 &< 0 \end{aligned}$$

Nule kvadratnog trinoma x^2+x-6 su $x_1=2$ i $x_2=-3$ pa rješenje nejednačine je

$$\mathcal{R} = (-3,2)\,.$$

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_1 &= (-\infty.-3] \cap (-3,2) \\ \mathcal{R}_1 &= \varnothing. \end{aligned}$$

2) $x \in (-3,1]$
Sada je

$$\left|x^2+2x-3\right| = -\left(x^2+2x-3\right) = -x^2-2x+3$$

pa nejednačina ima oblik

$$\begin{aligned} -x^2-2x+3 &< 3x+3 \\ -x^2-5x &< 0 \end{aligned}$$

Nule kvadratnog trinoma $-x^2-5x$ su $x_1=-5$ i $x_2=0$ pa rješenje nejednačine je

$$\mathcal{R} = (-\infty,-5) \cup (0,+\infty)$$

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_2 &= ((-\infty,-5) \cup (0,+\infty)) \cap (-3,1] \\ \mathcal{R}_2 &= (0,1]\,. \end{aligned}$$

3) $x \in (1,+\infty)$
Sada je

$$\left|x^2+2x-3\right| = x^2+2x-3$$

pa nejednačina ima oblik

$$\begin{aligned} x^2+2x-3 &< 3x+3 \\ x^2+x-6 &< 0 \end{aligned}$$

Nule kvadratnog trinoma $x^2 + x - 6$ su $x_1 = 2$ i $x_2 = -3$ pa rješenje nejednačine je

$$\mathcal{R} = (-3, 2).$$

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_3 &= (1, +\infty) \cap (-3, 2) \\ \mathcal{R}_3 &= (1, 2). \end{aligned}$$

Konačno rješenje nejednačine je

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \\ \mathcal{R} &= (0, 2). \end{aligned}$$

Zadatak 3.5 *Riješiti nejednačinu*

$$\left|x^2 - 2x - 3\right| + 2 - 2x \geq |x - 4| + x^2.$$

Rješenje:

$$\begin{aligned} x^2 - 2x - 3 &= 0 \implies x_1 = -1, x_2 = 3 \\ x - 4 &= 0 \implies x = 4. \end{aligned}$$

Posmatrajmo tablicu

$x:$ $-\infty$	-1	3	4	$+\infty$
$x^2 - 2x - 3$	$+$	$-$	$+$	$+$
$x - 4$	$-$	$-$	$-$	$+$

Imamo četiri slučaja:

1) $x \in (-\infty, -1]$

Sada je

$$\begin{aligned} \left|x^2 - 2x - 3\right| &= x^2 - 2x - 3 \\ x - 4 &= -(x - 4) = 4 - x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2 - 2x - 3 + 2 - 2x &\geq 4 - x + x^2 \\ -3x &\geq 5 \quad / : (-3) \\ x &\leq -\frac{5}{3} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, -\frac{5}{3}\right]$.

Rješenje početne nejednačine u ovom slučaju je

$$\mathcal{R}_1 = (-\infty, -1] \cap \left(-\infty, -\frac{5}{3}\right].$$

Dakle,

$$\mathcal{R}_1 = \left(-\infty, -\frac{5}{3}\right].$$

2) $x \in (-1, 3]$

Sada je

$$\begin{aligned} \left|x^2 - 2x - 3\right| &= -\left(x^2 - 2x - 3\right) = -x^2 + 2x + 3 \\ x - 4 &= -(x-4) = 4 - x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} -x^2 + 2x + 3 + 2 - 2x &\geq 4 - x + x^2 \\ -2x^2 + x + 1 &\geq 0 \end{aligned}$$

pa je rješenje kvadratne nejednačine $x \in \left[-\frac{1}{2}, 1\right]$.

Rješenje početne nejednačine u ovom slučaju je

$$\mathcal{R}_2 = (-1, 3] \cap \left[-\frac{1}{2}, 1\right].$$

Dakle,

$$\mathcal{R}_2 = \left[-\frac{1}{2}, 1\right].$$

3) $x \in (3, 4]$

Sada je

$$\begin{aligned} \left|x^2 - 2x - 3\right| &= x^2 - 2x - 3 \\ x - 4 &= -(x-4) = 4 - x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2 - 2x - 3 + 2 - 2x &\geq 4 - x + x^2 \\ -3x &\geq 5 \qquad / : (-3) \\ x &\leq -\frac{5}{3} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, -\frac{5}{3}\right]$.

Rješenje početne nejednačine u ovom slučaju je

$$\mathcal{R}_3 = (3,4] \cap \left(-\infty, -\frac{5}{3}\right].$$

Dakle,

$$\mathcal{R}_3 = \varnothing.$$

4) $\quad x \in (4,+\infty)$
Sada je

$$\begin{aligned} \left|x^2-2x-3\right| &= x^2-2x-3 \\ x-4 &= x-4 \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2-2x-3+2-2x &\geq x-4+x^2 \\ -5x &\geq -3 \qquad /:(-5) \\ x &\leq \frac{3}{5} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, \frac{3}{5}\right]$.
Rješenje početne nejednačine u ovom slučaju je

$$\mathcal{R}_4 = (4,+\infty) \cap \left(-\infty, \frac{3}{5}\right].$$

Dakle,

$$\mathcal{R}_4 = \varnothing.$$

Konačno rješenje nejednačine je

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \\ \mathcal{R} &= \left(-\infty, -\frac{5}{3}\right] \cup \left[-\frac{1}{2}, 1\right]. \end{aligned}$$

Zadatak 3.6 *Riješiti nejednačinu*

$$\left|x^2-4x\right| + 3 \geq x^2 + |x-5|.$$

Rješenje:

$$\begin{aligned} x^2-4x &= 0 \Longrightarrow x_1 = 0, x_2 = 4 \\ x-5 &= 0 \Longrightarrow x = 5. \end{aligned}$$

Posmatrajmo tablicu

$x:$ $-\infty$	0	4	5	$+\infty$
x^2-4x	$+$	$-$	$+$	$+$
$x-5$	$-$	$-$	$-$	$+$

Imamo četiri slučaja:

1) $x \in (-\infty, 0]$

Sada je

$$\begin{aligned} \left|x^2-4x\right| &= x^2-4x \\ |x-5| &= -(x-5) = 5-x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2-4x+3 &\geq x^2+5-x \\ -3x &\geq 2 \quad /:(-3) \\ x &\leq -\frac{2}{3} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, -\frac{2}{3}\right]$.

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_1 &= (-\infty, 0] \cap \left(-\infty, -\frac{2}{3}\right] \\ \mathcal{R}_1 &= \left(-\infty, -\frac{2}{3}\right]. \end{aligned}$$

2) $x \in (0, 4]$

Sada je

$$\begin{aligned} \left|x^2-4x\right| &= -\left(x^2-4x\right) = -x^2+4x \\ |x-5| &= -(x-5) = 5-x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} -x^2+4x+3 &\geq x^2+5-x \\ -2x^2+5x-2 &\geq 0 \end{aligned}$$

pa je rješenje kvadratne nejednačine $x \in \left[\frac{1}{2}, 2\right]$.

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_2 &= (0, 4] \cap \left[\frac{1}{2}, 2\right] \\ \mathcal{R}_2 &= \left[\frac{1}{2}, 2\right]. \end{aligned}$$

3) $x \in (4, 5]$
Sada je

$$\begin{aligned} \left|x^2 - 4x\right| &= x^2 - 4x \\ |x-5| &= -(x-5) = 5 - x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2 - 4x + 3 &\geq x^2 + 5 - x \\ -3x &\geq 2 \quad /:(-3) \\ x &\leq -\frac{2}{3} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, -\frac{2}{3}\right]$.
Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_3 &= (4,5] \cap \left(-\infty, -\frac{2}{3}\right] \\ \mathcal{R}_3 &= \varnothing. \end{aligned}$$

4) $x \in (5, +\infty)$
Sada je

$$\begin{aligned} \left|x^2 - 4x\right| &= x^2 - 4x \\ |x-5| &= x - 5 \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2 - 4x + 3 &\geq x^2 + x - 5 \\ -5x &\geq -8 \quad /:(-8) \\ x &\leq \frac{8}{5} \end{aligned}$$

pa je rješenje kvadratne nejednačine $x \in \left(-\infty, \frac{8}{5}\right]$.
Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_4 &= \left(-\infty, \frac{8}{5}\right] \cap (5, +\infty) \\ \mathcal{R}_4 &= \varnothing. \end{aligned}$$

Konačno rješenje nejednačine je

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \\ \mathcal{R} &= \left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, 2\right]. \end{aligned}$$

Zadatak 3.7 *Riješiti nejednačinu*

$$\left|x^2-8x+12\right|+4x-1\geq x^2+|x-7|.$$

Rješenje:

$$\begin{aligned} x^2-8x+12 &= 0 \Longrightarrow x_1=2, x_2=6 \\ x-7 &= 0 \Longrightarrow x=7. \end{aligned}$$

Posmatrajmo tablicu

$x:$ $-\infty$	2	6	7	$+\infty$
$x^2-8x+12$	$+$	$-$	$+$	$+$
$x-7$	$-$	$-$	$-$	$+$

Imamo četiri slučaja:

1) $x\in(-\infty,2]$

Sada je

$$\begin{aligned} \left|x^2-8x+12\right| &= x^2-8x+12 \\ |x-7| &= -(x-7)=7-x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2-8x+12+4x-1 &\geq x^2+7-x \\ -3x &\geq -4 \quad /:(-4) \\ x &\leq \frac{4}{3} \end{aligned}$$

pa je rješenje nejednačine $x\in\left(-\infty,\frac{4}{3}\right]$.

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_1 &= (-\infty,2]\cap\left(-\infty,\frac{4}{3}\right] \\ \mathcal{R}_1 &= \left(-\infty,\frac{4}{3}\right]. \end{aligned}$$

2) $x\in(2,6]$

Sada je

$$\begin{aligned} \left|x^2-8x+12\right| &= -\left(x^2-8x+12\right)=-x^2+8x-12 \\ |x-7| &= -(x-7)=7-x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} -x^2+8x-12+4x-1 &\geq x^2+7-x \\ -2x^2+13x-20 &\geq 0 \end{aligned}$$

pa je rješenje nejednačine $x \in \left[\frac{5}{2}, 4\right]$.

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_2 &= (2,6] \cap \left[\frac{5}{2}, 4\right] \\ \mathcal{R}_2 &= \left[\frac{5}{2}, 4\right]. \end{aligned}$$

3) $x \in (6,7]$

Sada je

$$\begin{aligned} \left|x^2-8x+12\right| &= x^2-8x+12 \\ |x-7| &= -(x-7) = 7-x \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2-8x+12+4x-1 &\geq x^2+7-x \\ -3x &\geq -4 \quad / : (-4) \\ x &\leq \frac{4}{3} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, \frac{4}{3}\right]$.

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_3 &= (6,7] \cap \left(-\infty, \frac{4}{3}\right] \\ \mathcal{R}_3 &= \varnothing. \end{aligned}$$

4) $x \in (7,+\infty)$

Sada je

$$\begin{aligned} \left|x^2-8x+12\right| &= x^2-8x+12 \\ |x-7| &= x-7 \end{aligned}$$

pa nejednačina ima oblik

$$\begin{aligned} x^2-8x+12+4x-1 &\geq x^2+x-7 \\ -5x &\geq -18 \quad / : (-5) \\ x &\leq \frac{18}{5} \end{aligned}$$

pa je rješenje nejednačine $x \in \left(-\infty, \frac{18}{5}\right]$.

Rješenje početne nejednačine je

$$\begin{aligned} \mathcal{R}_4 &= (7, +\infty) \cap \left(-\infty, \frac{18}{5}\right] \\ \mathcal{R}_4 &= \varnothing. \end{aligned}$$

Konačno rješenje nejednačine je

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \\ \mathcal{R} &= \left(-\infty, \frac{4}{3}\right] \cup \left[\frac{5}{2}, 4\right]. \end{aligned}$$

4 Binomni obrazac

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$T_{k+1} = \binom{n}{k} a^{n-k} b^k$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}$$

Zadatak 4.1 *Naći trinaesti član u razvoju binoma*

$$\left(\sqrt{2} + \sqrt[3]{3}\right)^{15}.$$

Rješenje:
Budući da tražimo trinaesti član u razvoju binoma, onda uzimamo da je k=12 i uvrštavajući u formulu za opći član razvoja binoma, dobijamo:

$$\begin{aligned} T_{13} &= \binom{15}{12} \left(\sqrt{2}\right)^{15-12} \cdot \left(\sqrt[3]{3}\right)^{12} = \binom{15}{12} \left(\sqrt{2}\right)^{3} \cdot \left(3^{\frac{1}{3}}\right)^{12} = \\ &= \binom{15}{12} 2\sqrt{2} \cdot 3^4 = 73710\sqrt{2}. \end{aligned}$$

Zadatak 4.2 *Odrediti koeficijent uz* x^8 *u razvoju binoma* $(x+3)^{12}$.

Rješenje:
Opći član razvoja binoma $(x+3)^{12}$ je

$$T_{k+1} = \binom{12}{k} x^{12-k} \cdot 3^k$$

pa je jasno da mora biti $12 - k = 8$ jer tražimo koeficijent uz x^8, odakle je $k = 4$. Traženi koeficijent je

$$\binom{12}{4} \cdot 3^4 = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 81 = 40095.$$

Zadatak 4.3 *U razvoju binoma*

$$\left(\frac{2}{x} + 3x\right)^{15}$$

odrediti koeficijent uz x^7.

Rješenje:
Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k} a^{n-k} b^k = \binom{15}{k} \left(\frac{2}{x}\right)^{15-k} \cdot (3x)^k = \\ &= \binom{15}{k} \left(2x^{-1}\right)^{15-k} \cdot 3^k x^k = \binom{15}{k} 2^{15-k} x^{-15+k} \cdot 3^k x^k = \\ &= \binom{15}{k} 2^{15-k} \cdot 3^k \cdot x^{-15+2k} \end{aligned}$$

Budući da tražimo koeficijent uz x^7, onda mora biti

$$-15 + 2k = 7$$

odakle je $k = 11$, pa je traženi koeficijent uz x^7

$$\binom{15}{k} 2^{15-k} \cdot 3^k = \binom{15}{11} 2^{15-11} \cdot 3^{11} = \binom{15}{4} 2^4 \cdot 3^{11}.$$

Zadatak 4.4 *Naći član koji sadrži* x^7 *u razvoju binoma*

$$\left(\sqrt[3]{x^2} - \sqrt{x}\right)^{12}.$$

Rješenje:
Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k} a^{n-k} b^k = \binom{12}{k} \left(-\sqrt[3]{x^2}\right)^{12-k} \cdot \left(\sqrt{x}\right)^k = \\ &= \binom{12}{k} \left(x^{\frac{2}{3}}\right)^{12-k} \cdot \left(-x^{\frac{1}{2}}\right)^k = \binom{12}{k} x^{\frac{24-2k}{3}} \cdot (-1)^k \cdot x^{\frac{k}{2}} = \\ &= \binom{12}{k} \cdot (-1)^k \cdot x^{\frac{24-2k}{3} + \frac{k}{2}} = \binom{12}{k} \cdot (-1)^k \cdot x^{\frac{48-k}{6}}. \end{aligned}$$

Sada je jasno da mora biti

$$\frac{48-k}{6} = 7$$

odakle je $k = 6$.

Dakle, sedmi član razvoja binoma $\left(\sqrt[3]{x^2} - \sqrt{x}\right)^{12}$ sadrži x^7.

Zadatak 4.5 *U razvoju binoma*

$$\left(\sqrt{x}+\frac{1}{\sqrt[3]{x}}\right)^{16}$$

naći član koji sadrži x^3.

Rješenje:
Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k}a^{n-k}b^k = \binom{16}{k}\left(\sqrt{x}\right)^{16-k}\cdot\left(\frac{1}{\sqrt[3]{x}}\right)^k = \\ &= \binom{16}{k}\left(x^{\frac{1}{2}}\right)^{16-k}\cdot\left(x^{-\frac{1}{3}}\right)^k = \binom{16}{k}x^{\frac{16-k}{2}}\cdot x^{-\frac{k}{3}} = \\ &= \binom{16}{k}x^{\frac{16-k}{2}-\frac{k}{3}} = \binom{16}{k}x^{\frac{48-5k}{6}}. \end{aligned}$$

Sada je jasno da mora biti

$$\frac{48-5k}{6}=3$$

odakle je $k=6$.

Dakle, sedmi član razvoja binoma $\left(\sqrt{x}+\frac{1}{\sqrt[3]{x}}\right)^{16}$ sadrži x^3.

Zadatak 4.6 *Odrediti onaj član razvoja binoma*

$$\left(\sqrt[3]{x^2}-x^{-1}\right)^{15}$$

koji ne sadrži x.

Rješenje:
Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k}a^{n-k}b^k = \binom{15}{k}\left(\sqrt[3]{x^2}\right)^{15-k}\cdot\left(-x^{-1}\right)^k = \\ &= \binom{15}{k}\left(x^{\frac{2}{3}}\right)^{15-k}\cdot(-1)^k\cdot x^{-k} = \binom{15}{k}x^{\frac{30-2k}{3}}\cdot(-1)^k\cdot x^{-k} = \\ &= \binom{15}{k}(-1)^k\cdot x^{\frac{30-2k}{3}-k} = \binom{15}{k}(-1)^k\cdot x^{\frac{30-5k}{3}}. \end{aligned}$$

Budući da tražimo član koji ne sadrži x, onda mora biti

$$\frac{30-5k}{3}=0$$

odakle je $k=6$.

Dakle, sedmi član razvoja binoma $\left(\sqrt[3]{x^2}-x^{-1}\right)^{15}$ ne sadrži x.

Zadatak 4.7 *Odrediti onaj član razvoja binoma*

$$\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)^8$$

koji ne sadrži x.

Rješenje:
Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k}a^{n-k}b^k = \binom{8}{k}\left(\sqrt{x}\right)^{8-k}\cdot\left(-\frac{3}{\sqrt{x}}\right)^k = \\ &= \binom{8}{k}\left(x^{\frac{1}{2}}\right)^{8-k}\cdot\left(-3x^{-\frac{1}{2}}\right)^k = \binom{8}{k}x^{\frac{8-k}{2}}\cdot(-3)^k\cdot x^{-\frac{k}{2}} = \\ &= \binom{8}{k}(-3)^k\cdot x^{\frac{8-k}{2}-\frac{k}{2}} = \binom{8}{k}(-3)^k\cdot x^{\frac{8-2k}{2}} = \binom{8}{k}(-3)^k\cdot x^{4-k}. \end{aligned}$$

Budući da tražimo član koji ne sadrži x, onda mora biti

$$4-k=0$$

odakle je $k=4$.

Dakle, peti član razvoja binoma $\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)^8$ ne sadrži x.

Zadatak 4.8 *Naći srednji član u razvoju binoma*

$$\left(\frac{2}{x}-\sqrt{x}\right)^{16}.$$

Rješenje:
Razvoj binoma $\left(\frac{2}{x}-\sqrt{x}\right)^{16}$ ima 17 članova pa je srednji član deveti član, pa ćemo u formulu za opći član razvoja binoma uvrstiti $k=8$. Tako imamo

$$\begin{aligned} T_{k+1} &= \binom{n}{k}a^{n-k}b^k = \binom{16}{8}\left(\frac{2}{x}\right)^{16-8}\cdot\left(-\sqrt{x}\right)^8 = \\ &= \binom{16}{8}\left(\frac{2}{x}\right)^8\cdot\left(-x^{\frac{1}{2}}\right)^8 = \binom{16}{8}2^8\cdot x^{-8}\cdot(-1)^8\cdot x^4 = \\ &= \binom{16}{8}2^8\cdot x^{-4} = 3294720x^{-4}. \end{aligned}$$

Zadatak 4.9 *Odrediti peti član u razvoju binoma*

$$\left(2x\sqrt{x}-\sqrt[3]{x}\right)^8.$$

Rješenje:
Budući da tražimo peti član u razvoju binoma, onda ćemo u formulu za opći član razvoja binoma uvrstiti $k = 4$ i tako dobijamo

$$\begin{aligned}
T_{k+1} &= \binom{n}{k} a^{n-k} b^k = \binom{8}{4} \left(2x\sqrt{x}\right)^{8-4} \cdot \left(-\sqrt[3]{x}\right)^4 = \\
&= \binom{8}{4} \left(2x^{\frac{3}{2}}\right)^4 \cdot \left(-x^{\frac{1}{3}}\right)^4 = \binom{8}{4} 2^4 \cdot x^6 \cdot (-1)^4 x^{\frac{4}{3}} = \\
&= \binom{8}{4} 2^4 \cdot x^{6+\frac{4}{3}} = 1120 x^{\frac{22}{3}}.
\end{aligned}$$

Zadatak 4.10 *Naći trinaesti član u razvoju binoma*

$$\left(9x - \frac{1}{\sqrt{3x}}\right)^n$$

ako je binomni koeficiet trećeg člana jednak 105.

Rješenje:
Binomni koeficijent trećeg člana je $\binom{n}{2}$ pa iz uslova zadatka imamo

$$\begin{aligned}
\binom{n}{2} &= 105 \implies \frac{n \cdot (n-1)}{1 \cdot 2} = 105 \implies \\
&\implies n \cdot (n-1) = 210 \implies n^2 - n = 210 \\
&\implies n^2 - n - 210 = 0.
\end{aligned}$$

Rješenja ove kvadratne jednačine su $n_1 = 15, n_2 = -14$ pa uzimamo samo prvo rješenje a drugo odbacujemo jer je negativno. Sada binom ima oblik

$$\left(9x - \frac{1}{\sqrt{3x}}\right)^{15}.$$

Budući da tražimo trinaesti član razvoja binoma, u formulu za opći član razvoja binoma uvrstit ćemo $k = 12$, i tako dobijamo

$$\begin{aligned}
T_{13} &= \binom{15}{12} (9x)^{15-12} \cdot \left(-\frac{1}{\sqrt{3x}}\right)^{12} = \\
&= \binom{15}{12} (9x)^3 \cdot \left(-3^{-\frac{1}{2}} x^{-\frac{1}{2}}\right)^{12} = \binom{15}{12} 9^3 x^3 \cdot (-1)^{12} \cdot 3^{-6} x^{-6} = \\
&= \binom{15}{3} 3^6 x^3 \cdot 3^{-6} x^{-6} = \binom{15}{3} x^{-3} = 455 x^{-3}.
\end{aligned}$$

Zadatak 4.11 *Naći redni broj onog člana razvoja binoma*

$$\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$$

koji sadrži a i b na isti eksponent.

Rješenje:
Opći član razvoja binoma je

$$\begin{aligned}T_{k+1} &= \binom{n}{k}a^{n-k}b^k=\binom{21}{k}\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-k}\cdot\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{k}=\\ &= \binom{21}{k}\left(a^{\frac{1}{3}}b^{-\frac{1}{6}}\right)^{21-k}\cdot\left(b^{\frac{1}{2}}a^{-\frac{1}{6}}\right)^{k}=\binom{21}{k}a^{\frac{21-k}{3}}\cdot b^{-\frac{21-k}{6}}\cdot b^{\frac{k}{2}}\cdot a^{-\frac{k}{6}}=\\ &= \binom{21}{k}a^{\frac{21-k}{3}-\frac{k}{6}}\cdot b^{\frac{k}{2}-\frac{21-k}{6}}=\binom{21}{k}a^{\frac{42-3k}{6}}\cdot b^{\frac{4k-21}{6}}.\end{aligned}$$

Budući da zahtijevamo da a i b budu na isti eksponent, tada mora biti

$$\frac{42-3k}{6}=\frac{4k-21}{6}$$

odakle dobijamo da je $k=9$.

Dakle, deseti član razvoja binoma sadrži a i b na isti eksponent.

Zadatak 4.12 *Naći vrijednost x u izrazu*

$$\left(\left(\sqrt{x}\right)^{\frac{1}{\log x+1}}+\sqrt[12]{x}\right)^{6}$$

čiji je četvrti član razvoja binoma jednak 200.

Rješenje:
Prvo, odredimo definiciono područje. Jano je da mora biti $x>0$ i $\log x+1\neq 0$, tj. $x\neq 10^{-1}$ pa je

$$\mathcal{D}=\left(0,10^{-1}\right)\cup\left(10^{-1},+\infty\right).$$

Ako u formulu za opći član razvoja binoma uvrstimo $k=3$, dobićemo da je četvrti član razvoja binoma je

$$\begin{aligned}T_4 &= \binom{6}{3}\left(\left(\sqrt{x}\right)^{\frac{1}{\log x+1}}\right)^{6-3}\cdot\left(\sqrt[12]{x}\right)^3=\binom{6}{3}\left(\left(x^{\frac{1}{2}}\right)^{\frac{1}{\log x+1}}\right)^{3}\cdot\left(x^{\frac{1}{12}}\right)^{3}=\\ &= \binom{6}{3}x^{\frac{3}{2(\log x+1)}}\cdot x^{\frac{1}{4}}=\binom{6}{3}x^{\frac{3}{2(\log x+1)}+\frac{1}{4}}\end{aligned}$$

pa po uslovu zadatka imamo

$$\begin{aligned}\binom{6}{3}x^{\frac{3}{2(\log x+1)}+\frac{1}{4}} &= 200\\ 20x^{\frac{3}{2(\log x+1)}+\frac{1}{4}} &= 200\\ x^{\frac{3}{2(\log x+1)}+\frac{1}{4}} &= 10.\end{aligned}$$

Logaritmiranjem ovog izraza, dobijamo

$$\log x^{\frac{3}{2(\log x+1)}+\frac{1}{4}} = \log 10$$

tj.

$$\begin{aligned}\left(\frac{3}{2(\log x+1)}+\frac{1}{4}\right)\log x &= 1\\ \frac{6+\log x+1}{4(\log x+1)}\cdot\log x &= 1\\ \frac{7+\log x}{4(\log x+1)}\cdot\log x &= 1\\ (7+\log x)\log x &= 4(\log x+1)\\ 7\log x+\log^2 x &= 4\log x+4\\ \log^2 x+3\log x-4 &= 0.\end{aligned}$$

Uvodeći smjenu $\log x = t$, dobijamo kvadratnu jednačinu $t^2+3t-4=0$ čija su rješenja $t_1=-4$ i $t_2=1$.

Sada imamo dva slučaja:

$$\begin{aligned}t_1 &= -4 \Longrightarrow \log x=-4 \Longrightarrow x=10^{-4}=0,001\in\mathcal{D}\\ t_1 &= 1 \Longrightarrow \log x=1 \Longrightarrow x=10\in\mathcal{D}.\end{aligned}$$

Zadatak 4.13 *Izračunati član razvoja binoma*

$$\left(4\sqrt[5]{x}+\frac{\sqrt[3]{x}}{2}\right)^n$$

koji sadrži $x^2\sqrt[5]{x^4}$ *ako je zbir prva tri binomna koeficijenta jednak* 56.

Rješenje:

Iz uslova zadatka imamo

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=56$$

odakle je

$$\begin{aligned} 1+n+\frac{n(n-1)}{2} &= 56 \\ 2+2n+n(n-1) &= 112 \\ n^2+n-110 &= 0. \end{aligned}$$

Rješenja posljednje kvadratne jednačine su $n_1 = 10$ i $n_2 = -11$ od kojih u obzir dolazi samo prvo rješenje jer je drugo negativno.

Binom sada ima oblik

$$\left(4\sqrt[5]{x}+\frac{\sqrt[3]{x}}{2}\right)^{10}.$$

Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k}a^{n-k}b^k = \binom{10}{k}\left(4\sqrt[5]{x}\right)^{10-k}\cdot\left(\frac{\sqrt[3]{x}}{2}\right)^k = \\ &= \binom{10}{k}4^{10-k}\left(x^{\frac{1}{5}}\right)^{10-k}\cdot\left(2^{-1}x^{\frac{1}{3}}\right)^k = \\ &= \binom{10}{k}4^{10-k}x^{\frac{10-k}{5}}\cdot 2^{-k}x^{\frac{k}{3}} = \binom{10}{k}4^{10-k}\cdot 2^{-k}\cdot x^{\frac{10-k}{5}+\frac{k}{3}} = \\ &= \binom{10}{k}4^{10-k}\cdot 2^{-k}\cdot x^{\frac{30+2k}{15}}. \end{aligned}$$

Kako je $x^2\sqrt[5]{x^4} = x^2x^{\frac{4}{5}} = x^{2+\frac{4}{5}} = x^{\frac{14}{5}}$, onda je jasno da mora biti

$$\frac{30+2k}{15} = \frac{14}{5}$$

odakle je $k = 6$.

Dakle, sedmi član razvoja binoma ima traženu osobinu.

Zadatak 4.14 *Naći za koju vrijednost x u razvoju binoma*

$$\left(\sqrt{2^x}+\frac{1}{\sqrt{2^{x-1}}}\right)^n$$

zbir trećeg i petog člana iznosi 135*, ako je zbir binomnih koeficijenata tri posljednja člana jednak* 22*.*

Rješenje:

Binomni koeficijenti tri posljednja člana jednaki su binomnim koeficijentima prva tri člana. Po uslovu zadatka imamo

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=22$$

odakle je

$$\begin{aligned}1+n+\frac{n(n-1)}{2} &= 22\\ 2+2n+n(n-1) &= 44\\ n^2+n-42 &= 0.\end{aligned}$$

Rješenja posljednje kvadratne jednačine su $n_1=6$ i $n_2=-7$ od kojih u obzir dolazi samo prvo rješenje jer je drugo negativno.

Binom sada ima oblik

$$\left(\sqrt{2^x}+\frac{1}{\sqrt{2^{x-1}}}\right)^6.$$

Treći član ($k=2$) razvoja binoma je

$$\begin{aligned}T_3 &= \binom{6}{2}\left(\sqrt{2^x}\right)^{6-2}\cdot\left(\frac{1}{\sqrt{2^{x-1}}}\right)^2=\\ &= \binom{6}{2}\left(2^{\frac{x}{2}}\right)^4\cdot\left(2^{-\frac{x-1}{2}}\right)^2=\binom{6}{2}2^{2x}\cdot 2^{1-x}=\\ &= \binom{6}{2}2^{1+x}\end{aligned}$$

a peti član ($k=4$) razvoja binoma je

$$\begin{aligned}T_5 &= \binom{6}{4}\left(\sqrt{2^x}\right)^{6-4}\cdot\left(\frac{1}{\sqrt{2^{x-1}}}\right)^4=\\ &= \binom{6}{4}\left(2^{\frac{x}{2}}\right)^2\cdot\left(2^{-\frac{x-1}{2}}\right)^4=\binom{6}{4}2^{x}\cdot 2^{2-2x}=\\ &= \binom{6}{4}2^{2-x}.\end{aligned}$$

Sada je, po uslovu zadatka,

$$\begin{aligned}\binom{6}{2}2^{1+x}+\binom{6}{4}2^{2-x} &= 135\\ 15\cdot 2^{1+x}+15\cdot 2^{2-x} &= 135\\ 2^{1+x}+2^{2-x} &= 9\\ 2\cdot 2^x+\frac{4}{2^x} &= 9\\ 2\cdot(2^x)^2-9\cdot 2^x-4 &= 0.\end{aligned}$$

Uvedimo smjenu $t=2^x$, dobijamo kvadratnu jednačinu

$$2t^2-9t-4=0$$

čija su rješenja $t_1=\frac{1}{2}$ i $t_2=4$.

Sad imamo

$$\begin{aligned}t_1 &= \frac{1}{2} \Longrightarrow 2^x=\frac{1}{2}=2^{-1} \Longrightarrow x=-1\\ t_1 &= 4 \Longrightarrow 2^x=4=2^2 \Longrightarrow x=2.\end{aligned}$$

Zadatak 4.15 *Odrediti za koju vrijednost x četvrti član razvoja binoma*

$$\left(\sqrt{2^{x-1}}+\frac{1}{\sqrt[3]{2^x}}\right)^n$$

je dvadeset pet puta veći od eksponenta binoma ako je binomni koeficijent četvrtog člana pet puta veći od binomnog koeficijenta drugog člana.

Rješenje:

Prema uslovu zadatka imamo

$$\binom{n}{3}=5\binom{n}{1}$$

odakle je

$$\begin{aligned}\frac{n\cdot(n-1)\cdot(n-2)}{1\cdot 2\cdot 3} &= 5n\\ \frac{n\cdot(n-1)\cdot(n-2)}{6} &= 5n\\ (n-1)\cdot(n-2) &= 30n\\ n^2-3n-28 &= 0.\end{aligned}$$

Rješenja posljednje kvadratne jednačine su $n_1 = 7$ i $n_2 = -4$ pa u obzir dolazi samo prvo rješenje a drugo odbacujemo jer je negativno.

Binom sada izgleda

$$\left(\sqrt{2^{x-1}} + \frac{1}{\sqrt[3]{2^x}}\right)^7.$$

Četvrti član ($k = 3$) razvoja ovog binoma je

$$\begin{aligned} T_4 &= \binom{7}{3}\left(\sqrt{2^{x-1}}\right)^{7-3}\left(\frac{1}{\sqrt[3]{2^x}}\right)^3 = \binom{7}{3}\left(2^{\frac{x-1}{2}}\right)^4\left(2^{-\frac{x}{3}}\right)^3 = \\ &= 35 \cdot 2^{2x-2} \cdot 2^{-x} = 35 \cdot 2^{x-2} \end{aligned}$$

Prema uslovu zadatka imamo

$$\begin{aligned} 35 \cdot 2^{x-2} &= 140 \\ 2^{x-2} &= 4 \\ 2^{x-2} &= 2^2 \\ x - 2 &= 2 \end{aligned}$$

pa je $x = 4$.

Zadatak 4.16 *Odnos koefcijenata petog i trećeg člana u razvoju binoma*

$$\left(x\sqrt{x^{-1}} - \sqrt[5]{\frac{1}{x^2\sqrt{x}}}\right)^n$$

je 14:3. Odrediti sedmi član razvoja.

Rješenje:

Opći član razvoja binoma je

$$\begin{aligned} T_{k+1} &= \binom{n}{k}a^{n-k}b^k = \binom{n}{k}\left(x\sqrt{x^{-1}}\right)^{n-k}\left(-\sqrt[5]{\frac{1}{x^2\sqrt{x}}}\right)^k = \\ &= \binom{n}{k}\left(xx^{-\frac{1}{2}}\right)^{n-k}\left(-\sqrt[5]{\frac{1}{x^2x^{\frac{1}{2}}}}\right)^k = \binom{n}{k}\left(x^{\frac{1}{2}}\right)^{n-k}\left(-\sqrt[5]{\frac{1}{x^{\frac{5}{2}}}}\right)^k = \\ &= \binom{n}{k}\left(x^{\frac{1}{2}}\right)^{n-k}\left(-\sqrt[5]{x^{-\frac{5}{2}}}\right)^k = \binom{n}{k}\left(x^{\frac{1}{2}}\right)^{n-k}\left(-\left(x^{-\frac{5}{2}}\right)^{\frac{1}{5}}\right)^k = \\ &= \binom{n}{k}\left(x^{\frac{1}{2}}\right)^{n-k}\left(-x^{-\frac{1}{2}}\right)^k = \binom{n}{k}x^{\frac{n-k}{2}}(-1)^k x^{-\frac{k}{2}} = \\ &= \binom{n}{k}(-1)^k x^{\frac{n-k}{2}-\frac{k}{2}} = \binom{n}{k}(-1)^k x^{\frac{n-2k}{2}}. \end{aligned}$$

Koeficijent trećeg člana ($k = 2$) je

$$\binom{n}{k}(-1)^k = \binom{n}{2}(-1)^2 = \binom{n}{2}.$$

Koeficijent petog člana ($k = 4$) je

$$\binom{n}{k}(-1)^k = \binom{n}{4}(-1)^4 = \binom{n}{4}.$$

Po uslovu zadatka imamo

$$\binom{n}{4} : \binom{n}{2} = 14 : 3$$

pa je

$$\begin{aligned}
3 \cdot \binom{n}{4} &= 14 \cdot \binom{n}{2} \\
3 \cdot \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{1 \cdot 2 \cdot 3 \cdot 4} &= 14 \cdot \frac{n \cdot (n-1)}{1 \cdot 2} \\
\frac{(n-2) \cdot (n-3)}{4} &= 14 \quad / \cdot 4 \\
(n-2) \cdot (n-3) &= 56 \\
n^2 - 5n - 50 &= 0
\end{aligned}$$

a rješenja posljednje kvadratne jednačine su $n_1 = -5$ i $n_2 = 10$. U obzir dolazi samo pozitivno rješenje pa binom sada izgleda

$$\left(x\sqrt{x^{-1}} - \sqrt[5]{\frac{1}{x^2\sqrt{x}}} \right)^{10}.$$

Na osnovu pokazanog, opći član razvoja ovog binoma je

$$T_{k+1} = \binom{n}{k}(-1)^k x^{\frac{n-2k}{2}}$$

pa ako u ovu formulu uvrstimo da je $n = 10$ i $k = 6$, dobićemo da je sedmi član razvoja ovog binoma

$$T_7 = \binom{10}{6}(-1)^6 x^{\frac{10-12}{2}} = \binom{10}{6} x^{-1}.$$

5 Matematička indukcija

Zadatak 5.1 *Dokazati da za svaki prirodan broj n vrijedi*

$$1+3+5+\cdots+(2n-1)=n^2.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n=1$:
$\mathcal{L}\equiv 2\cdot 1-1=2-1=1$
$\mathcal{D}\equiv 1^2=1$
Dakle, jednakost vrijedi za $n=1$.
2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$1+3+5+\cdots+(2k-1)=k^2.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$1+3+5+\cdots+(2(k+1)-1)=(k+1)^2.$$

Dokaz

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+(2(k+1)-1)= \\ &= k^2+2k+1=(k+1)^2.\end{aligned}$$

Dakle, jednakost vrijedi i za $n=k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.
Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n}(2k-1)=n^2.$$

Zadatak 5.2 *Dokazati da za svaki prirodan broj n vrijedi*

$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n=1$:
$\mathcal{L}\equiv 1^3=1$
$\mathcal{D}\equiv\left(\frac{1(1+1)}{2}\right)^2=1^2=1$
Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k + 1$, tj. da vrijedi

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$$

Dokaz:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} = \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \\
&= \frac{(k+1)^2(k+2)^2}{4} = \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n = k + 1$.
4) Jednakost vrijedi za svaki prirodan broj n.
Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Zadatak 5.3 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)\cdot(2n+1)} = \frac{n}{2n+1}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n = 1$:
$\mathcal{L} \equiv \frac{1}{1\cdot 3} = \frac{1}{3}$
$\mathcal{D} \equiv \frac{1}{2\cdot 1+1} = \frac{1}{3}$
Dakle, jednakost vrijedi za $n = 1$.

2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)\cdot(2k+1)} = \frac{k}{2k+1}.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k+1$, tj. da vrijedi

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2(k+1)-1)\cdot(2(k+1)+1)} = \frac{(k+1)}{2(k+1)+1}.$$

Dokaz:

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)\cdot(2k+1)} + \frac{1}{(2(k+1)-1)\cdot(2(k+1)+1)} =$$

$$\begin{aligned}
&= \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)\cdot(2(k+1)+1)} = \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)\cdot(2k+3)} = \frac{k(2k+3)+1}{(2k+1)\cdot(2k+3)} = \\
&= \frac{2k^2+3k+1}{(2k+1)\cdot(2k+3)} = \frac{(2k+1)\cdot(k+1)}{(2k+1)\cdot(2k+3)} = \frac{(k+1)}{(2k+3)} = \frac{(k+1)}{2(k+1)+1}.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n = k+1$.
Rješenja kvadratne jednačine $2k^2 + 3k + 1 = 0$ su brojevi $k_1 = -\frac{1}{2}$ i $k_2 = -1$, pa je $2k^2 + 3k + 1 = 2 \cdot \left(k + \frac{1}{2}\right) \cdot (k+1) = (2k+1)\cdot(k+1)$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \frac{1}{(2k-1)\cdot(2k+1)} = \frac{n}{2n+1}.$$

Zadatak 5.4 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \cdots + \frac{n^2}{(2n-1)\cdot(2n+1)} = \frac{n(n+1)}{2(2n+1)}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n = 1$:

$\mathcal{L} \equiv \frac{1^2}{1 \cdot 3} = \frac{1}{3}$

$\mathcal{D} \equiv \frac{1 \cdot (1+1)}{2 \cdot (2 \cdot 1+1)} = \frac{1}{3}$

Dakle, jednakost vrijedi za $n = 1$.

2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$\frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \cdots + \frac{k^2}{(2k-1) \cdot (2k+1)} = \frac{k(k+1)}{2(2k+1)}.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k+1$, tj. da vrijedi

$$\frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \cdots + \frac{(k+1)^2}{(2(k+1)-1) \cdot (2(k+1)+1)} = \frac{(k+1)((k+1)+1)}{2(2(k+1)+1)}.$$

Dokaz:

$$\frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \cdots + \frac{k^2}{(2k-1) \cdot (2k+1)} + \frac{(k+1)^2}{(2(k+1)-1) \cdot (2(k+1)+1)} =$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2(k+1)-1)(2(k+1)+1)} =$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} =$$

$$= \frac{k(k+1)(2k+3) + 2(k+1)^2}{2(2k+1)(2k+3)} = \frac{(k+1)(k(2k+3) + 2(k+1))}{2(2k+1)(2k+3)} =$$

$$= \frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)} = \frac{(k+1)(2k^2+5k+2)}{2(2k+1)(2k+3)} =$$

$$= \frac{(k+1)(k+2)(2k+1)}{2(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)} = \frac{(k+1)((k+1)+1)}{2(2(k+1)+1)}.$$

Dakle, jednakost vrijedi i za $n = k+1$.

Rješenja kvadratne jednačine $2k^2 + 5k + 1 = 0$ su brojevi $k_1 = -\frac{1}{2}$ i $k_2 = -2$, pa je $2k^2 + 5k + 1 = 2 \cdot \left(k + \frac{1}{2}\right) \cdot (k+2) = (2k+1) \cdot (k+2)$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \frac{k^2}{(2k-1) \cdot (2k+1)} = \frac{n(n+1)}{2(2n+1)}.$$

Zadatak 5.5 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2n+1}{n^2\left(n+1\right)^2}=1-\frac{1}{\left(n+1\right)^2}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n=1$:

$\mathcal{L}\equiv\frac{2\cdot1+1}{1^2(1+1)^2}=\frac{3}{4}$

$\mathcal{D}\equiv1-\frac{1}{(1+1)^2}=1-\frac{1}{4}=\frac{3}{4}$.

Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2k+1}{k^2\left(k+1\right)^2}=1-\frac{1}{\left(k+1\right)^2}.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2\left(k+1\right)+1}{\left(k+1\right)^2\left(\left(k+1\right)+1\right)^2}=1-\frac{1}{\left(\left(k+1\right)+1\right)^2}.$$

Dokaz:

$$\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2k+1}{k^2\left(k+1\right)^2}+\frac{2\left(k+1\right)+1}{\left(k+1\right)^2\left(\left(k+1\right)+1\right)^2}=$$

$$\begin{aligned}
&= 1-\frac{1}{\left(k+1\right)^2}+\frac{2\left(k+1\right)+1}{\left(k+1\right)^2\left(\left(k+1\right)+1\right)^2}=\\
&= 1-\frac{1}{\left(k+1\right)^2}+\frac{2k+3}{\left(k+1\right)^2\left(k+2\right)^2}=\\
&= 1-\frac{\left(k+2\right)^2-\left(2k+3\right)}{\left(k+1\right)^2\left(k+2\right)^2}=1-\frac{k^2+4k+4-2k-3}{\left(k+1\right)^2\left(k+2\right)^2}=\\
&= 1-\frac{k^2+2k+1}{\left(k+1\right)^2\left(k+2\right)^2}=1-\frac{\left(k+1\right)^2}{\left(k+1\right)^2\left(k+2\right)^2}=\\
&= 1-\frac{1}{\left(k+2\right)^2}=1-\frac{1}{\left(\left(k+1\right)+1\right)^2}.
\end{aligned}$$

Dakle, jednakost vrijedi i za $n=k+1$.

4) Jednakost vrijedi za svaki prirodan broj n.

Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n}\frac{2k+1}{k^2\left(k+1\right)^2}=1-\frac{1}{\left(n+1\right)^2}.$$

Zadatak 5.6 *Dokazati da za svaki prirodan broj n vrijedi*

$$\frac{5}{1\cdot 2}+\frac{12}{2\cdot 3}+\cdots+\frac{2n^2+2n+1}{n\cdot(n+1)}=\frac{n\,(2n+3)}{n+1}.$$

Rješenje:

1) Provjerimo da li jednakost vrijedi za $n=1$:

$\mathcal{L}\equiv\frac{5}{1\cdot 2}=\frac{5}{2}$

$\mathcal{D}\equiv\frac{1(2\cdot 1+3)}{1+1}=\frac{5}{2}$

Dakle, jednakost vrijedi za $n=1$.

2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$\frac{5}{1\cdot 2}+\frac{12}{2\cdot 3}+\cdots+\frac{2k^2+2k+1}{k\cdot(k+1)}=\frac{k\,(2k+3)}{k+1}.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$\frac{5}{1\cdot 2}+\frac{12}{2\cdot 3}+\cdots+\frac{2\,(k+1)^2+2\,(k+1)+1}{(k+1)\cdot((k+1)+1)}=\frac{(k+1)\,(2\,(k+1)+3)}{(k+1)+1}$$

Dokaz:

$$\frac{5}{1\cdot 2}+\frac{12}{2\cdot 3}+\cdots+\frac{2k^2+2k+1}{k\cdot(k+1)}+\frac{2\,(k+1)^2+2\,(k+1)+1}{(k+1)\cdot((k+1)+1)}=$$

$$
\begin{aligned}
&= \frac{k(2k+3)}{k+1} + \frac{2(k+1)^2 + 2(k+1) + 1}{(k+1)\cdot((k+1)+1)} = \\
&= \frac{k(2k+3)}{k+1} + \frac{2(k^2+2k+1) + 2(k+1) + 1}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^2+3k}{k+1} + \frac{2k^2+4k+2+2k+2+1}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^2+3k}{k+1} + \frac{2k^2+6k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{(2k^2+3k)(k+2) + 2k^2+6k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^3+4k^2+3k^2+6k+2k^2+6k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^3+9k^2+12k+5}{(k+1)\cdot(k+2)} = \frac{2k^3+2k^2+7k^2+7k+5k+5}{(k+1)\cdot(k+2)} = \\
&= \frac{2k^2(k+1)+7k(k+1)+5(k+1)}{(k+1)\cdot(k+2)} = \\
&= \frac{(2k^2+7k+5)\cdot(k+1)}{(k+1)\cdot(k+2)} = \frac{2k^2+7k+5}{k+2} = \\
&= \frac{2k^2+2k+5k+5}{k+2} = \frac{(k+1)(2k+5)}{k+2} = \\
&= \frac{(k+1)(2(k+1)+3)}{(k+1)+1}
\end{aligned}
$$

Dakle, jednakost vrijedi i za $n = k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.
Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \frac{2k^2+2k+1}{k\cdot(k+1)} = \frac{n(2n+3)}{n+1}.$$

Zadatak 5.7 *Dokazati da za svaki prirodan broj* n *vrijedi*

$$17 \mid 5^{n+3} + 11^{3n+1}.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n = 1$:

$$5^{1+3} + 11^{3\cdot 1+1} = 5^4 + 11^4 = 625 + 14641 = 15266 = 17 \cdot 898$$

Dakle, jednakost vrijedi za $n = 1$.
2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$17 \mid 5^{k+3} + 11^{3k+1} \implies 5^{k+3} + 11^{3k+1} = 17A.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k + 1$, tj. da vrijedi

$$17 \mid 5^{(k+1)+3} + 11^{3(k+1)+1}$$

Dokaz:

$$\begin{aligned}
5^{(k+1)+3} + 11^{3(k+1)+1} &= 5^{k+4} + 11^{3k+4} = 5^{k+3} \cdot 5 + 11^{3k+1} \cdot 11^3 = \\
&= 5^{k+3} \cdot 5 + 11^{3k+1} \cdot 1331 = \\
&= 5^{k+3} \cdot 5 + 11^{3k+1} \cdot (1326 + 5) = \\
&= 5^{k+3} \cdot 5 + 11^{3k+1} \cdot 5 + 11^{3k+1} \cdot 1326 = \\
&= 5 \cdot \left(5^{k+3} + 11^{3k+1}\right) + 11^{3k+1} \cdot 1326 = \\
&= 5 \cdot 17A + 11^{3k+1} \cdot 17 \cdot 78 = \\
&= 17\left(5 \cdot A + 11^{3k+1} \cdot 78\right) = 17B.
\end{aligned}$$

gdje je $B = 5 \cdot A + 11^{3k+1} \cdot 78$.
Dakle, jednakost vrijedi i za $n = k + 1$.
4) Jednakost vrijedi za svaki prirodan broj n.

Zadatak 5.8 *Dokazati da za svaki prirodan broj n vrijedi*

$$19 \mid 7 \cdot 5^{2n} + 12 \cdot 6^n.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n = 1$:

$$7 \cdot 5^{2\cdot 1} + 12 \cdot 6^1 = 7 \cdot 25 + 12 \cdot 6 = 175 + 72 = 247 = 19 \cdot 13.$$

Dakle, jednakost vrijedi za $n = 1$.
2) Pretpostavimo da tvrdnja vrijedi za $n = k$, tj. neka vrijedi

$$19 \mid 7 \cdot 5^{2n} + 12 \cdot 6^n \implies 7 \cdot 5^{2n} + 12 \cdot 6^n = 19A.$$

3) Dokažimo da tvrdnja vrijedi i za $n = k + 1$, tj. da vrijedi

$$19 \mid 7 \cdot 5^{2(n+1)} + 12 \cdot 6^{(n+1)}$$

Dokaz:

$$\begin{aligned}7\cdot 5^{2(n+1)}+12\cdot 6^{(n+1)} &= 7\cdot 5^{2n+2}+12\cdot 6^{n+1}=7\cdot 5^{2n}\cdot 25+12\cdot 6^{n}\cdot 6=\\ &= 7\cdot 5^{2n}\cdot(19+6)+12\cdot 6^{n}\cdot 6=\\ &= 7\cdot 5^{2n}\cdot 19+7\cdot 5^{2n}\cdot 6+12\cdot 6^{n}\cdot 6=\\ &= 7\cdot 5^{2n}\cdot 19+6\cdot\left(7\cdot 5^{2n}+12\cdot 6^{n}\right)=\\ &= 7\cdot 5^{2n}\cdot 19+6\cdot 19A=19\left(7\cdot 5^{2n}+6A\right)=19B.\end{aligned}$$

gdje je $B=7\cdot 5^{2n}+6A$.
Dakle, jednakost vrijedi i za $n=k+1$.
4) Jednakost vrijedi za svaki prirodan broj n.

Zadatak 5.9 *Dokazati da za svaki prirodan broj n vrijedi*

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin\left(2n-1\right)x=\frac{\sin^2 nx}{\sin x}.$$

Rješenje:
1) Provjerimo da li jednakost vrijedi za $n=1$:
$\mathcal{L}\equiv\sin x$
$\mathcal{D}\equiv\frac{\sin^2 x}{\sin x}=\sin x$
Dakle, jednakost vrijedi za $n=1$.
2) Pretpostavimo da tvrdnja vrijedi za $n=k$, tj. neka vrijedi

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin\left(2k-1\right)x=\frac{\sin^2 kx}{\sin x}.$$

3) Dokažimo da tvrdnja vrijedi i za $n=k+1$, tj. da vrijedi

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin\left(2\left(k+1\right)-1\right)x=\frac{\sin^2\left(k+1\right)x}{\sin x}$$

Dokaz:

$$\sin x+\sin 3x+\sin 5x+\cdots+\sin\left(2k-1\right)x+\sin\left(2\left(k+1\right)-1\right)x=$$

$$
\begin{aligned}
&= \frac{\sin^2 kx}{\sin x} + \sin(2k+1)x = \frac{\sin^2 kx + \sin x \cdot \sin(2k+1)x}{\sin x} = \\
&= \frac{\sin^2 kx + \frac{1}{2}\left[\cos 2kx - \cos(2kx+2x)\right]}{\sin x} = \\
&= \frac{\frac{2\sin^2 kx + \cos 2kx - \cos(2kx+2x)}{2}}{\sin x} = \frac{2\sin^2 kx + \cos 2kx - \cos 2(k+1)x}{2\sin x} = \\
&= \frac{2\sin^2 kx + \cos^2 kx - \sin^2 kx - \cos^2(k+1)x + \sin^2(k+1)x}{2\sin x} = \\
&= \frac{\sin^2 kx + \cos^2 kx - \cos^2(k+1)x + \sin^2(k+1)x}{2\sin x} = \\
&= \frac{1 - \cos^2(k+1)x + \sin^2(k+1)x}{2\sin x} = \frac{\sin^2(k+1)x + \sin^2(k+1)x}{2\sin x} = \\
&= \frac{2\sin^2(k+1)x}{2\sin x} = \frac{\sin^2(k+1)x}{\sin x}.
\end{aligned}
$$

Dakle, jednakost vrijedi i za $n = k + 1$.
4) Jednakost vrijedi za svaki prirodan broj n.
Napomena: Jednakost kraće možemo zapisati

$$\sum_{k=1}^{n} \sin(2k-1)x = \frac{\sin^2 nx}{\sin x}.$$

6 Kompleksni brojevi

Zadatak 6.1 *Kompleksni broj $z = 3$ zapisati u trigonometrijskom obliku.*

Rješenje:

Vrijedi $z = 3 = 3 + 0 \cdot i$ pa je $\mathrm{Re}(z) = 3 > 0$ i $Im(z) = 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio na realnoj osi. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z| = \sqrt{(\mathrm{Re}(z))^2 + (Im(z))^2} = \sqrt{3^2 + 0^2} = 3 \\ tg\varphi &= \frac{Im(z)}{\mathrm{Re}(z)} = \frac{0}{3} = 0 \implies \varphi = 0. \end{aligned}$$

Zato je

$$z = 3\left(\cos 0 + i \cdot \sin 0\right).$$

Zadatak 6.2 *Kompleksni broj $z = \sqrt{2} - i \cdot \sqrt{2}$ zapisati u trigonometrijskom obliku.*

Rješenje:

Vrijedi $z = \sqrt{2} - i \cdot \sqrt{2}$ pa je $\mathrm{Re}(z) = \sqrt{2} > 0$ i $Im(z) = -\sqrt{2} < 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u četvrtom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z| = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2 \\ tg\varphi &= \frac{-\sqrt{2}}{\sqrt{2}} = -1 \implies \varphi = -\frac{\pi}{4} \quad ili \quad \varphi = \frac{7\pi}{4} \end{aligned}$$

Zato je

$$z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4}\right)\right)$$

ili

$$z = 2\left(\cos\frac{7\pi}{4} + i \cdot \sin\frac{7\pi}{4}\right).$$

Zadatak 6.3 *Kompleksni broj $z = -1 + i$ zapisati u trigonometrijskom obliku.*

Rješenje:

Vrijedi $z = -1 + i$ pa je $\mathrm{Re}(z) = -1 < 0$ i $Im(z) = 1 > 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u drugom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2} \\ tg\varphi &= \frac{1}{-1} = -1 \Longrightarrow \varphi = \frac{3\pi}{4}. \end{aligned}$$

Zato je

$$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \cdot \sin\frac{3\pi}{4}\right).$$

Zadatak 6.4 *Naći realni i imaginarni dio kompleksnog broja*

$$z = \frac{1-3i}{1+i} - \frac{i}{2+i}.$$

Rješenje:

Kako je

$$\begin{aligned} \frac{1-3i}{1+i} - \frac{i}{2+i} &= \frac{(1-3i)\cdot(2+i) - i\cdot(1+i)}{(1+i)\cdot(2+i)} = \\ &= \frac{2+i-6i-3i^2-i-i^2}{2+i+2i+i^2} = \\ &= \frac{2-6i-4i^2}{2+3i+i^2} = \frac{2-6i+4}{2+3i-1} = \\ &= \frac{6-6i}{1+3i} = \frac{6-6i}{1+3i}\cdot\frac{1-3i}{1-3i} = \\ &= \frac{(6-6i)\cdot(1-3i)}{1^2-(3i)^2} = \frac{6-18i-6i+18i^2}{1-9i^2} = \\ &= \frac{6-24i-18}{1+9} = \frac{-12-24i}{10} = -\frac{12}{10} - \frac{24}{10}i = \\ &= -\frac{6}{5} - \frac{12}{5}i \end{aligned}$$

tada je

$$\begin{aligned} \mathrm{Re}(z) &= -\frac{6}{5} \\ Im(z) &= -\frac{12}{5}. \end{aligned}$$

Zadatak 6.5 *Ako je* $z_1 = \sqrt{2} - i \cdot \sqrt{2}$ *i* $z_2 = -1 + i$, *izračunati* $z_1 \cdot z_2$ *i* $\frac{z_1}{z_2}$.

Rješenje:

Kompleksne brojeve z_1 i z_2 zapišimo u tigonometrijskom obliku.

Vrijedi $z_1 = \sqrt{2} - i \cdot \sqrt{2}$ pa je $\mathrm{Re}(z_1) = \sqrt{2} > 0$ i $Im(z_1) = -\sqrt{2} < 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u četvrtom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z_1| = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2 \\ tg\varphi &= \frac{-\sqrt{2}}{\sqrt{2}} = -1 \implies \varphi = -\frac{\pi}{4}. \end{aligned}$$

Zato je

$$z_1 = 2\left(\cos\left(-\frac{\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4}\right)\right).$$

Vrijedi $z_2 = -1 + i$ pa je $\mathrm{Re}(z_2) = -1 < 0$ i $Im(z_2) = 1 > 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u drugom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z_2| = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2} \\ tg\varphi &= \frac{1}{-1} = -1 \implies \varphi = \frac{3\pi}{4}. \end{aligned}$$

Zato je

$$z_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \cdot \sin\frac{3\pi}{4}\right).$$

Sada je

$$\begin{aligned} z_1 \cdot z_2 &= 2\left(\cos\left(-\frac{\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4}\right)\right) \cdot \sqrt{2}\left(\cos\frac{3\pi}{4} + i \cdot \sin\frac{3\pi}{4}\right) = \\ &= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4} + \frac{3\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4} + \frac{3\pi}{4}\right)\right) = \\ &= 2\sqrt{2}\left(\cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2}\right) = 2\sqrt{2}\,(0 + i \cdot 1) = 2\sqrt{2}i. \end{aligned}$$

i

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\left(\cos\left(-\frac{\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4}\right)\right)}{\sqrt{2}\left(\cos\frac{3\pi}{4} + i \cdot \sin\frac{3\pi}{4}\right)} = \\ &= \frac{2}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4} - \frac{3\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4} - \frac{3\pi}{4}\right)\right) = \\ &= \sqrt{2}\,(\cos(-\pi) + i \cdot \sin(-\pi)) = \\ &= \sqrt{2}\,(-1 + i \cdot 0) = -\sqrt{2}. \end{aligned}$$

Zadatak 6.6 *Izračunati*

$$\left(\sqrt{3}+i\right)^{17}.$$

Rješenje:
Kompleksni broj $z=\sqrt{3}+i$ zapišimo u trigonometrijskom obliku.
Vrijedi $z=\sqrt{3}+i$ pa je $\mathrm{Re}(z)=\sqrt{3}>0$ i $Im(z)=1>0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u prvom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned}
\rho &= |z|=\sqrt{\left(\sqrt{3}\right)^2+1^2}=\sqrt{3+1}=\sqrt{4}=2\\
tg\varphi &= \frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}\Longrightarrow \varphi=\frac{\pi}{6}.
\end{aligned}$$

Zato je

$$z=2\left(\cos\frac{\pi}{6}+i\cdot\sin\frac{\pi}{6}\right).$$

Sada je

$$\begin{aligned}
z^{17} &= \left(2\left(\cos\frac{\pi}{6}+i\cdot\sin\frac{\pi}{6}\right)\right)^{17}=2^{17}\left(\cos 17\frac{\pi}{6}+i\cdot\sin 17\frac{\pi}{6}\right)=\\
&= 2^{17}\left(\cos\frac{5\pi+12\pi}{6}+i\cdot\sin\frac{5\pi+12\pi}{6}\right)=\\
&= 2^{17}\left(\cos\left(\frac{5\pi}{6}+2\pi\right)+i\cdot\sin\left(\frac{5\pi}{6}+2\pi\right)\right)=\\
&= 2^{17}\left(\cos\frac{5\pi}{6}+i\cdot\sin\frac{5\pi}{6}\right)=2^{17}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\\
&= 2^{17}\left(\frac{-\sqrt{3}+i}{2}\right)=2^{16}\left(-\sqrt{3}+i\right).
\end{aligned}$$

Zadatak 6.7 *Izračunati*

$$\sqrt{-7+24i}.$$

Rješenje:
Budući da je korijen kompleksnog broja kompleksan broj, tada vrijedi

$$\sqrt{-7+24i}=x+iy.$$

Odredimo x i y.
Kvadriranjem dobijamo

$$\begin{aligned} -7+24i &= (x+iy)^2 \\ -7+24i &= x^2+2xyi+i^2y^2 \\ -7+24i &= x^2+2xyi-y^2 \\ -7+24i &= x^2-y^2+2xyi \end{aligned}$$

a odavdje je

$$\left.\begin{aligned} x^2-y^2=-7 \\ 2xy=24 \end{aligned}\right\}.$$

Riješimo ovaj sistem.
Iz druge jednačine je $y=\frac{12}{x}$, pa ako to uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} x^2-\left(\frac{12}{x}\right)^2 &= -7 \\ x^2-\frac{144}{x^2} &= -7 \quad / \cdot x^2 \\ x^4-144 &= -7x^2 \\ x^4+7x^2-144 &= 0. \end{aligned}$$

Uvodeći smjenu $t=x^2$, dobijamo kvadratnu jednačinu $t^2+7t-144=0$, čija su rješenja $t_1=-16$ i $t_2=9$, ali u obzir dolazi samo drugo rješenje $t_2=9$ jer zbog smjene $t=x^2>0$.

Znači, $x_1=-3$ ili $x_2=3$.

Ako je $x_1=-3$, tada je $y_1=\frac{12}{-3}=-4$.

Ako je $x_2=3$, tada je $y_1=\frac{12}{3}=4$.

Dakle,

$$\sqrt{-7+24i}=-3-4i \quad ili \quad \sqrt{-7+24i}=3+4i.$$

Zadatak 6.8 *Izračunati*

$$\sqrt[4]{-8+8\sqrt{3}i}.$$

Rješenje:

Kompleksni broj $z=-8+8\sqrt{3}i$ zapišimo u trigonometrijskom obliku.

Vrijedi $z=-8+8\sqrt{3}i$ pa je $\mathrm{Re}(z)=-8<0$ i $Im(z)=8\sqrt{3}>0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u drugom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z|=\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2}=\sqrt{64+64\cdot 3}=\sqrt{64+192}=\sqrt{256}=16 \\ tg\varphi &= \frac{8\sqrt{3}}{-8}=-\sqrt{3} \Longrightarrow \varphi=\frac{2\pi}{3}. \end{aligned}$$

Zato je

$$z = 16\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right).$$

Sada je

$$\omega_k = \sqrt[4]{-8+8\sqrt{3}i} = \sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+2k\pi}{4} + i\cdot\sin\frac{\frac{2\pi}{3}+2k\pi}{4}\right)$$

gdje je $k = 0, 1, 2, 3$.
Za $k = 0$, dobijamo

$$\omega_0 = \sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+0}{4} + i\cdot\sin\frac{\frac{2\pi}{3}+0}{4}\right) = 2\left(\cos\frac{\pi}{6} + i\cdot\sin\frac{\pi}{6}\right).$$

Za $k = 1$, dobijamo

$$\omega_1 = \sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{4} + i\cdot\sin\frac{\frac{2\pi}{3}+2\pi}{4}\right) = 2\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right).$$

Za $k = 2$, dobijamo

$$\omega_1 = \sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+4\pi}{4} + i\cdot\sin\frac{\frac{2\pi}{3}+4\pi}{4}\right) = 2\left(\cos\frac{7\pi}{6} + i\cdot\sin\frac{7\pi}{6}\right).$$

Za $k = 3$, dobijamo

$$\omega_1 = \sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+6\pi}{4} + i\cdot\sin\frac{\frac{2\pi}{3}+6\pi}{4}\right) = 2\left(\cos\frac{5\pi}{3} + i\cdot\sin\frac{5\pi}{3}\right).$$

Zadatak 6.9 *Riješiti jednačinu*

$$z^5 - 1 - i = 0.$$

Rješenje:
Jasno je da vrijedi

$$\begin{aligned} z^5 - 1 - i &= 0 \\ z^5 &= 1 + i \\ z &= \sqrt[5]{1+i}. \end{aligned}$$

Kompleksni broj $z = 1 + i$ zapišimo u trigonometrijskom obliku.

Vrijedi $z = 1 + i$ pa je $\mathrm{Re}(z) = 1 > 0$ i $Im(z) = 1 > 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u prvom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned}\rho &= |z| = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2} \\ tg\varphi &= \frac{1}{1} = 1 \implies \varphi = \frac{\pi}{4}.\end{aligned}$$

Zato je

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}\right).$$

Rješenje jednačine je

$$\sqrt[5]{\sqrt{2}\left(\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}\right)} = \sqrt[10]{2}\sqrt[5]{\left(\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}\right)}$$

a to su kompleksni brojevi

$$\omega_k = \sqrt[10]{2}\left(\cos\frac{\frac{\pi}{4} + 2k\pi}{5} + i \cdot \sin\frac{\frac{\pi}{4} + 2k\pi}{5}\right)$$

gdje je $k = 0, 1, 2, 3, 4$.
Za $k = 0$, dobijamo

$$\omega_0 = \sqrt[10]{2}\left(\cos\frac{\frac{\pi}{4} + 0}{5} + i \cdot \sin\frac{\frac{\pi}{4} + 0}{5}\right) = \sqrt[10]{2}\left(\cos\frac{\pi}{20} + i \cdot \sin\frac{\pi}{20}\right).$$

Za $k = 1$, dobijamo

$$\omega_1 = \sqrt[10]{2}\left(\cos\frac{\frac{\pi}{4} + 2\pi}{5} + i \cdot \sin\frac{\frac{\pi}{4} + 2\pi}{5}\right) = \sqrt[10]{2}\left(\cos\frac{9\pi}{20} + i \cdot \sin\frac{9\pi}{20}\right).$$

Za $k = 2$, dobijamo

$$\omega_2 = \sqrt[10]{2}\left(\cos\frac{\frac{\pi}{4} + 4\pi}{5} + i \cdot \sin\frac{\frac{\pi}{4} + 4\pi}{5}\right) = \sqrt[10]{2}\left(\cos\frac{17\pi}{20} + i \cdot \sin\frac{17\pi}{20}\right).$$

Za $k = 3$, dobijamo

$$\begin{aligned}\omega_3 &= \sqrt[10]{2}\left(\cos\frac{\frac{\pi}{4} + 6\pi}{5} + i \cdot \sin\frac{\frac{\pi}{4} + 6\pi}{5}\right) = \sqrt[10]{2}\left(\cos\frac{25\pi}{20} + i \cdot \sin\frac{25\pi}{20}\right) = \\ &= \sqrt[10]{2}\left(\cos\frac{5\pi}{4} + i \cdot \sin\frac{5\pi}{4}\right).\end{aligned}$$

Za $k = 4$, dobijamo

$$\omega_4 = \sqrt[10]{2}\left(\cos\frac{\frac{\pi}{4} + 8\pi}{5} + i \cdot \sin\frac{\frac{\pi}{4} + 8\pi}{5}\right) = \sqrt[10]{2}\left(\cos\frac{33\pi}{20} + i \cdot \sin\frac{33\pi}{20}\right).$$

Zadatak 6.10 *Riješiti jednačinu*

$$2z^3 - \sqrt{3} + i = 0.$$

Rješenje:

$$\begin{aligned} 2z^3 - \sqrt{3} + i &= 0 \\ 2z^3 &= \sqrt{3} - i \\ z^3 &= \frac{\sqrt{3}}{2} - \frac{1}{2}i \\ z &= \sqrt[3]{\frac{\sqrt{3}}{2} - \frac{1}{2}i}. \end{aligned}$$

Kompleksni broj $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ zapišimo u trigonometrijskom obliku.

Vrijedi $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ pa je $\mathrm{Re}(z) = \frac{\sqrt{3}}{2} > 0$ i $Im(z) = -\frac{1}{2} < 0$, pa ako bismo ovaj kompleksni broj predstavili u Gausovoj ravni, on bi se nalazio u četvrtom kvadrantu. Izračunajmo modul i argument ovog kompleksnog broja:

$$\begin{aligned} \rho &= |z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \\ tg\varphi &= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \Longrightarrow \varphi = -\frac{\pi}{6}. \end{aligned}$$

Zato je

$$z = \cos\left(-\frac{\pi}{6}\right) + i \cdot \sin\left(-\frac{\pi}{6}\right).$$

Rješenje jednačine je

$$\sqrt[3]{\cos\left(-\frac{\pi}{6}\right) + i \cdot \sin\left(-\frac{\pi}{6}\right)}$$

a to su kompleksni brojevi

$$\omega_k = \left(\cos\frac{-\frac{\pi}{6} + 2k\pi}{3} + i \cdot \sin\frac{-\frac{\pi}{6} + 2k\pi}{3}\right)$$

gdje je $k = 0, 1, 2$.
Za $k = 0$, dobijamo

$$\omega_0 = \left(\cos\frac{-\frac{\pi}{6} + 0}{3} + i \cdot \sin\frac{-\frac{\pi}{6} + 0}{3}\right) = \cos\left(-\frac{\pi}{18}\right) + i \cdot \sin\left(-\frac{\pi}{18}\right).$$

Za $k = 1$, dobijamo

$$\omega_1 = \left(\cos \frac{-\frac{\pi}{6} + 2\pi}{3} + i \cdot \sin \frac{-\frac{\pi}{6} + 2\pi}{3}\right) = \cos \frac{11\pi}{18} + i \cdot \sin \frac{11\pi}{18}.$$

Za $k = 2$, dobijamo

$$\omega_2 = \left(\cos \frac{-\frac{\pi}{6} + 4\pi}{3} + i \cdot \sin \frac{-\frac{\pi}{6} + 4\pi}{3}\right) = \cos \frac{23\pi}{18} + i \cdot \sin \frac{23\pi}{18}.$$

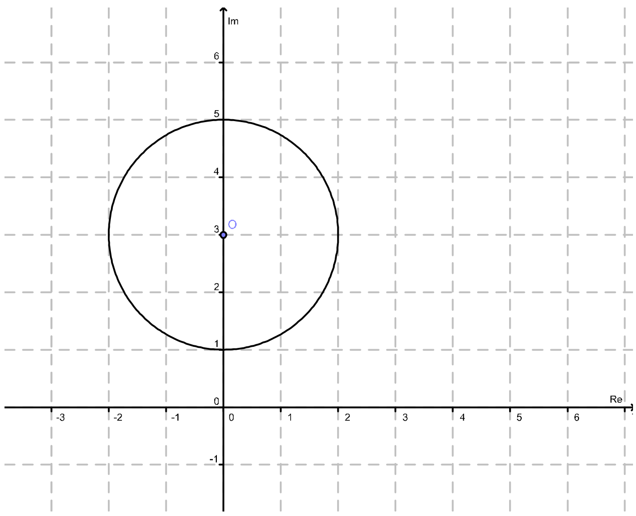
Zadatak 6.11 *Odrediti skup tačaka koji zadovoljava jednačinu*

$$|z - 3i| = 2.$$

Rješenje:
Neka je $z = x + iy$.

$$\begin{aligned} |x + iy - 3i| &= 2 \\ |x + (y - 3)\, i| &= 2 \\ \sqrt{x^2 + (y-3)^2} &= 2 \\ x^2 + (y-3)^2 &= 2^2 \end{aligned}$$

a jednačina $x^2 + (y-3)^2 = 2^2$ je jednačina kružnice sa centrom u tački $(0, 3)$ i poluprečnikom 2.



Zadatak 6.12 *Odrediti skup tačaka koji zadovoljava nejednačine*

$$3 \leq |z + i| \leq 5.$$

Rješenje:
Neka je $z = x + iy$.
Posmatrajmo prvo nejednakost $|z + i| \geq 3$. Sada je

$$\begin{aligned} |z+i| &\geq 3 \\ |x+iy+i| &\geq 3 \\ |x+(y+1)\,i| &\geq 3 \\ \sqrt{x^2+(y+1)^2} &\geq 3 \\ x^2+(y+1)^2 &\geq 3^2. \end{aligned}$$

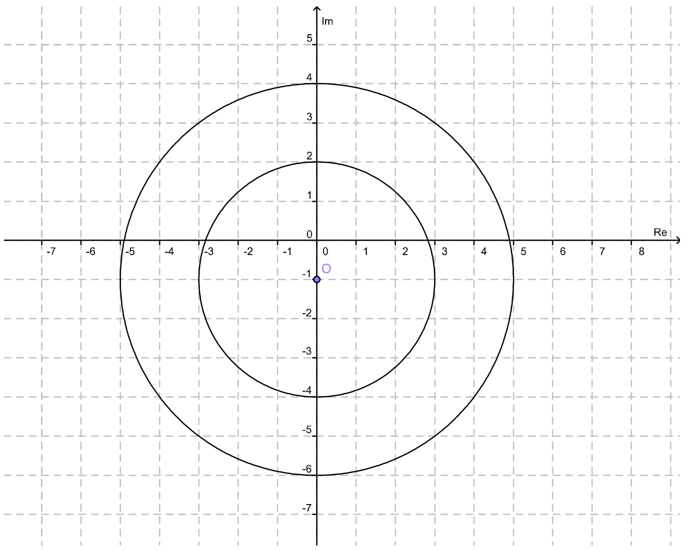
Jednačina $x^2 + (y+1)^2 = 3^2$ je jednačina kružnice sa centrom u tački $(0, -1)$ i poluprečnikom 3, a nejednakost $x^2 + (y+1)^2 \geq 3^2$ zadovoljavaju sve tačke van te kružnice uključujući i tačke na kružnici (jer je znak $\geq$).

Posmatrajmo sada drugu nejednakost

$$\begin{aligned} |z+i| &\leq 5 \\ |x+iy+i| &\leq 5 \\ |x+(y+1)\,i| &\leq 5 \\ \sqrt{x^2+(y+1)^2} &\leq 5 \\ x^2+(y+1)^2 &\leq 5^2. \end{aligned}$$

Jednačina $x^2 + (y+1)^2 = 5^2$ je jednačina kružnice sa centrom u tački $(0, -1)$ i poluprečnikom 5, a nejednakost $x^2 + (y+1)^2 \leq 5^2$ zadovoljavaju sve tačke unutar te kružnice uključujući i tačke na kružnici (jer je znak $\leq$).

Skup tačaka koji zadovoljava nejednačine $3 \leq |z + i| \leq 5$ će biti sve tačke van te kružnice uključujući i tačke na kružnici $x^2 + (y+1)^2 = 3^2$ i sve tačke unutar te kružnice uključujući i tačke na kružnici $x^2 + (y+1)^2 = 5^2$, tj. tačke koje pripadaju kružnom prstenu.



Zadatak 6.13 *Odrediti skup tačaka koji zadovoljava nejednačine*

$$1 \leq |z| < 5 \quad i \quad \frac{\pi}{4} < \varphi \leq \frac{5\pi}{4}.$$

Rješenje:
Neka je $z = x + iy$.
Posmatrajmo nejednakost $|z| \geq 1$.

$$\begin{aligned} |z| &\geq 1 \\ \sqrt{x^2+y^2} &\geq 1 \\ x^2+y^2 &\geq 1^2. \end{aligned}$$

Jednačina $x^2 + y^2 = 1^2$ je jednačina kružnice sa centrom u tački $(0,0)$ i poluprečnikom 1, pa datu nejednačinu zadovoljavaju sve tačke van kružnice ali uključujući i tačke na kružnici.

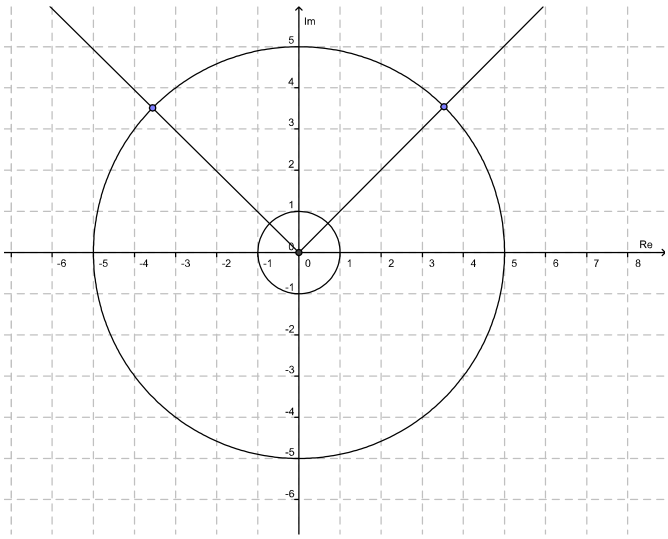
Posmatrajmo nejednakost $|z| < 5$.

$$\begin{aligned} |z| &< 5 \\ \sqrt{x^2+y^2} &< 5 \\ x^2+y^2 &< 5^2. \end{aligned}$$

Jednačina $x^2 + y^2 = 5^2$ je jednačina kružnice sa centrom u tački $(0,0)$ i poluprečnikom 5, pa datu nejednačinu zadovoljavaju sve tačke unutar kružnice ali ne uključujući i tačke na kružnici.

Tačke koje zadovoljavaju uvjet $\frac{\pi}{4} < \varphi \leq \frac{5\pi}{4}$ su tačke koje se nalaze između pravih određenih jednačinama $x = \frac{\pi}{4}$ i $x = \frac{5\pi}{4}$ ali uključujući i tačke na pravoj $x = \frac{\pi}{4}$.

Znači, skup tačaka koje zadovoljavaju postavljene uvjete čine tačke na dijelu kružnog prstena omeđenog dvjema pravima, ali uključujući tačke na većoj kružnici i tačke na pravoj $x = \frac{\pi}{4}$.



7 Determinante

Determinante drugog reda računamo po formuli

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = a \cdot d - b \cdot c.$$

Zadatak 7.1 *Izračunati*

$$\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix}.$$

Rješenje:

$$\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 1 = 4 - 3 = 1.$$

Zadatak 7.2 *Izračunati*

$$\begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix}.$$

Rješenje:

$$\begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -2 \cdot 4 - 3 \cdot (-1) = -8 + 3 = -5.$$

Determinante trećeg reda računamo na dva načina: Sarusovim pravilom ili LaPlasovim razvojem determinante po odabranoj vrsti ili koloni (prilikom LaPlasovog razvoja determinante veoma je povoljno odabrati vrstu ili kolonu koja ima nule i po njoj izvršiti razvoj).

Zadatak 7.3 *Izračunati*

$$\begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & 6 & 1 \end{vmatrix}.$$

Rješenje:
Prvi način - Sarusovo pravilo

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & 6 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & 6 & 1 \end{vmatrix} \begin{matrix} 1 & -2 \\ 4 & 1 \\ 2 & 6 \end{matrix} = \\ &= (1 + 8 + 72) - (6 - 12 - 8) = 95. \end{aligned}$$

Drugi način - LaPlasov razvoj (npr. po prvoj vrsti)

$$\begin{vmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & 6 & 1 \end{vmatrix} \begin{aligned} &= 1\begin{vmatrix} 1 & -2 \\ 6 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} = \\ &= 1\,(1+12) + 2\,(4+4) + 3\,(24-2) = \\ &= 13 + 16 + 66 = 95. \end{aligned}$$

Determinante četvrtog i većeg reda računamo LaPlasovim razvojem determinante po odabranoj vrsti ili koloni, i time red determinante smanjujemo za jedan.

Zadatak 7.4 *Izračunati*

$$\begin{vmatrix} 4 & 2 & 3 & 0 \\ 2 & 0 & 5 & 3 \\ -1 & -2 & 4 & 2 \\ 3 & 1 & 1 & -1 \end{vmatrix}.$$

Rješenje:
Izvršimo LaPlasov razvoj ove determinante po prvoj vrsti

$$\begin{vmatrix} 4 & 2 & 3 & 0 \\ 2 & 0 & 5 & 3 \\ -1 & -2 & 4 & 2 \\ 3 & 1 & 1 & -1 \end{vmatrix} \begin{aligned} &= 4\begin{vmatrix} 0 & 5 & 3 \\ -2 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -1 \end{vmatrix} + \\ &\quad +3\begin{vmatrix} 2 & 0 & 3 \\ -1 & -2 & 2 \\ 3 & 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 & 5 \\ -1 & -2 & 4 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 4\cdot(-18) - 2\cdot(-26) + 3\cdot 15 - 0 = 25 \end{aligned}$$

Zadatak 7.5 *Izračunati determinantu*

$$\begin{vmatrix} a & b & a & b \\ b & a & a & b \\ a & b & a & a \\ b & a & b & a \end{vmatrix}.$$

Rješenje:

Koristeći osobine determinanti, dobijamo

$$
\begin{aligned}
\begin{vmatrix} a & b & a & b \\ b & a & a & b \\ a & b & a & a \\ b & a & b & a \end{vmatrix}
&= \begin{vmatrix} a-b & b & a-b & b \\ b-a & a & a-b & b \\ a-b & b & 0 & a \\ b-a & a & b-a & a \end{vmatrix}
\begin{array}{l} \textit{Od prve kolone oduzeli drugu.} \\ \textit{Od treće kolone oduzeli četvrtu.} \\ \textit{Drugu i četvrtu kolonu prepisali.} \end{array} \\
&= \begin{vmatrix} a-b & b & a-b & b \\ -(a-b) & a & a-b & b \\ a-b & b & 0 & a \\ -(a-b) & a & -(a-b) & a \end{vmatrix}
\begin{array}{l} \textit{Iz prve i treće kolone} \\ \textit{izvlačimo zajednički} \\ \textit{faktor } (a-b)\,. \end{array} \\
&= (a-b)^2 \begin{vmatrix} 1 & b & 1 & b \\ -1 & a & 1 & b \\ 1 & b & 0 & a \\ -1 & a & -1 & a \end{vmatrix}
\begin{array}{l} \textit{Od druge kolone oduzmimo} \\ \textit{četvrtu.} \end{array} \\
&= (a-b)^2 \begin{vmatrix} 1 & 0 & 1 & b \\ -1 & a-b & 1 & b \\ 1 & -(a-b) & 0 & a \\ -1 & 0 & -1 & a \end{vmatrix}
\begin{array}{l} \textit{Iz druge kolone izvlačimo} \\ \textit{faktor } (a-b)\,. \end{array} \\
&= (a-b)^3 \begin{vmatrix} 1 & 0 & 1 & b \\ -1 & 1 & 1 & b \\ 1 & -1 & 0 & a \\ -1 & 0 & -1 & a \end{vmatrix}
\begin{array}{l} \textit{Razvoj} \\ \textit{po drugoj koloni} \end{array} \\
&= (a-b)^3 \left(\begin{vmatrix} 1 & 1 & b \\ 1 & 0 & a \\ -1 & -1 & a \end{vmatrix} + \begin{vmatrix} 1 & 1 & b \\ -1 & 1 & b \\ -1 & -1 & a \end{vmatrix} \right) \\
&= (a-b)^3 (-a-b+a-a+a+b-b+b+a+b) = \\
&= (a-b)^3 (a+b)\,.
\end{aligned}
$$

Zadatak 7.6 *Izračunati determinantu*

$$
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.
$$

Rješenje:

Koristeći osobine determinanti, dobijamo

$$\begin{aligned}
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \begin{array}{l} \textit{Od druge vrste oduzmimo prvu} \\ \textit{Od treće vrste oduzmimo prvu} \end{array} \\
&= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} \textit{ Razvoj po prvoj koloni} \\
&= \begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix} \\
&= (b-a)(b-c) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \begin{array}{l} \textit{Izvlačimo faktore } (b-a) \textit{ i} \\ (b-a) \end{array} \\
&= (b-a)(b-c)(c-a).
\end{aligned}$$

Zadatak 7.7 *Dokazati*

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = 4(a-b)(b-c)(a-c).$$

Rješenje:

Množenjem elemenata prve kolone sa -1 i dodavanjem drugoj i trećoj koloni, dobijamo

$$\begin{aligned}
\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} &= \begin{vmatrix} a^2 & 2a+1 & 4a+4 \\ b^2 & 2b+1 & 4b+4 \\ c^2 & 2c+1 & 4c+4 \end{vmatrix} = \\
&= 4 \cdot \begin{vmatrix} a^2 & 2a+1 & a+1 \\ b^2 & 2b+1 & b+1 \\ c^2 & 2c+1 & c+1 \end{vmatrix} = \\
&= 4 \cdot \begin{vmatrix} a^2 & a & a+1 \\ b^2 & b & b+1 \\ c^2 & c & c+1 \end{vmatrix} =
\end{aligned}$$

$$
\begin{aligned}
&= 4\cdot\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \\
&= 4\cdot\begin{vmatrix} a^2 & a & 1 \\ b^2-a^2 & b-a & 0 \\ c^2-a^2 & c-a & 0 \end{vmatrix} = \\
&= 4\cdot\begin{vmatrix} a^2 & a & 1 \\ (b-a)(b+a) & b-a & 0 \\ (b-c)(b+c) & c-a & 0 \end{vmatrix} = \\
&= 4(b-a)(b-c)\begin{vmatrix} a^2 & a & 1 \\ b+a & 1 & 0 \\ b+c & 1 & 0 \end{vmatrix} = \\
&= 4(b-a)(b-c)\begin{vmatrix} b+a & 1 \\ b+c & 1 \end{vmatrix} = \\
&= 4(a-b)(b-c)(a-c).
\end{aligned}
$$

8 Matrice. Operacije sa matricama.

Zadatak 8.1 *Date su matrice*

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix}.$$

Izračunati matrice $A + B, A - B, A \cdot B$.

Rješenje: Matrice A i B su formata 3×3 pa su moguće tražene operacije nad njima.

$$\begin{aligned}
A + B &= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 11 \\ 6 & 2 & 10 \\ 6 & 3 & 17 \end{bmatrix} \\
A - B &= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -1 \\ 0 & 2 & 2 \\ -4 & -1 & -3 \end{bmatrix} \\
A \cdot B &= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 39 & 8 & 78 \\ 39 & 9 & 86 \\ 39 & 13 & 80 \end{bmatrix}.
\end{aligned}$$

Zadatak 8.2 *Data je matrica*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Izračunati $P(A) = 3A^2 - 5A - 2E$, *gdje je* E *jedinična matrica.*

Rješenje:

$$\begin{aligned}
P(A) &= 3A^2 - 5A - 2E = \\
&= 3\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \\
&= 3\begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} 21 & 12 \\ 18 & 21 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -2 \\ 3 & 14 \end{bmatrix}.
\end{aligned}$$

Zadatak 8.3 *Dat je polinom* $P(x) = 2x - x^{-2} - 3$ *i matrica*

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Izračunati $P(A)$.

Rješenje:
Jasno je da vrijedi $P(A) = 2A - A^{-2} - 3E$, gdje je E jedinična matrica. Prvo ćemo izračunati inverznu matricu matrice A po formuli

$$A^{-1} = \frac{1}{\det A} \cdot adjA.$$

Imamo

$$\det A = \begin{vmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{vmatrix}\begin{matrix} 4 & -1 \\ 0 & -1 \\ 0 & 0 \end{matrix} = -8.$$

Kofaktori matrice A su:

$$A_{11} = \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = -2 \quad A_{12} = -\begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} = 0 \quad A_{13} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix} = 2 \quad A_{22} = \begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix} = 8 \quad A_{23} = -\begin{vmatrix} 4 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{31} = \begin{vmatrix} -1 & -2 \\ -1 & 3 \end{vmatrix} = -5 \quad A_{32} = -\begin{vmatrix} 4 & -2 \\ 0 & 3 \end{vmatrix} = -12 \quad A_{33} = \begin{vmatrix} 4 & -1 \\ 0 & 1 \end{vmatrix} = -4$$

pa je adjungovana matrica matrice A

$$adjA = \begin{bmatrix} -2 & 2 & -5 \\ 0 & 8 & -12 \\ 0 & 0 & -4 \end{bmatrix}.$$

Sada je

$$A^{-1} = -\frac{1}{8}\begin{bmatrix} -2 & 2 & -5 \\ 0 & 8 & -12 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

i

$$\begin{aligned} A^{-2} &= A^{-1} \cdot A^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{5}{8} \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{32} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

Konačno, imamo da je

$$
\begin{aligned}
P(A) &= 2A - A^{-2} - 3E = \\
&= 2\begin{bmatrix} 4 & -1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{32} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} 8 & -2 & -4 \\ 0 & -2 & 6 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{32} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \\
&= \begin{bmatrix} \frac{79}{16} & -\frac{35}{16} & -\frac{131}{32} \\ 0 & -6 & \frac{27}{4} \\ 0 & 0 & \frac{3}{4} \end{bmatrix}.
\end{aligned}
$$

Zadatak 8.4 *Dat je polinom* $P(x) = -2 + 3x + x^{-2}$ *i matrica*

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Izračunati $P(A)$.

Rješenje:
Jasno je da vrijedi $P(A) = -2E+3A+A^{-2}$, gdje je E jedninična matrica. Prvo ćemo izračunati inverznu matricu matrice A po formuli

$$A^{-1} = \frac{1}{\det A} \cdot adjA.$$

Imamo

$$\det A = \begin{vmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{vmatrix} = \left|\begin{array}{ccc|cc} 1 & -1 & -2 & 1 & -1 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & -3 & 0 & 0 \end{array}\right. = -6.$$

Kofaktori matrice A su:

$$
\begin{array}{lll}
A_{11} = \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} = -6 & A_{12} = -\begin{vmatrix} 0 & 1 \\ 0 & -3 \end{vmatrix} = 0 & A_{13} = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{21} = -\begin{vmatrix} -1 & -2 \\ 0 & -3 \end{vmatrix} = -3 & A_{22} = \begin{vmatrix} 1 & -2 \\ 0 & -3 \end{vmatrix} = -3 & A_{23} = -\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0 \\
A_{31} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = 3 & A_{32} = -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = -1 & A_{33} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2
\end{array}
$$

pa je adjungovana matrica matrice A

$$adjA = \begin{bmatrix} -6 & -3 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Sada je

$$A^{-1} = -\frac{1}{6}\begin{bmatrix} -6 & -3 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

i

$$\begin{aligned} A^{-2} &= A^{-1}\cdot A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}\cdot\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \\ &= \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{36} \\ 0 & 0 & \frac{1}{9} \end{bmatrix}. \end{aligned}$$

Konačno, imamo da je

$$\begin{aligned} P(A) &= -2E + 3A + A^{-2} = \\ &= -2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{36} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & -3 & -6 \\ 0 & 6 & 3 \\ 0 & 0 & -9 \end{bmatrix} + \begin{bmatrix} 1 & \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{36} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \\ &= \begin{bmatrix} 2 & -\frac{9}{4} & -\frac{25}{4} \\ 0 & \frac{17}{4} & \frac{109}{36} \\ 0 & 0 & -\frac{98}{9} \end{bmatrix}. \end{aligned}$$

Zadatak 8.5 *Dokazati da za matricu*

$$A(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

vrijedi $A(t)\cdot A(r) = A(t+r), t, r \in \mathbb{R}$.

Rješenje:

Kako je

$$A\,(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \quad i \quad A\,(x) = \begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix}$$

tada je

$$\begin{aligned} A(t)\cdot A(r) &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\cdot\begin{bmatrix} \cos r & -\sin r \\ \sin r & \cos r \end{bmatrix} = \\ &= \begin{bmatrix} \cos t\cdot\cos r - \sin t\cdot\sin r & -\cos t\cdot\sin r - \sin t\cdot\cos r \\ \sin t\cdot\cos r + \cos t\cdot\sin r & -\sin t\cdot\sin r + \cos t\cdot\cos r \end{bmatrix} = \\ &= \begin{bmatrix} \cos t\cdot\cos r - \sin t\cdot\sin r & -(\cos t\cdot\sin r + \sin t\cdot\cos r) \\ \sin t\cdot\cos r + \cos t\cdot\sin r & \cos t\cdot\cos r - \sin t\cdot\sin r \end{bmatrix} = \\ &= \begin{bmatrix} \cos(t+r) & -\sin(t+r) \\ \sin(t+r) & \cos(t+r) \end{bmatrix} = A(t+r). \end{aligned}$$

Zadatak 8.6 *Date su matrice*

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Izračunati matrice $P = AB^{-1} + A^{-2} + 2E$ *i* $Q = 2A - 3B^{-1} + A^{-1}B$, *gdje je* E *jedinična matrica.*

Rješenje:

Izračunajmo inverznu matricu matrice A po formuli

$$A^{-1} = \frac{1}{\det A}\cdot adjA.$$

Imamo

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}\begin{matrix} 2 & 0 \\ 4 & 3 \\ 1 & 1 \end{matrix} = \\ &= (6+0-4) - (-3+4+0) = 2-1 = 1. \end{aligned}$$

Kofaktori matrice A su

$$A_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1 \quad A_{12} = -\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = -2 \quad A_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{21} = -\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = -1 \quad A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \quad A_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -2$$

$$A_{31} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 \quad A_{32} = -\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = -8 \quad A_{33} = \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} = 6$$

pa je adjungovana matrica

$$adjA = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix}$$

Inverzna matrica matrice A je

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix}.$$

Izračunajmo inverznu matricu matrice B po formuli

$$B^{-1} = \frac{1}{\det B} \cdot adjB.$$

Imamo

$$\begin{aligned} \det B &= \begin{vmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix} \begin{matrix} 0 & 1 \\ 4 & 2 \\ 1 & 0 \end{matrix} = \\ &= (0+3+0) - (2+0-4) = 3+2 = 5. \end{aligned}$$

Kofaktori matrice B su

$$\begin{array}{lll} B_{11} = \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2 & B_{12} = -\begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} = 7 & B_{13} = \begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} = -2 \\ B_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1 & B_{22} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 & B_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \\ B_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 & B_{32} = -\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 4 & B_{33} = \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} = -4 \end{array}$$

pa je adjungovana matrica

$$adjB = \begin{bmatrix} -2 & 1 & 1 \\ 7 & -1 & 4 \\ -2 & 1 & -4 \end{bmatrix}$$

Inverzna matrica matrice B je

$$B^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 1 & 1 \\ 7 & -1 & 4 \\ -2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{1}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \end{bmatrix}.$$

Izračunajmo matricu A^{-2}

$$\begin{aligned} A^{-2} &= A^{-1}\cdot A^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} = \\ &= \begin{bmatrix} 6 & -10 & 29 \\ -16 & 27 & -78 \\ 11 & -19 & 55 \end{bmatrix}. \end{aligned}$$

Sada je

$$\begin{aligned} P &= AB^{-1} + A^{-2} + 2E = \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{1}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \end{bmatrix} + \begin{bmatrix} 6 & -10 & 29 \\ -16 & 27 & -78 \\ 11 & -19 & 55 \end{bmatrix} + \\ &\quad + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{38}{5} & -\frac{49}{5} & \frac{151}{5} \\ -\frac{71}{5} & \frac{148}{5} & -\frac{382}{5} \\ \frac{58}{5} & -\frac{94}{5} & \frac{286}{5} \end{bmatrix} \end{aligned}$$

i

$$\begin{aligned} Q &= 2A - 3B^{-1} + A^{-1}B = \\ &= 2\cdot \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} - 3\cdot \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{7}{5} & -\frac{1}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \end{bmatrix} + \\ &\quad + \begin{bmatrix} 1 & -1 & 3 \\ -2 & 3 & -8 \\ 1 & -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 4 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & -2 \\ 8 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} -\frac{6}{5} & \frac{3}{5} & \frac{3}{5} \\ \frac{21}{5} & -\frac{3}{5} & \frac{12}{5} \\ -\frac{6}{5} & \frac{3}{5} & -\frac{12}{5} \end{bmatrix} + \begin{bmatrix} -1 & -1 & -5 \\ 4 & 4 & 15 \\ -2 & -3 & -11 \end{bmatrix} = \\ &= \begin{bmatrix} \frac{21}{5} & -\frac{8}{5} & -\frac{38}{5} \\ \frac{39}{5} & \frac{53}{5} & \frac{83}{5} \\ \frac{6}{5} & -\frac{8}{5} & -\frac{33}{5} \end{bmatrix}. \end{aligned}$$

Zadatak 8.7 *Riješiti matričnu jednačinu*

$$X\,(A+E) = 2A - E$$

gdje je

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

i E jedinična matrica.

Rješenje:
Neka je

$$\begin{aligned} P &= A + E = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \\ Q &= 2A - E = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -7 \end{bmatrix}. \end{aligned}$$

Sada matrična jednačina ima oblik

$$XP = Q$$

koja se rješava na sljedeći način

$$\begin{aligned} XP &= Q \quad / \cdot P^{-1} \\ XPP^{-1} &= QP^{-1} \\ XE &= QP^{-1} \end{aligned}$$

pa je rješenje matrica $X = QP^{-1}$.

Izračunajmo matricu P^{-1} po formuli

$$P^{-1} = \frac{1}{\det P} \cdot adjP$$

Determinanta matrice P je

$$\det P = \begin{vmatrix} 2 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -12$$

a kofaktori matrice P su

$$\begin{aligned} &P_{11} = \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} = -6 \quad && P_{12} = -\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} = 0 \quad && P_{13} = \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} = 0 \\ &P_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = -2 \quad && P_{22} = \begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix} = -4 \quad && P_{23} = -\begin{vmatrix} 2 & 11 \\ 0 & 0 \end{vmatrix} = 0 \\ &P_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -7 \quad && P_{32} = -\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = -2 \quad && P_{33} = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 6 \end{aligned}$$

pa je

$$adjP = \begin{bmatrix} -6 & -2 & -7 \\ 0 & -4 & -2 \\ 0 & 0 & 6 \end{bmatrix}.$$

Dakle,

$$P^{-1} = \frac{1}{-12}\begin{bmatrix} -6 & -2 & -7 \\ 0 & -4 & -2 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{7}{12} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Rješenje matrične jednačine je matrica

$$\begin{aligned} X &= QP^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -7 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{7}{12} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{7}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{bmatrix}. \end{aligned}$$

Zadatak 8.8 *Riješiti matričnu jednačinu*

$$X(2A - 3E) = 2E - A$$

gdje je

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

i E jedinična matrica.

Neka je

$$\begin{aligned} P &= 2A - 3E = \begin{bmatrix} 2 & -2 & -4 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & -9 \end{bmatrix} \\ Q &= 2E - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix}. \end{aligned}$$

Sada matrična jednačina ima oblik

$$XP = Q$$

koja se rješava na sljedeći način

$$\begin{aligned} XP &= Q \quad / \cdot P^{-1} \\ XPP^{-1} &= QP^{-1} \\ XE &= QP^{-1} \end{aligned}$$

pa je rješenje matrica $X = QP^{-1}$.

Izračunajmo matricu P^{-1} po formuli

$$P^{-1} = \frac{1}{\det P} \cdot adjP$$

Determinanta matrice P je

$$\det P = \begin{vmatrix} -1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & -9 \end{vmatrix} = 9$$

a kofaktori matrice P su

$$P_{11} = \begin{vmatrix} 1 & 2 \\ 0 & -9 \end{vmatrix} = -9 \qquad P_{12} = -\begin{vmatrix} 0 & 2 \\ 0 & -9 \end{vmatrix} = 0 \qquad P_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$P_{21} = -\begin{vmatrix} -2 & -4 \\ 0 & -9 \end{vmatrix} = -18 \quad P_{22} = \begin{vmatrix} -1 & -4 \\ 0 & -9 \end{vmatrix} = 9 \quad P_{23} = -\begin{vmatrix} -1 & -2 \\ 0 & 0 \end{vmatrix} = 0$$

$$P_{31} = \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} = 0 \qquad P_{32} = -\begin{vmatrix} -1 & -4 \\ 0 & 2 \end{vmatrix} = 2 \quad P_{33} = \begin{vmatrix} -1 & -2 \\ 0 & 1 \end{vmatrix} = -1$$

pa je

$$adjP = \begin{bmatrix} -9 & -18 & 0 \\ 0 & 9 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Dakle,

$$P^{-1} = \frac{1}{9}\begin{bmatrix} -9 & -18 & 0 \\ 0 & 9 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}.$$

Rješenje matrične jednačine je matrica

$$\begin{aligned} X &= QP^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix} = \\ &= \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & \frac{1}{9} \\ 0 & 0 & -\frac{5}{9} \end{bmatrix}. \end{aligned}$$

Zadatak 8.9 *Riješiti matričnu jednačinu*

$$(A-2B)\,X = 2A - B + E$$

ako je

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \quad i \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

i E jedinična matrica.

Rješenje:
Neka je

$$\begin{aligned}
P &= A-2B = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - 2\cdot\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 1 & -4 \\ 0 & -2 & -1 \\ 0 & 0 & -9 \end{bmatrix} \\
Q &= 2A-B+E = 2\cdot\begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} 2 & -2 & -4 \\ 0 & 4 & 2 \\ 0 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & -8 \end{bmatrix}.
\end{aligned}$$

Sada matrična jednačina ima oblik

$$PX = Q$$

koja se rješava na sljedeći način

$$\begin{aligned}
P^{-1}\cdot /\quad PX &= Q \quad /\cdot P^{-1} \\
PP^{-1}X &= P^{-1}Q \\
EX &= P^{-1}Q
\end{aligned}$$

pa je rješenje matrica $X = P^{-1}Q$.
Izračunajmo matricu P^{-1} po formuli

$$P^{-1} = \frac{1}{\det P}\cdot adjP$$

Determinanta matrice P je

$$\det P = \begin{vmatrix} -3 & 1 & -4 \\ 0 & -2 & -1 \\ 0 & 0 & -9 \end{vmatrix} = -54$$

a kofaktori matrice P su

$$P_{11} = \begin{vmatrix} -2 & -1 \\ 0 & -9 \end{vmatrix} = 18 \quad P_{12} = -\begin{vmatrix} 0 & -1 \\ 0 & -9 \end{vmatrix} = 0 \quad P_{13} = \begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} = 0$$
$$P_{21} = -\begin{vmatrix} 1 & -4 \\ 0 & -9 \end{vmatrix} = 9 \quad P_{22} = \begin{vmatrix} -3 & -4 \\ 0 & -9 \end{vmatrix} = 27 \quad P_{23} = -\begin{vmatrix} -3 & 1 \\ 0 & 0 \end{vmatrix} = 0$$
$$P_{31} = \begin{vmatrix} 1 & -4 \\ -2 & -1 \end{vmatrix} = -9 \quad P_{32} = -\begin{vmatrix} -3 & -4 \\ 0 & -1 \end{vmatrix} = -3 \quad P_{33} = \begin{vmatrix} -3 & 1 \\ 0 & -2 \end{vmatrix} = 6$$

pa je

$$adj P = \begin{bmatrix} 18 & 9 & -9 \\ 0 & 27 & -3 \\ 0 & 0 & 6 \end{bmatrix}.$$

Dakle,

$$P^{-1} = \frac{1}{-54}\begin{bmatrix} 18 & 9 & -9 \\ 0 & 27 & -3 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{18} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}.$$

Rješenje matrične jednačine je matrica

$$\begin{aligned} X &= P^{-1}Q = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{18} \\ 0 & 0 & -\frac{1}{9} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & -8 \end{bmatrix} = \\ &= \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ 0 & -\frac{3}{2} & -\frac{51}{54} \\ 0 & 0 & \frac{8}{9} \end{bmatrix}. \end{aligned}$$

Zadatak 8.10 *Riješiti matričnu jednačinu*

$$AX^{-1} = A - X^{-1}$$

ako je

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Rješenje:
Matrična jednačina

$$AX^{-1} = A - X^{-1}$$

je ekvivalentna jednačini

$$AX = A + E$$

jer

$$\begin{aligned} AX^{-1} &= A - X^{-1} \\ AX^{-1} + X^{-1} &= A \\ (A+E)\,X^{-1} &= A \quad / \cdot X \\ (A+E)\,X^{-1}X &= AX \\ A + E &= AX. \end{aligned}$$

Neka je

$$B = A + E = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Sada matriča jednačina ima oblik

$$AX = B$$

koja se rješava na sljedeći način

$$\begin{aligned} A^{-1} \cdot /\ AX &= B \\ A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B. \end{aligned}$$

Inverzna matrica matrice A je matrica

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

pa je rješenje matrične jednačine matrica

$$X = A^{-1}B = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} \end{bmatrix}.$$

Zadatak 8.11 *Matricu*

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

predstaviti kao zbir jedne simetrične i jedne antisimetrične matrice.

Rješenje:

Kvadratna matrica je simetrična ako su joj elementi simetrični u odnosu na glavnu dijagonalu, tj. ako je $a_{ij} = a_{ji}$ i vrijedi

$$A_S = \frac{1}{2}\left(A + A^T\right).$$

Kvadratna matrica je antisimetrična ako je jednaka svojoj negativnoj transponovanoj matrici i vrijedi

$$A_K = \frac{1}{2}\left(A - A^T\right).$$

Kako je

$$A^T = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

onda je

$$\begin{aligned} A_S &= \frac{1}{2}\left(A + A^T\right) = \frac{1}{2}\left(\begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \\ A_k &= \frac{1}{2}\left(A - A^T\right) = \frac{1}{2}\left(\begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Zadatak 8.12 *Odrediti rang matrice*

$$A = \begin{bmatrix} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned} A &= \begin{bmatrix} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & 1 & 3 & -1 \\ 4 & -3 & -2 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 7 & -2 & 7 \end{bmatrix} \begin{array}{r} V_1 \ \textit{prepisana} \\ 2 \cdot V_1 + V_2 \\ 4 \cdot V_1 - V_3 \end{array} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 13 & 0 \end{bmatrix} \begin{array}{l} V_1 \ \textit{prepisana} \\ V_2 \ \textit{prepisana} \\ 7 \cdot V_2 - 3 \cdot V_3 \end{array} \end{aligned}$$

Dakle, $rang A = 3$.

Zadatak 8.13 *Odrediti rang matrice*

$$A = \begin{bmatrix} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned} A &= \begin{bmatrix} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \\ 3 & -2 & 1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 1 & -8 & 7 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ 2 \cdot V_1 + V_2 \\ 3 \cdot V_1 + V_3 \end{array} \\ &\sim \begin{bmatrix} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \begin{array}{r} V_1 \; prepisana \\ V_2 \; prepisana \\ V_2 - V_3 \end{array} \end{aligned}$$

Dakle, $rang A = 2$.

Zadatak 8.14 *Odrediti rang matrice*

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 & 2 \\ -1 & 2 & -3 & -1 & -4 \\ 2 & -1 & 3 & 1 & 4 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ V_1 - V_2 \\ V_1 + V_3 \\ 2 \cdot V_1 - V_4 \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & -1 & 3 & 1 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ V_1 + V_3 \\ V_4 \ prepisana \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & -4 & -4 \end{bmatrix} \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ V_3 \ prepisana \\ V_3 - V_4 \end{array}
\end{aligned}$$

Dakle, $rangA = 4$.

Zadatak 8.15 *Odrediti rang matrice*

$$A = \begin{bmatrix} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{bmatrix}.$$

Rješenje:

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 2 & 3 & -4 & 1 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ V_1 + V_2 \\ 2V_1 - V_3 \\ 3V_1 - V_4 \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & 0 & 5 & 25 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ V_2 \; prepisana \\ V_2 + 4V_3 \\ V_3 - V_4 \end{array}
\end{aligned}
$$

Dakle, $rangA = 3$.

Zadatak 8.16 *Odrediti rang matrice*

$$
A = \begin{bmatrix} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{bmatrix}.
$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & -3 & -1 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & -5 & 7 & 3 & 5 \\ 0 & -1 & 5 & 5 & 9 \end{bmatrix} \begin{matrix} V_1 \ prepisana \\ 2V_1 - V_2 \\ 2V_1 + V_3 \\ 3V_1 - V_4 \end{matrix} \\
&\sim \begin{bmatrix} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & 0 & 18 & 22 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} V_1 \ prepisana \\ V_2 \ prepisana \\ 5V_2 - V_3 \\ V_2 - V_3 \end{matrix}
\end{aligned}$$

Dakle, $rangA = 3$.

Zadatak 8.17 *U zavisnosti od realnog parametra λ odrediti rang matrice*

$$A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}.$$

Rješenje:

$$\begin{aligned}
A &= \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{bmatrix}
\end{aligned}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda^2 \end{bmatrix} \begin{matrix} V_1 \; prepisana \\ V_2 - V_1 \\ V_3 - \lambda V_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 0 & 2-\lambda-\lambda^2 \end{bmatrix} \begin{matrix} V_1 \; prepisana \\ V_2 \; prepisana \\ V_3 + V_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 0 & (1-\lambda)(2+\lambda) \end{bmatrix}$$

Diskusija:

1. ako je $\lambda = 1$ tada je $rangA = 1$ jer je

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. ako je $\lambda = -2$ tada je $rangA = 2$ jer je

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3. ako je $\lambda \neq 1$ i $\lambda \neq -2$ tada je $rangA = 3$.

Zadatak 8.18 *U zavisnosti od realnog parametra λ odrediti rang matrice*

$$A = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 4 & 10 & 1 & \lambda \\ 7 & 17 & 3 & 1 \\ 2 & 4 & 3 & 2 \end{bmatrix}.$$

Rješenje:

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 4 & 3 \\ 4 & 10 & 1 & \lambda \\ 7 & 17 & 3 & 1 \\ 2 & 4 & 3 & 2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 6 & -15 & \lambda-12 \\ 0 & 10 & -25 & -20 \\ 0 & 2 & -5 & -4 \end{bmatrix} \begin{array}{r} V_1\ prepisana \\ V_2-4V_1 \\ V_3-7V_1 \\ V_4-2V_1 \end{array} \\
&\sim \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 2 & -5 & -4 \\ 0 & 6 & -15 & \lambda-12 \\ 0 & 10 & -25 & -20 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 1 & 4 & 3 \\ 0 & 2 & -5 & -4 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{r} V_1\ prepisana \\ V_2\ prepisana \\ V_3-3V_2 \\ V_4-5V_2 \end{array}
\end{aligned}
$$

Diskusija:

1. ako je $\lambda = 0$ tada je $rangA = 2$.
2. ako je $\lambda \neq 0$ tada je $rangA = 3$.

9 Sistemi linearnih jednačina

Zadatak 9.1 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r} 4x-3y-2z=1 \\ -2x+y+3z=-1 \\ x+y-z=2 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 4 & -3 & -2 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \\ A_p &= \left[\begin{array}{rrr|r} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{rrr|r} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ -2 & 1 & 3 & -1 \\ 4 & -3 & -2 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 7 & -2 & 7 \end{array}\right] \begin{array}{r} V_1 \ prepisana \\ 2\cdot V_1+V_2 \\ 4\cdot V_1-V_3 \end{array} \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 13 & 0 \end{array}\right] \begin{array}{l} V_1 \ prepisana \\ V_2 \ prepisana \\ 7\cdot V_2-3\cdot V_3 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A)=3$ i $rang\,(A_p)=3$ pa je sistem saglasan. Formirajmo novi sistem

$$\left.\begin{array}{r} x+y-z=2 \\ 3y+z=3 \\ 13z=0 \end{array}\right\}.$$

Iz treće jednačine vidimo da je $z=0$ i ako to uvrstimo u drugu jednačinu dobićemo

$$\begin{aligned} 3y+0 &= 3 \\ 3y &= 3 \Longrightarrow y=1. \end{aligned}$$

Uvrštavajući $z = 0$ i $y = 1$ u prvu jednačinu, dobijamo

$$\begin{aligned} x + 1 - 0 &= 2 \\ x + 1 &= 2 \Longrightarrow x = 2. \end{aligned}$$

Rješenje sistema je

$$(x, y, z) = (1, 1, 0).$$

Zadatak 9.2 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r} 3x - 2y + z = 2 \\ -x + y - 3z = -3 \\ 2x - y - 2z = -1 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 3 & -2 & 1 \\ -1 & 1 & -3 \\ 2 & -1 & -2 \end{bmatrix} \\ A_p &= \left[\begin{array}{rrr|r} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{rrr|r} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \\ 3 & -2 & 1 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 1 & -8 & 7 \end{array}\right] \begin{array}{r} V_1 \ prepisana \\ 2 \cdot V_1 + V_2 \\ 3 \cdot V_1 + V_3 \end{array} \\ &\sim \left[\begin{array}{rrr|r} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] . \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ V_2 - V_3 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 2$ i $rang\,(A_p) = 2$ pa je sistem saglasan. Formirajmo novi sistem

$$\left.\begin{array}{r} -x + y - 3z = -3 \\ y - 8z = -7 \end{array}\right\}.$$

Ovaj sistem ima tri nepoznate a dvije jednačine. Zato jednu nepoznatu biramo proizvoljno.

Neka je, recimo, $z = \alpha$ gdje je $\alpha \in \mathbb{R}$.

Ako $z = \alpha$ uvrstimo u drugu jednačinu, dobijamo

$$y - 8\alpha = -7 \implies y = -7 + 8\alpha.$$

Ako $y = -7 + 8\alpha$ i $z = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} -x - 7 + 8\alpha - 3\alpha &= -3 \\ -x - 7 + 5\alpha &= -3 \\ -x &= 4 - 5\alpha \implies x = -4 + 5\alpha \end{aligned}$$

Rješenje sistema je

$$(x, y, z) = (-4 + 5\alpha, -7 + 8\alpha, \alpha), \qquad \alpha \in \mathbb{R}.$$

Zadatak 9.3 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{aligned} 2x - y + 3z + t = 4 \\ x + y + z + t = 3 \\ -x + 2y - 3z - t = -4 \\ x + y + 2z - 2t = 3 \end{aligned}\right\}.$$

Rješenje:

Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 2 & -3 & -1 \\ 1 & 1 & 2 & -2 \end{bmatrix} \\ A_p &= \left[\begin{array}{cccc|c} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{array}\right] \end{aligned}$$

Odredimo rang proširene matrice:

$$
\begin{aligned}
A_p &= \left[\begin{array}{rrrr|r} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 & 2 \\ -1 & 2 & -3 & -1 & -4 \\ 2 & -1 & 3 & 1 & 4 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \end{array}\right] \begin{array}{r} V_1\ prepisana \\ V_1 - V_2 \\ V_1 + V_3 \\ 2 \cdot V_1 - V_4 \end{array} \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & -1 & 3 & 1 \end{array}\right] \begin{array}{r} V_1\ prepisana \\ V_2\ prepisana \\ V_1 + V_3 \\ V_4\ prepisana \end{array} \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right] \begin{array}{r} V_1\ prepisana \\ V_2\ prepisana \\ V_3\ prepisana \\ V_3 - V_4 \end{array}
\end{aligned}
$$

Odavdje vidimo da je $rang\,(A) = 4$ i $rang\,(A_p) = 4$ pa je sistem saglasan. Formirajmo novi sistem

$$
\left.\begin{aligned} x + y + z + t &= 3 \\ 3y - 2z &= -1 \\ -z - t &= -3 \\ -4t &= -4 \end{aligned}\right\}.
$$

Iz četvrte jednačine vidimo da je $t = 1$.
Ako $t = 1$ uvrstimo u treću jednačinu dobićemo

$$
\begin{aligned} -z - 1 &= -3 \\ -z &= -2 \Longrightarrow z = 2. \end{aligned}
$$

Ako $z = 2$ i $t = 1$ uvrstimo u drugu jednačinu, dobićemo

$$\begin{aligned} 3y - 4 &= -1 \\ 3y &= 3 \implies y = 1. \end{aligned}$$

Ako $y = 1$, $z = 2$ i $t = 1$ uvrstimo u prvu jednačinu, dobićemo

$$x + 1 + 2 + 1 = 3 \implies x = -1.$$

Rješenje sistema je

$$(x, y, z, t) = (-1, 1, 2, 1).$$

Zadatak 9.4 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{aligned} 2x + 3y - 4z + t = 2 \\ x + y - z + 3t = 4 \\ -x + 3y - 2z + 2t = 2 \\ 3x + 4y - 5z + 4t = 6 \end{aligned}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 2 & 3 & -4 & 1 \\ 1 & 1 & -1 & 3 \\ -1 & 3 & -2 & 2 \\ 3 & 4 & -5 & 4 \end{bmatrix} \\ A_p &= \left[\begin{array}{cccc|c} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned}
A_p &= \left[\begin{array}{rrrr|r} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 2 & 3 & -4 & 1 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \end{array}\right] \begin{array}{r} V_1 \ prepisana \\ V_1 + V_2 \\ 2V_1 - V_3 \\ 3V_1 - V_4 \end{array} \\
&\sim \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & 0 & 5 & 25 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ V_2 + 4V_3 \\ V_3 - V_4 \end{array}
\end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 3$ i $rang\,(A_p) = 3$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{array}{r} x + y - z + 3t = 4 \\ 4y - 3z + 5t = 6 \\ 5z + 25t = 30 \end{array}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t = \alpha \ (\alpha \in \mathbb{R})$.
Iz treće jednačine imamo

$$\begin{aligned} 5z + 25\alpha &= 30 \\ z + 5\alpha &= 6 \Longrightarrow z = 6 - 5\alpha. \end{aligned}$$

Ako $z = 6 - 5\alpha$ i $t = \alpha$ uvrstimo u drugu jednačinu, dobijamo

$$\begin{aligned} 4y - 3(6 - 5\alpha) + 5\alpha &= 6 \\ 4y - 18 + 15\alpha + 5\alpha &= 6 \\ 4y &= 24 - 20\alpha \Longrightarrow y = 6 - 5\alpha. \end{aligned}$$

Ako $y = 6 - 5\alpha, z = 6 - 5\alpha$ i $t = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} x + 6 - 5\alpha - 6 + 5\alpha + 3\alpha &= 4 \\ x + 3\alpha &= 4 \Longrightarrow x = 4 - 3\alpha. \end{aligned}$$

Rješenje sistema je

$$(x, y, z, t) = (4 - 3\alpha, 6 - 5\alpha, 6 - 5\alpha, \alpha) \qquad (\alpha \in \mathbb{R}).$$

Zadatak 9.5 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{aligned} 3x - 2y + 2z &= 3 \\ x + y - z &= 1 \\ 4x - y + z &= 4 \\ 2x - 3y + 3z &= 2 \end{aligned}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 3 & -2 & 2 \\ 1 & 1 & -1 \\ 4 & -1 & 1 \\ 2 & -3 & 3 \end{bmatrix} \\ A_p &= \left[\begin{array}{ccc|c} 3 & -2 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 4 & -1 & 1 & 4 \\ 2 & -3 & 3 & 2 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{ccc|c} 3 & -2 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 4 & -1 & 1 & 4 \\ 2 & -3 & 3 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & -2 & 2 & 3 \\ 4 & -1 & 1 & 4 \\ 2 & -3 & 3 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 - 3V_1 \\ V_3 - 4V_1 \\ V_4 - 2V_1 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ V_3 - V_2 \\ V_4 - V_2 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 2$ i $rang\,(A_p) = 2$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{aligned} x + y - z = 1 \\ -5y + 5z = 0 \end{aligned}\right\}.$$

Ovaj sistem ima tri nepoznate a dvije jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $z = \alpha \ (\alpha \in \mathbb{R})$.
Iz druge jednačine imamo

$$\begin{aligned} -5y + 5\alpha &= 0 \\ -5y &= -5\alpha \Longrightarrow y = \alpha. \end{aligned}$$

Ako $y = \alpha$ i $z = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} x + y - z &= 1 \\ x + \alpha - \alpha &= 1 \Longrightarrow x = 1. \end{aligned}$$

Rješenje sistema je

$$(x, y, z) = (1, \alpha, \alpha) \qquad\qquad (\alpha \in \mathbb{R}).$$

Zadatak 9.6 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{aligned} 3x - 2y + z + t = 3 \\ 2x + y - 2z - t = 0 \\ x + 2y - z + t = 3 \\ 3x + 3y - 3z = 3 \end{aligned}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 3 & -2 & 1 & 1 \\ 2 & 1 & -2 & -1 \\ 1 & 2 & -1 & 1 \\ 3 & 3 & -3 & 0 \end{bmatrix} \\ A_p &= \left[\begin{array}{cccc|c} 3 & -2 & 1 & 1 & 3 \\ 2 & 1 & -2 & -1 & 0 \\ 1 & 2 & -1 & 1 & 3 \\ 3 & 3 & -3 & 0 & 3 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$
\begin{aligned}
A_p &= \left[\begin{array}{rrrr|r} 3 & -2 & 1 & 1 & 3 \\ 2 & 1 & -2 & -1 & 0 \\ 1 & 2 & -1 & 1 & 3 \\ 3 & 3 & -3 & 0 & 3 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 2 & 1 & -2 & -1 & 0 \\ 3 & -2 & 1 & 1 & 3 \\ 3 & 3 & -3 & 0 & 3 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 0 & -3 & -6 \\ 0 & -8 & 4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -6 \end{array}\right] \begin{array}{r} V_1 \ prepisana \\ V_2 - 2V_1 \\ V_3 - 3V_1 \\ V_4 - 3V_1 \end{array} \\
&\sim \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 0 & -3 & -6 \\ 0 & 0 & -12 & -18 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \ prepisana \\ V_2 \ prepisana \\ 8V_2 - 3V_3 \\ V_2 - V_4 \end{array}
\end{aligned}
$$

Odavdje vidimo da je $rang\,(A) = 3$ i $rang\,(A_p) = 3$ pa je sistem saglasan. Formirajmo novi sistem

$$
\left.\begin{array}{r} x + 2y - z + t = 3 \\ -3y - 3t = -6 \\ -12z - 18t = -30 \end{array}\right\} \iff \left.\begin{array}{r} x + 2y - z + t = 3 \\ y + t = 2 \\ 2z + 3t = 5 \end{array}\right\}.
$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t = \alpha \ (\alpha \in \mathbb{R})$.
Iz treće jednačine imamo

$$
\begin{aligned}
2z + 3\alpha &= 5 \\
2z &= 5 - 3\alpha \implies z = \frac{5-3\alpha}{2}.
\end{aligned}
$$

Ako $z = \frac{5-3\alpha}{2}$ i $t = \alpha$ uvrstimo u drugu jednačinu, dobijamo

$$
y + \alpha = 2 \implies y = 2 - \alpha.
$$

Ako $y = 2 - \alpha$, $z = \frac{5-3\alpha}{2}$ i $t = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned}
x + 2(2-\alpha) - \frac{5-3\alpha}{2} + \alpha &= 3\\
x + 4 - 2\alpha - \frac{5-3\alpha}{2} + \alpha &= 3\\
x + \frac{8 - 4\alpha - 5 + 3\alpha + 2\alpha}{2} &= 3\\
x + \frac{3+\alpha}{2} &= 3\\
x &= 3 - \frac{3+\alpha}{2} \Longrightarrow x = \frac{3-\alpha}{2}.
\end{aligned}$$

Rješenje sistema je

$$(x, y, z, t) = \left(\frac{3-\alpha}{2}, 2-\alpha, \frac{5-3\alpha}{2}, \alpha\right) \qquad (\alpha \in \mathbb{R}).$$

Zadatak 9.7 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{c}
2x + 2y - z + 3t = 6\\
3x + y - 3z + t = 2\\
5x + 3y - 4z + 4t = 8\\
x - y - 2z - 2t = -4
\end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 2 & -1 & 3\\ 3 & 1 & -3 & 1\\ 5 & 3 & -4 & 4\\ 1 & -1 & -2 & -2 \end{bmatrix}\\
A_p &= \left[\begin{array}{rrrr|r} 2 & 2 & -1 & 3 & 6\\ 3 & 1 & -3 & 1 & 2\\ 5 & 3 & -4 & 4 & 8\\ 1 & -1 & -2 & -2 & 4 \end{array}\right].
\end{aligned}$$

Odredimo rang proširene matrice:

$$
\begin{aligned}
A_p &= \left[\begin{array}{cccc|c} 2 & 2 & -1 & 3 & 6 \\ 3 & 1 & -3 & 1 & 2 \\ 5 & 3 & -4 & 4 & 8 \\ 1 & -1 & -2 & -2 & -4 \end{array}\right] \\
&\sim \left[\begin{array}{cccc|c} 1 & -1 & -2 & -2 & -4 \\ 2 & 2 & -1 & 3 & 6 \\ 3 & 1 & -3 & 1 & 2 \\ 5 & 3 & -4 & 4 & 8 \end{array}\right] \\
&\sim \left[\begin{array}{cccc|c} 1 & -1 & -2 & -2 & -4 \\ 0 & -4 & -3 & -7 & -14 \\ 0 & -4 & -3 & -7 & -14 \\ 0 & -8 & -6 & -14 & -28 \end{array}\right] \begin{array}{r} V_1 \; prepisana \\ 2V_1 - V_2 \\ 3V_1 - V_3 \\ 5V_1 - 3V_4 \end{array} \\
&\sim \left[\begin{array}{cccc|c} 1 & -1 & -2 & -2 & -4 \\ 0 & -4 & -3 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \; prepisana \\ 2V_1 - V_2 \\ V_2 - V_3 \\ 2V_2 - V_4 \end{array}
\end{aligned}
$$

Odavdje vidimo da je $rang\,(A) = 2$ i $rang\,(A_p) = 2$ pa je sistem saglasan. Formirajmo novi sistem

$$
\left.\begin{array}{r} x - y - 2z - 2t = -4 \\ -4y - 3z - 7t = -14 \end{array}\right\}.
$$

Ovaj sistem ima četiri nepoznate a dvije jednačine.
Znači, dvije nepoznate uzimamo proizvoljno.
Neka je, npr. $z = \alpha, t = \beta \; (\alpha, \beta \in \mathbb{R})$.
Iz druge jednačine imamo

$$
\begin{aligned}
-4y - 3\alpha - 7\beta &= -14 \\
-4y &= -14 + 3\alpha + 7\beta \Longrightarrow y = \frac{14 - 3\alpha - 7\beta}{4}.
\end{aligned}
$$

Ako $y = \frac{14-3\alpha-7\beta}{4}$, $z = \alpha, t = \beta$ uvrstimo u prvu jednačinu, dobijamo

$$
\begin{aligned}
x - \frac{14-3\alpha-7\beta}{4} - 2\alpha - 2\beta &= -4 \\
x + \frac{-14+3\alpha+7\beta-8\alpha-8\beta}{4} &= -4 \\
x + \frac{-14-5\alpha-\beta}{4} &= -4 \\
x &= -4 - \frac{-14-5\alpha-\beta}{4} \\
x &= \frac{-16+14+5\alpha+\beta}{4} \Longrightarrow x = \frac{-2+5\alpha+\beta}{4}.
\end{aligned}
$$

Rješenje sistema je

$$
(x,y,z,t) = \left(\frac{-2+5\alpha+\beta}{4}, \frac{14-3\alpha-7\beta}{4}, \alpha, \beta\right) \qquad (\alpha, \beta \in \mathbb{R}).
$$

Zadatak 9.8 *Ispitati saglasnost sistema i u slučaju saglasnosti riješiti sistem matričnom metodom*

$$
\left.
\begin{aligned}
2x - y + z - 3t = -1 \\
x - y + 3z + t = 4 \\
-2x - 3y + z + t = -3 \\
3x - 2y + 4z - 2t = 3
\end{aligned}
\right\}.
$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 1 & -3 \\ 1 & -1 & 3 & 1 \\ -2 & -3 & 1 & 1 \\ 3 & -2 & 4 & -2 \end{bmatrix} \\
A_p &= \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right].
\end{aligned}
$$

Odredimo rang proširene matrice:

$$\begin{aligned}
A_p &= \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & -3 & -1 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & -5 & 7 & 3 & 5 \\ 0 & -1 & 5 & 5 & 9 \end{array}\right] \begin{array}{r} V_1\ prepisana \\ 2V_1 - V_2 \\ 2V_1 + V_3 \\ 3V_1 - V_4 \end{array} \\
&\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & 0 & 18 & 22 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1\ prepisana \\ V_2\ prepisana \\ 5V_2 - V_3 \\ V_2 - V_3 \end{array}
\end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 3$ i $rang\,(A_p) = 3$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{aligned} x - y + 3z + t &= 4 \\ -y + 5z + 5t &= 9 \\ 18z + 22t &= 40 \end{aligned}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t = \alpha \ (\alpha \in \mathbb{R})$.
Sada sistem ima oblik

$$\left.\begin{aligned} x - y + 3z &= 4 - \alpha \\ -y + 5z &= 9 - 5\alpha \\ 9z &= 20 - 11\alpha \end{aligned}\right\}.$$

Ovaj sistem je ekvivalentan matričnoj jednačini

$$AX = B$$

gdje su

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 4 - \alpha \\ 9 - 5\alpha \\ 20 - 11\alpha \end{bmatrix}.$$

Rješenje matrične jednačine je matrica $X = A^{-1}B$.
Izračunajmo matricu A^{-1} na osnovu formule

$$A^{-1} = \frac{1}{\det A} adj A.$$

Determinanta matrice A je

$$\det A = \begin{vmatrix} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 9 \end{vmatrix} = -9$$

Kofaktori matrice A su

$$A_{11} = \begin{vmatrix} -1 & 5 \\ 0 & 9 \end{vmatrix} = -9 \quad A_{12} = -\begin{vmatrix} 0 & 5 \\ 0 & 9 \end{vmatrix} = 0 \quad A_{13} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$
$$A_{21} = -\begin{vmatrix} -1 & 3 \\ 0 & 9 \end{vmatrix} = 9 \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 9 \end{vmatrix} = 9 \quad A_{23} = -\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0$$
$$A_{31} = \begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix} = -2 \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \quad A_{33} = \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} = -1$$

pa je adjungovana matrica

$$adj A = \begin{bmatrix} -9 & 9 & -2 \\ 0 & 9 & -5 \\ 0 & 0 & -1 \end{bmatrix}.$$

Inverzna matrica matrice A je

$$A^{-1} = \frac{1}{-9}\begin{bmatrix} -9 & 9 & -2 \\ 0 & 9 & -5 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{2}{9} \\ 0 & -1 & \frac{5}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix}.$$

Sada je

$$X = A^{-1}B = \begin{bmatrix} 1 & -1 & \frac{2}{9} \\ 0 & -1 & \frac{5}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} \cdot \begin{bmatrix} 4-\alpha \\ 9-5\alpha \\ 20-11\alpha \end{bmatrix} = \begin{bmatrix} \frac{-5+14\alpha}{9} \\ \frac{19-10\alpha}{9} \\ \frac{20-11\alpha}{9} \end{bmatrix}$$

pa je

$$\begin{aligned} x &= \frac{-5+14\alpha}{9} \\ y &= \frac{19-10\alpha}{9} \\ z &= \frac{20-11\alpha}{9}, \qquad \alpha \in \mathbb{R} \end{aligned}$$

rješenje sistema.

Zadatak 9.9 *Ispitati saglasnost sistema i u slučaju saglasnosti riješiti sistem metodom determinanti*

$$\left.\begin{array}{r} 2x-y+z-3t=-1 \\ x-y+3z+t=4 \\ -2x-3y+z+t=-3 \\ 3x-2y+4z-2t=3 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 2 & -1 & 1 & -3 \\ 1 & -1 & 3 & 1 \\ -2 & -3 & 1 & 1 \\ 3 & -2 & 4 & -2 \end{bmatrix} \\ A_p &= \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right] \\ &\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & -3 & -1 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right] \\ &\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & -5 & 7 & 3 & 5 \\ 0 & -1 & 5 & 5 & 9 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ 2V_1-V_2 \\ 2V_1+V_3 \\ 3V_1-V_4 \end{array} \\ &\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & 0 & 18 & 22 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ 5V_2-V_3 \\ V_2-V_3 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A)=3$ i $rang\,(A_p)=3$ pa je sistem saglasan.

Formirajmo novi sistem

$$\left.\begin{array}{r} x-y+3z+t=4 \\ -y+5z+5t=9 \\ 18z+22t=40 \end{array}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t=\alpha\ (\alpha\in\mathbb{R})$.
Sada sistem ima oblik

$$\left.\begin{array}{r} x-y+3z=4-\alpha \\ -y+5z=9-5\alpha \\ 9z=20-11\alpha \end{array}\right\}.$$

Izračunajmo determinante D, D_x, D_y, D_z :

$$\begin{aligned}
D &= \begin{vmatrix} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 9 \end{vmatrix} = -9 \\
D_x &= \begin{vmatrix} 4-\alpha & -1 & 3 \\ 9-5\alpha & -1 & 5 \\ 20-11\alpha & 0 & 9 \end{vmatrix} = 5-14\alpha \\
D_y &= \begin{vmatrix} 1 & 4-\alpha & 3 \\ 0 & 9-5\alpha & 5 \\ 0 & 20-11\alpha & 9 \end{vmatrix} = -19+10\alpha \\
D_z &= \begin{vmatrix} 1 & -1 & 4-\alpha \\ 0 & -1 & 9-5\alpha \\ 0 & 0 & 20-11\alpha \end{vmatrix} = -20+11\alpha
\end{aligned}$$

pa je

$$\begin{aligned}
x &= \frac{D_x}{D} = \frac{-5+14\alpha}{9} \\
y &= \frac{D_y}{D} = \frac{19-10\alpha}{9} \\
z &= \frac{D_z}{D} = \frac{20-11\alpha}{9}, \qquad \alpha\in\mathbb{R}
\end{aligned}$$

rješenje sistema.

Zadatak 9.10 *U zavisnosti od realnog parametra λ diskutovati rješenja sistema*

$$\left.\begin{array}{r}\lambda x+y+z=1\\ x+\lambda y+z=\lambda\\ x+y+\lambda z=\lambda^2\end{array}\right\}.$$

Rješenje:
Izračunajmo determinante D, D_x, D_y, D_z :

$$\begin{aligned}
D &= \begin{vmatrix}\lambda & 1 & 1\\ 1 & \lambda & 1\\ 1 & 1 & \lambda\end{vmatrix} = \begin{vmatrix}\lambda & 1 & 1\\ 1 & \lambda & 1\\ 1 & 1 & \lambda\end{vmatrix}\begin{matrix}\lambda & 1\\ 1 & \lambda\\ 1 & 1\end{matrix} = \\
&= \left(\lambda^3+1+1\right)-(\lambda+\lambda+\lambda)=\lambda^3-3\lambda+2= \\
&= \lambda^3-\lambda-2\lambda+2=\lambda\left(\lambda^2-1\right)-2(\lambda-1)= \\
&= \lambda(\lambda-1)(\lambda+1)-2(\lambda-1)= \\
&= (\lambda-1)[\lambda(\lambda+1)-2]=(\lambda-1)\left[\lambda^2+\lambda-2\right]= \\
&= (\lambda-1)\left[\lambda^2+2\lambda-\lambda-2\right]=(\lambda-1)[\lambda(\lambda+2)-(\lambda+2)]= \\
&= (\lambda-1)(\lambda-1)(\lambda+2)=(\lambda-1)^2(\lambda+2).\\
D_x &= \left|\begin{matrix}1 & 1 & 1\\ \lambda & \lambda & 1\\ \lambda^2 & 1 & \lambda\end{matrix}\right|\begin{matrix}1 & 1\\ \lambda & \lambda\\ \lambda^2 & 1\end{matrix} = \\
&= \left(\lambda^2+\lambda^2+\lambda\right)-\left(\lambda^3+1+\lambda^2\right)= \\
&= -\lambda^3+\lambda^2+\lambda-1=-\left(\lambda^3-\lambda^2\right)+(\lambda-1)= \\
&= -\lambda^2(\lambda-1)+(\lambda-1)=(\lambda-1)\left(1-\lambda^2\right)= \\
&= (\lambda-1)(1-\lambda)(1+\lambda)=-(\lambda-1)(\lambda-1)(\lambda+1)= \\
&= -(\lambda-1)^2(\lambda+1).\\
D_y &= \left|\begin{matrix}\lambda & 1 & 1\\ 1 & \lambda & 1\\ 1 & \lambda^2 & \lambda\end{matrix}\right|\begin{matrix}\lambda & 1\\ 1 & \lambda\\ 1 & \lambda^2\end{matrix} = \\
&= \left(\lambda^3+1+\lambda^2\right)-\left(\lambda+\lambda^3+\lambda\right)= \\
&= \lambda^2-2\lambda+1=(\lambda-1)^2.
\end{aligned}$$

$$\begin{aligned}
D_z &= \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & \lambda \\ 1 & 1 & \lambda^2 \end{vmatrix} \begin{matrix} \lambda & 1 \\ 1 & \lambda \\ 1 & 1 \end{matrix} = \\
&= \left(\lambda^4 + \lambda + 1\right) - \left(\lambda + \lambda^2 + \lambda^2\right) = \\
&= \lambda^4 - 2\lambda^2 + 1 = \left(\lambda^2 - 1\right)^2 = \\
&= \left((\lambda - 1)(\lambda + 1)\right)^2 = (\lambda - 1)^2 (\lambda + 1)^2 .
\end{aligned}$$

Dakle,

$$\begin{aligned}
D &= (\lambda - 1)^2 (\lambda + 2) \\
D_x &= -(\lambda - 1)^2 (\lambda + 1) \\
D_y &= (\lambda - 1)^2 \\
D_z &= (\lambda - 1)^2 (\lambda + 1)^2
\end{aligned}$$

Diskusija:

1. Ako je $D \neq 0$, tj. $\lambda \neq 1$ i $\lambda \neq -2$, tada je sistem saglasan i rješenja su data sa

$$\begin{aligned}
x &= \frac{D_x}{D} = \frac{-(\lambda - 1)^2 (\lambda + 1)}{(\lambda - 1)^2 (\lambda + 2)} = -\frac{\lambda + 1}{\lambda + 2} \\
y &= \frac{D_y}{D} = \frac{(\lambda - 1)^2}{(\lambda - 1)^2 (\lambda + 2)} = \frac{1}{\lambda + 2} \\
z &= \frac{D_z}{D} = \frac{(\lambda - 1)^2 (\lambda + 1)^2}{(\lambda - 1)^2 (\lambda + 2)} = \frac{(\lambda + 1)^2}{\lambda + 2}.
\end{aligned}$$

2. Ako je $D = 0$, tj. $\lambda = 1$ ili $\lambda = -2$, tada je sistem neodređen.

* Za $\lambda = 1$ imamo da je $D = D_x = D_y = D_z = 0$ pa sistem ima beskonačno mnogo rješenja.
Uvrštavajući $\lambda = 1$ u početni sistem, dobijamo da je sistem ekvivalentan samo jednoj jednačini $x + y + z = 1$. Budući da imamo jednu jednačinu a tri nepoznate, onda dvije nepoznate biramo proizvoljno. Neka su npr. $y = \alpha$ i $z = \beta$, $(\alpha, \beta \in \mathbb{R})$. Tada je $x = 1 - \alpha - \beta$, pa je rješenje sistema

$$(x, y, z) = (1 - \alpha - \beta, \alpha, \beta) \qquad (\alpha, \beta \in \mathbb{R}) .$$

** Za $\lambda = -2$ imamo da je $D = 0$ ali je $D_x = 9$ pa sistem nema rješenja.

Zadatak 9.11 *U zavisnosti od realnog parametra a diskutovati rješenja sistema*

$$\left.\begin{array}{r} (a-1)\,x + y - z = 2a-1 \\ (2a-5)\,x + 4y - 5z = a+2 \\ (2a-3)\,x + y + (a-3)\,z = a+1 \end{array}\right\}.$$

Rješenje:
Izračunajmo determinante D, D_x, D_y, D_z :

$$\begin{aligned}
D &= \begin{vmatrix} a-1 & 1 & -1 \\ 2a-5 & 4 & -5 \\ 2a-3 & 1 & a-3 \end{vmatrix} = 2a(a-2) \\
D_x &= \begin{vmatrix} 2a-1 & 1 & -1 \\ a+2 & 4 & -5 \\ a+1 & 1 & a-3 \end{vmatrix} = (a-2)\,(7a-5) \\
D_y &= \begin{vmatrix} a-1 & 2a-1 & -1 \\ 2a-5 & a+2 & -5 \\ 2a-3 & a+1 & a-3 \end{vmatrix} = -a\,(a-2)\,(3a-1) \\
D_z &= \begin{vmatrix} a-1 & 1 & 2a-1 \\ 2a-5 & 4 & a+2 \\ 2a-3 & 1 & a+1 \end{vmatrix} = -\,(a-2)\,(9a-5)\,.
\end{aligned}$$

Diskusija:

1. Ako je $D \neq 0$ tj $2a(a-2) \neq 0$ a to je slučaj kada je $a \neq 0$ i $a \neq 2$, tada je sistem saglasan i rješenja su data sa

$$\begin{aligned}
x &= \frac{D_x}{D} = \frac{(a-2)\,(7a-5)}{2a(a-2)} = \frac{7a-5}{2a} \\
y &= \frac{D_y}{D} = \frac{-a\,(a-2)\,(3a-1)}{2a(a-2)} = -\frac{3a-1}{2} \\
z &= \frac{D_z}{D} = \frac{-\,(a-2)\,(9a-5)}{2a(a-2)} = -\frac{9a-5}{2a}.
\end{aligned}$$

2. Ako je $D = 0$, tj. $a = 0$ ili $a = 2$, tada je sistem neodređen.

* Za $a = 2$ imamo da je $D = D_x = D_y = D_z = 0$ pa sistem ima beskonačno mnogo rješenja.
Uvrštavajući $a = 2$ u početni sistem, dobijamo da je sistem jednačina

$$\left.\begin{array}{r} x + y - z = 3 \\ -x + 4y - 5z = 4 \\ x + y - z = 3 \end{array}\right\}.$$

Ispitajmo saglasnost ovog sistema: Proširena matrica sistema je

$$A_p = \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & -5 & 4 \\ 1 & 1 & -1 & 3 \end{bmatrix}.$$

Odredimo njen rang:

$$\begin{aligned} A_p &= \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & -5 & 4 \\ 1 & 1 & -1 & 3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 5 & -6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{r} V_1 \; prepisana \\ V_1 + V_2 \\ V_1 - V_2 \end{array} \end{aligned}$$

Formirajmo novi sistem

$$\left. \begin{array}{r} x + y - z = 3 \\ 5y - 6z = 7 \end{array} \right\}.$$

Ovaj sistem ima tri nepoznate a dvije jednačine, pa jednu nepoznatu biramo proizvoljno. Neka je, recimo $z = \alpha, \alpha \in \mathbb{R}$. Sada, iz druge jednačine imamo

$$5y - 6\alpha = 7 \implies y = \frac{7 - 6\alpha}{5}.$$

Uvrštavajući $y = \frac{7-6\alpha}{5}$ i $z = \alpha$ u prvu jednačinu dobijamo

$$x + \frac{7 - 6\alpha}{5} - \alpha = 3 \implies x = \frac{8 - \alpha}{5}.$$

Dakle, rješenje sistema je

$$(x, y, z) = \left(\frac{8 - \alpha}{5}, \frac{7 - 6\alpha}{5}, \alpha \right)$$

** Za $a = 0$ imamo da je $D = 0$ ali je $D_x \neq 0$ pa sistem nema rješenja.

10 Limes funkcije

Zadatak 10.1

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{4x^2-3x+1}{2x^2-2x+3} &= \lim_{x\to\pm\infty}\frac{4x^2-3x+1:x^2}{2x^2-2x+3:x^2}=\lim_{x\to\pm\infty}\frac{4-\frac{3}{x}+\frac{1}{x^2}}{2-\frac{2}{x}+\frac{3}{x^2}}=\\ &= \frac{4-0+0}{2-0+0}=2.\end{aligned}$$

Zadatak 10.2

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4.$$

Zadatak 10.3

$$\lim_{x\to 5}\frac{x^2-7x+10}{x^2-9x+20}=\lim_{x\to 5}\frac{(x-5)(x-2)}{(x-5)(x-4)}=\lim_{x\to 5}\frac{x-2}{x-4}=\frac{5-2}{5-4}=3.$$

Zadatak 10.4

$$\begin{aligned}\lim_{x\to\infty}\left(x-\sqrt{x^2-10x}\right) &= \lim_{x\to\infty}\frac{x-\sqrt{x^2-10x}}{1}\cdot\frac{x+\sqrt{x^2-10x}}{x+\sqrt{x^2-10x}}=\\ &= \lim_{x\to\infty}\frac{x^2-\left(\sqrt{x^2-10x}\right)^2}{x+\sqrt{x^2-10x}}=\lim_{x\to\infty}\frac{x^2-x^2+10x}{x+\sqrt{x^2-10x}}=\\ &= \lim_{x\to\infty}\frac{10x}{x+\sqrt{x^2-10x}}\frac{:x}{:x}=\lim_{x\to\infty}\frac{10}{1+\frac{\sqrt{x^2-10x}}{x}}=\\ &= \lim_{x\to\infty}\frac{10}{1+\sqrt{\frac{x^2-10x}{x^2}}}=\lim_{x\to\infty}\frac{10}{1+\sqrt{1-\frac{10}{x}}}=\\ &= \frac{10}{1+\sqrt{1-0}}=\frac{10}{2}=5.\end{aligned}$$

Zadatak 10.5

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{x^2} &= \lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{x^2}\cdot\frac{\left(\sqrt[3]{1+x^2}\right)^2+\sqrt[3]{1+x^2}+1}{\left(\sqrt[3]{1+x^2}\right)^2+\sqrt[3]{1+x^2}+1}=\\ &= \lim_{x\to 0}\frac{\left(\sqrt[3]{1+x^2}\right)^3-1^3}{x^2\cdot\left(\sqrt[3]{1+x^2}\right)^2+\sqrt[3]{1+x^2}+1}=\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0}\frac{1+x^2-1}{x^2\cdot\left(\left(\sqrt[3]{1+x^2}\right)^2+\sqrt[3]{1+x^2}+1\right)} = \\
&= \lim_{x\to 0}\frac{x^2}{x^2\cdot\left(\left(\sqrt[3]{1+x^2}\right)^2+\sqrt[3]{1+x^2}+1\right)} = \\
&= \lim_{x\to 0}\frac{x^2}{x^2\cdot\left(\left(\sqrt[3]{1+x^2}\right)^2+\sqrt[3]{1+x^2}+1\right)} = \\
&= \lim_{x\to 0}\frac{1}{\sqrt[3]{1+x^2}^2+\sqrt[3]{1+x^2}+1} = \frac{1}{3}.
\end{aligned}
$$

Zadatak 10.6

$$
\begin{aligned}
\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} &= \begin{bmatrix} smjena \\ x=t^6 \\ x\to 1 \implies t\to 1\end{bmatrix} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \\
&= \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{2}{3}.
\end{aligned}
$$

Zadatak 10.7

$$
\begin{aligned}
\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2} &= \lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \\
&= \lim_{x\to 3}\frac{(x-3)\cdot\left(\sqrt{x+1}+2\right)}{\left(\sqrt{x+1}\right)^2-2^2} = \\
&= \lim_{x\to 3}\frac{(x-3)\cdot\left(\sqrt{x+1}+2\right)}{x+1-4} = \\
&= \lim_{x\to 3}\frac{(x-3)\cdot\left(\sqrt{x+1}+2\right)}{x-3} = \\
&= \lim_{x\to 3}\left(\sqrt{x+1}+2\right) = 4.
\end{aligned}
$$

Zadatak 10.8

$$
\begin{aligned}
\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49} &= \lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}\cdot\frac{2+\sqrt{x-3}}{2+\sqrt{x-3}} = \\
&= \lim_{x\to 7}\frac{2^2-\left(\sqrt{x-3}\right)^2}{(x^2-49)\left(2+\sqrt{x-3}\right)} = \\
&= \lim_{x\to 7}\frac{4-(x-3)}{(x^2-49)\left(2+\sqrt{x-3}\right)} =
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to 7}\frac{7-x}{(x-7)(x+7)\left(2+\sqrt{x-3}\right)}= \\
&= \lim_{x\to 7}\frac{-(x-7)}{(x-7)(x+7)\left(2+\sqrt{x-3}\right)}= \\
&= \lim_{x\to 7}\frac{1}{(x+7)\left(2+\sqrt{x-3}\right)}= \\
&= -\frac{1}{14\cdot 4}=-\frac{1}{56}.
\end{aligned}
$$

Zadatak 10.9

$$
\begin{aligned}
\lim_{x\to 4}\frac{x^2-16}{\sqrt{x+5}-3} &= \lim_{x\to 4}\frac{x^2-16}{\sqrt{x+5}-3}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}= \\
&= \lim_{x\to 4}\frac{\left(x^2-16\right)\left(\sqrt{x+5}+3\right)}{\left(\sqrt{x+5}\right)^2-3^2}= \\
&= \lim_{x\to 4}\frac{(x-4)(x+4)\left(\sqrt{x+5}+3\right)}{x+5-9}= \\
&= \lim_{x\to 4}\frac{(x-4)(x+4)\left(\sqrt{x+5}+3\right)}{x-4}= \\
&= \lim_{x\to 4}(x+4)\left(\sqrt{x+5}+3\right)=8\cdot 6=48.
\end{aligned}
$$

Zadatak 10.10

$$
\begin{aligned}
\lim_{x\to 3}\frac{x^3-27}{\sqrt{12-x}-3} &= \lim_{x\to 3}\frac{x^3-3^3}{\sqrt{12-x}-3}\cdot\frac{\sqrt{12-x}+3}{\sqrt{12-x}+3}= \\
&= \lim_{x\to 3}\frac{\left(x^3-3^3\right)\left(\sqrt{12-x}+3\right)}{\left(\sqrt{12-x}\right)^2-3^2}= \\
&= \lim_{x\to 3}\frac{(x-3)\left(x^2+3x+9\right)\left(\sqrt{12-x}+3\right)}{12-x-9}= \\
&= \lim_{x\to 3}\frac{(x-3)\left(x^2+3x+9\right)\left(\sqrt{12-x}+3\right)}{3-x}= \\
&= \lim_{x\to 3}\frac{(x-3)\left(x^2+3x+9\right)\left(\sqrt{12-x}+3\right)}{-(x-3)}= \\
&= -\lim_{x\to 3}\left(x^2+3x+9\right)\left(\sqrt{12-x}+3\right)= \\
&= -27\cdot 6=-162.
\end{aligned}
$$

Zadatak 10.11

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^{x} &= \lim_{x\to+\infty}\left(1+\frac{x+1}{x-1}-1\right)^{x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{x+1-x+1}{x-1}\right)^{x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x} = \\
&= e^{\lim\limits_{x\to+\infty}\frac{2x}{x-1}} = e^{2}.
\end{aligned}$$

Zadatak 10.12

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^{3x} &= \lim_{x\to+\infty}\left(1+\frac{x^2+5x+4}{x^2-3x+7}-1\right)^{3x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{x^2+5x+4-x^2+3x-7}{x^2-3x+7}\right)^{3x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^{3x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}\cdot\frac{8x-3}{x^2-3x+7}\cdot 3x} = \\
&= e^{\lim\limits_{x\to+\infty}\frac{8x-3}{x^2-3x+7}\cdot 3x} = e^{\lim\limits_{x\to+\infty}\frac{24x^2-9x}{x^2-3x+7}} = \\
&= e^{\lim\limits_{x\to+\infty}\frac{24-\frac{9}{x}}{1-\frac{3}{x}+\frac{7}{x^2}}} = e^{24}.
\end{aligned}$$

Zadatak 10.13

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x^2+x+1}{x^2-x+1}\right)^{2x} &= \lim_{x\to+\infty}\left(1+\frac{x^2+x+1}{x^2-x+1}-1\right)^{2x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{x^2+x+1-x^2+x-1}{x^2-x+1}\right)^{2x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{2x}{x^2-x+1}\right)^{2x} = \\
&= \lim_{x\to+\infty}\left(1+\frac{2x}{x^2-x+1}\right)^{\frac{x^2-x+1}{2x}\cdot\frac{2x}{x^2-x+1}\cdot 2x} = \\
&= e^{\lim\limits_{x\to+\infty}\frac{2x}{x^2-x+1}\cdot 2x} = e^{\lim\limits_{x\to+\infty}\frac{4x^2}{x^2-x+1}} = e^{4}.
\end{aligned}$$

Zadatak 10.14

$$\lim_{x\to 0}\frac{\sin 7x}{x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot 7=7\cdot\lim_{x\to 0}\frac{\sin 7x}{7x}=7\cdot 1=7.$$

Zadatak 10.15

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 6x}{\sin 3x} &= \lim_{x\to 0}\frac{\frac{\sin 6x}{6x}\cdot 6x}{\frac{\sin 3x}{3x}\cdot 3x}=\lim_{x\to 0}\frac{\frac{\sin 6x}{6x}\cdot 6}{\frac{\sin 3x}{3x}\cdot 3}=\\ &= \frac{\lim\limits_{x\to 0}\frac{\sin 6x}{6x}\cdot 6}{\lim\limits_{x\to 0}\frac{\sin 3x}{3x}\cdot 3}=\frac{6}{3}=2.\end{aligned}$$

Zadatak 10.16

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{\sqrt{x+4}-2} &= \lim_{x\to 0}\frac{\sin 3x}{\sqrt{x+4}-2}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}=\\ &= \lim_{x\to 0}\frac{\sin 3x\left(\sqrt{x+4}+2\right)}{\left(\sqrt{x+4}\right)^2-2^2}=\\ &= \lim_{x\to 0}\frac{\sin 3x\left(\sqrt{x+4}+2\right)}{x+4-4}=\\ &= \lim_{x\to 0}\frac{\sin 3x\left(\sqrt{x+4}+2\right)}{x}=\\ &= \lim_{x\to 0}\frac{\sin 3x\left(\sqrt{x+4}+2\right)}{3x}\cdot 3=\\ &= 3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\left(\sqrt{x+4}+2\right)=\\ &= 3\cdot 1\cdot 4=12.\end{aligned}$$

Zadatak 10.17

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sin 2x} &= \lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sin 2x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\\ &= \lim_{x\to 0}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{\sin 2x\left(\sqrt{x+1}+1\right)}=\\ &= \lim_{x\to 0}\frac{\left(\sqrt{x+1}\right)^2-1^2}{\sin 2x\left(\sqrt{x+1}+1\right)}=\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0}\frac{x+1-1}{\sin 2x\left(\sqrt{x+1}+1\right)} = \\
&= \lim_{x\to 0}\frac{x}{\sin 2x\left(\sqrt{x+1}+1\right)} = \\
&= \lim_{x\to 0}\frac{x}{\frac{\sin 2x}{2x}\cdot 2x\cdot\left(\sqrt{x+1}+1\right)} = \\
&= \lim_{x\to 0}\frac{1}{\frac{\sin 2x}{2x}\cdot 2\cdot\left(\sqrt{x+1}+1\right)} = \\
&= \frac{1}{\lim\limits_{x\to 0}\frac{\sin 2x}{2x}}\cdot\frac{1}{2\cdot\left(\sqrt{x+1}+1\right)} = \\
&= \frac{1}{1}\cdot\frac{1}{2\cdot 2} = \frac{1}{4}.
\end{aligned}
$$

Zadatak 10.18

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+9}-3} &= \lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+9}-3}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \\
&= \lim_{x\to 0}\frac{\sin 4x\left(\sqrt{x+9}+3\right)}{\left(\sqrt{x+9}\right)^2-3^2} = \\
&= \lim_{x\to 0}\frac{\sin 4x\left(\sqrt{x+9}+3\right)}{x+9-9} = \\
&= \lim_{x\to 0}\frac{\sin 4x\left(\sqrt{x+9}+3\right)}{x} = \\
&= \lim_{x\to 0}\frac{\sin 4x\left(\sqrt{x+9}+3\right)}{4x}\cdot 4 = \\
&= 4\cdot\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\lim_{x\to 0}\left(\sqrt{x+9}+3\right) = \\
&= 4\cdot 1\cdot 6 = 24.
\end{aligned}
$$

Zadatak 10.19

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x} &= \lim_{x\to 0}\frac{\frac{\sin 5x}{5x}\cdot 5x-\frac{\sin 3x}{3x}\cdot 3x}{\frac{\sin x}{x}\cdot x}= \\
&= \lim_{x\to 0}\frac{x\left(5\frac{\sin 5x}{5x}-3\frac{\sin 3x}{3x}\right)}{x\frac{\sin x}{x}}= \\
&= \lim_{x\to 0}\frac{5\frac{\sin 5x}{5x}-3\frac{\sin 3x}{3x}}{\frac{\sin x}{x}}= \\
&= \frac{5\lim\limits_{x\to 0}\frac{\sin 5x}{5x}-3\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}= \\
&= \frac{5-3}{1}=2
\end{aligned}$$

Zadatak 10.20

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{2\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{x\cdot x}= \\
&= 2\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{\frac{x}{2}\cdot\frac{x}{2}\cdot 4}= \\
&= \frac{2}{4}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}.
\end{aligned}$$

Zadatak 10.21

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2} &= \lim_{x\to 0}\frac{2\sin^2\frac{x^2}{2}}{x^2\sin x^2}=\lim_{x\to 0}\frac{2\sin\frac{x^2}{2}\cdot\sin\frac{x^2}{2}}{\frac{x^2}{2}\cdot\frac{x^2}{2}\cdot 4\cdot\frac{\sin x^2}{x^2}}= \\
&= \frac{2}{4}\cdot\lim_{x\to 0}\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x^2}{2}}{\frac{x^2}{2}}\cdot\frac{1}{\lim\limits_{x\to 0}\frac{\sin x^2}{x^2}}= \\
&= \frac{1}{2}.
\end{aligned}$$

Zadatak 10.22

$$\begin{aligned}
\lim_{x\to+\infty}x\cdot\left[\ln(x+1)-\ln(x-1)\right] &= \lim_{x\to+\infty}x\cdot\ln\left(\frac{x+1}{x-1}\right)= \\
&= \lim_{x\to+\infty}\ln\left(\frac{x+1}{x-1}\right)^x= \\
&= \ln\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x=
\end{aligned}$$

$$
\begin{aligned}
&= \ln \lim_{x\to+\infty}\left(1+\frac{x+1}{x-1}-1\right)^{x} = \\
&= \ln \lim_{x\to+\infty}\left(1+\frac{x+1-x+1}{x-1}\right)^{x} = \\
&= \ln \lim_{x\to+\infty}\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x} = \\
&= \ln e^{\lim\limits_{x\to+\infty}\frac{2x}{x-1}} = \ln e^{2} = 2\ln e = 2.
\end{aligned}
$$

Zadatak 10.23

$$
\lim_{x\to+\infty} 3x\cdot\left[\ln\left(x^2+5x+4\right)-\ln\left(x^2-3x+7\right)\right] =
$$

$$
\begin{aligned}
&= \lim_{x\to+\infty} 3x\cdot\ln\left(\frac{x^2+5x+4}{x^2-3x+7}\right) = \\
&= \lim_{x\to+\infty} \ln\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^{3x} = \\
&= \ln \lim_{x\to+\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^{3x} = \\
&= \ln \lim_{x\to+\infty}\left(1+\frac{x^2+5x+4-x^2+3x-7}{x^2-3x+7}\right)^{3x} = \\
&= \ln \lim_{x\to+\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^{3x} = \\
&= \ln \lim_{x\to+\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}\cdot\frac{8x-3}{x^2-3x+7}\cdot 3x} = \\
&= \ln e^{\lim\limits_{x\to+\infty}\frac{8x-3}{x^2-3x+7}\cdot 3x} = \ln e^{\lim\limits_{x\to+\infty}\frac{24x^2-9x}{x^2-3x+7}} = \\
&= \ln e^{\lim\limits_{x\to+\infty}\frac{24-\frac{9}{x}}{1-\frac{3}{x}+\frac{7}{x^2}}} = \ln e^{24} = 24\ln e = 24.
\end{aligned}
$$

Zadatak 10.24

$$
\lim_{x\to+\infty} 3x\cdot\left[\ln(x+1)+\ln(x-4)-\ln\left(x^2+5x+6\right)\right] =
$$

$$\begin{aligned}
&= \lim_{x\to+\infty} 3x\cdot\ln\left(\frac{(x+1)(x-4)}{x^2+5x+6}\right)= \\
&= \lim_{x\to+\infty} 3x\cdot\ln\left(\frac{x^2-3x-4}{x^2+5x+6}\right)= \\
&= \lim_{x\to+\infty} \ln\left(\frac{x^2-3x-4}{x^2+5x+6}\right)^{3x}= \\
&= \ln\lim_{x\to+\infty}\left(1+\frac{x^2-3x-4}{x^2+5x+6}-1\right)^{3x}= \\
&= \ln\lim_{x\to+\infty}\left(1+\frac{x^2-3x-4-x^2-5x-6}{x^2+5x+6}\right)^{3x}= \\
&= \ln\lim_{x\to+\infty}\left(1+\frac{-8x-10}{x^2+5x+6}\right)^{3x}= \\
&= \ln\lim_{x\to+\infty}\left(1+\frac{-8x-10}{x^2+5x+6}\right)^{\frac{x^2+5x+6}{-8x-10}\cdot\frac{-8x-10}{x^2+5x+6}\cdot 3x}= \\
&= \ln e^{\lim\limits_{x\to+\infty}\frac{-8x-10}{x^2+5x+6}\cdot 3x}=\ln e^{\lim\limits_{x\to+\infty}\frac{-24x^2-30x}{x^2+5x+6}}=\ln e^{-24}=-24.
\end{aligned}$$

Zadatak 10.25

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x+3}{x+2}\right)^{\frac{x^2+x+1}{x+1}} &= \lim_{x\to+\infty}\left(1+\frac{x+3}{x+2}-1\right)^{\frac{x^2+x+1}{x+1}}= \\
&= \lim_{x\to+\infty}\left(1+\frac{x+3-x-2}{x+2}\right)^{\frac{x^2+x+1}{x+1}}= \\
&= \lim_{x\to+\infty}\left(1+\frac{1}{x+2}\right)^{\frac{x^2+x+1}{x+1}}= \\
&= \lim_{x\to+\infty}\left(1+\frac{1}{x+2}\right)^{(x+2)\cdot\frac{1}{x+2}\cdot\frac{x^2+x+1}{x+1}}= \\
&= e^{\lim\limits_{x\to+\infty}\frac{1}{x+2}\cdot\frac{x^2+x+1}{x+1}}=e^{\lim\limits_{x\to+\infty}\frac{x^2+x+1}{x^2+3x+2}}=e^1=e.
\end{aligned}$$

Zadatak 10.26

$$\begin{aligned}
\lim_{x\to2}\left(\frac{x}{2}\right)^{\frac{1}{x-2}} &= \left[\begin{array}{c} smjena \\ t=\frac{1}{x-2}\implies x=2+\frac{1}{t} \\ x\to2\implies t\to\infty\end{array}\right]=\lim_{x\to+\infty}\left(1+\frac{1}{2t}\right)^t= \\
&= \lim_{x\to+\infty}\left(1+\frac{1}{2t}\right)^{2t\cdot\frac{1}{2}}=e^{\frac{1}{2}}.
\end{aligned}$$

Zadatak 10.27

$$\begin{aligned}
\lim_{x\to 0}\frac{e^{ax}-1}{x} &= \begin{bmatrix} smjena \\ t=e^{ax}-1 \implies e^{ax}=t+1 \\ \implies ax=\ln(1+t) \implies x=\frac{1}{a}\ln(1+t) \\ x\to 0 \implies t\to 0 \end{bmatrix} = \\
&= \lim_{t\to 0}\frac{t}{\frac{1}{a}\ln(1+t)} = a\cdot\lim_{t\to 0}\frac{t}{\ln(1+t)} = a\cdot\lim_{t\to 0}\frac{1}{\frac{\ln(1+t)}{t}} = \\
&= a\cdot\frac{1}{\lim\limits_{t\to 0}\frac{\ln(1+t)}{t}} = a\cdot\frac{1}{\lim\limits_{t\to 0}\frac{1}{t}\cdot\ln(1+t)} = \\
&= a\cdot\frac{1}{\lim\limits_{t\to 0}\ln(1+t)^{\frac{1}{t}}} = a\cdot\frac{1}{\ln\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}}} = \\
&= a\cdot\frac{1}{\ln e} = a.
\end{aligned}$$

Zadatak 10.28 *Izračunati lijevi i desni limes funkcije*

$$f(x)=\frac{3x-x^2}{x-4}$$

u tački $x=4$.

$$\begin{aligned}
L(4) &= \lim_{x\to 4^-}\frac{3x-x^2}{x-4} = \begin{bmatrix} smjena \\ x=4-\varepsilon \\ \varepsilon>0,\varepsilon\to 0 \end{bmatrix} = \lim_{\varepsilon\to 0}\frac{3(4-\varepsilon)-(4-\varepsilon)^2}{4-\varepsilon-4} = \\
&= \lim_{\varepsilon\to 0}\frac{12-3\varepsilon-16+8\varepsilon-\varepsilon^2}{-\varepsilon} = \lim_{\varepsilon\to 0}\frac{-\varepsilon^2+5\varepsilon-4}{-\varepsilon} = \\
&= \lim_{\varepsilon\to 0}\left(-\varepsilon-5+\frac{4}{\varepsilon}\right) = -0-5+\lim_{\varepsilon\to 0}\frac{4}{\varepsilon} = +\infty. \\
D(4) &= \lim_{x\to 4^+}\frac{3x-x^2}{x-4} = \begin{bmatrix} smjena \\ x=4+\varepsilon \\ \varepsilon>0,\varepsilon\to 0 \end{bmatrix} = \lim_{\varepsilon\to 0}\frac{3(4+\varepsilon)-(4+\varepsilon)^2}{4+\varepsilon-4} = \\
&= \lim_{\varepsilon\to 0}\frac{12+3\varepsilon-16-8\varepsilon-\varepsilon^2}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{-\varepsilon^2-5\varepsilon-4}{\varepsilon} = \\
&= \lim_{\varepsilon\to 0}\left(-\varepsilon-5-\frac{4}{\varepsilon}\right) = -0-5-\lim_{\varepsilon\to 0}\frac{4}{\varepsilon} = -\infty.
\end{aligned}$$

Zadatak 10.29 *Izračunati lijevi i desni limes funkcije*

$$\ln\left(x^2-1\right)$$

u tački $x = -1$.

$$\begin{aligned}
L(-1) &= \lim_{x\to -1^-} \ln\left(x^2-1\right) = \ln \lim_{x\to -1^-}\left(x^2-1\right) = \begin{bmatrix} smjena \\ x=-1-\varepsilon \\ \varepsilon>0, \varepsilon\to 0 \end{bmatrix} = \\
&= \ln\lim_{\varepsilon\to 0}\left((-1-\varepsilon)^2-1\right) = \ln\lim_{\varepsilon\to 0}\left(1+2\varepsilon+\varepsilon^2-1\right) = \\
&= \ln\lim_{\varepsilon\to 0}\left(2\varepsilon+\varepsilon^2\right) = -\infty. \\
D(1) &= \lim_{x\to 1^+} \ln\left(x^2-1\right) = \ln \lim_{x\to 1^+}\left(x^2-1\right) = \begin{bmatrix} smjena \\ x=1+\varepsilon \\ \varepsilon>0, \varepsilon\to 0 \end{bmatrix} = \\
&= \ln\lim_{\varepsilon\to 0}\left((1+\varepsilon)^2-1\right) = \ln\lim_{\varepsilon\to 0}\left(1+2\varepsilon+\varepsilon^2-1\right) = \\
&= \ln\lim_{\varepsilon\to 0}\left(2\varepsilon+\varepsilon^2\right) = -\infty.
\end{aligned}$$

11 Izvod funkcije

Zadatak 11.1 *Po definiciji naći izvod funkcije*

$$f(x) = 3x^2 - 2x - 1.$$

Rješenje:
Kako je

$$\begin{aligned} f(x) &= 3x^2 - 2x - 1 \\ f(x+h) &= 3(x+h)^2 - 2(x+h) - 1 = \\ &= 3\left(x^2 + 2xh + h^2\right) - 2x - 2h - 1 = \\ &= 3x^2 + 6xh + 3h^2 - 2x - 2h - 1 \end{aligned}$$

tada je

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f\left(x+h\right) - f(x)}{h} = \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 1 - 3x^2 + 2x + 1}{h} = \\ &= \lim_{h\to 0} \frac{6xh + 3h^2 - 2h}{h} = \lim_{h\to 0}\left(6x + 3h - 2\right) = 6x - 2. \end{aligned}$$

Zadatak 11.2 *Po definiciji naći izvod funkcije*

$$f(x) = \sqrt{3x+1}.$$

Rješenje:
Kako je

$$\begin{aligned} f(x) &= \sqrt{3x+1} \\ f(x+h) &= \sqrt{3(x+h)+1} = \sqrt{3x+3h+1} \end{aligned}$$

tada je

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f\left(x+h\right) - f(x)}{h} = \\ &= \lim_{h\to 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} = \\ &= \lim_{h\to 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3x+3h+1} + \sqrt{3x+1}}{\sqrt{3x+3h+1} + \sqrt{3x+1}} = \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0}\frac{\left(\sqrt{3x+3h+1}\right)^2-\left(\sqrt{3x+1}\right)^2}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}=\\
&= \lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}=\\
&= \lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}=\\
&= \lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}=\frac{3}{2\sqrt{3x+1}}.
\end{aligned}
$$

Zadatak 11.3 *Po definiciji naći izvod funkcije*

$$f(x)=e^{4x-3}.$$

Rješenje:
Kako je

$$
\begin{aligned}
f(x) &= e^{4x-3}\\
f(x+h) &= e^{4(x+h)-3}=e^{4x+4h-3}
\end{aligned}
$$

tada je

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f\,(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{4x+4h-3}-e^{4x-3}}{h}=\\
&= \lim_{h\to 0}\frac{e^{4x-3}\left(e^{4h}-1\right)}{h}=e^{4x-3}\cdot\lim_{h\to 0}\frac{e^{4h}-1}{h}=\\
&= e^{4x-3}\cdot 4=4e^{4x-3}
\end{aligned}
$$

jer je

$$
\begin{aligned}
\lim_{x\to 0}\frac{e^{ax}-1}{x} &= \left[\begin{array}{c} smjena\\ t=e^{ax}-1\implies e^{ax}=t+1\\ \implies ax=\ln(1+t)\implies x=\frac{1}{a}\ln(1+t)\\ x\to 0\implies t\to 0\end{array}\right]=\\
&= \lim_{t\to 0}\frac{t}{\frac{1}{a}\ln(1+t)}=a\cdot\lim_{t\to 0}\frac{t}{\ln(1+t)}=a\cdot\lim_{t\to 0}\frac{1}{\frac{\ln(1+t)}{t}}=\\
&= a\cdot\frac{1}{\lim\limits_{t\to 0}\frac{\ln(1+t)}{t}}=a\cdot\frac{1}{\lim\limits_{t\to 0}\frac{1}{t}\cdot\ln(1+t)}=\\
&= a\cdot\frac{1}{\lim\limits_{t\to 0}\ln(1+t)^{\frac{1}{t}}}=a\cdot\frac{1}{\ln\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}}}=\\
&= a\cdot\frac{1}{\ln e}=a.
\end{aligned}
$$

Zadatak 11.4 *Po definiciji naći izvod funkcije*

$$f(x) = \ln(7x).$$

Rješenje:
Kako je

$$\begin{aligned} f(x) &= \ln(7x) \\ f(x+h) &= \ln(7(x+h)) = \ln(7x+7h) \end{aligned}$$

tada je

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(7x+7h)-\ln(7x)}{h} = \\ &= \lim_{h\to 0}\frac{\ln\frac{7x+7h}{7x}}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0}\frac{1}{h}\cdot\ln\left(1+\frac{h}{x}\right) = \\ &= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \\ &= \ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}\cdot\frac{h}{x}\cdot\frac{1}{h}} = \ln e^{\lim\limits_{h\to 0}\frac{1}{x}} = \ln e^{\frac{1}{x}} = \frac{1}{x}. \end{aligned}$$

Zadatak 11.5 *Po definiciji naći izvod funkcije*

$$f(x) = \sin(2x).$$

Rješenje:
Kako je

$$\begin{aligned} f(x) &= \sin(2x) \\ f(x+h) &= \sin(2(x+h)) = \sin(2x+2h) \end{aligned}$$

tada je

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(2x+2h)-\sin(2x)}{h} = \\ &= \lim_{h\to 0}\frac{2\sin\frac{2x+2h-2x}{2}\cos\frac{2x+2h+2x}{2}}{h} = \\ &= \lim_{h\to 0}\frac{2\sin h\cos(2x+h)}{h} = 2\cdot\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\cos(2x+h) = \\ &= 2\cos(2x). \end{aligned}$$

Zadatak 11.6 *Naći izvod funkcije*

$$y = \frac{3}{2}x^3 - \frac{1}{4}x^2 - x - \frac{6}{5}.$$

Rješenje:

$$y' = \frac{3}{2}3x^2 - \frac{1}{4}2x - 1 = \frac{9}{2}x^2 - \frac{1}{2}x - 1.$$

Zadatak 11.7 *Naći izvod funkcije*

$$y = -\frac{4}{5}\sqrt[3]{x^5} - \frac{1}{7}x^2 + x\sqrt{x} + 4.$$

Rješenje:
Kako je

$$y = -\frac{4}{5}x^{\frac{5}{3}} - \frac{1}{7}x^2 + x^{\frac{3}{2}} + 4$$

tada je

$$y' = -\frac{4}{5}\frac{5}{3}x^{\frac{2}{3}} - \frac{1}{7}2x + \frac{3}{2}x^{\frac{1}{2}} = -\frac{4}{3}x^{\frac{2}{3}} - \frac{2}{7}x + \frac{3}{2}x^{\frac{1}{2}}.$$

Zadatak 11.8 *Naći izvod funkcije*

$$y = x^3 \sin x.$$

Rješenje:
Neka je

$$\begin{aligned} f &= x^3 \Longrightarrow f' = 3x^2 \\ g &= \sin x \Longrightarrow g' = \cos x \end{aligned}$$

pa primjenom pravila za izvod proizvoda, dobijamo

$$y' = f'g + g'f = 3x^2 \sin x + x^3 \cos x = x^2 \left(3 \sin x + x \cos x\right).$$

Zadatak 11.9 *Naći izvod funkcije*

$$y = \frac{x^2 + 1}{x^2 - 1}.$$

Rješenje:
Neka je

$$\begin{aligned} f &= x^2+1 \Longrightarrow f' = 2x \\ g &= x^2-1 \Longrightarrow g' = 2x \end{aligned}$$

pa primjenom pravila za izvod količnika, dobijamo

$$\begin{aligned} y' &= \frac{f'g-g'f}{g^2} = \frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2-1\right)^2} = \\ &= \frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2} = \frac{-4x}{\left(x^2-1\right)^2}. \end{aligned}$$

Zadatak 11.10 *Naći izvod funkcije*

$$y = \frac{\sin x - \cos x}{\sin x + \cos x}.$$

Rješenje:
Neka je

$$\begin{aligned} f &= \sin x - \cos x \Longrightarrow f' = \cos x + \sin x \\ g &= \sin x + \cos x \Longrightarrow g' = \cos x - \sin x \end{aligned}$$

pa primjenom pravila za izvod količnika, dobijamo

$$\begin{aligned} y' &= \frac{f'g-g'f}{g^2} = \\ &= \frac{(\cos x+\sin x)(\sin x+\cos x)-(\cos x-\sin x)(\sin x-\cos x)}{(\sin x+\cos x)^2} \\ &= \frac{\cos x\sin x+\cos^2 x+\sin^2 x+\sin x\cos x-\cos x\sin x}{(\sin x+\cos x)^2} \\ &\quad+\frac{\cos^2 x+\sin^2 x-\sin x\cos x}{(\sin x+\cos x)^2} \\ &= \frac{2\cos^2 x+2\sin^2 x}{\sin^2 x+2\sin x\cos x+\cos^2 x} = \frac{2\left(\cos^2 x+\sin^2 x\right)}{\sin^2 x+\cos^2 x+2\sin x\cos x} \\ &= \frac{2}{1+\sin 2x}. \end{aligned}$$

Zadatak 11.11 *Naći izvod funkcije*

$$y = \left(1-5x^2\right)^{10}.$$

Rješenje:
Kako je

$$y = \left(1-5x^2\right)^{10} = \left[\begin{array}{c} smjena \\ u = 1-5x^2 \implies u' = -10x \end{array}\right] = u^{10}$$

tada je

$$y' = 10u^9 \cdot u' = 10\left(1-5x^2\right)^9 \cdot (-10x) = -100x\left(1-5x^2\right)^9.$$

Zadatak 11.12 *Naći izvod funkcije*

$$y = \sqrt[3]{\left(1+x^3\right)^2}.$$

Rješenje:
Kako je

$$y = \sqrt[3]{\left(1+x^3\right)^2} = \left(1+x^3\right)^{\frac{2}{3}} = \left[\begin{array}{c} smjena \\ u = 1+x^3 \implies u' = 3x^2 \end{array}\right] = u^{\frac{2}{3}}$$

tada je

$$y' = \frac{2}{3}u^{-\frac{1}{3}} \cdot u' = \frac{2}{3}\left(1+x^3\right)^{-\frac{1}{3}} \cdot 3x^2 = \frac{2x^2}{\sqrt[3]{1+x^3}}$$

Zadatak 11.13 *Naći izvod funkcije*

$$y = e^{4x^2-x-2}.$$

Rješenje:
Kako je

$$y = e^{4x^2-x-2} = \left[\begin{array}{c} smjena \\ u = 4x^2-x-2 \implies u' = 8x-1 \end{array}\right] = e^u$$

tada je

$$y' = e^u u' = e^{4x^2-x-2}(8x-1).$$

Zadatak 11.14 *Naći izvod funkcije*

$$y = \ln\left(x^2+1\right).$$

Rješenje:
Kako je

$$y = \ln\left(x^2+1\right) = \left[\begin{array}{c} smjena \\ u = x^2+1 \implies u' = 2x \end{array}\right] = \ln u$$

tada je

$$y' = \frac{1}{u}\cdot u' = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}.$$

Zadatak 11.15 *Naći izvod funkcije*

$$y = \ln\frac{x^2+1}{x^2-1}.$$

Rješenje:
Kako je

$$y = \ln\frac{x^2+1}{x^2-1} = \left[\begin{array}{c} smjena \\ u = \frac{x^2+1}{x^2-1} \implies u' = \frac{-4x}{(x^2-1)^2} \end{array}\right] = \ln u$$

tada je

$$\begin{aligned} y' &= \frac{1}{u}\cdot u' = \frac{1}{\frac{x^2+1}{x^2-1}}\cdot\frac{-4x}{\left(x^2-1\right)^2} = \\ &= \frac{x^2-1}{x^2+1}\cdot\frac{-4x}{\left(x^2-1\right)^2} = \frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)} = \\ &= \frac{-4x}{x^4-1}. \end{aligned}$$

Zadatak 11.16 *Naći prvi i drugi izvod funkcije*

$$y = \frac{3x-x^2}{x-4}$$

Rješenje:
Neka je

$$\begin{aligned} f &= 3x - x^2 \implies f' = 3-2x \\ g &= x-4 \implies g' = 1 \end{aligned}$$

pa primjenom pravila za izvod količnika, dobijamo

$$
\begin{aligned}
y' &= \frac{f'g - g'f}{g^2} \\
&= \frac{(3-2x)(x-4) - 1 \cdot (3x - x^2)}{(x-4)^2} = \\
&= \frac{3x - 12 - 2x^2 + 8x - 3x + x^2}{(x-4)^2} = \\
&= \frac{-x^2 + 8x - 12}{(x-4)^2}
\end{aligned}
$$

Neka je

$$
\begin{aligned}
f &= -x^2 + 8x - 12 \implies f' = -2x + 8 \\
g &= (x-4)^2 \implies g' = 2(x-4)
\end{aligned}
$$

pa primjenom pravila za izvod količnika, dobijamo
Drugi izvod funkcije je

$$
\begin{aligned}
y'' &= (y')' = \frac{f'g - g'f}{g^2} = \\
&= \frac{(-2x+8)(x-4)^2 - 2(x-4)(-x^2 + 8x - 12)}{\left((x-4)^2\right)^2} = \\
&= \frac{(x-4) \cdot [(-2x+8)(x-4) - 2(-x^2 + 8x - 12)]}{(x-4)^4} = \\
&= \frac{-2x^2 + 8x + 8x - 32 + 2x^2 - 16x + 24}{(x-4)^3} = \\
&= \frac{-8}{(x-4)^3}.
\end{aligned}
$$

Zadatak 11.17 *Naći prvi i drugi izvod funkcije*

$$
y = \frac{x^2 - 2x + 1}{x^2 + 1}
$$

Rješenje:
Neka je

$$
\begin{aligned}
f &= x^2 - 2x + 1 \implies f' = 2x - 2 \\
g &= x^2 + 1 \implies g' = 2x
\end{aligned}
$$

pa primjenom pravila za izvod količnika, dobijamo

$$\begin{aligned} y' &= \frac{f'g - g'f}{g^2} = \frac{(2x-2)(x^2+1) - 2x(x^2-2x+1)}{(x^2+1)^2} = \\ &= \frac{2x^3 + 2x - 2x^2 - 2 - 2x^3 + 4x^2 - 2x}{(x^2+1)^2} = \\ &= \frac{2x^2-2}{(x^2+1)^2} \end{aligned}$$

Neka je

$$\begin{aligned} f &= 2x^2 - 2 \Longrightarrow f' = 4x \\ g &= \left(x^2+1\right)^2 \Longrightarrow g' = 2\left(x^2+1\right) \cdot 2x = 4x\left(x^2+1\right) \end{aligned}$$

pa primjenom pravila za izvod količnika, dobijamo
Drugi izvod funkcije je

$$\begin{aligned} y'' &= \frac{f'g - g'f}{g^2} = \\ &= \frac{4x(x^2+1)^2 - 4x(x^2+1)(2x^2-2)}{\left((x^2+1)^2\right)^2} = \\ &= \frac{4x(x^2+1)^2 - (2x^2-2) \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4} = \\ &= \frac{(x^2+1)\left[4x(x^2+1) - 4x(2x^2-2)\right]}{(x^2+1)^4} = \\ &= \frac{4x^3 + 4x - 8x^3 + 8x}{(x^2+1)^3} = \frac{-4x^3 + 12x}{(x^2+1)^3} = \\ &= \frac{-4x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

Zadatak 11.18 *Naći prvi i drugi izvod funkcije*

$$y = \frac{x^3}{x^2-3}.$$

Neka je

$$\begin{aligned} f &= x^3 \Longrightarrow f' = 3x^2 \\ g &= x^2 - 3 \Longrightarrow g' = 2x \end{aligned}$$

pa primjenom pravila za izvod količnika, dobijamo da je prvi izvod funkcije

$$\begin{aligned}
y' &= \frac{f'g-g'f}{g^2}= \\
&= \frac{3x^2\left(x^2-3\right)-2x\cdot x^3}{\left(x^2-3\right)^2}= \\
&= \frac{3x^4-9x^2-2x^4}{\left(x^2-3\right)^2}= \\
&= \frac{x^4-9x^2}{\left(x^2-3\right)^2}
\end{aligned}$$

Neka je

$$\begin{aligned}
f &= x^4-9x^2 \Longrightarrow f'=4x^3-18x \\
g &= \left(x^2-3\right)^2 \Longrightarrow g'=2\left(x^2-3\right)\cdot 2x
\end{aligned}$$

pa primjenom pravila za izvod količnika, dobijamo da je drugi izvod funkcije

$$\begin{aligned}
y'' &= \frac{f'g-g'f}{g^2}= \\
&= \frac{\left(4x^3-18x\right)\left(x^2-3\right)^2-2\left(x^2-3\right)\cdot 2x\cdot\left(x^4-9x^2\right)}{\left(\left(x^2-3\right)^2\right)^2}= \\
&= \frac{\left(x^2-3\right)\left[\left(4x^3-18x\right)\left(x^2-3\right)-4x\left(x^4-9x^2\right)\right]}{\left(x^2-3\right)^4}= \\
&= \frac{4x^5-12x^3-18x^3+54x-4x^5+36x^3}{\left(x^2-3\right)^3}= \\
&= \frac{6x^3+54x}{\left(x^2-3\right)^3}
\end{aligned}$$

Zadatak 11.19 *Naći prvi i drugi izvod funkcije*

$$y=e^{2x}\sin 3x.$$

Rješenje:

$$\begin{aligned}
y' &= 2e^{2x}\sin 3x+e^{2x}\cos 3x\cdot 3=2e^{2x}\sin 3x+3e^{2x}\cos 3x \\
y'' &= 2\cdot 2e^{2x}\sin 3x+2\cdot e^{2x}\cos 3x\cdot 3+3\cdot 2e^{2x}\cos 3x+3e^{2x}\left(-\sin 3x\right)\cdot 3= \\
&= 4e^{2x}\sin 3x+6e^{2x}\cos 3x+6e^{2x}\cos 3x-9e^{2x}\sin 3x= \\
&= 12e^{2x}\cos 3x-5e^{2x}\sin 3x=e^{2x}\left(12\cos 3x-5\sin 3x\right).
\end{aligned}$$

Zadatak 11.20 *Naći prvi i drugi izvod funkcije*

$$y = x \arccos x - \sqrt{1-x^2}$$

Rješenje:

$$\begin{aligned}
y' &= 1 \cdot \arccos x + x \cdot \frac{-1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \\
&= \arccos x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \\
&= \arccos x. \\
y'' &= -\frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

Logaritamski izvod

Zadatak 11.21 *Naći izvod funkcije*

$$y = x^x.$$

Rješenje:
Logaritmovanjem lijeve i desne strane dobijamo

$$\ln y = \ln x^x \implies \ln y = x \cdot \ln x$$

pa je

$$\begin{aligned}
(\ln y)' &= (x \cdot \ln x)' \\
\frac{1}{y} \cdot y' &= 1 \cdot \ln x + x \cdot \frac{1}{x} \\
\frac{1}{y} \cdot y' &= \ln x + 1 \\
y' &= y\,(\ln x + 1) \\
y' &= x^x\,(\ln x + 1)\,.
\end{aligned}$$

Zadatak 11.22 *Naći izvod funkcije*

$$y = x^{x^2}.$$

Rješenje:
Logaritmovanjem lijeve i desne strane dobijamo

$$\ln y = \ln x^{x^2} \implies \ln y = x^2 \cdot \ln x$$

pa je

$$\begin{aligned}
(\ln y)' &= \left(x^2 \cdot \ln x\right)' \\
\frac{1}{y} \cdot y' &= 2x \cdot \ln x + x^2 \cdot \frac{1}{x} \\
\frac{1}{y} \cdot y' &= 2x \ln x + x \\
y' &= y\,(2x \ln x + x) \\
y' &= x^{x^2}\,(2x \ln x + x)\,.
\end{aligned}$$

Zadatak 11.23 *Naći izvod funkcije*

$$y = \left(\frac{x}{x+1}\right)^x.$$

Rješenje:
Logaritmovanjem lijeve i desne strane dobijamo

$$\ln y = \ln\left(\frac{x}{x+1}\right)^x \implies \ln y = x \cdot \ln\left(\frac{x}{x+1}\right)$$

pa je

$$\begin{aligned}
(\ln y)' &= \left(x \cdot \ln\left(\frac{x}{x+1}\right)\right)' \\
\frac{1}{y} \cdot y' &= 1 \cdot \ln\left(\frac{x}{x+1}\right) + x \cdot \frac{1}{\frac{x}{x+1}} \cdot \left(\frac{x}{x+1}\right)' \\
\frac{1}{y} \cdot y' &= \ln\left(\frac{x}{x+1}\right) + x \cdot \frac{x+1}{x} \cdot \frac{x+1-x}{(x+1)^2} \\
\frac{1}{y} \cdot y' &= \ln\left(\frac{x}{x+1}\right) + x \cdot \frac{x+1}{x} \cdot \frac{1}{(x+1)^2} \\
\frac{1}{y} \cdot y' &= \ln\left(\frac{x}{x+1}\right) + \frac{1}{x+1} \\
y' &= y\left(\ln\left(\frac{x}{x+1}\right) + \frac{1}{x+1}\right) \\
y' &= \left(\frac{x}{x+1}\right)^x\left(\ln\left(\frac{x}{x+1}\right) + \frac{1}{x+1}\right).
\end{aligned}$$

Izvod funkcije zadane u parametarskom obliku

$$\left.\begin{array}{r} x = x(t) \\ y = y(t) \end{array}\right\} \Longrightarrow y'_x = \frac{y'_t}{x'_t}$$

Zadatak 11.24 *Naći izvod funkcije*

$$\left.\begin{array}{r} x(t) = \cos t \\ y(t) = t + \sin t \end{array}\right\}.$$

Rješenje:
Kako je

$$\begin{aligned} y'_t &= 1 + \cos t \\ x'_t &= -\sin t \end{aligned}$$

tada je

$$y'_x = \frac{y'_t}{x'_t} = \frac{1 + \cos t}{-\sin t} = -\frac{1 + \cos t}{\sin t}.$$

Zadatak 11.25 *Naći izvod funkcije*

$$\left.\begin{array}{r} x(t) = e^t \sin t \\ y(t) = e^t \cos t \end{array}\right\}.$$

Rješenje:
Kako je

$$\begin{aligned} y'_t &= e^t \cos t - e^t \sin t = e^t (\cos t - \sin t) \\ x'_t &= e^t \sin t + e^t \cos t = e^t (\cos t + \sin t) \end{aligned}$$

tada je

$$y'_x = \frac{y'_t}{x'_t} = \frac{e^t (\cos t - \sin t)}{e^t (\cos t + \sin t)} = \frac{\cos t - \sin t}{\cos t + \sin t}.$$

Jednačine tangente i normale

Jednačine tangente i normale na krivu $y = y(x)$ u tački $M(x_0, y_0)$:

$$\begin{aligned} T &: \quad y - y_0 = y'(x_0)(x - x_0) \\ N &: \quad y - y_0 = -\frac{1}{y'(x_0)}(x - x_0) \end{aligned}$$

Zadatak 11.26 *Odrediti jednačinu tangente na krivu* $y = 2x^2 - 3x + 2$ *u tački* $M(2, y)$

Rješenje:
Prvo odredimo koordinate tačke M:

$$y(2) = 2 \cdot 4 - 3 \cdot 2 + 2 = 8 - 6 + 2 = 4$$

pa je $M(2, 4)$.
Prvi izvod funkcije je

$$y' = 4x - 3$$

i

$$y'(2) = 8 - 3 = 5$$

pa jednačina tangente glasi

$$\begin{aligned} y - 4 &= 5(x - 2) \\ y - 4 &= 5x - 10 \\ y &= 5x - 6. \end{aligned}$$

Zadatak 11.27 *Odrediti jednačinu tangente na krivu* $y = x^4 - x^2 + 3$ *u tački* $M(1, y)$

Rješenje:
Prvo odredimo koordinate tačke M:

$$y(1) = 1 - 1 + 3 = 3$$

pa je $M(1, 3)$.
Prvi izvod funkcije je

$$y' = 4x^3 - 2x$$

i

$$y'(1) = 4 - 2 = 2$$

pa jednačina tangente glasi

$$\begin{aligned} y - 3 &= 2(x - 1) \\ y - 3 &= 2x - 2 \\ y &= 2x + 1. \end{aligned}$$

Zadatak 11.28 *U kojoj tački krive* $y = \sqrt{1 + x^2}$ *je njena tangenta paralelna sa pravom* $y = \frac{1}{2}x + 1$.

Rješenje:

Koeficijent pravca prave $y = \frac{1}{2}x + 1$ je $\frac{1}{2}$, a po uslovu zadatka tangenta na krivu mora biti paralelna sa ovom pravom pa zato i njen koeficijent pravca mora biti $\frac{1}{2}$.

Prvi izvod funkcije je

$$y' = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$$

pa po uslovu zadatka, vrijedi

$$\begin{aligned}
\frac{x}{\sqrt{1+x^2}} &= \frac{1}{2} \\
&\Longrightarrow 2x = \sqrt{1+x^2} \qquad /^2 \\
&\Longrightarrow 4x^2 = 1 + x^2 \\
&\Longrightarrow 3x^2 = 1 \\
&\Longrightarrow x^2 = \frac{1}{3} \\
&\Longrightarrow x_1 = -\sqrt{\frac{1}{3}} = -\frac{\sqrt{3}}{3}, \qquad x_1 = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}.
\end{aligned}$$

Kako je

$$\begin{aligned}
y\left(-\frac{\sqrt{3}}{3}\right) &= \sqrt{1 + \left(-\frac{\sqrt{3}}{3}\right)^2} = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
y\left(\frac{\sqrt{3}}{3}\right) &= \sqrt{1 + \left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\end{aligned}$$

onda su tražene tačke

$$A\left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right) \quad i \quad B\left(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right).$$

L'Hospitalovo pravilo

Zadatak 11.29

$$\lim_{x\to 1} \frac{x^5 - 3x^4 + 2x^2 + x - 1}{x^7 - 4x + 3} = \left(\frac{0}{0}\right) \stackrel{LP}{=} \lim_{x\to 1} \frac{5x^4 - 12x^3 + 4x + 1}{7x^6 - 4} = -\frac{2}{3}.$$

Zadatak 11.30

$$\begin{aligned}\lim_{x\to+\infty}\frac{e^x}{x^5+4x^2+3} &= \left(\frac{+\infty}{+\infty}\right)\overset{LP}{=}\lim_{x\to+\infty}\frac{e^x}{5x^4+8x}\left(=\frac{+\infty}{+\infty}\right)\overset{LP}{=}\\ &= \lim_{x\to+\infty}\frac{e^x}{20x^3+8}\left(=\frac{+\infty}{+\infty}\right)\overset{LP}{=}\\ &= \lim_{x\to+\infty}\frac{e^x}{60x^2}\left(=\frac{+\infty}{+\infty}\right)\overset{LP}{=}\\ &= \lim_{x\to+\infty}\frac{e^x}{120x}\left(=\frac{+\infty}{+\infty}\right)\overset{LP}{=}\\ &= \lim_{x\to+\infty}\frac{e^x}{120}=+\infty.\end{aligned}$$

Zadatak 11.31

$$\lim_{x\to0}\frac{\ln(\cos x)}{x}=\left(\frac{0}{0}\right)\overset{LP}{=}\lim_{x\to0}\frac{\frac{1}{\cos x}\cdot(-\sin x)}{1}=-\lim_{x\to0}tgx=0.$$

Zadatak 11.32

$$\begin{aligned}\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x-\sin x} &= \left(\frac{0}{0}\right)\overset{LP}{=}\lim_{x\to0}\frac{e^x+e^{-x}-2}{1-\cos x}\left(=\frac{0}{0}\right)\overset{LP}{=}\\ &= \lim_{x\to0}\frac{e^x-e^{-x}}{\sin x}\left(=\frac{0}{0}\right)\overset{LP}{=}\\ &= \lim_{x\to0}\frac{e^x+e^{-x}}{\cos x}=2\end{aligned}$$

Zadatak 11.33

$$\lim_{x\to0}x\ln x=(0\cdot\infty)=\lim_{x\to0}\frac{\ln x}{\frac{1}{x}}\overset{LP}{=}\lim_{x\to0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=-\lim_{x\to0}x=0.$$

Zadatak 11.34

$$\begin{aligned}\lim_{x\to\pi}(\pi-x)\,tg\frac{x}{2} &= \lim_{x\to\pi}\frac{tg\frac{x}{2}}{\frac{1}{\pi-x}}\left(=\frac{\infty}{\infty}\right)\overset{LP}{=}\lim_{x\to\pi}\frac{\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}}{\frac{1}{(\pi-x)^2}}=\\ &= \lim_{x\to\pi}\frac{(\pi-x)^2}{2\cos^2\frac{x}{2}}\left(=\frac{0}{0}\right)\overset{LP}{=}\lim_{x\to\pi}\frac{2(\pi-x)\cdot(-1)}{4\cos\frac{x}{2}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}}=\\ &= \lim_{x\to\pi}\frac{\pi-x}{\cos\frac{x}{2}\sin\frac{x}{2}}\left(=\frac{0}{0}\right)\overset{LP}{=}\\ &= \lim_{x\to\pi}\frac{-1}{-\sin\frac{x}{2}\cdot\frac{1}{2}\sin\frac{x}{2}+\cos\frac{x}{2}\cos\frac{x}{2}\cdot\frac{1}{2}}=\\ &= \lim_{x\to\pi}\frac{1}{\frac{1}{2}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)}=\frac{1}{\frac{1}{2}}=2.\end{aligned}$$

Zadatak 11.35

$$\begin{aligned}\lim_{x\to 0} x^x &= \lim_{x\to 0} e^{x\ln x} = e^{\lim\limits_{x\to 0} x\ln x} = e^{\lim\limits_{x\to 0}\frac{\ln x}{\frac{1}{x}}} = e^{\lim\limits_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{-\lim\limits_{x\to 0} x} = \\ &= e^0 = 1\end{aligned}$$

12 Ispitivanje osobina i crtanje grafika funkcije

Zadatak 12.1 *Ispitati osobine i nacrtati grafik funkcije*

$$f(x) = x^3 - 3x + 2.$$

Rješenje:
1. *Definiciono područje:* $\mathcal{D} = \mathbb{R}$.
2. *Nule i znak:*

$$\begin{aligned} f(x) &= 0 \iff \\ &\iff x^3 - 3x + 2 = 0 \\ &\iff x^3 - 4x + x + 2 = 0 \\ &\iff x\left(x^2 - 4\right) + x + 2 = 0 \\ &\iff x(x-2)(x+2) + x + 2 = 0 \\ &\iff (x+2)[x(x-2)+1] = 0 \\ &\iff (x+2)\left(x^2 - 2x + 1\right) = 0 \\ &\iff (x+2)(x-1)^2 = 0 \\ &\iff x_1 = -2, \quad x_2 = 1. \end{aligned}$$

Znak:

$x:$ $-\infty$	-2	1	$+\infty$
$x+2$	$-$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$
$(x+2)(x-1)^2$	$-$	$+$	$+$

Dakle,

$$\begin{aligned} y &< 0 \quad za \quad x \in (-\infty, -2) \\ y &> 0 \quad za \quad x \in (2, +\infty). \end{aligned}$$

3. *Parnost:*

$$\begin{aligned} f(-x) &= (-x)^3 - 3(-x) + 2 = -x^3 + 3x + 2 = \\ &= -\left(x^3 - 3x - 2\right) \neq f(-x) \\ f(-x) &= (-x)^3 - 3(-x) + 2 = -x^3 + 3x + 2 = \\ &= -\left(x^3 - 3x - 2\right) \neq -f(x). \end{aligned}$$

Funkcija nije ni parna ni neparna.
Funkcija nije periodična.
4. *Asimptote:*

Nema vertikalnih asimptota jer je $\mathcal{D} = \mathbb{R}$.
Budući da je

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(x^3 - 3x + 2\right) = \infty$$

funkcija nema horizontalnih asimptota.
Budući da je

$$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3 - 3x + 2}{x} = \lim_{x\to\infty} \left(x^2 - 3 + \frac{2}{x}\right) = +\infty$$

funkcija nema kosih asimptota.
5. *Ekstremi i tok funkcije:*
Prvi izvod funkcije je

$$y' = 3x^2 - 3$$

pa je

$$\begin{aligned} y' &= 0 \\ \iff& 3x^2 - 3 = 0 \\ \iff& x^2 - 1 = 0 \\ \iff& x_1 = -1, \quad x_2 = 1. \end{aligned}$$

Iz

x : $-\infty$	-1	1	$+\infty$
$3x^2 - 3$	$+$	$-$	$+$
y'	$+$	$-$	$+$
y	$\nearrow$	$\searrow$	$\nearrow$

zaključujemo da
za $x = -1$ funkcija ima maksimum

$$y_{\max} = y(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$$

za $x = 1$ funkcija ima mininum

$$y_{\min} = y(-1) = 1^3 - 3 \cdot 1 + 2 = 1 - 3 + 2 = 0.$$

6. *Konveksnost i konkavnost. Prevojne tačke.*
Drugi izvod funkcije je

$$y'' = 6x$$

pa je

$$\begin{aligned} y'' &= 0 \\ \iff& x = 0. \end{aligned}$$

Iz

$x:$ $-\infty$	0	$+\infty$
y''	$-$	$+$
y	$\cap$	$\cup$

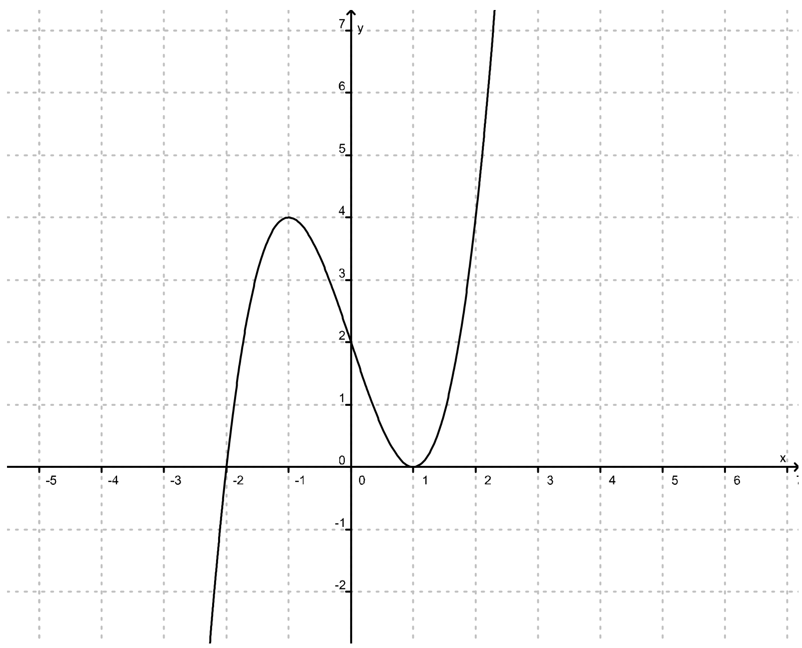
zaključujemo da je

funkcija konkavna na intervalu $(-\infty, 0)$

funkcija konveksna na intervalu $(0, +\infty)$.

Funkcija ima prevojnu tačku $P(0, 2)$ jer je $y(0) = 2$.

7. *Grafik funkcije*:



Zadatak 12.2 *Ispitati osobine i nacrtati grafik funkcije*

$$f(x) = \frac{3x - x^2}{x - 4}.$$

Rješenje:

1. *Definiciono područje:*

Budući da je racionalna funkcija, tada mora da vrijedi

$$x - 4 \neq 0 \implies x \neq 4$$

pa je

$$\mathcal{D} = (-\infty, 4) \cup (4, +\infty)$$

2. *Nule i znak:*

$$
\begin{aligned}
f(x) &= 0 \iff \\
&\iff \frac{3x-x^2}{x-4}=0 \\
&\iff 3x-x^2=0 \\
&\iff x\,(3-x)=0 \\
&\iff x_1=0, \quad x_2=3.
\end{aligned}
$$

Znak:

$x:$ $-\infty$	0	3	4	$+\infty$
$3x-x^2$	$-$	$+$	$-$	$-$
$x-4$	$-$	$-$	$-$	$+$
$\frac{3x-x^2}{x-4}$	$+$	$-$	$+$	$-$

Dakle,

$$
\begin{aligned}
y &< 0 \quad za \quad x\in(0,3)\cup(4,+\infty) \\
y &> 0 \quad za \quad x\in(-\infty,0)\cup(3,4)\,.
\end{aligned}
$$

3. *Parnost:*

$$
\begin{aligned}
f(-x) &= \frac{3x-x^2}{x-4}=\frac{3(-x)-(-x)^2}{(-x)-4}=\frac{-3x-x^2}{-x-4}= \\
&= \frac{-\left(3x+x^2\right)}{-\left(x+4\right)}=\frac{3x+x^2}{x+4}\neq f(-x) \\
f(-x) &= \frac{3x-x^2}{x-4}=\frac{3(-x)-(-x)^2}{(-x)-4}=\frac{-3x-x^2}{-x-4}= \\
&= \frac{-\left(3x+x^2\right)}{-\left(x+4\right)}=\frac{3x+x^2}{x+4}\neq -f(x)
\end{aligned}
$$

Funkcija nije ni parna ni neparna.
Funkcija nije periodična.
4. *Asimptote:*

Budući da funkcija nije definisana za $x = 4$, ispitajmo lijevi i desni limes funkcije u toj tački:

$$\begin{aligned}
L(4) &= \lim_{x\to 4^-} \frac{3x-x^2}{x-4} = \begin{bmatrix} smjena \\ x = 4-\varepsilon \\ \varepsilon > 0, \varepsilon \to 0 \end{bmatrix} = \lim_{\varepsilon\to 0} \frac{3\left(4-\varepsilon\right)-\left(4-\varepsilon\right)^2}{4-\varepsilon-4} = \\
&= \lim_{\varepsilon\to 0} \frac{12-3\varepsilon-16+8\varepsilon-\varepsilon^2}{-\varepsilon} = \lim_{\varepsilon\to 0} \frac{-\varepsilon^2+5\varepsilon-4}{-\varepsilon} = \\
&= \lim_{\varepsilon\to 0} \left(-\varepsilon-5+\frac{4}{\varepsilon}\right) = -0-5+\lim_{\varepsilon\to 0}\frac{4}{\varepsilon} = +\infty. \\
D(4) &= \lim_{x\to 4^+} \frac{3x-x^2}{x-4} = \begin{bmatrix} smjena \\ x = 4+\varepsilon \\ \varepsilon > 0, \varepsilon \to 0 \end{bmatrix} = \lim_{\varepsilon\to 0} \frac{3\left(4+\varepsilon\right)-\left(4+\varepsilon\right)^2}{4+\varepsilon-4} = \\
&= \lim_{\varepsilon\to 0} \frac{12+3\varepsilon-16-8\varepsilon-\varepsilon^2}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{-\varepsilon^2-5\varepsilon-4}{\varepsilon} = \\
&= \lim_{\varepsilon\to 0} \left(-\varepsilon-5-\frac{4}{\varepsilon}\right) = -0-5-\lim_{\varepsilon\to 0}\frac{4}{\varepsilon} = -\infty.
\end{aligned}$$

Dakle, funkcija ima vertikalnu asimptotu $x = 4$.

Budući da je

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{3x-x^2}{x-4} \overset{L.P}{=} \lim_{x\to-\infty} \frac{-2x}{1} = +\infty \\
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{3x-x^2}{x-4} \overset{L.P}{=} \lim_{x\to+\infty} \frac{-2x}{1} = -\infty
\end{aligned}$$

funkcija nema horizontalnih asimptota.

Budući da je

$$\begin{aligned}
k &= \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{3x-x^2}{x-4}}{x} = \lim_{x\to\infty} \frac{3x-x^2}{x^2-4x} = \\
&= \lim_{x\to\infty} \frac{3x-x^2 : x^2}{x^2-4x : x^2} = \lim_{x\to\infty} \frac{\frac{3}{x}-1}{1-\frac{4}{x^2}} = -1 \\
n &= \lim_{x\to\infty} \left(f(x)-kx\right) = \lim_{x\to\infty} \left(\frac{3x-x^2}{x-4}+x\right) = \\
&= \lim_{x\to\infty} \frac{3x-x^2+x^2-4x}{x-4} = \lim_{x\to\infty} \frac{-x}{x-4} = -1
\end{aligned}$$

funkcija ima kosu asimptotu $y = -x-1$.

5. *Ekstremi i tok funkcije:*

Prvi izvod funkcije je

$$\begin{aligned} y' &= \frac{(3-2x)(x-4)-(3x-x^2)\cdot 1}{(x-4)^2} = \\ &= \frac{3x-12-2x^2+8x-3x+x^2}{(x-4)^2} = \\ &= \frac{-x^2+8x-12}{(x-4)^2} \end{aligned}$$

pa je

$$\begin{aligned} y' &= 0 \\ \iff & \frac{-x^2+8x-12}{(x-4)^2} = 0 \\ \iff & -x^2+8x-12 = 0 \\ \iff & x_1 = 2, \quad x_2 = 6. \end{aligned}$$

Iz

$x: \quad -\infty$	2	4	6	$+\infty$
$-x^2+8x-12$	$-$	$+$	$+$	$-$
$(x-4)^2$	$+$	$+$	$+$	$+$
y'	$-$	$+$	$+$	$-$
y	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$

zaključujemo da
za $x=2$ funkcija ima minimum

$$y_{\min} = y(2) = \frac{6-4}{-2} = -1$$

za $x=6$ funkcija ima maksimum

$$y_{\max} = y(6) = \frac{18-36}{2} = -9$$

6. *Konveksnost i konkavnost. Prevojne tačke.*
Drugi izvod funkcije je

$$\begin{aligned} y'' &= \frac{(-2x+8)(x-4)^2-(-x^2+8x-12)\cdot 2(x-4)}{\left((x-4)^2\right)^2} = \\ &= \frac{(x-4)\cdot\left[(-2x+8)(x-4)-2(-x^2+8x-12)\right]}{(x-4)^4} = \\ &= \frac{-2x^2+8x+8x-32+2x^2-16x+24}{(x-4)^3} = \\ &= \frac{-8}{(x-4)^3} \end{aligned}$$

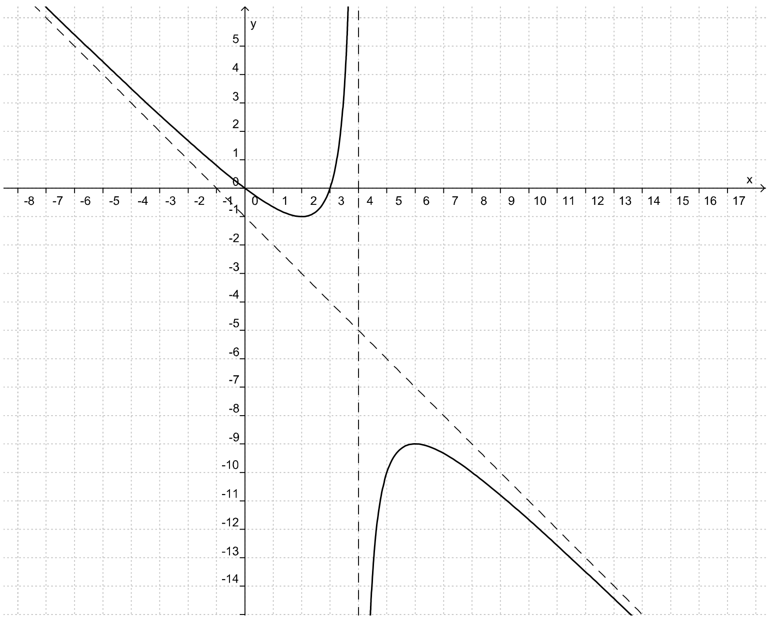
pa je

$$\begin{aligned} y'' &= 0 \\ \iff & \frac{-8}{(x-4)^3} = 0 \end{aligned}$$

a posljednja jednačina nema rješenja.
Iz

$x: \quad -\infty$	4	$+\infty$
y''	$-$	$+$
y	$\cup$	$\cap$

zaključujemo da je
funkcija konkavna na intervalu $(4, +\infty)$
funkcija konveksna na intervalu $(-\infty, 4)$.
Funkcija nema prevojnih tačaka jer $4 \notin \mathcal{D}$.
7. *Grafik funkcije*:



Zadatak 12.3 *Ispitati osobine i nacrtati grafik funkcije*

$$f(x) = \frac{x^2 - 2x + 1}{x^2 + 1}.$$

Rješenje:
1. *Definiciono područje:*

Budući da je racionalna funkcija, tada mora da vrijedi $x^2+1 \neq 0$ a to vrijedi za svaki realan broj x
pa je $\mathcal{D}=\mathbb{R}$.
2. *Nule i znak:*

$$\begin{aligned} f(x) &= 0 \iff \\ &\iff \frac{x^2-2x+1}{x^2+1}=0 \\ &\iff x^2-2x+1=0 \\ &\iff (x-1)^2=0 \\ &\iff x=1. \end{aligned}$$

Znak:

$x:$ $-\infty$	1	$+\infty$
x^2-2x+1	$+$	$+$
x^2+1	$+$	$+$
$\frac{x^2-2x+1}{x^2+1}$	$+$	$+$

Dakle,

$$y>0 \quad za \quad \forall x \in \mathbb{R}$$

3. *Parnost:*

$$\begin{aligned} f(-x) &= \frac{x^2-2x+1}{x^2+1} = \frac{(-x)^2-2(-x)+1}{(-x)^2+1} = \\ &= \frac{x^2+2x+1}{x^2+1} \neq f(-x) \\ f(-x) &= \frac{x^2-2x+1}{x^2+1} = \frac{(-x)^2-2(-x)+1}{(-x)^2+1} = \\ &= \frac{x^2+2x+1}{x^2+1} \neq -f(x) \end{aligned}$$

Funkcija nije ni parna ni neparna.
Funkcija nije periodična.
4. *Asimptote:*
Budući da je $\mathcal{D}=\mathbb{R}$ funkcija nema vertikalnih asimptota.
Budući da je

$$\begin{aligned} \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^2-2x+1}{x^2+1} \overset{L.P}{=} \lim_{x\to-\infty} \frac{2x}{2x} = 1 \\ \lim_{x\to+\infty} f(x) &= \lim_{x\to-\infty} \frac{x^2-2x+1}{x^2+1} \overset{L.P}{=} \lim_{x\to-\infty} \frac{2x}{2x} = 1 \end{aligned}$$

funkcija ima horizontalnu asimptotu $y = 1$.

Budući da je

$$\begin{aligned} k &= \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{x^2-2x+1}{x^2+1}}{x} = \lim_{x\to\infty} \frac{x^2-2x+1}{x^3+x} \overset{L.P}{=} \\ &= \lim_{x\to\infty} \frac{2x-2}{3x^2+1} \overset{L.P}{=} \lim_{x\to\infty} \frac{2}{6x} = 0 \end{aligned}$$

funkcija nema kosih asimptota.

5. *Ekstremi i tok funkcije:*

Prvi izvod funkcije je

$$\begin{aligned} y' &= \frac{(2x-2)\left(x^2+1\right) - \left(x^2-2x+1\right)2x}{\left(x^2+1\right)^2} = \\ &= \frac{2x^3+2x-2x^2-2-2x^3+4x^2-2x}{\left(x^2+1\right)^2} = \\ &= \frac{2x^2-2}{\left(x^2+1\right)^2} \end{aligned}$$

pa je

$$\begin{aligned} y' &= 0 \\ &\Longleftrightarrow \frac{2x^2-2}{\left(x^2+1\right)^2} = 0 \\ &\Longleftrightarrow 2x^2-2 = 0 \\ &\Longleftrightarrow x_1 = -1, \quad x_2 = 1. \end{aligned}$$

Iz

$x:$ $-\infty$	-1	1	$+\infty$
$2x^2-2$	$+$	$-$	$+$
$\left(x^2+1\right)^2$	$+$	$+$	$+$
y'	$+$	$-$	$+$
y	$\nearrow$	$\searrow$	$\nearrow$

zaključujemo da

za $x = -1$ funkcija ima maksimum

$$y_{\max} = y(-1) = \frac{(-1)^2 - 2(-1) + 1}{(-1)^2 + 1} = 2$$

za $x = 1$ funkcija ima minimum

$$y_{\min} = y(1) = \frac{1^2 - 2 \cdot 1 + 1}{1^2 + 1} = 0.$$

6. *Konveksnost i konkavnost. Prevojne tačke.*
Drugi izvod funkcije je

$$\begin{aligned}
y'' &= \frac{4x\left(x^2+1\right)^2-\left(2x^2-2\right)\cdot 2\left(x^2+1\right)\cdot 2x}{\left(\left(x^2+1\right)^2\right)^2}= \\
&= \frac{4x\left(x^2+1\right)^2-\left(2x^2-2\right)\cdot 2\left(x^2+1\right)\cdot 2x}{\left(x^2+1\right)^4}= \\
&= \frac{\left(x^2+1\right)\left[4x\left(x^2+1\right)-4x\left(2x^2-2\right)\right]}{\left(x^2+1\right)^4}= \\
&= \frac{4x^3+4x-8x^3+8x}{\left(x^2+1\right)^3}=\frac{-4x^3+12x}{\left(x^2+1\right)^3}= \\
&= \frac{-4x\left(x^2-3\right)}{\left(x^2+1\right)^3}
\end{aligned}$$

pa je

$$\begin{aligned}
y'' &= 0 \\
\Longleftrightarrow & \quad \frac{-4x\left(x^2-3\right)}{\left(x^2+1\right)^3}=0 \\
\Longleftrightarrow & \quad -4x\left(x^2-3\right)=0 \\
\Longleftrightarrow & \quad x_1=0, \quad x_2=-\sqrt{3}, \quad x_3=\sqrt{3}
\end{aligned}$$

Iz

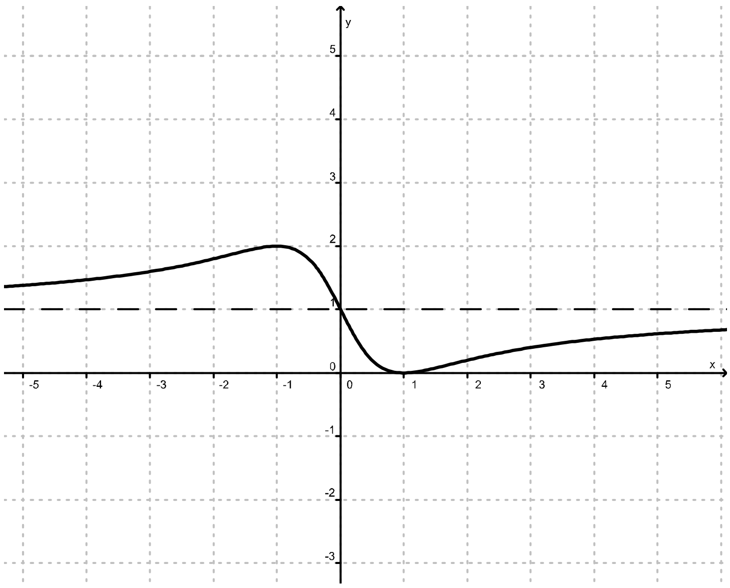
$x:$ $-\infty$	$-\sqrt{3}$	0	$\sqrt{3}$	$+\infty$
$-4x$	$+$	$+$	$-$	$-$
x^2-3	$+$	$-$	$-$	$+$
$\left(x^2+1\right)^3$	$+$	$+$	$+$	$+$
y''	$+$	$-$	$+$	$-$
y	$\cup$	$\cap$	$\cup$	$\cap$

zaključujemo da je
funkcija konkavna na intervalu $\left(-\sqrt{3},0\right)\cup\left(\sqrt{3},+\infty\right)$
funkcija konveksna na intervalu $\left(-\infty,-\sqrt{3}\right)\cup\left(0,\sqrt{3}\right)$.
Funkcija ima prevojne tačke za $x_1=0, \quad x_2=-\sqrt{3}, \quad x_3=\sqrt{3}$

$$
\begin{aligned}
y(0) &= \frac{0^2 - 2\cdot 0 + 1}{0^2+1} = 1 \\
y\left(-\sqrt{3}\right) &= \frac{\left(-\sqrt{3}\right)^2 - 2\cdot\left(-\sqrt{3}\right)+1}{\left(-\sqrt{3}\right)^2+1} = 1 + \frac{\sqrt{3}}{2} \\
y\left(\sqrt{3}\right) &= \frac{\left(\sqrt{3}\right)^2 - 2\cdot\left(\sqrt{3}\right)+1}{\left(\sqrt{3}\right)^2+1} = 1 - \frac{\sqrt{3}}{2}
\end{aligned}
$$

pa su prevojne tačke $P_1\,(0,1)\,, P_2\left(-\sqrt{3}, 1+\frac{\sqrt{3}}{2}\right), P_3\left(\sqrt{3}, 1-\frac{\sqrt{3}}{2}\right)$.

7. *Grafik funkcije*:



Zadatak 12.4 *Ispitati osobine i nacrtati grafik funkcije*

$$f(x) = \frac{x^3}{x^2-3}.$$

Rješenje:

1. *Definiciono područje:*

Budući da je racionalna funkcija, onda mora biti $x^2 - 3 \neq 0$ tj $x \neq \pm\sqrt{3}$ pa je

$$D = \left(-\infty, -\sqrt{3}\right) \cup \left(-\sqrt{3}, \sqrt{3}\right) \cup \left(\sqrt{3}, +\infty\right).$$

2. *Nule i znak:*

$$\begin{aligned} f(x) &= 0 \Longleftrightarrow \\ &\Longleftrightarrow \frac{x^3}{x^2-3} = 0 \\ &\Longleftrightarrow x^3 \\ &\Longleftrightarrow x = 0. \end{aligned}$$

Znak:

$x:$ $-\infty$	$-\sqrt{3}$	0	$\sqrt{3}$	$+\infty$
x^3	$-$	$-$	$+$	$+$
x^2-3	$+$	$-$	$-$	$+$
y	$-$	$+$	$-$	$+$

Dakle,

$$\begin{aligned} y &< 0 \quad za \quad x \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right) \\ y &> 0 \quad za \quad x \in \left(-\sqrt{3}, 0\right) \cup \left(\sqrt{3}, +\infty\right). \end{aligned}$$

3. *Parnost:*

$$f(-x) = \frac{(-x)^3}{(-x)^2-3} = \frac{-x^3}{x^2-3} = -\frac{x^3}{x^2-3} = -f(x)$$

Funkcija je neparna pa je grafik funkcije simetričan u odnosu na koordinatni početak..

Funkcija nije periodična.

4. *Asimptote:*

Funkcija nije definisana za $x = -\sqrt{3}$ i $x = \sqrt{3}$. Ispitajmo lijevi i desni limes funkcije u ovim tačkama:

$$\begin{aligned}
L\left(-\sqrt{3}\right) &= \lim_{x\to-\sqrt{3}^-}\frac{x^3}{x^2-3} = \begin{bmatrix} smjena \\ x=-\sqrt{3}-\varepsilon \\ \varepsilon>0, \varepsilon\to 0\end{bmatrix} = \\
&= \lim_{\varepsilon\to 0}\frac{\left(-\sqrt{3}-\varepsilon\right)^3}{\left(-\sqrt{3}-\varepsilon\right)^2-3} = \lim_{\varepsilon\to 0}\frac{-3\sqrt{3}-9\varepsilon-3\sqrt{3}\varepsilon-\varepsilon^3}{3+2\sqrt{3}\varepsilon+\varepsilon^2-3} = \\
&= \lim_{\varepsilon\to 0}\frac{-3\sqrt{3}-9\varepsilon-3\sqrt{3}\varepsilon-\varepsilon^3}{2\sqrt{3}\varepsilon+\varepsilon^2} = -\infty \\
D\left(-\sqrt{3}\right) &= \lim_{x\to-\sqrt{3}^+}\frac{x^3}{x^2-3} = \begin{bmatrix} smjena \\ x=-\sqrt{3}+\varepsilon \\ \varepsilon>0, \varepsilon\to 0\end{bmatrix} = \\
&= \lim_{\varepsilon\to 0}\frac{\left(-\sqrt{3}+\varepsilon\right)^3}{\left(-\sqrt{3}+\varepsilon\right)^2-3} = \lim_{\varepsilon\to 0}\frac{3\sqrt{3}-9\varepsilon+3\sqrt{3}\varepsilon-\varepsilon^3}{3-2\sqrt{3}\varepsilon+\varepsilon^2-3} = \\
&= \lim_{\varepsilon\to 0}\frac{3\sqrt{3}-9\varepsilon+3\sqrt{3}\varepsilon-\varepsilon^3}{-2\sqrt{3}\varepsilon+\varepsilon^2} = +\infty
\end{aligned}$$

$$\begin{aligned}
L\left(\sqrt{3}\right) &= \lim_{x\to\sqrt{3}^-}\frac{x^3}{x^2-3} = \begin{bmatrix} smjena \\ x=\sqrt{3}-\varepsilon \\ \varepsilon>0, \varepsilon\to 0\end{bmatrix} = \\
&= \lim_{\varepsilon\to 0}\frac{\left(\sqrt{3}-\varepsilon\right)^3}{\left(\sqrt{3}-\varepsilon\right)^2-3} = \lim_{\varepsilon\to 0}\frac{3\sqrt{3}-9\varepsilon+3\sqrt{3}\varepsilon-\varepsilon^3}{3-2\sqrt{3}\varepsilon+\varepsilon^2-3} = \\
&= \lim_{\varepsilon\to 0}\frac{3\sqrt{3}-9\varepsilon+3\sqrt{3}\varepsilon-\varepsilon^3}{-2\sqrt{3}\varepsilon+\varepsilon^2} = -\infty \\
D\left(-\sqrt{3}\right) &= \lim_{x\to\sqrt{3}^+}\frac{x^3}{x^2-3} = \begin{bmatrix} smjena \\ x=\sqrt{3}+\varepsilon \\ \varepsilon>0, \varepsilon\to 0\end{bmatrix} = \\
&= \lim_{\varepsilon\to 0}\frac{\left(\sqrt{3}+\varepsilon\right)^3}{\left(\sqrt{3}+\varepsilon\right)^2-3} = \lim_{\varepsilon\to 0}\frac{3\sqrt{3}+9\varepsilon+3\sqrt{3}\varepsilon+\varepsilon^3}{3+2\sqrt{3}\varepsilon+\varepsilon^2-3} = \\
&= \lim_{\varepsilon\to 0}\frac{3\sqrt{3}+9\varepsilon+3\sqrt{3}\varepsilon+\varepsilon^3}{2\sqrt{3}\varepsilon+\varepsilon^2} = +\infty
\end{aligned}$$

Budući da je

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x^3}{x^2-3} \overset{L.P.}{=} \lim_{x\to+\infty} \frac{3x^2}{2x} \overset{L.P.}{=} \lim_{x\to+\infty} \frac{6x}{2} = \lim_{x\to+\infty} 3x = +\infty \\
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^3}{x^2-3} \overset{L.P.}{=} \lim_{x\to-\infty} \frac{3x^2}{2x} \overset{L.P.}{=} \lim_{x\to-\infty} \frac{6x}{2} = \lim_{x\to-\infty} 3x = -\infty
\end{aligned}$$

funkcija nema horizontalnih asimptota.

Budući da je

$$\begin{aligned}
k &= \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{x^3}{x^2-3}}{x} = \lim_{x\to\infty} \frac{x^3}{x^3-3x} = 1 \\
n &= n = \lim_{x\to\infty} \left(f(x) - kx\right) = \lim_{x\to\infty} \left(\frac{x^3}{x^2-3} - x\right) = \\
&= \lim_{x\to\infty} \frac{x^3 - x^3 + 3x}{x^2-3} = \lim_{x\to\infty} \frac{3x}{x^2-3} \overset{L.P.}{=} \lim_{x\to\infty} \frac{3}{2x} = 0
\end{aligned}$$

funkcija ima kosu asimptotu $y = x$

5. *Ekstremi i tok funkcije:*

Prvi izvod funkcije je

$$\begin{aligned}
y' &= \frac{3x^2\left(x^2-3\right) - x^3 \cdot 2x}{\left(x^2-3\right)^2} = \\
&= \frac{3x^4 - 9x^2 - 2x^4}{\left(x^2-3\right)^2} = \\
&= \frac{x^4 - 9x^2}{\left(x^2-3\right)^2}
\end{aligned}$$

pa je

$$\begin{aligned}
y' &= 0 \\
&\Longleftrightarrow \frac{x^4 - 9x^2}{\left(x^2-3\right)^2} = 0 \\
&\Longleftrightarrow x^4 - 9x^2 = 0 \\
&\Longleftrightarrow x^2\left(x^2 - 9\right) \\
&\Longleftrightarrow x_1 = 0, \quad x_2 = -3, \quad x_3 = 3.
\end{aligned}$$

Iz

$x:$ $-\infty$	-3	$-\sqrt{3}$	0	$\sqrt{3}$	3	$+\infty$
x^2	$+$	$+$	$+$	$+$	$+$	$+$
x^2-9	$+$	$-$	$-$	$-$	$-$	$+$
$\left(x^2-3\right)^2$	$+$	$+$	$+$	$+$	$-$	$+$
y'	$+$	$-$	$-$	$-$	$-$	$+$
y	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$

zaključujemo da
za $x = -3$ funkcija ima maksimum

$$y_{\max} = y(-3) = \frac{(-3)^3}{(-3)^2 - 3} = -\frac{9}{2}$$

za $x = 3$ funkcija ima mininum

$$y_{\min} = y(3) = \frac{3^3}{3^2 - 3} = \frac{9}{2}.$$

6. *Konveksnost i konkavnost. Prevojne tačke.*
Drugi izvod funkcije je

$$\begin{aligned} y'' &= \frac{(4x^3 - 18x)(x^2-3)^2 - (x^4 - 9x^2)\cdot 2(x^2-3)\cdot 2x}{\left((x^2-3)^2\right)^2} = \\ &= \frac{(x^2-3)\left[(4x^3-18x)(x^2-3) - 4x(x^4-9x^2)\right]}{(x^2-3)^4} = \\ &= \frac{4x^5 - 12x^3 - 18x^3 + 54x - 4x^5 + 36x^3}{(x^2-3)^3} = \\ &= \frac{6x^3+54x}{(x^2-3)^3} \end{aligned}$$

pa je

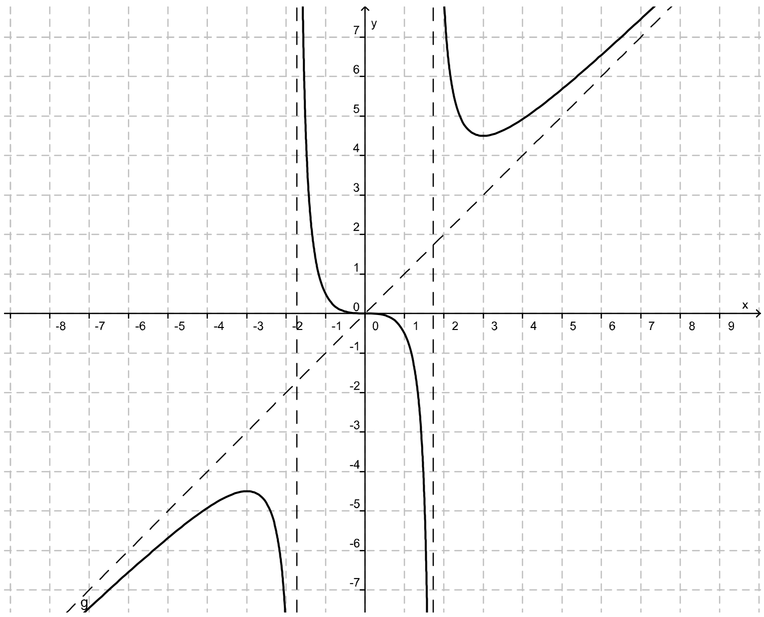
$$\begin{aligned} y'' &= 0 \\ &\iff \frac{6x^3+54x}{(x^2-3)^3} = 0 \\ &\iff 6x^3 + 54x = 0 \\ &\iff 6x\left(x^2+9\right) = 0 \\ &\iff x = 0. \end{aligned}$$

Iz

$x:$ $-\infty$	$-\sqrt{3}$	0	$\sqrt{3}$	$+\infty$
$6x$	$-$	$-$	$+$	$+$
x^2+9	$+$	$+$	$+$	$+$
$(x^2-3)^3$	$+$	$-$	$-$	$+$
y''	$-$	$+$	$-$	$+$
y	$\cap$	$\cup$	$\cap$	$\cup$

zaključujemo da je

funkcija konkavna na intervalu $\left(-\infty,-\sqrt{3}\right)\cup\left(0,\sqrt{3}\right)$
funkcija konveksna na intervalu $\left(-\sqrt{3},0\right)\cup\left(\sqrt{3},+\infty\right)$.
Funkcija ima prevojnu tačku $P(0,0)$ jer je $y(0)=0$.
7. *Grafik funkcije*:



Zadatak 12.5 *Ispitati osobine i nacrtati grafik funkcije*

$$f(x)=\left(3-x^2\right)e^{-x}.$$

Rješenje:
1. *Definiciono područje:* $\mathcal{D}=\mathbb{R}$.
2. *Nule i znak:*

$$\begin{aligned} f(x) &= 0 \iff \\ &\iff 3-x^2=0 \\ &\iff x_1=-\sqrt{3}, \quad x_2=\sqrt{3}. \end{aligned}$$

Znak:

x : $-\infty$	$-\sqrt{3}$	$\sqrt{3}$	$+\infty$
$3-x^2$	$-$	$+$	$-$
e^{-x}	$+$	$+$	$+$
$(3-x^2)\,e^{-x}$	$-$	$+$	$-$

Dakle,

$$\begin{aligned} y &< 0 \quad za \quad x\in\left(-\infty,-\sqrt{3}\right)\cup\left(\sqrt{3},+\infty\right) \\ y &> 0 \quad za \quad x\in\left(-\sqrt{3},\sqrt{3}\right). \end{aligned}$$

3. *Parnost:*

$$\begin{aligned}
f(-x) &= \left(3-(-x)^2\right)e^{-(-x)} = \left(3-x^2\right)e^{x} \neq f(-x)\\
f(-x) &= \left(3-(-x)^2\right)e^{-(-x)} = \left(3-x^2\right)e^{x} \neq -f(x)
\end{aligned}$$

Funkcija nije ni parna ni neparna.
Funkcija nije periodična.
4. *Asimptote:*
Nema vertikalnih asimptota jer je $\mathcal{D}=\mathbb{R}$.
Budući da je

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty}\left(3-x^2\right)e^{-x} = \lim_{x\to+\infty}\frac{3-x^2}{e^x} \overset{L.P.}{=} \lim_{x\to+\infty}\frac{-2x}{e^x} \overset{L.P.}{=} \lim_{x\to+\infty}\frac{-2}{e^x} = 0\\
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty}\left(3-x^2\right)e^{-x} = (-\infty)\cdot(+\infty) = -\infty
\end{aligned}$$

funkcija ima desnu horizontalnu asimptotu $y=0$.
Ovdje smo iskoristili da je

$$\begin{aligned}
\lim_{x\to+\infty} e^x &= +\infty\\
\lim_{x\to-+\infty} e^x &= 0.
\end{aligned}$$

Budući da je

$$\begin{aligned}
k &= \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\frac{3-x^2}{e^x}}{x} = \lim_{x\to+\infty}\frac{3-x^2}{xe^x} \overset{L.P.}{=} \lim_{x\to+\infty}\frac{-2x}{e^x+xe^x} \overset{L.P.}{=}\\
&= \lim_{x\to+\infty}\frac{-2}{e^x+e^x+xe^x} = 0\\
k &= \lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\frac{3-x^2}{e^x}}{x} = \lim_{x\to-\infty}\frac{3-x^2}{xe^x} = \lim_{x\to-\infty}\frac{3-x^2}{x}e^{-x} =\\
&= \lim_{x\to-\infty}\left(\frac{3}{x}-x\right)e^{-x} = +\infty.
\end{aligned}$$

funkcija nema kosih asimptota.
5. *Ekstremi i tok funkcije:*
Prvi izvod funkcije je

$$\begin{aligned}
y' &= -2xe^{-x} + \left(3-x^2\right)e^{-x}\cdot(-1) =\\
&= e^{-x}\left(x^2-2x-3\right)
\end{aligned}$$

pa je

$$\begin{aligned}
y' &= 0\\
&\Longleftrightarrow e^{-x}\left(x^2-2x-3\right)=0\\
&\Longleftrightarrow x^2-2x-3=0\\
&\Longleftrightarrow x_1=-1, \quad x_2=3.
\end{aligned}$$

Iz

$x:$ $-\infty$	-1	3	$+\infty$
e^{-x}	$+$	$+$	$+$
x^2-2x-3	$+$	$-$	$+$
y'	$+$	$-$	$+$
y	$\nearrow$	$\searrow$	$\nearrow$

zaključujemo da
za $x=-1$ funkcija ima maksimum

$$y_{\max}=y(-1)=(3-1)\,e=2e$$

za $x=3$ funkcija ima mininum

$$y_{\min}=y(3)=(3-9)\,e^{-3}=-6e^{-3}$$

6. *Konveksnost i konkavnost. Prevojne tačke.*
Drugi izvod funkcije je

$$\begin{aligned} y'' &= e^{-x}\cdot(-1)\left(x^2-2x-3\right)+e^{-x}\,(2x-2)= \\ &= -e^{-x}\left(x^2-2x-3-2x+2\right)= \\ &= -e^{-x}\left(x^2-4x-1\right) \end{aligned}$$

pa je

$$\begin{aligned} y'' &= 0 \\ &\Longleftrightarrow -e^{-x}\left(x^2-4x-1\right)=0 \\ &\Longleftrightarrow x^2-4x-1=0 \\ &\Longleftrightarrow x_1=2-\sqrt{5},\quad x_2=2+\sqrt{5}. \end{aligned}$$

Iz

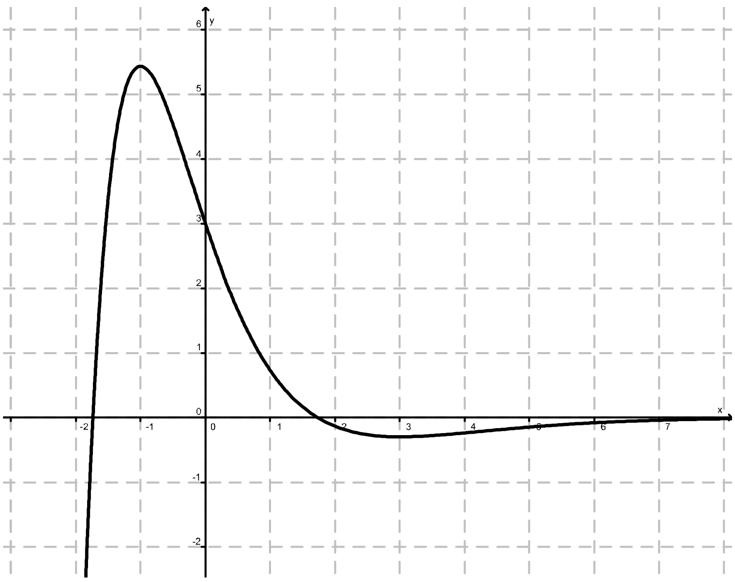
$x:$ $-\infty$	$2-\sqrt{5}$	$2+\sqrt{5}$	$+\infty$
$-e^{-x}$	$-$	$-$	$-$
x^2-4x-1	$+$	$-$	$+$
y''	$-$	$+$	$-$
y	$\cap$	$\cup$	$\cap$

zaključujemo da je
funkcija konkavna na intervalu $\left(-\infty,2-\sqrt{5}\right)\cup\left(2+\sqrt{5},+\infty\right)$
funkcija konveksna na intervalu $\left(2-\sqrt{5},2+\sqrt{5}\right)$.
Funkcija ima prevojne tačke

$$P_1(2-\sqrt{5},\left(-6+4\sqrt{5}\right)e^{-2+\sqrt{5}})$$

$$P_1(2+\sqrt{5},\left(-6-4\sqrt{5}\right)e^{-2-\sqrt{5}}).$$

7. *Grafik funkcije*:



Zadatak 12.6 *Ispitati osobine i nacrtati grafik funkcije*

$$f(x) = \ln\left(x^2 - 1\right).$$

Rješenje:

1. *Definiciono područje:*

Budući da je u pitanje logaritamska funkcija, onda mora biti

$$x^2 - 1 > 0 \iff x \in (-\infty, -1) \cup (1, +\infty)$$

pa je $\mathcal{D} = (-\infty, -1) \cup (1, +\infty)$.

2. *Nule i znak:*

$$\begin{aligned} f(x) &= 0 \iff \\ &\iff \ln\left(x^2 - 1\right) = 0 \\ &\iff x^2 - 1 = 1 \\ &\iff x^2 = 2 \\ &\iff x_1 = -\sqrt{2}, \quad x_2 = \sqrt{2}. \end{aligned}$$

Znak:

$x:$ $\quad -\infty$	$-\sqrt{2}$	-1	1	$\sqrt{2}$	$+\infty$
$\ln(x^2-1)$	$+$	$-$		$-$	$+$
y	$+$	$-$		$-$	$+$

Dakle,

$$\begin{aligned} y &< 0 \quad za \quad x \in \left(-\sqrt{2}, -1\right) \cup \left(1, \sqrt{2}\right) \\ y &> 0 \quad za \quad x \in \left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, +\infty\right). \end{aligned}$$

3. *Parnost:*

$$f(-x) = \ln\left((-x)^2 - 1\right) = \ln\left(x^2 - 1\right) = f(x)$$

Funkcija je parna pa je grafik simetričan u odnosu na y osu.
Funkcija nije periodična.
4. *Asimptote:*

Budući da je $\mathcal{D} = (-\infty, -1) \cup (1, +\infty)$, onda možemo ispitati samo lijevi limes u $x = -1$ i desni limes u $x = 1$

$$\begin{aligned} L(-1) &= \lim_{x \to -1^-} \ln\left(x^2 - 1\right) = \ln \lim_{x \to -1^-} \left(x^2 - 1\right) = \begin{bmatrix} smjena \\ x = -1 - \varepsilon \\ \varepsilon > 0, \varepsilon \to 0 \end{bmatrix} = \\ &= \ln \lim_{\varepsilon \to 0} \left((-1 - \varepsilon)^2 - 1\right) = \ln \lim_{\varepsilon \to 0} \left(1 + 2\varepsilon + \varepsilon^2 - 1\right) = \\ &= \ln \lim_{\varepsilon \to 0} \left(2\varepsilon + \varepsilon^2\right) = -\infty. \end{aligned}$$

$$\begin{aligned} D(1) &= \lim_{x \to 1^+} \ln\left(x^2 - 1\right) = \ln \lim_{x \to 1^+} \left(x^2 - 1\right) = \begin{bmatrix} smjena \\ x = 1 + \varepsilon \\ \varepsilon > 0, \varepsilon \to 0 \end{bmatrix} = \\ &= \ln \lim_{\varepsilon \to 0} \left((1 + \varepsilon)^2 - 1\right) = \ln \lim_{\varepsilon \to 0} \left(1 + 2\varepsilon + \varepsilon^2 - 1\right) = \\ &= \ln \lim_{\varepsilon \to 0} \left(2\varepsilon + \varepsilon^2\right) = -\infty. \end{aligned}$$

Budući da je

$$\begin{aligned} \lim_{x \to +\infty} f(x) &= \lim_{x \to +\infty} \ln\left(x^2 - 1\right) = \ln \lim_{x \to +\infty} \left(x^2 - 1\right) = +\infty \\ \lim_{x \to -\infty} f(x) &= \lim_{x \to -\infty} \ln\left(x^2 - 1\right) = \ln \lim_{x \to -\infty} \left(x^2 - 1\right) = +\infty \end{aligned}$$

funkcija nema horizontalnih asimptota.

Budući da je

$$\begin{aligned} k &= \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{\ln\left(x^2 - 1\right)}{x} \overset{L.P.}{=} \lim_{x \to \pm\infty} \frac{\frac{1}{x^2 - 1} \cdot 2x}{1} = \lim_{x \to \pm\infty} \frac{2x}{x^2 - 1} \overset{L.P.}{=} \\ &= \lim_{x \to \pm\infty} \frac{2}{2x} = \lim_{x \to \pm\infty} \frac{1}{x} = 0 \end{aligned}$$

funkcija nema kosih asimptota.
5. *Ekstremi i tok funkcije:*
Prvi izvod funkcije je

$$y' = \frac{1}{x^2-1} \cdot 2x = \frac{2x}{x^2-1}$$

pa je

$$\begin{aligned} y' &= 0 \\ \iff& \frac{2x}{x^2-1} = 0 \\ \iff& 2x = 0 \\ \iff& x = 0. \end{aligned}$$

Ali $0 \notin \mathcal{D}$ pa funkcija nema stacionarnih tačaka a samim tim ni ekstrema
Tok funkcije

$x:$ $-\infty$	$-\sqrt{2}$	-1	1	$\sqrt{2}$	$+\infty$
$2x$	$-$	$-$		$+$	$+$
x^2-1	$+$	$+$		$+$	$+$
y'	$-$	$-$		$+$	$+$
y	$\searrow$	$\searrow$		$\nearrow$	$\nearrow$

6. *Konveksnost i konkavnost. Prevojne tačke.*
Drugi izvod funkcije je

$$y'' = \frac{2(x^2-1) - 2x \cdot 2x}{x^2-1} = \frac{2x^2 - 2 - 4x^2}{x^2-1} = \frac{-2x^2-2}{x^2-1}$$

pa je

$$\begin{aligned} y'' &= 0 \\ \iff& \frac{-2x^2-2}{x^2-1} = 0 \\ \iff& -2x^2 - 2 = 0 \\ \iff& x^2 = -1 \end{aligned}$$

a posljednja jednačina nema realnih rješenje pa nema ni prevojnih tačaka
Iz

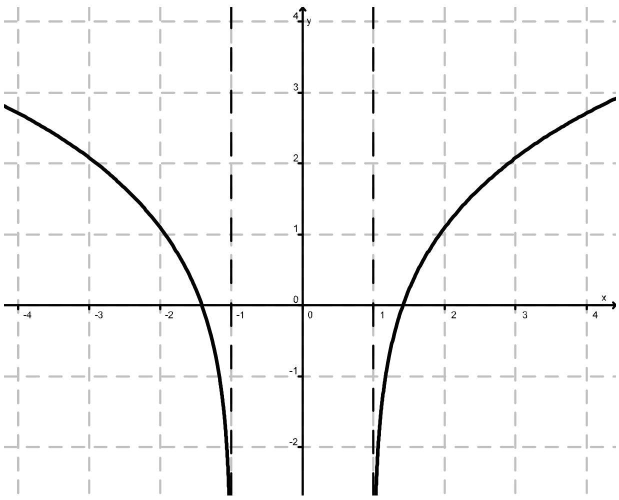
$x:$ $-\infty$	$-\sqrt{2}$	-1	1	$\sqrt{2}$	$+\infty$
$-2x^2-2$	$-$	$-$		$-$	$-$
x^2-1	$+$	$+$		$+$	$+$
y''	$-$	$-$		$-$	$-$
y	$\cap$	$\cap$		$\cap$	$\cap$

zaključujemo da je

funkcija konkavna na skupu $\mathcal{D}$.

Funkcija nema prevojnih tačaka.

7. *Grafik funkcije*:



13 Neodređeni i određeni integrali. Primjena određenog integrala.

14 Parcijalni izvodi. Ekstremi funkcije dvije promjenljive.

Zadatak 14.1 *Pokazati da za funkciju* $z = \ln\sqrt{x^2+y^2}$ *vrijedi*

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

Rješenje:
Kako je

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \left(\ln\sqrt{x^2+y^2}\right)'_x = \frac{1}{\sqrt{x^2+y^2}}\cdot\left(\sqrt{x^2+y^2}\right)'_x = \\
&= \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{x^2+y^2} \\
\frac{\partial z}{\partial y} &= \left(\ln\sqrt{x^2+y^2}\right)'_y = \frac{1}{\sqrt{x^2+y^2}}\cdot\left(\sqrt{x^2+y^2}\right)'_y = \\
&= \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y = \frac{y}{x^2+y^2}
\end{aligned}$$

tada je

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y\frac{x}{x^2+y^2} - x\frac{y}{x^2+y^2} = \frac{xy}{x^2+y^2} - \frac{xy}{x^2+y^2} = 0.$$

Zadatak 14.2 *Naći prvi i drugi totalni diferencijal funkcije*

$$z = 2x^2 - 3xy - y^2.$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 4x - 3y \\
\frac{\partial z}{\partial y} &= -3x - 2y
\end{aligned}$$

pa je prvi totalni diferencijal funkcije z

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (4x-3y)\,dx + (-3x-2y)\,dy.$$

Parcijalni izvodi drugog reda funkcije z su

$$\frac{\partial^2 z}{\partial x^2} = 4, \qquad \frac{\partial^2 z}{\partial x \partial y} = -3, \qquad \frac{\partial^2 z}{\partial y^2} = -2$$

pa je drugi totalni diferencijal funkcije z

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dxdy + \frac{\partial^2 z}{\partial y^2} dy^2 = 4dx^2 - 6dxdy - 2dy^2.$$

Zadatak 14.3 *Naći prvi i drugi totalni diferencijal funkcije*

$$z = \ln\left(x^2 + y\right).$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{x^2 + y} \cdot 2x = \frac{2x}{x^2 + y} \\ \frac{\partial z}{\partial y} &= \frac{1}{x^2 + y} \end{aligned}$$

pa je prvi totalni diferencijal funkcije z

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{2x}{x^2 + y} dx + \frac{1}{x^2 + y} dy.$$

Parcijalni izvodi drugog reda funkcije z su

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{2\left(x^2 + y\right) - 2x \cdot 2x}{\left(x^2 + y\right)^2} = \frac{-2x^2 + 2y}{\left(x^2 + y\right)^2} \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{-2x \cdot 1}{x^2 + y} = \frac{-2x}{\left(x^2 + y\right)^2} \\ \frac{\partial^2 z}{\partial y^2} &= \frac{-1}{\left(x^2 + y\right)^2} \end{aligned}$$

pa je drugi totalni diferencijal funkcije z

$$\begin{aligned} d^2 z &= \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dxdy + \frac{\partial^2 z}{\partial y^2} dy^2 = \\ &= \frac{-2x^2 + 2y}{\left(x^2 + y\right)^2} dx^2 - \frac{4x}{\left(x^2 + y\right)^2} dxdy - \frac{1}{\left(x^2 + y\right)^2} dy^2. \end{aligned}$$

Zadatak 14.4 *Pokazati da za funkciju*

$$z = \ln\left(x^2 + xy + y^2\right)$$

vrijedi

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{x^2+xy+y^2}\cdot(2x+y) = \frac{2x+y}{x^2+xy+y^2} \\
\frac{\partial z}{\partial y} &= \frac{1}{x^2+xy+y^2}\cdot(x+2y) = \frac{x+2y}{x^2+xy+y^2}
\end{aligned}$$

pa je

$$\begin{aligned}
x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x\frac{2x+y}{x^2+xy+y^2} + y\frac{x+2y}{x^2+xy+y^2} = \\
&= \frac{2x^2+xy}{x^2+xy+y^2} + \frac{xy+2y^2}{x^2+xy+y^2} = \\
&= \frac{2x^2+xy+xy+2y^2}{x^2+xy+y^2} = \frac{2x^2+2xy+2y^2}{x^2+xy+y^2} = \\
&= \frac{2x^2+2xy+2y^2}{x^2+xy+y^2} = \frac{2\left(x^2+xy+y^2\right)}{x^2+xy+y^2} = 2.
\end{aligned}$$

Zadatak 14.5 *Pokazati da za funkciju*

$$u = (x-y)(y-z)(z-x)$$

vrijedi

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned}
\frac{\partial u}{\partial x} &= (y-z)(z-x-(x-y)) = (y-z)(z-x-x+y) = \\
&= (y-z)(y+z-2x) \\
\frac{\partial u}{\partial y} &= (z-x)(-(y-z)+x-y) = (z-x)(-y+z+x-y) = \\
&= (z-x)(z+x-2y) \\
\frac{\partial u}{\partial z} &= (x-y)(-(z-x)+y-z) = (x-y)(-z+x+y-z) = \\
&= (x-y)(x+y-2z).
\end{aligned}$$

pa je

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} &= (y-z)(y+z-2x)+(z-x)(z+x-2y)+\\ &\quad (x-y)(x+y-2z)\\ &= y^2+yz-2xy-zy-z^2+2zx+z^2+zx-2zy-xz-\\ &\quad -x^2+2xy+x^2+xy-2xz-xy-y^2+2yz\\ &= 0.\end{aligned}$$

Zadatak 14.6 *Ispitati ekstreme funkcije*

$$z=x^2+xy+y^2-2x-y.$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x+y-2\\ \frac{\partial z}{\partial y} &= x+2y-1\end{aligned}$$

pa stacionarne tačke dobijamo rješavanjem sistema jednačina

$$\left.\begin{aligned}2x+y-2=0\\ x+2y-1=0\end{aligned}\right\}.$$

Iz prve jednačine imamo $y=-2x+2$, pa uvrštavajući u drugu jednačinu, dobijamo

$$\begin{aligned}x+2(-2x+2)-1 &= 0\\ x-4x+4-1 &= 0\\ -3x+3 &= 0 \Longrightarrow x=1 \Longrightarrow y=-2\cdot 1+2=0\end{aligned}$$

pa je stacionarna tačka funkcije $M(1,0)$.
Parcijalni izvodi drugog reda funkcije z su

$$\begin{aligned}A &= \frac{\partial^2 z}{\partial x^2}=2\\ B &= \frac{\partial^2 z}{\partial x\partial y}=1\\ C &= \frac{\partial^2 z}{\partial y^2}=2\end{aligned}$$

pa je

$$\Delta = AC - B^2 = 2 \cdot 2 - 1 = 4 - 1 = 3.$$

Za tačku $M(1,0)$ imamo

$$\Delta(M) = 3 > 0$$

pa ekstrem funkcije postoji a kako je

$$A(M) = 2 > 0$$

to funkcija ima minimum u tački $M(1,0)$ i on iznosi

$$z_{\min} = z(1,0) = 1 + 0 + 0 - 2 - 0 = -1.$$

Zadatak 14.7 *Ispitati ekstreme funkcije*

$$z = x^3 + y^3 - 6xy.$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned} \frac{\partial z}{\partial x} &= 3x^2 - 6y \\ \frac{\partial z}{\partial y} &= 3y^2 - 6x \end{aligned}$$

pa stacionarne tačke dobijamo rješavanjem sistema jednačina

$$\left.\begin{array}{l} 3x^2 - 6y = 0 \\ 3y^2 - 6x = 0 \end{array}\right\} \iff \left.\begin{array}{l} x^2 - 2y = 0 \\ y^2 - 2x = 0 \end{array}\right\}.$$

Iz prve jednačine imamo $y = \frac{x^2}{2}$, pa uvrštavajući u drugu jednačinu, dobijamo

$$\begin{aligned} \left(\frac{x^2}{2}\right)^2 - 2x &= 0 \\ \frac{x^4}{4} - 2x &= 0/ \quad \cdot 4 \\ x^4 - 8x &= 0 \\ x\left(x^3 - 8\right) &= 0 \Longrightarrow x = 0 \vee x = 2. \end{aligned}$$

Sada imamo

$$\begin{aligned} x &= 0 \Longrightarrow y = 0 \\ x &= 2 \Longrightarrow y = 2 \end{aligned}$$

pa su stacionarne tačke

$$M_1(0,0), \qquad M_2\left(2,2\right).$$

Parcijalni izvodi drugog reda funkcije z su

$$\begin{aligned}
A &= \frac{\partial^2 z}{\partial x^2} = 6x \\
B &= \frac{\partial^2 z}{\partial x \partial y} = 6y \\
C &= \frac{\partial^2 z}{\partial y^2} = -6
\end{aligned}$$

pa je

$$\Delta = AC - B^2 = 6x \cdot 6x - 36 = 36x^2 - 36.$$

Za tačku $M_1(0,0)$ imamo

$$\Delta\left(M_1\right) = -36 < 0$$

pa funkcija nema ekstrema u tački $M_1(0,0)$.
Za tačku $M_2\left(2,2\right)$ imamo

$$\Delta\left(M_2\right) = 144 - 36 = 108 > 0$$

pa postoji ekstrem funkcije z u tački $M_2\left(2,2\right)$. Kako je

$$A\left(M_2\right) = 12 > 0$$

to funkcija ima minimum u tački $M_2\left(2,2\right)$ i on iznosi

$$z_{\min} = z(2,2) = 2^3 + 2^3 - 6 \cdot 2 \cdot 2 = -8.$$

Zadatak 14.8 *Ispitati ekstreme funkcije*

$$z = x^3 + 3xy^2 - 15x - 12y.$$

Rješenje:
Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 3x^2 + 3y^2 - 15 \\
\frac{\partial z}{\partial y} &= 6xy - 12
\end{aligned}$$

pa stacionarne tačke dobijamo rješavanjem sistema jednačina

$$\left.\begin{array}{r} 3x^2+3y^2-15=0 \\ 6xy-12=0 \end{array}\right\}.$$

Iz druge jednačine imamo

$$6xy-12=0 \implies xy=2 \implies y=\frac{2}{x}$$

pa uvrštavajući u prvu jednačinu sistema, dobijamo

$$\begin{aligned} 3x^2+3\left(\frac{2}{x}\right)^2-15 &= 0 \\ 3x^2+3\frac{4}{x^2}-15 &= 0 \quad / \quad \cdot x^2 \\ 3x^4-15x^2+12 &= 0 \\ x^4-5x^2+4 &= 0. \end{aligned}$$

Uvodeći smjenu $t=x^2$, dobijamo kvadratnu jednačinu

$$t^2-5t+4=0 \implies t_1=1 \quad i \quad t_2=4.$$

Sada je

$$\begin{aligned} t_1 &= 1 \implies x^2=1 \implies \begin{array}{l} x_1=-1 \implies y_1=-2 \\ x_2=1 \implies y_2=2 \end{array} \\ t_1 &= 4 \implies x^2=4 \implies \begin{array}{l} x_1=-2 \implies y_1=-1 \\ x_2=2 \implies y_2=1 \end{array} \end{aligned}$$

pa su stacionarne tačke

$$M_1(-1,-2), \quad M_2(1,2), \quad M_3(-2,-1), \quad M_4(2,1).$$

Parcijalni izvodi drugog reda funkcije z su

$$\begin{aligned} A &= \frac{\partial^2 z}{\partial x^2}=6x \\ B &= \frac{\partial^2 z}{\partial x \partial y}=6y \\ C &= \frac{\partial^2 z}{\partial y^2}=6x \end{aligned}$$

pa je

$$\Delta = AC - B^2 = 6x \cdot 6x - 6y = 36x^2 - 36y^2.$$

Za tačku $M_1(-1,-2)$ imamo

$$\Delta(M_1) = 36 - 144 = -108 < 0$$

pa ekstrem funkcije ne postoji.

Za tačku $M_2(1,2)$ imamo

$$\Delta(M_1) = 36 - 144 = -108 < 0$$

pa ekstrem funkcije ne postoji.

Za tačku $M_3(-2,-1)$ imamo

$$\Delta(M_3) = 144 - 36 = 108 > 0$$

pa ekstrem funkcije postoji a kako je

$$A(M_3) = -12 < 0$$

to funkcija ima maksimum u tački $M_3(-2,-1)$ i on iznosi

$$z_{\max} = z(-2,-1) = -8 - 6 + 30 + 12 = 28.$$

Za tačku $M_4(2,1)$ imamo

$$\Delta(M_4) = 144 - 36 = 108 > 0$$

pa ekstrem funkcije postoji a kako je

$$A(M_4) = 12 > 0$$

to funkcija ima minimum u tački $M_4(2,1)$ i on iznosi

$$z_{\min} = z(2,1) = 8 + 6 - 30 - 12 = -28.$$

Zadatak 14.9 *Ispitati ekstreme funkcije*

$$z = x^4 + y^4 - x^2 - 2xy - y^2.$$

Rješenje:

Parcijalni izvodi prvog reda funkcije z su

$$\begin{aligned} \frac{\partial z}{\partial x} &= 4x^3 - 2x - 2y \\ \frac{\partial z}{\partial y} &= 4y^3 - 2y - 2x \end{aligned}$$

pa stacionarne tačke dobijamo rješavanjem sistema jednačina

$$\left.\begin{array}{r} 4x^3 - 2x - 2y = 0 \\ 4y^3 - 2y - 2x = 0 \end{array}\right\} \iff \left.\begin{array}{r} 2x^3 - x - y = 0 \\ 2y^3 - y - x = 0 \end{array}\right\}.$$

Oduzimanjem ovih jednačina, dobijamo

$$x^3 = y^3 = 0 \implies x^3 = y^3 \implies x = y.$$

Vraćajući se u prvu jednačinu, dobijamo

$$\begin{aligned} 2x^3 - x - x &= 0 \\ 2x^3 - 2x &= 0 \\ x^3 - x &= 0 \\ x\left(x^2 - 1\right) &= 0 \implies x_1 = 0, \quad x_2 = -1, \quad x_3 = 1. \end{aligned}$$

Stacionarne tačke su

$$M_1(0,0), \quad M_2(-1,-1), \quad M_3(1,1).$$

Parcijalni izvodi drugog reda funkcije z su

$$\begin{aligned} A &= \frac{\partial^2 z}{\partial x^2} = 12x^2 - 2 \\ B &= \frac{\partial^2 z}{\partial x \partial y} = -2 \\ C &= \frac{\partial^2 z}{\partial y^2} = 12y^2 - 2 \end{aligned}$$

pa je

$$\Delta = AC - B^2 = \left(12x^2 - 2\right) \cdot \left(12y^2 - 2\right) - 4.$$

Za tačku $M_1(0,0)$ imamo

$$\Delta\left(M_1\right) = 4 - 4 = 0$$

pa su potrebna dodatna ispitivanja za ekstrem funkcije.

Za tačku $M_2(-1,-1)$ imamo

$$\Delta\left(M_2\right) = 10 \cdot 10 - 4 = 96 > 0$$

pa ekstrem funkcije postoji a kako je

$$A\left(M_2\right) = 10 > 0$$

to funkcija ima minimum u tački $M_2(-1,-1)$ i on iznosi

$$z_{\min} = (-1)^4 + (-1)^4 - (-1)^2 - 2(-1)(-1) - (-1)^2 = -2.$$

Za tačku $M_3(1,1)$ imamo

$$\Delta(M_2) = 10 \cdot 10 - 4 = 96 > 0$$

pa ekstrem funkcije postoji a kako je

$$A(M_2) = 10 > 0$$

to funkcija ima minimum u tački $M_3(1,1)$ i on iznosi

$$z_{\min} = 1^4 + 1^4 - 1^2 - 2 \cdot 1 \cdot 1 - 1^2 = -2.$$

Zadatak 14.10 *Ispitati ekstreme funkcije*

$$z = x^2 + y^2$$

uz uslov $3x + 2y = 6$.

Rješenje:
Lagranževa funkcija je

$$F(x,y,\lambda) = x^2 + y^2 + \lambda(3x + 2y - 6)$$

a parcijalni izvodi prvog reda funkcije F su

$$\begin{aligned} \frac{\partial F}{\partial x} &= 2x + 3\lambda \\ \frac{\partial F}{\partial y} &= 2y + 2\lambda \\ \frac{\partial F}{\partial \lambda} &= 3x + 2y - 6 \end{aligned}$$

pa stacionarne tačke i vrijednost λ dobijamo rješavanjem sistema jednačina

$$\left.\begin{aligned} 2x + 3\lambda = 0 \\ 2y + 2\lambda = 0 \\ 3x + 2y - 6 = 0 \end{aligned}\right\}.$$

Odredimo stacionarne tačke

$$\left.\begin{aligned} 2x + 3\lambda = 0 &\implies x = -\tfrac{3}{2}\lambda \\ 2y + 2\lambda = 0 &\implies y = -\lambda \end{aligned}\right\}$$

pa uvrštavajui u treću jednačinu sistema, dobijamo

$$\begin{aligned} 3x + 2y - 6 &= 0 \\ 3 \cdot \left(-\frac{3}{2}\lambda \right) + 2 \cdot (-\lambda) - 6 &= 0 \\ -9\lambda - 4\lambda - 12 &= 0 \\ -13\lambda &= 12 \Longrightarrow \lambda = -\frac{12}{13} \end{aligned}$$

pa dobijamo da je

$$\begin{aligned} x &= -\frac{3}{2}\lambda = -\frac{3}{2} \cdot \left(-\frac{12}{13} \right) = \frac{18}{13} \\ y &= -\lambda = \frac{12}{13} \end{aligned}$$

pa funkcija ima samo jednu stacionarnu tačku $M_1\left(\frac{18}{13}, \frac{12}{13}\right)$.

Parcijalni izvodi drugog reda funkcije F su

$$\begin{aligned} A &= \frac{\partial^2 F}{\partial x^2} = 2 \\ B &= \frac{\partial^2 F}{\partial x \partial y} = 0 \\ C &= \frac{\partial^2 F}{\partial y^2} = 2. \end{aligned}$$

pa je

$$\Delta(M_1) = AC - B^2 = 4 - 0 = 4 > 0$$

pa funkcija ima ekstrem u toj tački, a kako je

$$A(M_1) = 2 > 0$$

to funkcija ima minimum u tački $M_1\left(\frac{18}{13}, \frac{12}{13}\right)$ i on iznosi

$$z_{\min} = x^2 + y^2 = \left(\frac{18}{13} \right)^2 + \left(\frac{12}{13} \right)^2 = \frac{324}{169} + \frac{144}{169} = \frac{468}{169} = \frac{36}{13}.$$

Zadatak 14.11 *Ispitati ekstreme funkcije*

$$z = 6 - 4x - 3y$$

uz uslov $x^2 + y^2 = 1$.

Rješenje:
Lagranževa funkcija je

$$F(x,y,\lambda) = 6 - 4x - 3y + \lambda\left(x^2 + y^2 - 1\right)$$

a parcijalni izvodi prvog reda funkcije F su

$$\begin{aligned}
\frac{\partial F}{\partial x} &= -4 + 2\lambda x \\
\frac{\partial F}{\partial y} &= -3 + 2\lambda y \\
\frac{\partial F}{\partial \lambda} &= x^2 + y^2 - 1
\end{aligned}$$

pa stacionarne tačke i vrijednost λ dobijamo rješavanjem sistema jednačina

$$\left.\begin{aligned}
-4 + 2\lambda x = 0 \\
-3 + 2\lambda y = 0 \\
x^2 + y^2 - 1 = 0
\end{aligned}\right\}.$$

Iz prve jednačine imamo $x = \frac{2}{\lambda}$, a iz druge $y = \frac{3}{2\lambda}$. Uvrštavajući u treću jednačinu dobijamo

$$\begin{aligned}
\left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 - 1 &= 0 \\
\frac{4}{\lambda^2} + \frac{9}{4\lambda^2} - 1 &= 0/ \qquad \cdot 4\lambda^2 \\
-4\lambda^2 + 25 &= 0 \\
\lambda^2 &= \frac{25}{4} \Longrightarrow \lambda_1 = -\frac{5}{2}, \qquad \lambda_2 = \frac{5}{2}.
\end{aligned}$$

Sada je

$$\begin{aligned}
\lambda_1 &= -\frac{5}{2} \Longrightarrow x = -\frac{4}{5}, \quad y = -\frac{3}{5} \\
\lambda_2 &= \frac{5}{2} \Longrightarrow x = \frac{4}{5}, \quad y = \frac{3}{5}
\end{aligned}$$

pa su stacionarne tačke

$$M_1\left(-\frac{4}{5}, -\frac{3}{5}\right), \qquad M_2\left(\frac{4}{5}, \frac{3}{5}\right).$$

Parcijalni izvodi drugog reda funkcije F su

$$\begin{aligned} A &= \frac{\partial^2 F}{\partial x^2} = 2\lambda \\ B &= \frac{\partial^2 F}{\partial x \partial y} = 0 \\ C &= \frac{\partial^2 F}{\partial y^2} = 2\lambda. \end{aligned}$$

pa je

$$\Delta = AC - B^2 = 2\lambda \cdot 2\lambda - 0 = 4\lambda^2.$$

Za tačku $M_1\left(-\frac{4}{5}, -\frac{3}{5}\right), \lambda = -\frac{5}{2}$ imamo

$$\Delta\left(M_1\right) = 25 > 0$$

pa funkcija ima ekstrem u tački $M_1\left(-\frac{4}{5}, -\frac{3}{5}\right)$ a kako je

$$A\left(M_1\right) = -5 < 0$$

to funkcija ima maksimum u tački $M_1\left(-\frac{4}{5}, -\frac{3}{5}\right)$ i on iznosi

$$z_{\max} = z\left(-\frac{4}{5}, -\frac{3}{5}\right) = 6 - 4\left(-\frac{4}{5}\right) - 3\left(-\frac{3}{5}\right) = 6 + \frac{16}{5} + \frac{9}{5} = 11.$$

Za tačku $M_2\left(\frac{4}{5}, \frac{3}{5}\right), \lambda = \frac{5}{2}$ imamo

$$\Delta\left(M_2\right) = 25 > 0$$

pa funkcija ima ekstrem u tački $M_2\left(\frac{4}{5}, \frac{3}{5}\right)$ a kako je

$$A\left(M_2\right) = 5 > 0$$

to funkcija ima minimum u tački $M_2\left(\frac{4}{5}, \frac{3}{5}\right)$ i on iznosi

$$z_{\min} = z\left(\frac{4}{5}, \frac{3}{5}\right) = 6 - 4 \cdot \frac{4}{5} - 3 \cdot \frac{3}{5} = 6 - \frac{16}{5} - \frac{9}{5} = 1.$$

Zadatak 14.12 *Ispitati ekstreme funkcije*

$$z = xy$$

uz uslov $x^2 + y^2 = 8$.

Rješenje:
Lagranževa funkcija je

$$F(x,y,\lambda) = xy + \lambda\left(x^2 + y^2 - 8\right)$$

a parcijalni izvodi prvog reda funkcije F su

$$\begin{aligned} \frac{\partial F}{\partial x} &= y + 2\lambda x \\ \frac{\partial F}{\partial y} &= x + 2\lambda y \\ \frac{\partial F}{\partial \lambda} &= x^2 + y^2 - 8 \end{aligned}$$

pa stacionarne tačke i vrijednost λ dobijamo rješavanjem sistema jednačina

$$\left.\begin{array}{r} y + 2\lambda x = 0 \\ x + 2\lambda y = 0 \\ x^2 + y^2 - 8 = 0 \end{array}\right\}.$$

Odredimo stacionarne tačke

$$\left.\begin{array}{l} y + 2\lambda x = 0 \implies \lambda = -\frac{y}{2x} \\ x + 2\lambda y = 0 \implies \lambda = -\frac{x}{2y} \end{array}\right\} \implies -\frac{y}{2x} = -\frac{x}{2y} \implies x^2 = y^2 \implies x = \pm y$$

pa uvrštavanjem u treću jednačinu dobijamo

$$\begin{aligned} x^2 + y^2 - 8 &= 0 \\ 2x^2 - 8 &= 0 \\ x^2 &= 4 \implies x = \pm 2 \end{aligned}$$

pa su stacionarne tačke

$$M_1(2,2), \quad M_2(2,-2), \quad M_3(-2,2), \quad M_4(-2,-2).$$

Parcijalni izvodi drugog reda funkcije F su

$$\begin{aligned} A &= \frac{\partial^2 F}{\partial x^2} = 2\lambda \\ B &= \frac{\partial^2 F}{\partial x \partial y} = 1 \\ C &= \frac{\partial^2 F}{\partial y^2} = 2\lambda. \end{aligned}$$

pa je

$$\Delta = AC - B^2 = 2\lambda \cdot 2\lambda - 1 = 4\lambda^2 - 1.$$

Ispitajmo ekstreme funkcije preko drugog totalnog diferencijala. Drugi totalni diferencijal funkcije z je

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2 = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2.$$

Za tačku $M_1\left(2,2\right), \lambda = -\frac{1}{2}$ imamo

$$d^2z = d^2z = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2 = -dx^2 + 2dxdy - dy^2 = -\left(dx - dy\right)^2 < 0$$

pa funkcija ima maksimum u tački $M_1\left(2,2\right)$ i on iznosi

$$z_{\max} = z(2,2) = 2 \cdot 2 = 4.$$

Za tačku $M_2\left(2,-2\right), \lambda = \frac{1}{2}$ imamo

$$d^2z = d^2z = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2 = dx^2 + 2dxdy + dy^2 = \left(dx + dy\right)^2 > 0$$

pa funkcija ima minimum u tački $M_2\left(2,-2\right)$ i on iznosi

$$z_{\min} = z(2,-2) = 2 \cdot (-2) = -4.$$

Za tačku $M_3(-2,2), \lambda = \frac{1}{2}$ imamo

$$d^2z = d^2z = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2 = dx^2 + 2dxdy + dy^2 = \left(dx + dy\right)^2 > 0$$

pa funkcija ima minimum u tački $M_3(-2,2)$ i on iznosi

$$z_{\min} = z(2,-2) = 2 \cdot (-2) = -4.$$

Za tačku $M_4(-2,-2), \lambda = -\frac{1}{2}$ imamo

$$d^2z = d^2z = 2\lambda dx^2 + 2dxdy + 2\lambda dy^2 = -dx^2 + 2dxdy - dy^2 = -\left(dx - dy\right)^2 < 0$$

pa funkcija ima maksimum u tački $M_4(-2,-2)$ i on iznosi

$$z_{\max} = z(-2,-2) = (-2) \cdot (-2) = 4.$$

15 Diferencijalne jednačine