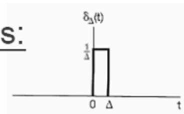
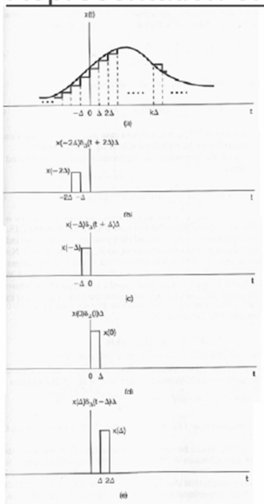
Konvolüsyon İntegrali

Sürekli Sistemler

- Representation of CT Signals by Impulses:





$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

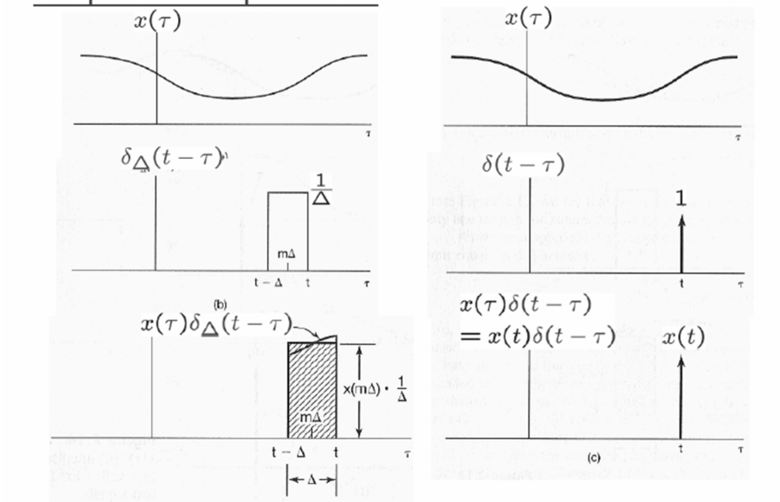
$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

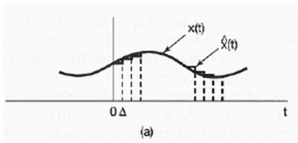
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

the sifting property of CT impulse

$x(t) =$ an integral of weighted, shifted impulses

- Graphical interpretation:



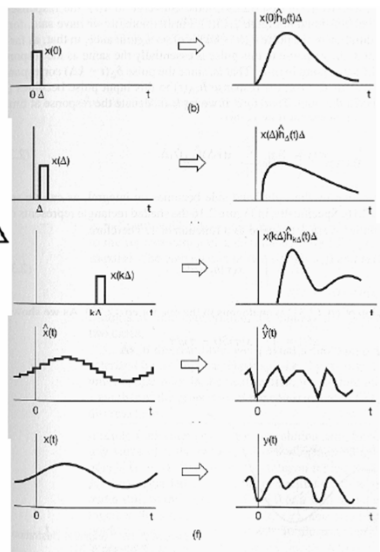


$x(k\Delta)$ → Linear System → $x(k\Delta)\ \widehat{h}_{k\Delta}(t)\ \Delta$

$$\widehat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\ \widehat{h}_{k\Delta}(t)\ \Delta$$

$$y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\ \widehat{h}_{k\Delta}(t)\ \Delta$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\ \ h_\tau(t)\ \ d\tau$$



- CT Unit Impulse Response & Convolution Integral:

$$\delta(t-\tau) \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_\tau(t)$$

$$x(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

$$\Longrightarrow \; y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

- If the linear system is also time-invariant $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$
  - Then,

$$h_\tau(t) = h_0(t-\tau) = h(t-\tau)$$

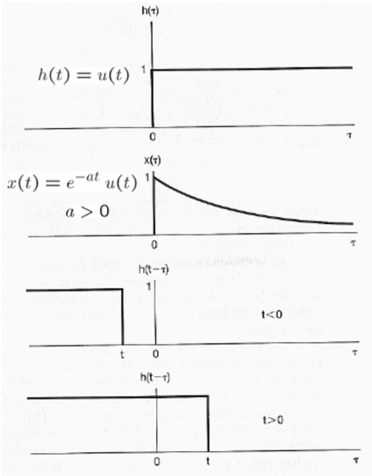
- Hence, for an LTI system,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

  - Known as the convolution of x(t) & h(t)
  - Referred as the convolution integral or the superposition integral

- Symbolically, $y(t) = x(t) * h(t) \qquad = h(t) * x(t)$

- Example 2.6: $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$



for $t < 0$, $x(\tau)\,h(t-\tau) = 0$

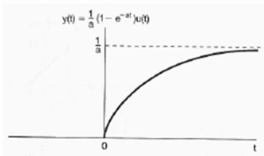
$$\Rightarrow\ y(t) = 0$$

for $t \geq 0$, $x(\tau)\,h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow\ y(t) = \int_0^t e^{-a\tau}\,d\tau$$

$$= -\frac{1}{a}e^{-a\tau}\Big|_0^t$$

$$= \frac{1}{a}(1 - e^{-at})$$

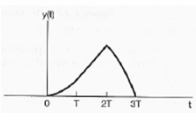


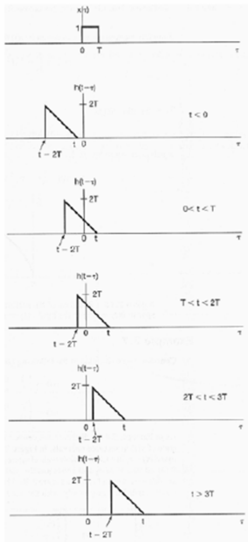
- Example 2.7: $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

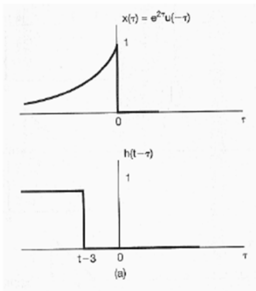


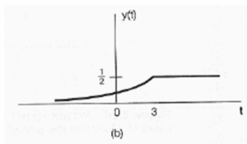


- <u>Example 2.8:</u>

$$x(t) = e^{2t}u(-t)$$
$$h(t) = u(t-3)$$

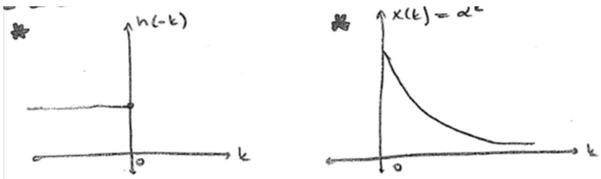
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

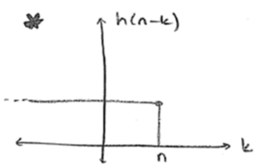


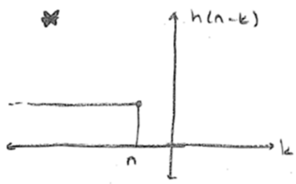


for $t-3 \leq 0$, $\quad y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau$
$$= \frac{1}{2}e^{2(t-3)}$$

for $t-3 \geq 0$, $\quad y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau$
$$= \frac{1}{2}$$







$n<0$

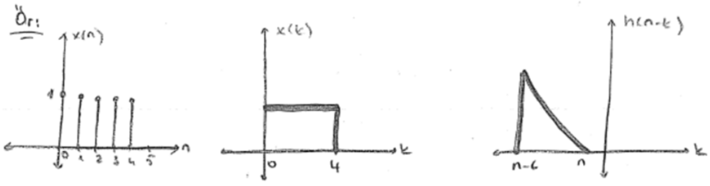
$y(0)=0$?

$y(n)=0$?

$n>0$

$$y(n) = \sum_{k=0}^{n} \alpha^k \cdot 1 = \frac{1-\alpha^{n+1}}{1-\alpha} \quad n \geq 0$$

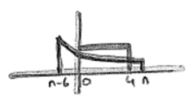
Ör:



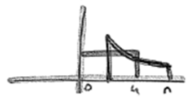
$x[n]=1 \quad 0 \leqslant n \leqslant 4$

$0 \leqslant n < 4 \Rightarrow y(n) = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$

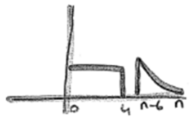
$4 \leqslant n < 6 \Rightarrow y(n) = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$



$6 \leqslant n < 10 \Rightarrow y(n) = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}$



$n \geqslant 11 \Rightarrow y(n) = 0$



Doğrusal Zamanla Değişmeyen Sistemlerin Özellikleri

- <u>Convolution Sum & Integral of LTI Systems:</u>

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[k] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

- <u>Properties of LTI Systems</u>

1. Commutative property
2. Distributive property
3. Associative property
4. With or without memory
5. Invertibility
6. Causality
7. Stability
8. Unit step response

- Commutative Property: $n-k=r$

$$x[n]*h[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k] \quad =\sum_{r=-\infty}^{+\infty}x[n-r]h[r]$$

$$=\sum_{r=-\infty}^{+\infty}h[r]x[n-r] \quad =h[n]*x[n]$$

$$x(t)*h(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau \qquad \begin{aligned} t-\tau&=\sigma \\ -d\tau&=d\sigma \end{aligned}$$

$$=\int_{+\infty}^{-\infty}x(t-\sigma)h(\sigma)(-d\sigma) \quad =\int_{-\infty}^{+\infty}x(t-\sigma)h(\sigma)d\sigma$$

$$=\int_{-\infty}^{+\infty}h(\sigma)x(t-\sigma)d\sigma \qquad =h(t)*x(t)$$

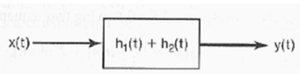
- Distributive Property:

$$x[n]*h[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$

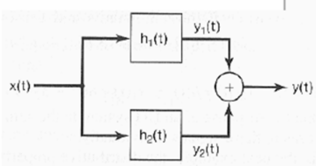
$$x(t)*h(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

$$x[n]*\left(h_1[n]+h_2[n]\right)=x[n]*h_1[n]+x[n]*h_2[n]$$

$$x(t)*\left(h_1(t)+h_2(t)\right)=x(t)*h_1(t)+x(t)*h_2(t)$$



(b)



(a)

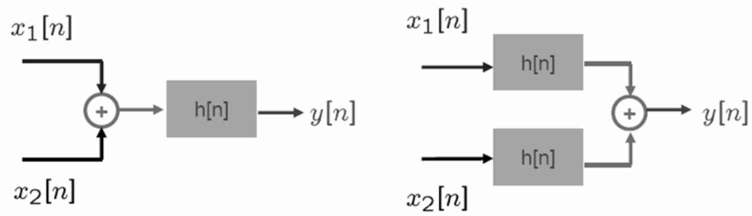
- Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

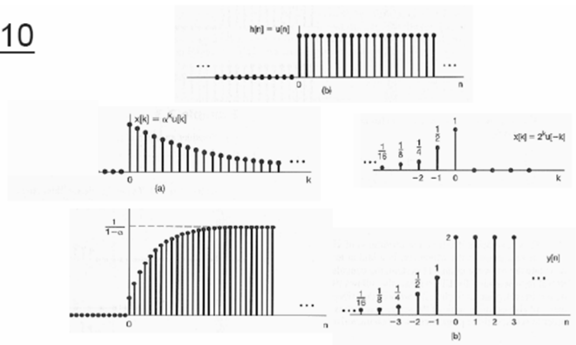
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\left(x_1[n] + x_2[n]\right) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$\left(x_1(t) + x_2(t)\right) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

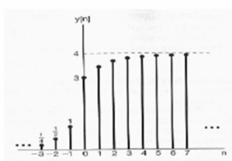


- Example 2.10



$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \left(x_1[n] + x_2[n]\right) * h[n] \\ &= x_1[n] * h[n] + x_2[n] * h[n] \end{aligned}$$



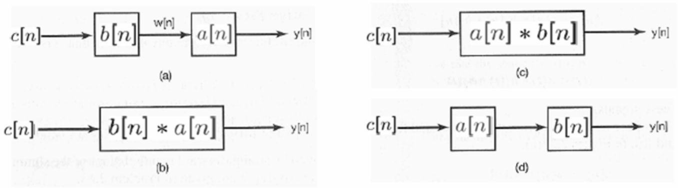
- <u>Associative Property:</u>

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$a[n] * \Big(b[n] * c[n]\Big) = \Big(a[n] * b[n]\Big) * c[n]$$

$$a(t) * \Big(b(t) * c(t)\Big) = \Big(a(t) * b(t)\Big) * c(t)$$



- <u>Memoryless:</u>
  - A DT LTI system is memoryless if $h[n] = 0$ for $n \neq 0$
  - The impulse response: $h[n] = K\delta[n], \quad K = h[0]$
  - The convolution sum: $y[n] = x[n] * h[n] = Kx[n]$
  - Similarly, for CT LTI system:

$$y(t) = x(t) * h(t) = Kx(t)$$

- <u>Invertibility:</u>

$$x(t) \to \boxed{\text{h1(t)}} \to y(t) \to \boxed{\text{h2(t)}} \to w(t)$$

$$y(t) = x(t) * h_1(t) \qquad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t)$$

$$x(t) \to \boxed{\text{Identity System } \delta(t)} \to x(t)$$

$$x(t) = x(t) * \delta(t)$$

$$\Longrightarrow h_2(t) * h_1(t) = \delta(t)$$

- <u>Example 2.11: Pure time shift</u>

$$x(t) \to \boxed{\text{h1(t)}} \to y(t) \to \boxed{\text{h2(t)}} \to w(t)$$

- $y(t) = x(t - t_0)$
  - delay if $t_0 > 0$
  - advance if $t_0 < 0$

$$\Rightarrow \quad h_1(t) = \delta(t - t_0) \quad \Rightarrow \quad x(t - t_0) = x(t) * \delta(t - t_0)$$

- $w(t) = x(t) = y(t + t_0)$

$$\Rightarrow \quad h_2(t) = \delta(t + t_0) \quad \Rightarrow \quad y(t + t_0) = y(t) * \delta(t + t_0)$$

$$\Rightarrow \quad h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

- Example 2.12

$$x[n] \rightarrow \boxed{h1[n]} \rightarrow y[n] \rightarrow \boxed{h2[n]} \rightarrow w[n]$$

$$h_1[n] = u[n]$$

$$\Rightarrow \; y[n] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k] \;=\; \sum_{k=-\infty}^{n} x[k]$$

$\Rightarrow$ a running-sum operation

- Its inverse is a first difference operation:

$$w[n] = y[n] - y[n-1] \;\Rightarrow\; h_2[n] = \delta[n] - \delta[n-1]$$

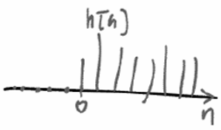
$$\Rightarrow \; h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

- Causality:
  - The output of a causal system depends only on the present and past values of the input to the system
  - Specifically, y[n] must not depend on x[k], for k > n

$$h[n-k] = 0, \qquad \text{for } k > n$$

$$h[n] = 0, \qquad \text{for } n < 0$$

  - It implies that the system is initially rest



  - A CT LTI system is causal if $h(t) = 0, \quad \text{for } t < 0$

- Convolution Sum & Integral

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k]$$

$$= \sum_{k=-\infty}^{n} x[k]\ h[n-k]$$

$$= \sum_{k=0}^{\infty} h[k]\ x[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\ h(t-\tau)\ d\tau$$

$$= \int_{0}^{\infty} h(\tau)\ x(t-\tau)\ d\tau$$

- Stability:
  - A system is stable
    if every bounded input produces a bounded output

$$x[n] \rightarrow \boxed{\text{Stable LTI}} \rightarrow y[n]$$

$|x[n]| < B$ for all n

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|$$

$$\Rightarrow |y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|$$

$$\Rightarrow |y[n]| \leq B \left(\sum_{k=-\infty}^{+\infty} |h[k]| \right)$$

if $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$ then, $y[n]$ is bounded

absolutely summable

- Stability:
  - For CT LTI stable system:

$$x(t) \rightarrow \boxed{\text{Stable LTI}} \rightarrow y(t)$$

$\left|x(t)\right| < B$ for all t $\qquad \left|y(t)\right| = \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right|$

$$\Rightarrow \left|y(t)\right| \leq \int_{-\infty}^{+\infty} \left|h(\tau)\right|\left|x(t-\tau)\right|d\tau$$

$$\Rightarrow \left|y(t)\right| \leq B\left(\int_{-\infty}^{+\infty} \left|h[(\tau)\right|d\tau\right)$$

if $\int_{-\infty}^{+\infty} \left|h(\tau)\right|d\tau < \infty$ then, $y(t)$ is bounded

absolutely integrable

- Example 2.13: Pure time shift

• $y[n] = x[n-n_0]$ & $h[n] = \delta[n-n_0]$

• $y(t) = x(t-t_0)$ & $h(t) = \delta(t-t_0)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left|h[n]\right| = \sum_{n=-\infty}^{+\infty} \left|\delta[n-n_0]\right| = 1 \quad \text{absolutely summable}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \left|h(\tau)\right| = \int_{-\infty}^{+\infty} \left|\delta(\tau-t_0)\right|d\tau = 1 \quad \text{absolutely integrable}$$

$\Rightarrow$ A (CT or DT) pure time shift is stable

Example 2.13: Accumulator

- $y[n] = \sum_{k=-\infty}^{n} x[k]$ & $h[n] = u[n]$
- $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ & $h(t) = u(t)$

$\Rightarrow \sum_{n=-\infty}^{+\infty} \left|h[n]\right| = \sum_{n=0}^{+\infty} \left|u[n]\right| = \infty$ NOT absolutely summable

$\Rightarrow \int_{-\infty}^{+\infty} \left|h(\tau)\right| = \int_{0}^{\infty} \left|u(\tau)\right| d\tau = \infty$ NOT absolutely integrable

$\Rightarrow$ A accumulator or integrator is NOT stable

Unit Step Response:

$h[n] = \delta[n] * h[n]$

- For an LTI system, its impulse response is:

$\delta[n] \rightarrow$ DT LTI $\rightarrow h[n]$ $\qquad$ $\delta(t) \rightarrow$ CT LTI $\rightarrow h(t)$

- Its unit step response is:

$u[n] \rightarrow$ DT LTI $\rightarrow s[n]$ $\qquad$ $u(t) \rightarrow$ CT LTI $\rightarrow s(t)$

$$\Rightarrow s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[n] \qquad \Rightarrow s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$\Rightarrow h[n] = s[n] - s[n-1] \qquad \Rightarrow h(t) = \frac{ds(t)}{dt}$$

Birim impuls cevabı ve karalılık

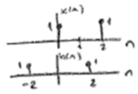
$x(n)$, $h(n)=\delta(n)$, $y(n)=\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$

$\cdots \delta(-1)x(n+1)+\delta(0)x(n)+\delta(1)x(n-1)+\cdots = x(n)$

$x(n)$, $h(n)=\delta(n-a)$, $y(n)=x(n-a)$

$\boxed{x(n) * \delta(n)=x(n)}$, $\boxed{x(n) * \delta(n-1)=x(n-1)}$

* $x(n) \longrightarrow \boxed{\delta(n-1)} \longrightarrow x(n-1)$
* $x(n-1) \longrightarrow \boxed{\delta(n+1)} \longrightarrow x(n)$

Ör: $x(n)=\delta(n)+\delta(n-2) \Rightarrow$ 

$h(n)=\delta(n-2)+\delta(n+2) \Rightarrow$

2 br sağa
2 br sola
kaydırarak
topla

$y(n)=?$ $\quad x(n) * h(n)$

$= x(n) * \left(\delta(n-2)+\delta(n+2)\right)$

$= x(n) * \delta(n-2) + x(n) * \delta(n+2)$

$= x(n-2)+x(n+2)$

$= \delta(n-2)+\delta(n-4)+\delta(n+2)+\delta(n)$