3. HAFTA

Doğrusal Zamanla Değişmeyen Sistemler

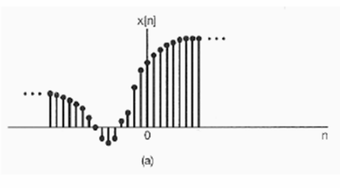
Konvolüsyon Toplamı

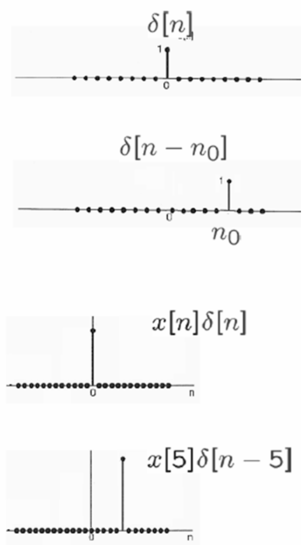
- Sample by Unit Impulse
  - For x[n]

$$x[n]\delta[n] = x[0]\delta[n]$$

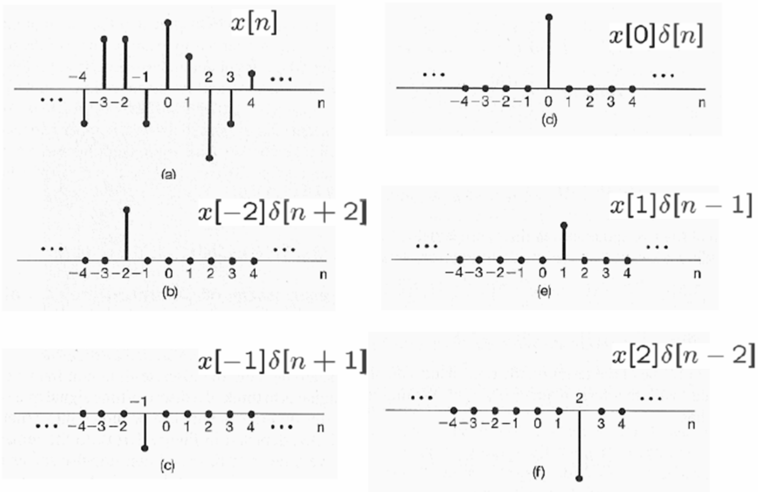
  - More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$





▪ Representation of DT Signals by Impulses:



▪ Representation of DT Signals by Impulses:

- More generally,

$$x[n] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2]$$

$$+ x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1]$$

$$+ x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

- The sifting property of the DT unit impulse
- x[n] = a superposition of scaled versions of shifted unit impulses δ[n-k]

- DT Unit Impulse Response & Convolution Sum:

$$\text{input} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \text{output}$$

$$\delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n]$$

$$\delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n]$$

$$\delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n]$$

$$\vdots$$

$$\delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n]$$

- DT Unit Impulse Response & Convolution Sum:

$$x[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow y[n]$$

$$x[0] \cdot \delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n] \cdot x[0]$$

$$x[1] \cdot \delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n] \cdot x[1]$$

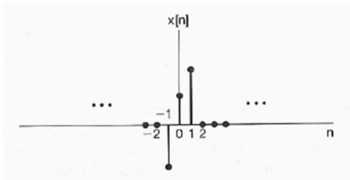
$$x[2] \cdot \delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n] \cdot x[2]$$

$$\vdots$$

$$x[k] \cdot \delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n] \cdot x[k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

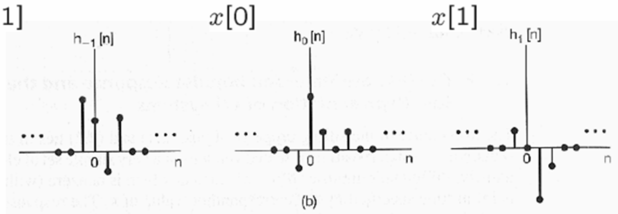
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \implies y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



(a)

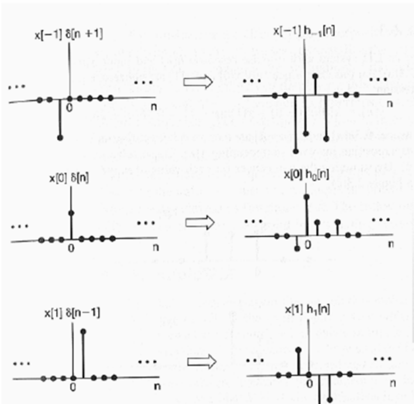
$\delta[n+1]$ $\delta[n]$ $\delta[n-1]$

$x[-1]$ $x[0]$ $x[1]$

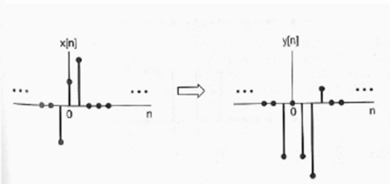


(b)

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \implies y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



(c)



(d)

$x[n] \longrightarrow$ Linear System $\longrightarrow y[n]$

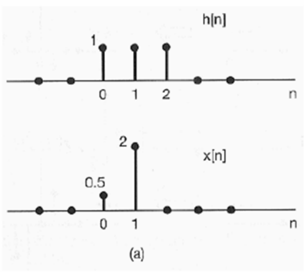
- If the linear system is also time-invariant
  - Then, $$h_k[n] = h_0[n-k] = h[n-k]$$
- Hence, for an LTI system,

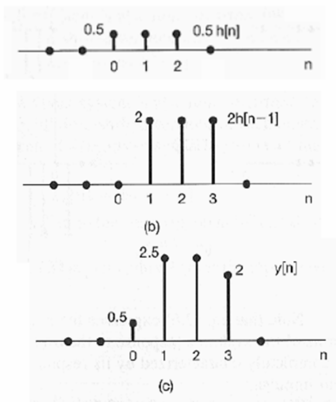
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

  - Known as the convolution of x[n] & h[n]
  - Referred as the convolution sum or superposition sum
- Symbolically, $y[n] = x[n] * h[n] = h[n] * x[n]$

- Example 2.1: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

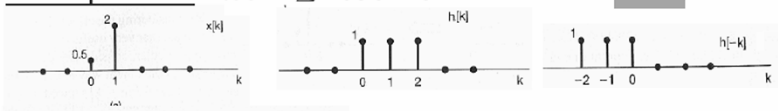
$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

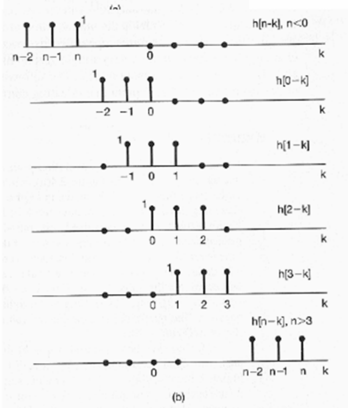




$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$
$$= 0.5h[n] + 2h[n-1]$$

- Example 2.2: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$





$y[n] = 0$ for $n < 0$

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k] = 0.5$$

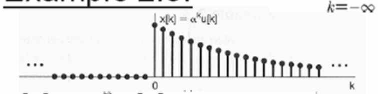
$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = 2.5$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 2.5$$

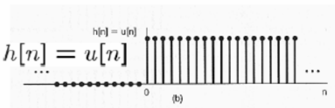
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 2.0$$

$y[n] = 0$ for $n > 3$

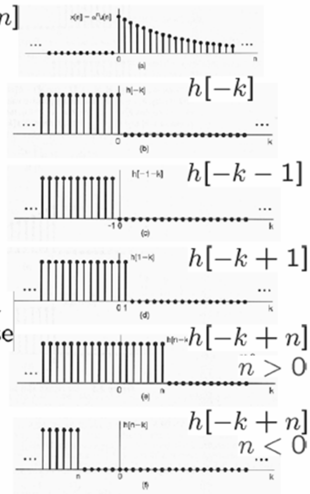
- Example 2.3: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$



$x[n] = \alpha^n\, u[n], \quad 0 < \alpha < 1$



$h[n] = u[n]$



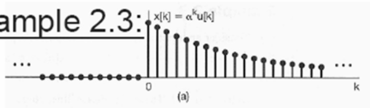
for $n < 0, \quad x[k]\, h[n-k] = 0 \quad \Rightarrow \quad y[n] = 0$

for $n \geq 0, \quad x[k]\, h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$

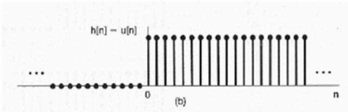
$$\Rightarrow \quad y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

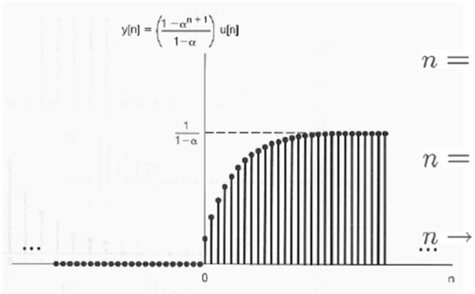
for all $n, \quad y[n] = \left(\dfrac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$

- Example 2.3:



$$x[n] \longrightarrow \boxed{\boldsymbol{h[n]}} \longrightarrow y[n]$$





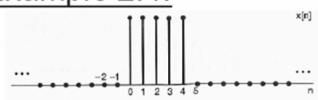
$$\alpha = \frac{7}{8}$$

$$n = 0 \quad y[0] = \frac{1-\frac{7}{8}}{1-\frac{7}{8}} = 1$$

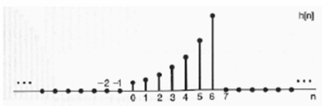
$$n = 1 \quad y[1] = \frac{1-(\frac{7}{8})^2}{1-\frac{7}{8}} = \frac{15}{8}$$

$$n \to \infty \quad y[n] = \frac{1-0}{1-\frac{7}{8}} = 8$$

- Example 2.4:

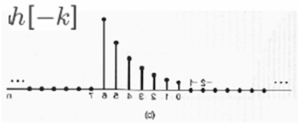


$$x[n] = 1,\ 0 \le n \le 4$$



$$h[n] = \alpha^n,\ 0 \le n \le 6$$

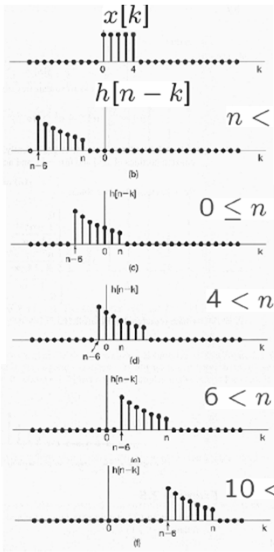
$h[-k]$



$$x[n] \longrightarrow \boxed{\boldsymbol{h[n]}} \longrightarrow y[n]$$

$x[k]$

$h[n-k]$



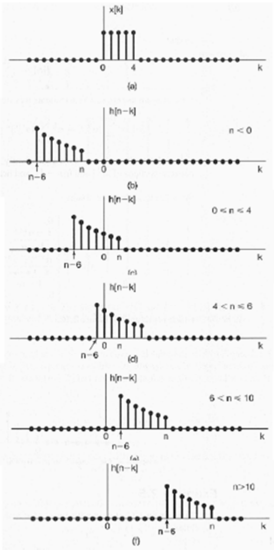
$n < 0$

$0 \le n \le 4$

$4 < n \le 6$

$6 < n \le 10$

$10 < n$



$$x[n] \longrightarrow \boxed{\boldsymbol{h[n]}} \longrightarrow y[n]$$

for $n < 0$, $\quad x[k]\,h[n-k] = 0 \;\Rightarrow\; y[n] = 0$

for $0 \le n \le 4$, $\quad x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

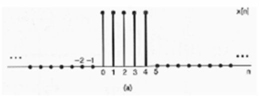
for $4 < n \le 6$, $\quad x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

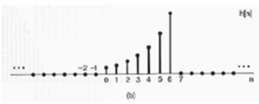
$$\Rightarrow y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

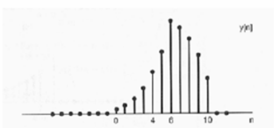
for $6 < n \le 10$, $\quad x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

for $n > 10$, $\quad y[n] = 0$







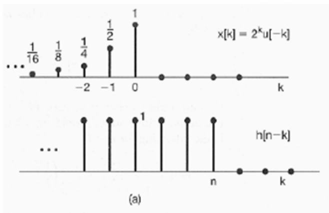
$$x[n] \longrightarrow \boxed{\boldsymbol{h[n]}} \longrightarrow y[n]$$

$$x[n] = 1,\; 0 \le n \le 4$$

$$h[n] = \alpha^{n},\; 0 \le n \le 6$$

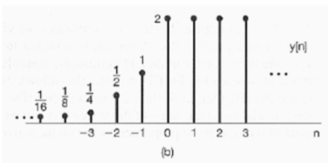
$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases}$$

- Example 2.5: $x[n] = 2^n u[-n]$ $h[n] = u[n]$ $x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$



for $n \geq 0$, $y[n] = \sum_{k=-\infty}^{0} x[k]\, h[n-k] = \sum_{k=-\infty}^{0} 2^k$

$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$



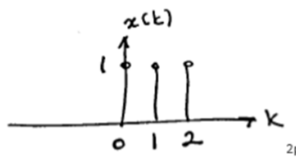
for $n < 0$, $y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] = \sum_{k=-\infty}^{n} 2^k$

$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

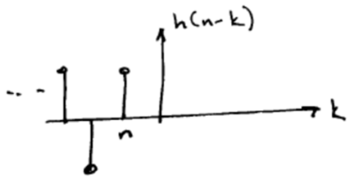
$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}$$

1. Birim darbe cevabı $h(n) = (-1)^n u(n)$ şeklinde verilen doğrusal zamanla değişmeyen sistemin $x(n) = u(n) - u(n-3)$ işaretine cevabı $y(n)$'yi hesaplayınız.

$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ 2p



2p



3p

$n < 0$ iken $y(n) = 0$ 3p
$n = 0$ iken $y(0) = 1$ 3p
$n = 1$ iken $y(1) = 0$ 3p
$n = 2$ iken $y(2) = 1$ 3p
$n = 3$ iken $y(3) = -1$ 3p
$n \geq 2$ iken $y(n) = (-1)^n$ 3p