

An elegant argument that $P \neq NP$

Craig Alan Feinstein

2712 Willow Glen Drive, Baltimore, Maryland 21209

E-mail: cafeinst@msn.com, BS”D

Abstract: In this note, we present an elegant argument that $P \neq NP$ by demonstrating that the Meet-in-the-Middle algorithm must have the fastest running-time of all deterministic and exact algorithms which solve the SUBSET-SUM problem on a classical computer.

Disclaimer: This article was authored by Craig Alan Feinstein in his private capacity. No official support or endorsement by the U.S. Government is intended or should be inferred.

“This one’s from *The Book!*” - Paul Erdős (1913-1996)

Consider the following problem: Let $\{s_1, \dots, s_n\}$ be a set of n integers and t be another integer. We want to determine whether there exists a subset of $\{s_1, \dots, s_n\}$ for which the sum of its elements equals t . We shall consider the sum of the elements of the empty set to be zero. This problem is called the SUBSET-SUM problem [2, 4]. Let

$$S_k^+ = \left\{ \sum_{i \in I^+} s_i \mid I^+ \subseteq \{1, \dots, k\} \right\}$$

and

$$S_k^- = \left\{ \sum_{i \in I^-} s_i \mid I^- \subseteq \{k+1, \dots, n\} \right\},$$

where $k \in \{1, \dots, n\}$. Notice that for any $k \in \{1, \dots, n\}$, the SUBSET-SUM problem is equivalent to the problem of determining whether set $S_k^+ + S_k^-$ intersects set $\{t\}$; therefore, for any $k \in \{1, \dots, n\}$, the SUBSET-SUM problem is equivalent to the problem of determining whether set S_k^+ intersects set $t - S_k^-$. Now consider the following algorithm for solving the SUBSET-SUM problem:

Meet-in-the-Middle Algorithm - Sort the sets $S_{\lfloor n/2 \rfloor}^+$ and $t - S_{\lfloor n/2 \rfloor}^-$ in ascending order. Compare the first elements in both of the lists. If they match, then output “YES”. If not, then compare the greater element with the next element in the other list. Continue this process until there is a match, in which case the computer outputs “YES”, or until one of the lists runs out of elements, in which case the computer outputs “NO”.

This algorithm takes $\Theta(\sqrt{2^n})$ time, since it takes $\Theta(\sqrt{2^n})$ steps to sort sets $S_{\lfloor n/2 \rfloor}^+$ and $t - S_{\lfloor n/2 \rfloor}^-$ and

$O(\sqrt{2^n})$ steps to compare elements from each of the two sets. It turns out that no deterministic and exact algorithm with a better worst-case running-time has ever been found since Horowitz and Sahni published this algorithm in 1974 [3, 5]. We give a simple proof that it is impossible for such an algorithm to exist:

Let $k \in \{1, \dots, n\}$. Then the SUBSET-SUM problem is to determine whether there exist sets $I^+ \subseteq \{1, \dots, k\}$ and $I^- \subseteq \{k+1, \dots, n\}$ such that

$$\sum_{i \in I^+} s_i = t - \sum_{i \in I^-} s_i.$$

There is nothing that can be done to make this equation simpler. Then since there are 2^k possible expressions on the left-hand side of this equation and 2^{n-k} possible expressions on the right-hand side of this equation, we can find a lower-bound for the worst-case running-time of an algorithm that solves the SUBSET-SUM problem by minimizing $2^k + 2^{n-k}$ subject to $k \in \{1, \dots, n\}$.

When we do this, we find that $2^k + 2^{n-k} = 2^{\lfloor n/2 \rfloor} + 2^{n-\lfloor n/2 \rfloor} = \Theta(\sqrt{2^n})$ is the solution, so it is impossible to solve the SUBSET-SUM problem in $o(\sqrt{2^n})$ time; thus, because the Meet-in-the-Middle algorithm achieves a running-time of $\Theta(\sqrt{2^n})$, we can conclude that $\Theta(\sqrt{2^n})$ is a tight lower-bound for the worst-case running-time of any deterministic and exact algorithm which solves SUBSET-SUM. And this conclusion implies that $P \neq NP$ [1, 2]. \square

References

- [1] P.B. Bovet and P. Crescenzi, *Introduction to the Theory of Complexity*, Prentice Hall, 1994.

- [2] T.H. Cormen, C.E. Leiserson, and R.L. Rivest, *Introduction to Algorithms*, McGraw-Hill, 1990.
- [3] E. Horowitz and S. Sahni, “Computing Partitions with Applications to the Knapsack Problem”, *Journal of the ACM*, vol. 21, no. 2, April 1974, pp 277-292.
- [4] A. Menezes, P. van Oorschot, and S. Vanstone, *Handbook of Applied Cryptography*, CRC Press, 1996.
- [5] G.J. Woeginger, “Exact Algorithms for NP-Hard Problems”, *Lecture Notes in Computer Science*, Springer-Verlag Heidelberg, Volume 2570, pp. 185-207, 2003.