Theory Homework 2.

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1.

(1) $f(x) = Xa \quad X \in \mathbb{R}^{d \times k}, a \in \mathbb{R}^{k \times 1}$

$Df(x)[H] = f(H) = Ha \quad H \in \mathbb{R}^{d \times k}$

$f(x) = XX^T \quad X \in \mathbb{R}^{d \times n}$

$Df(x)[H] = HX^T + XH^T \quad H \in \mathbb{R}^{d \times n}$

(2) $f(x) = XCX, \quad X \in \mathbb{R}^{d \times d}$

$\nabla_H f(x) = \nabla_H(x)CX + X(\nabla_H(g(x))), \quad g(x) = CX$

Since $g(x)$, and X are all linear mappings

$= HCX + XCH$

$f(x) = CXBX^TAX \quad X \in \mathbb{R}^{d \times d}, \{A, B, C\} \in \mathbb{R}^{d \times d}$

$\nabla_H f(x) = \nabla_H(CX)BX^TAX + CX\nabla_H(BX^TAX)$

$= CHBX^TAX + CX(\nabla_H(BX^T)AX + BX^T\nabla_H(AX))$

$= CHBX^TAX + CX(BH^TAX + BX^TAH)$

$= CHBX^TAX + CXBH^TAX + CXBX^TAH$

(3)

$$f(X) = (1, X_2)\begin{pmatrix} 1 & X_2^3 \\ \sin X_2 & X_1 \end{pmatrix} = (1+X_2\sin X_2,\ X_2^3+X_1X_2)$$

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

$$\lim_{n\to 0} \frac{1+(nH_2+X_2)\sin(nH_2+X_2)-(1+X_2\sin X_2)}{n} =$$

$$\lim_{n\to 0} \frac{(nH_2+X_2)(\sin(nH_2)\cos X_2+\cos nH_2 \sin X_2)-X_2\sin X_2}{n}$$

$$= \lim_{n\to 0} \frac{(nH_2+X_2)\sin X_2 - X_2\sin X_2 + X_2H_2\cos X_2 \cdot n}{n}$$

$$= H_2\sin X_2$$

$$\lim_{n\to 0} \frac{(X_2+nH_2)^3+(X_1+nH_1)(X_2+nH_2)-X_2^3-X_1X_2}{n}$$

$$= \lim_{n\to 0} \frac{3X_2^2 nH_2 + nH_1X_2 + nH_2X_1}{n}$$

$$= 3X_2^2H_2 + H_1X_2 + H_2X_1$$

$$\therefore \nabla_H f(X_1, X_2) = [H_2\sin X_2 + X_2\cos X_2 H_2,\ 3X_2^2H_2 + H_1X_2 + H_2X_1]$$

2.

3 dims

To maximize the margin, the hyperplane must be perpendicular to the line that connects two points

$\vec{x}_1 - \vec{x}_2 = (-6, -1, 5)$

so we assume the hyperplane to be:

$$-6w_1 - w_2 + 5w_3 + Bias. = 0$$

To calculate the bias term, we know hyperplane go through middle point of x_1, x_2

$$(\vec{x}_1 + \vec{x}_2) \cdot \frac{1}{2} = (-2, 1.5, -0.5)$$

$$-6 \cdot -2 - 1.5 + 5 \cdot (-0.5) + Bias = 0$$

$$\therefore Bias = -8$$

$$\therefore f(x) = \begin{bmatrix} -6 \\ -1 \\ 5 \end{bmatrix} \cdot \vec{x} - 8$$

5 dims:

$$\vec{w} = (1, 0, 1, -2, 6) - (3, 1, -3, 5, 1)$$

$$= (-2, -1, 4, -7, 5)$$

b can range from $(-46, 49)$

$$\therefore y = \begin{pmatrix} -2 \\ -1 \\ 4 \\ -7 \\ 5 \end{pmatrix} \cdot \vec{x} + b$$

5 dims:

I. $\vec{x_2} - \vec{x_3} = (1, 2, -3, 1, -1)$

$\vec{x_1} - \vec{x_3} = (-1, 1, 1, 2, 1)$

$\vec{w} := (1, 1, 1, 1, 1)$

II.

let $v = \vec{x_2} - \vec{x_3}$

if w not parallel to each other, since there is no real number k such that

$k\vec{v} = \vec{w}$

let $\vec{u} = \vec{x_1} - \vec{x_3}$

$\because \vec{u} \cdot \vec{w} \neq 0$

$\therefore \vec{w}$ is not orthogonal to line $x_1 - x_3$

III.

$z = \vec{x_2} - \vec{x_3} = (1, 2, -3, 1, -1)$

$$W_2 = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-3\\1\\-1\end{pmatrix} \cdot \frac{z}{1^2+2^2+(-3)^2+1^2+(-1)^2}$$

$$= \omega$$

$$W_2 = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

IV. $\vec{W_2} \cdot \vec{z} = 0$

V.

$\vec{x_1} \cdot \vec{w_2} = 11$

$\vec{x_2} \cdot \vec{w_2} = 7$

$\vec{x_3} \cdot \vec{w_2} = 7$

Therefore, to make the value symmetic I chose $b = \frac{11+7}{2} = 9$

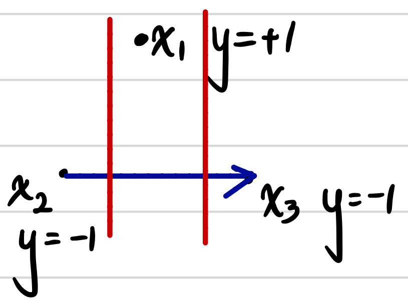
$\therefore \vec{x_1} \cdot \vec{w_2} - b = 2$

$\vec{x_2} \cdot \vec{w_2} - b = -2$

$\vec{x_3} \cdot \vec{w_2} - b = -2$

VI.

for 2 dimension case:



if $z = x_2 - x_3$ parallel to $\vec{w}$.

the hyperplane (line) that separates positive from negative need to be perpendicular to $\vec{x_2} - \vec{x_3}$

the red line is hyperplane: as shown above the hyperplane cannot separate positive & negative cases.

in conclusion, if x_1 is "between" x_2 x_3, the hyperplane cannot successfully separates.