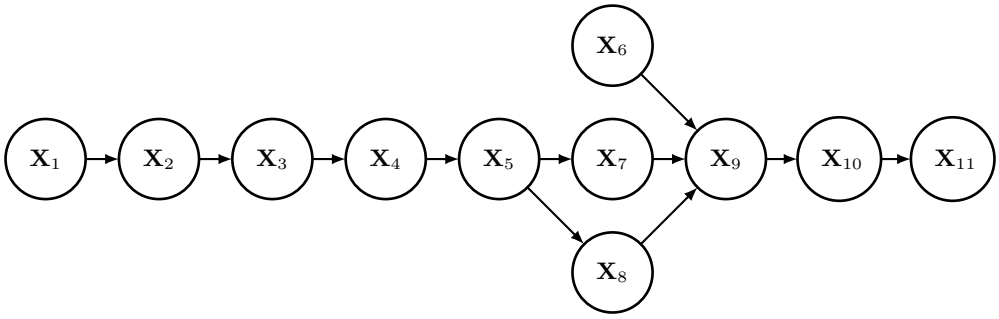
SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

01.112 Machine Learning, Spring 2018
Homework 5

Due 20 Apr 2018, 5pm

Sample Solutions

In this homework, we would like to look at the Bayesian Networks. You are given a Bayesian network as below. All nodes can take 2 different values: $\{1, 2\}$.



Question 1. Without knowing the actual value of any node, are node $\mathbf{X}_1$ and $\mathbf{X}_6$ independent of each other? What if we know the value of node $\mathbf{X}_7$ and $\mathbf{X}_{10}$? *(5 points)*

Answer Without knowing the actual value of any node, node X_1 and X_6 are independent of each other. This is because there does not exist any path from X_1 to X_6 that is open. Based on the Bayes' ball algorithm, X_1 and X_6 are independent of each other.

If we know the value of node X_7 and X_{10}, then the two variables X_1 and X_6 become dependent. This is because there exist a path connecting X_1 and X_6 that is open: $X_1 - X_2 - X_3 - X_4 - X_5 - X_8 - X_9 - X_{10} - X_9 - X_6$ or $X_1 - X_2 - X_3 - X_4 - X_5 - X_7 - X_5 - X_8 - X_9 - X_{10} - X_9 - X_6$. Based on the Bayes' ball algorithm, X_1 and X_6 are dependent.

Question 2. What is the *effective* number of parameters needed to for this Bayesian network? What would be the *effective* number of parameters for the same network if node $\mathbf{X}_3$ and $\mathbf{X}_9$ can take 4 different values: $\{1, 2, 3, 4\}$, and all other nodes can only take 3 different values: $\{1, 2, 3\}$? *(5 points)*

Answer The number of parameters correspond to the number of entries in the probability table of each node in the Bayesian network. Assume the number of values for node k to take is r_k. For a node i with parents pa_i, the number of rows is $\prod_{j \in pa_i} r_j$. The number of columns is r_i. However the values in the last column can be uniquely determined from the other columns since the values of each row sum to 1. This means for the node i there are $(r_i - 1) \prod_{j \in pa_i} r_j$ free/independet/effective parameters involved.

	$\mathbf{X}_1$	
	1	2
	0.5	0.5

$\mathbf{X}_1$	$\mathbf{X}_2$	
	1	2
1	0.2	0.8
2	0.3	0.7

$\mathbf{X}_2$	$\mathbf{X}_3$	
	1	2
1	0.3	0.7
2	0.3	0.7

$\mathbf{X}_3$	$\mathbf{X}_4$	
	1	2
1	0.1	0.9
2	0.5	0.5

$\mathbf{X}_4$	$\mathbf{X}_5$	
	1	2
1	0.5	0.5
2	0.6	0.4

	$\mathbf{X}_6$	
	1	2
	0.6	0.4

$\mathbf{X}_5$	$\mathbf{X}_7$	
	1	2
1	0.2	0.8
2	0.3	0.7

$\mathbf{X}_5$	$\mathbf{X}_8$	
	1	2
1	0.8	0.2
2	0.7	0.3

$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$	$\mathbf{X}_9$	
			1	2
1	1	1	0.8	0.2
1	1	2	0.1	0.9
1	2	1	0.9	0.1
1	2	2	0.7	0.3
2	1	1	0.3	0.7
2	1	2	0.2	0.8
2	2	1	0.2	0.8
2	2	2	0.9	0.1

$\mathbf{X}_9$	$\mathbf{X}_{10}$	
	1	2
1	0.8	0.2
2	0.8	0.2

$\mathbf{X}_{10}$	$\mathbf{X}_{11}$	
	1	2
1	0.7	0.3
2	0.8	0.2

Therefore in the initial Bayesian network, the number of free parameters is:
$1(X_1) + 2 \times 1(X_2) + 2 \times 1(X_3) + 2 \times 1(X_4) + 2 \times 1(X_5) + 1(X_6) + 2 \times 1(X_7) + 2 \times 1(X_8) + 2 \times 2 \times 2 \times 1(X_9) + 2 \times 1(X_{10}) + 2 \times 1(X_{11}) = 26$

If node X_3 and X_9 can take 4 different values: 1, 2, 3, 4, and all other nodes can only take 3 different values: 1, 2, 3, the number of free parameters is:
$2(X_1) + 3 \times 2(X_2) + 3 \times 3(X_3) + 4 \times 2(X_4) + 3 \times 2(X_5) + 2(X_6) + 3 \times 2(X_7) + 3 \times 2(X_8) + 3 \times 3 \times 3 \times 3(X_9) + 4 \times 2(X_{10}) + 3 \times 2(X_{11}) = 140$

Question 3. If we have the following probability tables for the nodes. Compute the following probabilities. Clearly write down all the necessary steps.

(a) Calculate the following conditional probability:

$$P(\mathbf{X}_3 = 2|\mathbf{X}_4 = 1)$$

(6 points)

Answer One standard approach is to start by computing the following marginal probability:

$$P(X_3, X_4) = \sum_{X_1, X_2} P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

Compute $P(X_3 = 1, X_4 = 1)$ and $P(X_3 = 2, X_4 = 1)$ respectively, and then compute $P(X_4 = 1) = P(X_3 = 1, X_4 = 1) + P(X_3 = 2, X_4 = 1)$. The conditional probability $P(X_3 = 2|X_4 = 1) = P(X_3 = 2, X_4 = 1)/P(X_4 = 1)$.

$$
\begin{aligned}
P(X_3=1,X_4=1) &= P(X_1=1)P(X_2=1|X_1=1)P(X_3=1|X_2=1)P(X_4=1|X_3=1) \\
&+P(X_1=1)P(X_2=2|X_1=1)P(X_3=1|X_2=2)P(X_4=1|X_3=1) \\
&+P(X_1=2)P(X_2=1|X_1=2)P(X_3=1|X_2=1)P(X_4=1|X_3=1) \\
&+P(X_1=2)P(X_2=2|X_1=2)P(X_3=1|X_2=2)P(X_4=1|X_3=1) \\
&= 0.5\times 0.2\times 0.3\times 0.1 + 0.5\times 0.9\times 0.3\times 0.1 + 0.5\times 0.3\times 0.3\times 0.1 \\
&+0.5\times 0.7\times 0.3\times 0.1 = 0.003 + 0.0135 + 0.0045 + 0.0105 = 0.0315
\end{aligned}
$$

$$
\begin{aligned}
P(X_3=2,X_4=1) &= P(X_1=1)P(X_2=1|X_1=1)P(X_3=2|X_2=1)P(X_4=1|X_3=2) \\
&+P(X_1=1)P(X_2=2|X_1=1)P(X_3=2|X_2=2)P(X_4=1|X_3=2) \\
&+P(X_1=2)P(X_2=1|X_1=2)P(X_3=2|X_2=1)P(X_4=1|X_3=2) \\
&+P(X_1=2)P(X_2=2|X_1=2)P(X_3=2|X_2=2)P(X_4=1|X_3=2) \\
&= 0.5\times 0.2\times 0.7\times 0.5 + 0.5\times 0.9\times 0.7\times 0.5 + 0.5\times 0.3\times 0.7\times 0.5 \\
&+0.5\times 0.7\times 0.7\times 0.5 = 0.035 + 0.1575 + 0.0525 + 0.1225 = 0.3675
\end{aligned}
$$

$$
P(X_3=2|X_4=1) = \frac{P(X_3=2,X_4=1)}{P(X_4=1)} = \frac{0.3675}{0.0315+0.3675} \approx 0.921 \tag{1}
$$

(b) Calculate the following conditional probability:

$$
P(\mathbf{X}_5=2|\mathbf{X}_3=1,\mathbf{X}_{11}=2,\mathbf{X}_1=1)
$$

(9 points)

(*Hint: find a short answer. The values in some of the probability tables may reveal some useful information.*)

Answer We can have the following two observations from the tables:

- Distribution of X_3 doesn't change no matter what values X_2 takes.
- Distribution of X_10 doesn't change no matter what values X_9 takes.

Thus, we have X_2 and X_3 are independent, X_9 and X_{10} are independent. There is no path connecting X_1 to X_5 and X_5 to X_{11}.

$$
\begin{aligned}
P(X_5|X_3,X_{11},X_1) &= P(X_5|X_3) \\
&= \frac{P(X_3,X_5)}{P(X_3)} \\
&= \frac{\sum_{X_4} P(X_3)P(X_4|X_3)P(X_5|X_4)}{P(X_3)} \\
&= \sum_{X_4} P(X_4|X_3)P(X_5|X_4)
\end{aligned}
$$

$$P(X_5 = 2|X_3 = 1, X_{11} = 2, X_1 = 1) = P(X_4 = 1|X_3 = 1)P(X_5 = 2|X_4 = 1) + P(X_4 = 2|X_3 = 1)P(X_5 = 2|X_4 = 2) = 0.1 \times 0.5 + 0.9 \times 0.4 = 0.41$$

Question 4.

(a) Now, assume we do not have any knowledge about the probability tables for the nodes in the network, but we have the following 12 observations/samples. Find a way to estimate the probability tables associated with the nodes $\mathbf{X}_3$ and $\mathbf{X}_9$ respectively. *(6 points)*

$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$	$\mathbf{X}_9$	$\mathbf{X}_{10}$	$\mathbf{X}_{11}$
1	1	2	2	2	1	1	1	1	1	1
1	2	1	1	2	1	1	1	1	1	2
2	2	2	1	2	2	1	1	2	2	1
1	1	2	1	2	1	1	2	1	2	2
1	2	1	1	1	1	2	2	2	1	1
2	2	1	2	1	2	2	1	1	1	2
2	1	2	2	1	2	1	2	2	2	1
2	2	2	1	2	1	2	2	1	2	2
1	1	1	1	2	2	1	1	1	1	1
1	1	1	1	2	1	1	1	2	1	2
1	2	1	2	2	1	2	1	2	1	2
2	2	1	2	1	2	2	2	1	1	1

Answer We can use the maximum likelihood estimation to find the optimal model parameters.

$$\theta_{X_3}(1) = \frac{\text{Count}(X_2 = 1, X_3 = 1)}{\text{Count}(X_2 = 1)} = 2/5$$
$$\theta_{X_3}(2) = \frac{\text{Count}(X_2 = 1, X_3 = 2)}{\text{Count}(X_2 = 1)} = 3/5$$
$$\theta_{X_3}(1) = \frac{\text{Count}(X_2 = 2, X_3 = 1)}{\text{Count}(X_2 = 2)} = 5/7$$
$$\theta_{X_3}(2) = \frac{\text{Count}(X_2 = 2, X_3 = 2)}{\text{Count}(X_2 = 2)} = 2/7$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 1, X_9 = 1)}{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 1)} = 2/3$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 1, X_9 = 2)}{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 1)} = 1/3$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 2, X_9 = 1)}{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 2)} = 1$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 2, X_9 = 2)}{\text{Count}(X_6 = 1, X_7 = 1, X_8 = 2)} = 0$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 1, X_9 = 1)}{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 1)} = 0$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 1, X_9 = 2)}{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 1)} = 1$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 2, X_9 = 1)}{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 2)} = 1/2$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 2, X_9 = 2)}{\text{Count}(X_6 = 1, X_7 = 2, X_8 = 2)} = 1/2$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 1, X_9 = 1)}{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 1)} = 1/2$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 1, X_9 = 2)}{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 1)} = 1/2$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 2, X_9 = 1)}{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 2)} = 0$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 2, X_9 = 2)}{\text{Count}(X_6 = 2, X_7 = 1, X_8 = 2)} = 1$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 1, X_9 = 1)}{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 1)} = 1$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 1, X_9 = 2)}{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 1)} = 0$$
$$\theta_{X_9}(1) = \frac{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 2, X_9 = 1)}{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 2)} = 1$$
$$\theta_{X_9}(2) = \frac{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 2, X_9 = 2)}{\text{Count}(X_6 = 2, X_7 = 2, X_8 = 2)} = 0$$

The resulting probability tables for X_3 and X_9 are:

X_2	X_3 = 1	X_3 = 2
1	2/5	3/5
2	5/7	2/7

X_6	X_7	X_8	X_9 = 1	X_9 = 2
1	1	1	2/3	1/3
1	1	2	1	0
1	2	1	0	1
1	2	2	1/2	1/2
2	1	1	1/2	1/2
2	1	2	0	1
2	2	1	1	0
2	2	2	1	0

(b) Based on the above observations, you would like to find a good Bayesian network structure to model the data. You started with the initial structure shown on the previous page, and decided to delete the edge between $\mathbf{X}_{10}$ and $\mathbf{X}_{11}$. Is the resulting new structure (after deleting the single edge between $\mathbf{X}_{10}$ and $\mathbf{X}_{11}$ from the original graph) better than the original structure in terms of BIC score? Clearly explain the reason. *(9 points)*

(Hint: Try to find a short answer.)

Answer Deletion of the edge between X_{10} and X_{11} will only change the probability table of the node X_11. Now let's see what happens to the probability table of node X_{11}.

Before deletion:

X_{10}	X_{11} = 1	X_{11} = 2
1	$1/2$	$1/2$
2	$1/2$	$1/2$

After deletion:

	X_{11} = 1	X_{11} = 2
	$1/2$	$1/2$

This means deleting this edge does not affect the Bayesian network's log-likelihood (when the model parameters are estimated using MLE from the data). However, the BIC scores for the two networks are different now. Specifically the new Bayesian network needs 1 less free parameters. Therefore the resulting new structure has a better (higher) BIC score.