

Solution of the Linear Ordering Problem (NP=P)

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We consider the following problem

$$\begin{aligned} \max \quad & \sum_{i=1, i \neq j}^n \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s. t.} \quad & 0 \leq x_{ij} \leq 1, \\ & x_{ij} + x_{ji} = 1, \\ & 0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1, i \neq j, i \neq k, j \neq k, i, j, k = 1, \dots, n. \end{aligned}$$

We denote the corresponding polytope by B_n . The polytope B_n has integer vertices corresponding to feasible solutions of the linear ordering problem as well as non-integer vertices. We denote the polytope of integer vertices as P_n .

Let us give an example of non-integer vertex in B_n and describe an exact facet cut. In what follows we will interested only in generating exact facet cuts.

Fig. 1 shows a graph interpretation of a non-integer vertex [1],

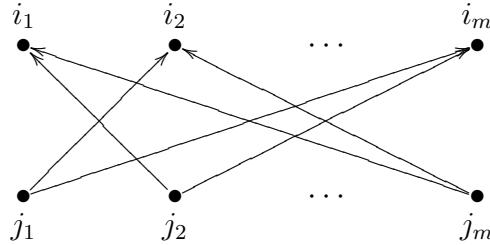


Fig. 1

where $\{i_1, \dots, i_m\}$, $\{j_1, \dots, j_m\} \subset \{1, \dots, n\}$; $\{i_1, \dots, i_m\} \cap \{j_1, \dots, j_m\} = \emptyset$; $x_{ilj_q} = 0$, $x_{j_qi_l} = 1$, $l \neq q$, $l, q = 1, \dots, m$; the other variables that are not shown at the Figure 1 are equal to $\frac{1}{2}$. This is the simplest non-integer vertex of the polytope B_n . For this vertex all adjacent integer vertices can be written as:

$$\alpha_k i_k j_k \beta_k, \quad \text{where } \alpha_k \text{ is any ordering from the set } \{j_1, \dots, j_m\} \setminus \{j_k\}, \\ \beta_k \text{ is any ordering from the set } \{i_1, \dots, i_m\} \setminus \{i_k\},$$

$$\alpha_{kp} i_k j_k i_p j_p \beta_{kp}, \quad \text{where } \alpha_{kp} \text{ is any ordering from the set } \{j_1, \dots, j_m\} \setminus \{j_k, j_p\}, \\ \beta_{kp} \text{ is any ordering from the set } \{i_1, \dots, i_m\} \setminus \{i_k, i_p\},$$

$$k \neq p, k, p = 1, \dots, m.$$

All adjacent integer vertices, which number is equal to

$$m [(m-1)!]^2 + \frac{m(m-1)}{2} [(m-2)!]^2$$

lie in one hyperplane

$$f(x) = 2 \sum_{l=1}^m x_{ilj_l} - \sum_{l=1}^m \sum_{q=1}^m x_{ilj_q} = 1.$$

This hyperplane for $f(x) \leq 1$ is a facet of the polytope P_n .

Our aim is to determine exact facet cuts for any non-integer vertex of B_n (and not only for them) in an analogous fashion.

Figures 2 and 3 show non-integer vertices of the polytope B_n :

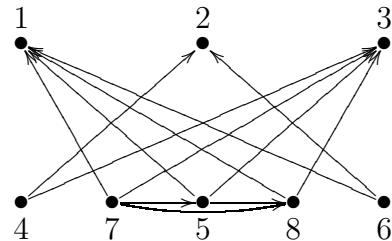


Fig. 2

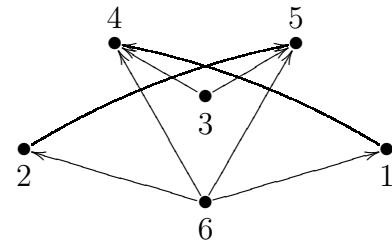


Fig. 3

Noninteger vertex at Figure 2 has an oriented chain 7583 of the length 3, and non-integer vertex at Figure 3 has an oriented 614 chain of the length

2. The oriented chain 758 at Figure 2 is independent, that is if we exchange the chain 758 with any other chain the rest of the graph does not change, while the chains 81 at Figure 2 and 614 at Figure 3 are dependent. We define corresponding dependent and independent oriented chains.

The following Theorems take place.

Theorem 1 *Let x^0 be a noninteger vertex in B_n and assume that in graph interpretation there is a graph vertex i which is the begin or the end of all adjacent arcs. Assume that the vertex i can be repeated arbitrarily many times such that each of the new vertices has the same location with the other part of the graph as the vertex i . Then the new noninteger vertex, corresponding to the new graph, is a noninteger vertex of B_n , and in the new graph the vertices i and new inserted vertices may be put in any order.*

Theorem 2 *Let x^0 be a noninteger vertex in B_n . Then there does not exist corresponding dependent oriented chains of the length 4.*

The polytope B_n has noninteger vertices whose fractional components are equal to $\frac{l}{r}$, $r \geq 2$, $l < r$, as well.

For $r = 2$ after matrix transformation we get in all cases the following non unimodular minimal standard matrix:

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

For $r = 3$ after matrix transformation we get in all cases the known combination of two minimal standard matrices:

$$\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

For any r after matrix transformation we get known combination of $r - 1$ minimal standard matrices.

Theorem 3 Let x^0 be a noninteger vertex of B_n , which has fractional components $\frac{l_1}{r_1}$, $l_1 < r_1$, $r_1 \geq 3$. Then we pass to an adjacent noninteger vertex with fractional components $\frac{l_2}{r_2}$, $r_2 < r_1$, $l_2 < r_2$, by changing an equality in a basis (thus changing one or more minimal standard matrices).

Let $I_1, I_2, \dots, I_s \subset \{1, \dots, n\}$, and assume that each set I_p , $p = 1, \dots, s$, corresponds to noninteger components of a vertex. Then for each I_p , $p = 1, \dots, s$, we can construct a facet cut. If $s = 1$ a noninteger vertex is called simple. A noninteger vertex is called complex if $s \geq 2$.

Thus we have given a general description of the polytope B_n .

Theorem 4 Let x^0 be a simple noninteger vertex of the polytope B_n with fractional components $\frac{1}{2}$, assume further that there does not exist dependent oriented chains with the length 3. Then all adjacent integer vertices lie in one hyperplane, this hyperplane is a facet of the polytope P_n , and it can be constructed by a polynomial algorithm.

Now we describe the principle for constructing facets.

Consider a noninteger vertex x^0 . It can be defined as the solution of the following system of the linear basic equalities

$$\begin{aligned} x_{i_l j_l} &= 0, \quad l = 1, \dots, q; \\ x_{i_l j_l} + x_{j_l k_l} - x_{i_l k_l} &= 0, \quad l = q + 1, \dots, \frac{n^2 - n}{2}. \end{aligned} \tag{1}$$

We introduce artificial variables $x_{j_l n+1} = 0$, $x_{i_l n+1} = 0$, into the first q equalities of the system (1):

$$x_{i_l j_l} + x_{j_l n+1} - x_{i_l n+1} = 0, \quad l = 1, \dots, q.$$

With the help of the notation

$$x_{i_l j_l} + x_{j_l k_l} - x_{i_l k_l} := x(i_l, j_l, k_l), \quad l = 1, \dots, \frac{n^2 - n}{2},$$

we rewrite the system (1) as follows:

$$x(i_l, j_l, k_l) = 0, \quad l = 1, \dots, \frac{n^2 - n}{2}.$$

We can determine $\frac{n^2-n}{2}$ linear independent adjacent integer vertices

$$x^q(i_s, j_s, k_s) = \delta^q(i_s, j_s, k_s), \quad s, q = 1, \dots, \frac{n^2-n}{2},$$

where δ^q is either 1 or 0. We can prove that all adjacent integer vertices lie in the hyperplane:

$$f(x) = \begin{vmatrix} x(i_1, j_1, k_1) & \dots & x(i_m, j_m, k_m) & 1 \\ \delta^1(i_1, j_1, k_1) & \dots & \delta^1(i_m, j_m, k_m) & 1 \\ \dots & & \dots & \\ \delta^m(i_1, j_1, k_1) & \dots & \delta^m(i_m, j_m, k_m) & 1 \end{vmatrix} = 0$$

where $m = \frac{n^2-n}{2}$.

Theorem 5 Let x^0 be a simple noninteger vertex of the polytope B_n with fractional components $\frac{1}{2}$, assume further that there exist τ dependent oriented chains with the length 3. Then all adjacent integer vertices lie in 2^τ hyperplanes, each of them is a facet of the polytope P_n , and can be constructed by a polynomial algorithm.

Theorem 6 Let x^0 be a simple noninteger vertex of the polytope B_n with fractional components $\frac{l}{r}$, $r \geq 3$, $l < r$. Then we can construct all minimal standard matrices and corresponding noninteger vertices with fractional components $\frac{1}{2}$. For every such vertex we can construct facet cuts.

Theorem 7 Let x^0 be a complex noninteger vertex of the polytope B_n , and I_1, I_2, \dots, I_s correspond to noninteger values. Then we can construct facet cuts for each set I_p , $p = 1, \dots, s$.

Assume we have generated facet cuts. Solving the problem again we get the noninteger vertex x^1 of the polytope

$$\begin{aligned} 0 &\leq x_{ij} \leq 1, \\ x_{ij} + x_{ji} &= 1, \\ 0 &\leq x_{ij} + x_{jk} - x_{ik} \leq 1, \quad i \neq j, \quad i \neq k, \quad j \neq k, \quad i, j, k = 1, \dots, n, \\ f_s^1 &\leq f_s(x) \leq f_s^2, \quad s = 1, \dots, q. \end{aligned}$$

Without loss of generality we may assume that the noninteger vertex x^1 satisfies the following linear system:

$$\begin{aligned} x(i_s, j_s, k_s) &= 1, \quad i = 1, \dots, p, \\ f_s(x) &= f_s^2, \quad s = 1, \dots, q. \end{aligned}$$

Now we find all adjacent integer vertices. If all of them lie in one hyperplane and this hyperplane is a facet of P_n then we generate this facet and re-solve the problem with a new facet. In case we cannot determine the facet we solve the auxiliary problem:

$$\begin{aligned} \max \quad & \left(\sum_{s=1}^p x(i_s, j_s, k_s) + \sum_{s=1}^q \frac{f_s(x) - f_s^1}{f_s^2 - f_s^1} \right), \\ \text{subject to} \quad & 0 \leq x_{ij} \leq 1, \\ & x_{ij} + x_{ji} = 1, \\ & 0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1, \quad i \neq j, \quad i \neq k, \quad j \neq k, \quad i, j, k = 1, \dots, n, \end{aligned}$$

With the solution of this problem we can construct the facet of the polytope P_n . In the case of theorems 5 and 6 we can construct the necessary facets by means of a polynomial algorithm.

References

- [1] Bolotashvili G., Kovalev M., Girlich E. New Facets of the linear ordering Polytope. SIAM J. Discrete Mathematics 12(3):326-336, 1999.