Mathematical Analysis Study Notes

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1 Numerical Sequences and Series

1.1 Series

Definition 1.1.1 Given a sequence $\{a_n\}$,we use the notation

$$\sum_{n=p}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. And we call the symbolic expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

an infinite series, or just a series.

With $\{a_n\}$ we associate a sequence $\{S_n\}$, where

$$S_n = \sum_{k=1}^{n} a_k.$$

The numbers S_n are called the partial sums of the series.

If $\{S_n\}$ converges to s, we say that the series converges, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of seires; If $\{S_n\}$ diverges, the series is said to diverge.

Remark 1.1.1 The above s is called the sum of the series; but it should be clearly understood that s is the limit of a sequence of sums, and is not obtained simply by addition.

Remark 1.1.2 It is clear that every theorem about sequences can be state in series. The Cauchy criterion can be restated in the following form.

Remark 1.1.3 For convenience of notation, we shall simply write Σa_n in place of $\sum\limits_{n=1}^{\infty}$.

Theorem 1.1.1 ***(Cauchy criterion)*** Σa_n converges $\iff$ for every $\varepsilon > 0$ there exists an integer N such that if $m \geq n \geq M$

$$|\sum_{k=n}^{m} a_k| \leq \varepsilon \tag{1.1}$$

It can be poved by the Cauchy criterion of the limit of sequences. And In particular, by taking $m = n$, (1.1) becomes

$$|a_n| \leq \varepsilon \quad (n \geq N).$$

So we can get the following vanishing condition.

Theorem 1.1.2 ***(Vanishing condition)*** If Σa_n converges, then $\lim_{n\to\infty} a_n = 0$.

Remark 1.1.4 The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of Σa_n. For instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges;and we will prove it in the *Example 1.1.2*.

Example 1.1.1 let $q \in \mathbb{R}$, discuss the convergence and divergence of the geometric series Σq^n.
Solution: We know that the partial sum of it is $S_n = \frac{q(1-q^n)}{1-q}$, so we have the series converges when $|q| < 1$ and diverges when $|q| \geq 1$.

Example 1.1.2 Determine the convergence and divergence of the harmonic series $\Sigma \frac{1}{n}$.
Solution: When $n \geq 1$, we have

$$\sum_{k=n+1} 2n \frac{1}{k} \geq \sum_{k=n+1} 2n \frac{1}{2n} = \frac{1}{2},$$

so by Cauchy criterion, we know the original series diverges.

1.2 Positive Series

Introduction

When studying a subject, we usually proceed from the specific to the general. Therefore, when researching series, we first focus on a specific type of series. It is called the positive series, which, as the name suggests, are series where each term is a positive number. But next, we will also reseach the non-negetive series meanwhile.

Theorem 1.2.1 ***(Fundamental Test)*** The positive series converges $\iff$ $\{S_n\}$ is bounded above.

Proof: Because $a_n > 0$, We conclude that this partial sum S_n is monotonically increasing with respect to n. It follows from the fact that a monotonically increasing and bounded above sequence must converge that $\{S_n\}$ converges.

Remark 1.2.1 The above test also holds for $a_n \geq 0$.

Theorem 1.2.2 ***(Cauchy Condensation Test)*** Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the series $\sum\limits_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof: By the *Theorem 1.2.1*, it suffices to consider boundness of the partial sums. let

$$S_n = \sum_{k=1}^{n} = a_1 + a_2 + \cdots + a_n,$$
$$T_m = \sum_{k=0}^{m} 2^k a^{2^k} = a_1 + 2a_2 + \cdots + 2^m a_{2^m}$$

When $2^m \leq n \leq 2^{m+1}$, we have

$$\begin{aligned} S_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^m} + \cdots + a_{2^{m+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^m a^{2^m} = T_m \end{aligned}$$

and

$$\begin{aligned} S_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{m-1}+1} + \cdots + a_{2^m}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{m-1}a_{2^m} = \frac{1}{2}T_m \end{aligned}$$

Thus, we have S_n is bounded above $\iff$ T_m is abounded above. This completes the proof.

Example 1.2.1 $\Sigma \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof: If $p \leq 0$, divergence follows from the vanishing condition of series.
If $p > 0$, *Theorem 1.2.2* is applicable, we are led to the series

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k} = \sum_{k=0}^{\infty} (2^{1-p})^k.$$

Now, $|2^{1-p}| < 1 \iff p > 1$, and the proof follows by the first example of the previous subsection.

Example 1.2.2 $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof: We consider the series

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k(\ln 2^k)^p} = (\ln 2)^{-p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

By *Example 1.2.2*, we can easily prove the original proposition.

Corollary 1.2.1 The above procedure may evidently be continued. In other words, we can also conclude

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^p}$$

converges if and only if $p > 1$, and so forth. For instance,

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))} \tag{1.2}$$

diverges, whreas

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2} \tag{1.3}$$

converges.

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From the above corollary, we may now observe that the series (1.2) differ very little from those of (1.3). Still, one diverges, and the other converges. If we continue the process which led us from *Example 1.2.1* to *Example 1.2.2*, and then to (1.2) and (1.3), we also get pairs of convergent and divergent series whose terms differ even less than those of (1.2) and (1.3).

Thus, one might be led to the conjecture that there is a limiting situation of some sort, i.e. a ”boundary” with all convergent series on one side, all divergent series on the other side. Or when n is sufficiently large, Is there a largest convergent series?

In the following, We shall show that this conjecture is false.

Proposition 1.2.1 Suppose positive seires Σa_n converges, then there must exist positive series Σb_n converges and satisfies $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = +\infty$.

Proof: Let

$$S_n = a_1 + \cdots + a_n,$$

$$T_n = a_n + a_{n+1} + \cdots = \sum_{k=n}^{\infty} a_k.$$

We initially prove that S_n converges $\iff$ T_n converges to 0. We can easily prove that T_n converges iff T_n converges to 0. Hence, next we use the Cauchy criterion to prove

$$S_n \ converges \iff T_n \ converges.$$

Since for $m > n$,

$$T_m - T_n = a_n + \cdots + a_{m-1} = S_{m-1} - S_{n-1}$$

So we have for every $\varepsilon > 0$, there exists $N_1 > 0$, such that for $m > n > N$, $|T_m - T_n| < \varepsilon$, if and only if for every $\varepsilon > 0$, there exists $N_1 > 0$, such that for $m > n > N$, $|S_m - S_n| < \varepsilon$. In other words, the above proposition holds.
By

$$a_n = T_n - T_{n+1} = (\sqrt{T_n} - \sqrt{T_{n+1}})(\sqrt{T_n} + \sqrt{T_{n+1}}),$$

we set $b_n = \sqrt{T_n} - \sqrt{T_{n+1}}$, and $B_n = \Sigma b_n = \sqrt{T_1} - \sqrt{T_n + 1}$.
Hence, Σb_n converges owing to the convergence of T_n, and we have

$$\frac{b_n}{a_n} = \frac{1}{\sqrt{T_n} + \sqrt{T_{n+1}}} \to \infty, \quad (n \to \infty)$$

Introduction

From the proof of *Theorem 1.2.2*, we can know that for the non-negetive series $\Sigma a_n, \Sigma b_n$, if $A_n \leq B_n$ and Σb_n converges, then Σa_n converges.

This is rooted from the *Theorem 1.2.1*, we can expand this to more general case.

Theorem 1.2.3 ***(Comparison Test)*** Suppose $\Sigma a_n, \Sigma b_n$ are non-negetive series, if there exists $C > 0, N \geq 1$, such that $a_n \leq C b_n$ for $n > N$, then if Σb_n converges, Σa_n also converges and if Σa_n diverges, Σb_n also diverges.

Remark 1.2.2 Whether analyzing a single series or comparing two series, we actually don't need to consider the finite initial terms. That is to say, we can only consider the case when n is sufficiently large, and we can even remove the finite terms from one series to compare it with another series.

1.3 The Root and Ratio Tests

Introduction

The above comparison test could generate other useful tests if we let b be some series whose convergence or divergence is known.

Sometimes if we cannot find some series which have some relationship with the original series, we can try some typical series like $\Sigma q^n, \Sigma \frac{1}{n^q}, \Sigma \frac{1}{n \ln n}$