

Puzzle: Zermelo-Fraenkel set theory is inconsistent

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Abstract: In this note, we present a puzzle. We prove that Zermelo-Fraenkel set theory is inconsistent by proving, using Zermelo-Fraenkel set theory, the false statement that any algorithm that determines whether any $n \times n$ matrix over \mathbb{F}_2 , the finite field of order 2, is nonsingular must run in exponential time in the worst-case scenario. The object of the puzzle is to find the error in the proof.

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In this note, we present a puzzle. Let M_n be the set of $n \times n$ matrices over \mathbb{F}_2 , the finite field of order 2. And let $f_i : M_n \rightarrow \{0, 1\}$, for $i = 1, \dots, m$, be m functions with the following special property: For any $j \in \{1, \dots, m\}$, there exist at least two $n \times n$ matrices, A and B , such that $f_i(A) = f_i(B) = 1$ for each $i = 1, \dots, j-1, j+1, \dots, m$, but $f_j(A) = 0$ and $f_j(B) = 1$. And let $f : M_n \rightarrow \{0, 1\}$ be defined as $f(A) = \prod_{i=1}^m f_i(A)$ for each $A \in M_n$. We shall now prove, using Zermelo-Fraenkel set theory [1], the following theorem, that we shall afterwards show is false:

Theorem: For any algorithm that computes $f(A)$ given any matrix $A \in M_n$, the algorithm must compute $f_i(A)$ for each $i = 1, \dots, m$ whenever $f_i(A) = 1$ for each $i = 1, \dots, m$, which takes at least m steps.

Proof: We use induction on m : For $m = 1$, the theorem is a tautology.

Assume true for $m = k$. We shall prove true for $m = k + 1$: Let Q be an algorithm that computes $f(A)$ given any matrix $A \in M_n$. Suppose that $f_{k+1}(A) = 1$. Then $f(A) = \prod_{i=1}^k f_i(A)$, so by the induction hypothesis, Q must compute $f_i(A)$ for each $i = 1, \dots, k$ if $f_i(A) = 1$ for each $i = 1, \dots, k$, which takes k steps. Suppose that $f_i(A) = 1$ for each $i = 1, \dots, k$. Then $f(A) = f_{k+1}(A)$, so by the special property of the functions f_i given above, Q must compute $f_{k+1}(A)$, which takes at least one step. Hence, whenever $f_i(A) = 1$ for each $i = 1, \dots, k + 1$, Q must compute $f_i(A)$ for each $i = 1, \dots, k + 1$, which takes at least $k + 1$ steps. \square

We can easily see that the above theorem is false when we let $m = 2^n - 1$ and we define functions $f_i :$

$M_n \rightarrow \{0, 1\}$, where each $i \in \{1, \dots, m\}$ corresponds to a vector $\mathbf{x} \in \mathbb{F}_2^n \setminus \{\mathbf{0}\}$ via a bijection $g : \{1, \dots, m\} \rightarrow \mathbb{F}_2^n \setminus \{\mathbf{0}\}$, such that $f_{g^{-1}(\mathbf{x})}(A) = 0$ if and only if $A\mathbf{x} = \mathbf{0}$. In this situation, it is not necessary for an algorithm to take at least $m = 2^n - 1$ steps in the worst-case scenario to compute $f(A) = \prod_{i=1}^m f_i(A)$, since computing $f(A)$ is equivalent to determining whether A is nonsingular and it is possible to determine in polynomial-time whether any matrix A is nonsingular via Gaussian elimination [2]. Hence, since we have proven, using Zermelo-Fraenkel set theory, a statement that is known to be false, we can conclude that Zermelo-Fraenkel set theory is inconsistent. Puzzle: Where is the error? (There is an error.)

References

- [1] Weisstein, Eric W. “Zermelo-Fraenkel Set Theory.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/Zermelo-FraenkelSetTheory.html>
- [2] Weisstein, Eric W. “Gaussian Elimination.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/GaussianElimination.html>