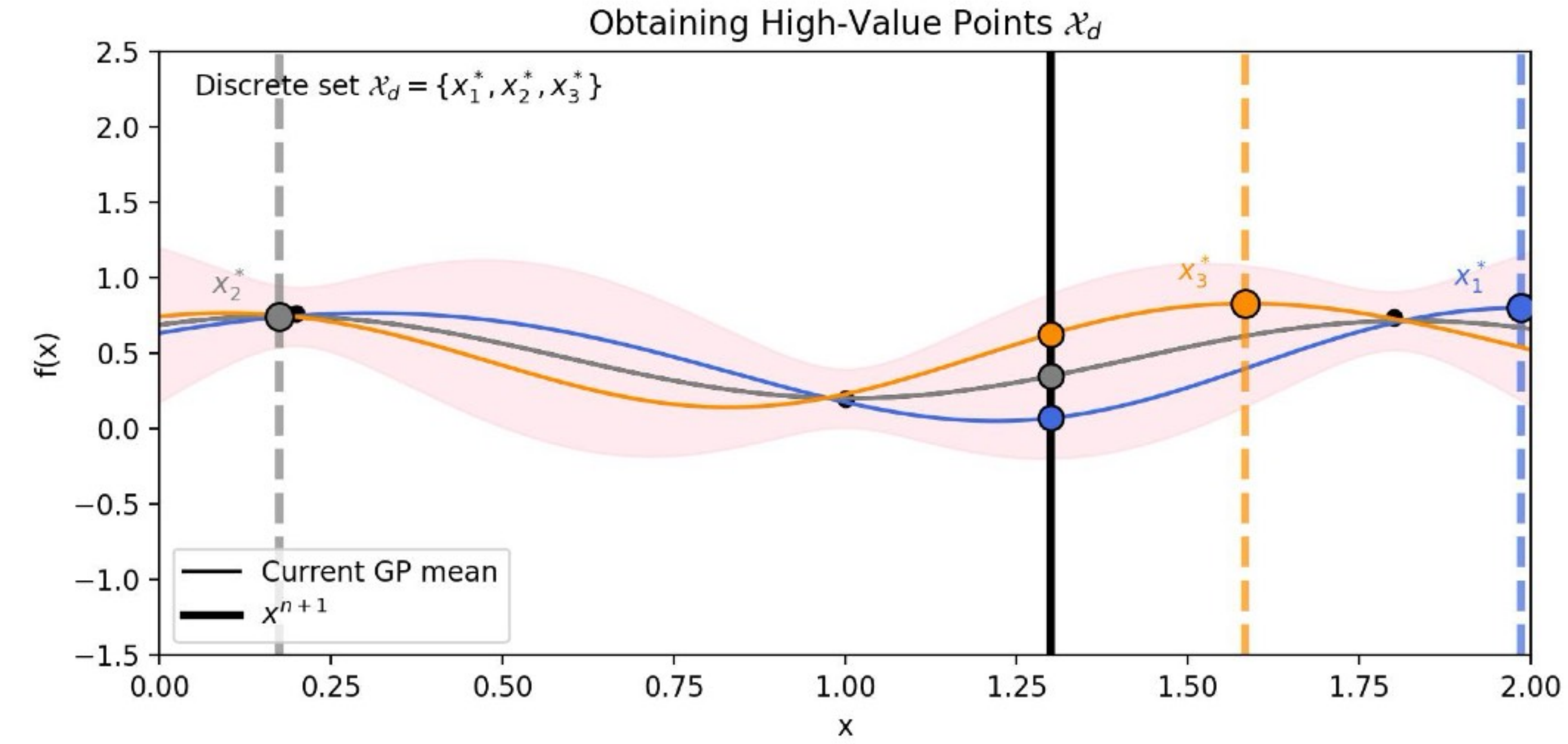
How to Compute ???

$$\mathrm{cKG}(x) = \mathbb{E}_{Z_c, Z_y}\left[\max_{x' \in \mathcal{X}} \left\{ \left(\mu_y^n(x') + \tilde{\sigma}_y(x', x^{n+1}) Z_y\right) \mathrm{PF}^{n+1}(x'; x^{n+1}, Z_c) \right\} - \mu_y^n(x_r) \mathrm{PF}^{n+1}(x_r^n; x^{n+1}, Z_c) \;\middle|\; x^{n+1} = x\right]$$

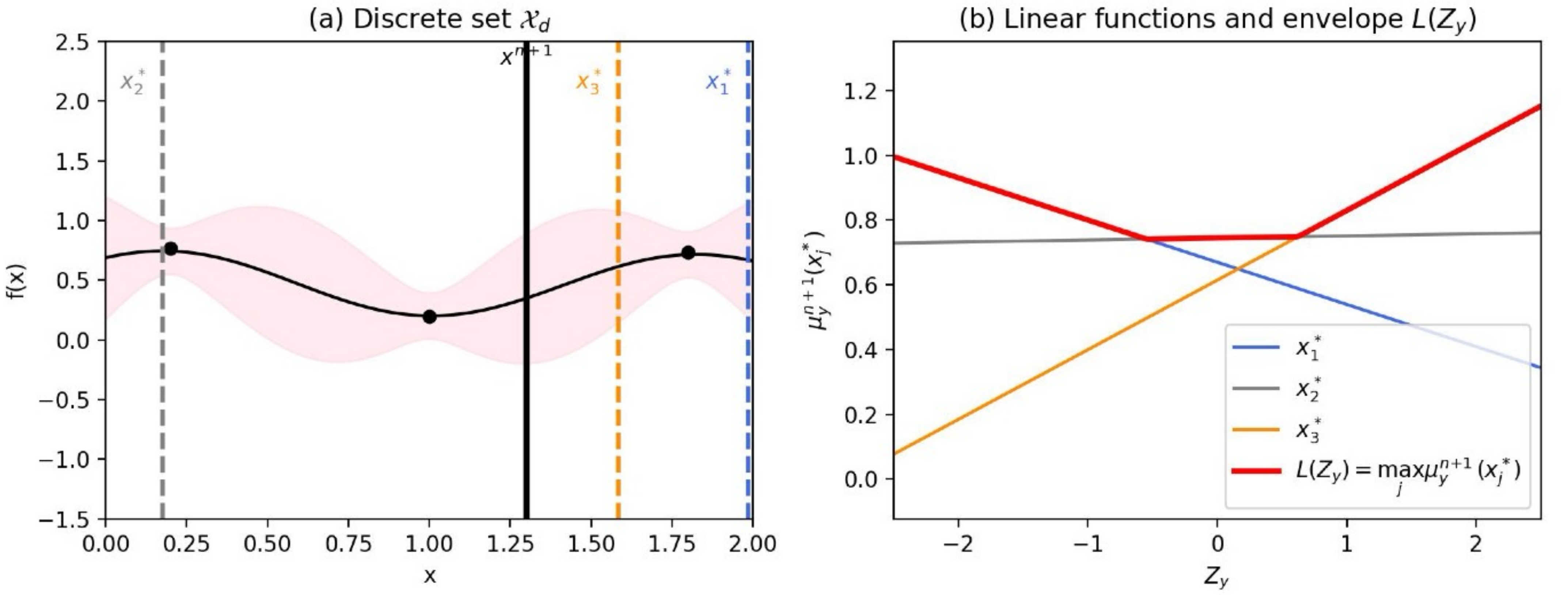
Since Z_c is fixed, we can compute this using the **high value linear envelope method Pearce et al. [2020]**

- Generate a discrete set $\mathcal{X}_d$ using the high value estimates
- Obtain a piecewise linear function for $max_{x \in \mathcal{X}}\ \mu_y^{n+1}(x)\,\mathrm{PF}^{n+1}(x; Z_c)$
- Integrate w.r.t Z_y to obtain $\mathbb{E}_{Z_y}\left[\max_{x' \in \mathcal{X}} \left\{ \left(\mu_y^n(x') + \tilde{\sigma}_y(x', x^{n+1})Z_y\right)\mathrm{PF}^{n+1}(x'; x^{n+1}, Z_c) \right\} \,\middle|\, x^{n+1} = x, Z_c \right]$

$$\mathrm{cKG}(x) = \mathbb{E}_{Z_c}\left[\mathbb{E}_{Z_y}\left[\max_{x' \in \mathcal{X}}\left\{\left[\mu_y^n(x') + \tilde{\sigma}_y(x', x^{n+1})Z_y\right]\underbrace{\mathrm{PF}^{n+1}(x'; x^{n+1}, Z_c)}_{\text{Condition(fix) on } Z_c}\right\} - \mu_y^n(x_r)\mathrm{PF}^{n+1}(x_r^n; x^{n+1}, Z_c) \;\middle|\; x^{n+1} = x, Z_c\right]\right]$$



Obtaining High-Value Points $\mathcal{X}_d$
Discrete set $\mathcal{X}_d = \{x_1^*, x_2^*, x_3^*\}$
x_2^*
x_3^*
x_1^*
f(x)
x
Current GP mean
x^{n+1}



(a) Discrete set $\mathcal{X}_d$
x_2^*
x^{n+1}
x_3^*
x_1^*
f(x)
x
(b) Linear functions and envelope $L(Z_y)$
$\mu_y^{n+1}(x_j^*)$
Z_y
x_1^*
x_2^*
x_3^*
$L(Z_y) = \max_j \mu_y^{n+1}(x_j^*)$

$$\mathrm{cKG}(x) = \frac{1}{n_c} \sum_{m=1}^{n_c} \mathrm{KG}_d(x^{n+1} = x; Z_c^m)\,.$$

How to Compute ???

$$\mathrm{cKG}(x) = \mathbb{E}_{Z_c,Z_y}\left[\max_{x'\in\mathcal{X}}\left\{\left(\mu_y^n(x') + \tilde{\sigma}_y(x',x^{n+1})Z_y\right)\mathrm{PF}^{n+1}(x';x^{n+1},Z_c)\right\} - \mu_y^n(x_r)\mathrm{PF}^{n+1}(x_r^n;x^{n+1},Z_c)\ \middle|\ x^{n+1}=x\right]$$

$$\Big\downarrow\ \text{Condition(fix) on } Z_c$$

$$\mathrm{cKG}(x) = \mathbb{E}_{Z_c}\left[\underbrace{\mathbb{E}_{Z_y}\left[\max_{x'\in\mathcal{X}}\left\{\left[\mu_y^n(x') + \tilde{\sigma}_y(x',x^{n+1})Z_y\right]\mathrm{PF}^{n+1}(x';x^{n+1},Z_c)\right\} - \mu_y^n(x_r)\mathrm{PF}^{n+1}(x_r^n;x^{n+1},Z_c)\ \middle|\ x^{n+1}=x,Z_c\right]}_{\mathrm{KG}_d(x^{n+1}=x;\,Z_c^m)}\right]$$

$$\mathrm{cKG}(x) = \frac{1}{n_c}\sum_{m=1}^{n_c}\mathrm{KG}_d(x^{n+1}=x;Z_c^m)\,.$$

Pseudocode

ALGORITHM 1: cKG computation.

Input: Sample x^{n+1}, size of Monte Carlo discretisations n_c and n_y

0. Initialise discretisation $X_d^0 = \{\}$ and set $n_z = n_c n_y$

1. Compute $x_r^n = \arg\max_{x \in \mathbb{X}} \mu_y^n(x)\mathrm{PF}^n(x)$

2. **for** j **in** $[1, \ldots, n_z]$ **:**

3. Generate $Z_y^j, Z_1^j, \ldots, Z_K^j \sim N(0,1)$

4. Compute $x_j^* = \max_{x \in X_d} \left\{\left[\mu_y^n(x) + \tilde{\sigma_y}(x, x^{n+1})Z_y^j\right]\mathrm{PF}^{n+1}(x; x^{n+1}, \mathbf{Z}_c^j)\right\}$

5. Update discretisation $X_d^j = X_d^{j-1} \cup \{x_j^*\}$

6. **for** m **in** $[1, \ldots, n_c]$ **:**

7. Compute $\mathrm{KG}_d(x^{n+1} = x; \mathbf{Z}_c^m)$ using X_d

8. Compute Monte Carlo estimation $\frac{1}{n_c}\sum_{m=1}^{n_c} \mathrm{KG}_d(x^{n+1}; \mathbf{Z}_c^m)$

9. **Return:** $\mathrm{cKG}(x^{n+1})$