Homework #8 Solution

Problems 1

An orthogonal set of signals is characterized by the property that the inner product of any pair of signals in the set is zero. Fig. 1 shows a pair of signals $s_1(t)$ and $s_2(t)$ that satisfy this condition. Construct the signal constellation for $s_1(t)$ and $s_2(t)$.

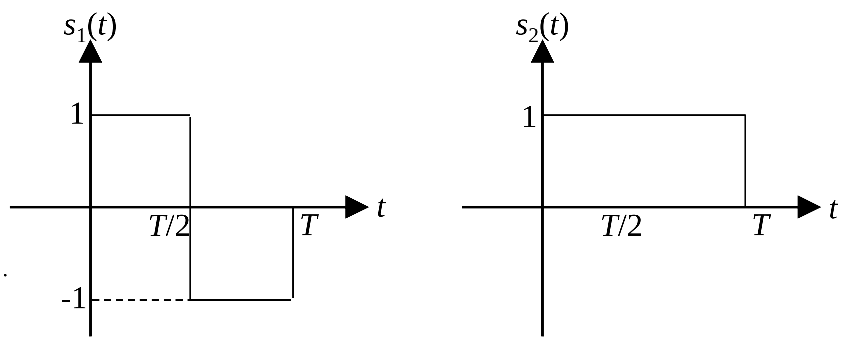

Fig. 1. Prob 1.

Ans:

Signals $s_1(t)$ and $s_2(t)$ are orthogonal to each other. The energy of $s_1(t)$ is

$$E_1 = \int_0^{T/2} 1^2 dt + \int_{T/2}^{T} (-1)^2 dt = T.$$

The energy of $s_2(t)$ is

$$E_2 = \int_0^{T} 1^2 dt = T.$$

To represent the orthogonal signals $s_1(t)$ and $s_2(t)$, we need two basis functions. The first basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}.$$

The second basis function is given by

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}.$$

The signal-space diagram for $s_1(t)$ and $s_2(t)$ is as shown in Fig. 2.

Problems 2

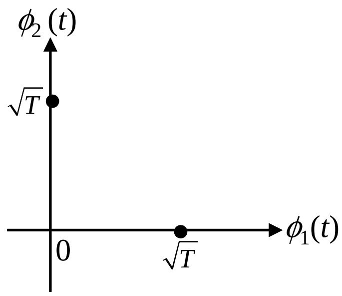

Fig. 2. Prob 1 solution.

Fig. 3 shows a pair of signals $s_1(t)$ and $s_2(t)$ that are orthogonal to each other over the observation interval $0 \leq t \leq 3T$. The received signal is defined by

$$x(t) = s_k(t) + w(t), \quad \text{for} \begin{cases} 0 \leq t \leq 3T, \\ k = 1, 2, \end{cases}$$

where $w(t)$ is white Gaussian noise of zero mean and power spectral density $\frac{N_0}{2}$.

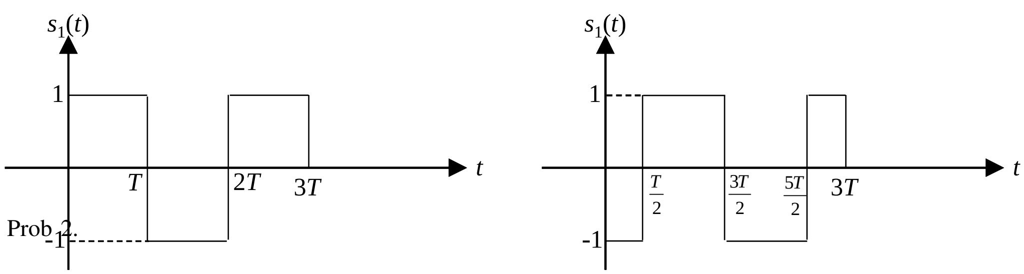

Fig. 3. Prob 2.

(a) Design a receiver that decides in favor of signals $s_1(t)$ or $s_2(t)$, assuming that these two signals are equiprobable.

Ans:

The matched filter for signal $s_1(t)$ is defined by the impulse response

$$h_1(t) = s_1(T - t).$$

The matched filter for signal $s_2(t)$ is defined by the impulse response

$$h_2(t) = s_2(T - t).$$

The matched filter receiver is shown in Fig. 4. The receiver decides in favor of $s_2(t)$ if, for the noisy received signal,

$$x(t) = s_k(t) + w(t), \quad \begin{cases} 0 \leq t \leq T, \\ k = 1, 2 \end{cases}$$

we find that $x_1 > x_2$. On the other hand, if $x_2 > x_1$, it decides in favor of $s_2(t)$. If $x_1 = x_2$, the decision is made by tossing a fair coin.

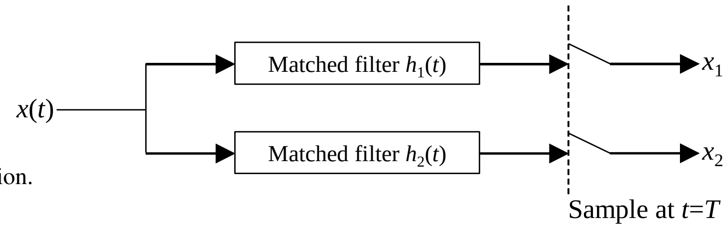

Fig. 4. Prob 2a solution.

(b) Calculate the average probability of symbol error incurred by this receiver for $E/N_0 = 4$, where E is the signal energy.

Ans:

Energy of signal $s_1(t)$ is given by

$$\begin{aligned} E_1 &= \int_0^T (1)^2 dt + \int_T^{2T} (-1)^2 dt + \int_{2T}^{3T} (1)^2 dt \\ &= 3T = E. \end{aligned}$$

Energy of signal $s_2(t)$ is

$$\begin{aligned} E_2 &= \int_0^{T/2} (-1)^2 dt + \int_{T/2}^{3T/2} (1)^2 dt + \int_{3T/2}^{5T/2} (-1)^2 dt + \int_{5T/2}^{3T} (1)^2 dt \\ &= 3T = E. \end{aligned}$$

The orthonormal basis functions for the signal-space diagram of these two orthogonal signals are given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{3T}}, \text{ and } \phi_2(t) = \frac{s_2(t)}{\sqrt{3T}}.$$

The signal-space diagram of signals $s_1(t)$ and $s_2(t)$ is shown in Fig. 5.

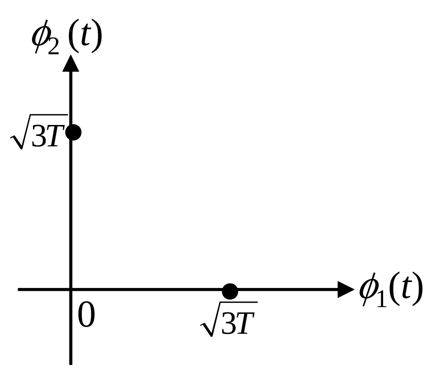

Fig. 5. Prob 2b solution.

Problems 3

In the Bayes test, applied to a binary hypothesis testing problem where we have to choose one of two possible hypotheses H_0 and H_1, we minimize the risk R defined by

$$\begin{aligned} R = {} & C_{00}p_0 Pr\,(\text{say } H_0|H_0 \text{ is true}) + C_{10}p_0 Pr\,(\text{say } H_1|H_0 \text{ is true}) \\ & + C_{11}p_1 Pr\,(\text{say } H_1|H_1 \text{ is true}) + C_{01}p_1 Pr\,(\text{say } H_0|H_1 \text{ is true})\,. \end{aligned}$$

The terms C_{00}, C_{10}, C_{11}, and C_{01} denote the costs assigned to the four possible outcomes of the experiment: The first subscript indicates the hypothesis chosen, and the second the hypothesis that is true. Assume that $C_{10} > C_{00}$ and $C_{01} > C_{11}$. The p_0 and p_1 denote the a priori probabilities of hypotheses H_0 and H_1, respectively.

(a) Given the observation vector $\mathbf{x}$, show that the partitioning of the observation space so as to minimize the risk R leads to the likelihood ratio test:

$$\begin{aligned}&\text{say } H_0 \text{ if } \Lambda(\mathbf{x}) < \lambda\\ &\text{say } H_1 \text{ if } \Lambda(\mathbf{x}) > \lambda,\end{aligned}$$

where $\Lambda(\mathbf{x})$ is the likelihood ratio

$$\Lambda(\mathbf{x}) = \frac{f_X(\mathbf{x}|H_1)}{f_X(\mathbf{x}|H_0)}$$

and λ is the threshold of the test defined by

$$\lambda = \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}.$$

Ans:

Let Z denote the total observation space, which is divided into two parts Z_0 and Z_1. Whenever an observation falls in Z_0, we say H_0, and whenever an observation falls in Z_1, we say H_1. Thus, expressing the risk R in terms of the conditional probability density functions and the decision regions, we may write

$$\begin{aligned}R = C_{00}p_0 \int_{Z_0} f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)d\mathbf{x} + C_{10}p_0 \int_{Z_1} f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)d\mathbf{x}\\ + C_{11}p_1 \int_{Z_1} f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)d\mathbf{x} + C_{01}p_1 \int_{Z_0} f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)d\mathbf{x}.\end{aligned} \tag{1}$$

Note that for an N-dimensional observation space, the integrals in (1) are N-fold integrals. To find the Bayes test, we must choose the decision regions Z_0 and Z_1 in such a manner that the risk will be minimized. Because we require that a decision be made, this means that we must assign each point $\mathbf{x}$ in the observation space Z to Z_0 or Z_1; thus

$$Z = Z_0 + Z_1.$$

Hence, we may rewrite (1) as

$$\begin{aligned}R = p_0C_{00} \int_{Z_0} f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)d\mathbf{x} + p_0C_{10} \int_{Z-Z_0} f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)d\mathbf{x}\\ + p_1C_{11} \int_{Z-Z_0} f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)d\mathbf{x} + p_1C_{01} \int_{Z_0} f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)d\mathbf{x}.\end{aligned} \tag{2}$$

We observe that

$$\int_Z f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)d\mathbf{x} = f_Z f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)d\mathbf{x} = 1.$$

Hence, (2) reduces to

$$\begin{aligned}R &= p_0C_{10} + p_1C_{11}\\ &+ \int_{Z_0} \left\{-\left[p_0(C_{10} - C_{00})f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)\right] + \left[p_1(C_{01} - C_{11})f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)\right]\right\} d\mathbf{x}\end{aligned} \tag{3}$$

The first two terms in (3) represent the fixed cost. The integral represents the cost controlled by those points $\mathbf{x}$ that we assign to Z_0. Since $C_{10} > C_{00}$ and $C_{01} > C_{11}$, we find that the two terms inside the square brackets are positive. Therefore, all values of $\mathbf{x}$ where the first term is larger than the second should be included in Z_0 because they contribute a negative amount to the integral. Similarly, all values of $\mathbf{x}$ where the second term is larger than the first should be excluded from Z_0 (i.e., assigned to Z_1) because they would contribute a positive amount to the integral. Values of $\mathbf{x}$ where the two terms are equal have no effect on the cost and may be assigned arbitrarily. Thus the decision regions are defined by the following statement: If

$$p_1(C_{01} - C_{11}) f_{\mathbf{x}|H_1}(\mathbf{x}|H_1) > p_0(C_{10} - C_{00}) f_{\mathbf{x}|H_0}(\mathbf{x}|H_0),$$

assign $\mathbf{x}$ to Z_1 and consequently say that H_1 is true. If the reverse is true, assign $\mathbf{x}$ to Z_0 and say H_0 is true.

Alternatively, we may write

$$\frac{f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)}{f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}.$$

The quantity on the left is the likelihood ratio

$$\Lambda(\mathbf{x}) = \frac{f_{\mathbf{x}|H_1}(\mathbf{x}|H_1)}{f_{\mathbf{x}|H_0}(\mathbf{x}|H_0)}.$$

Let

$$\lambda = \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}.$$

Thus, Bayes criterion yields a likelihood ratio test described by

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda.$$

(b) What are the cost values for which the Bayes' criterion reduces to the minimum probability of error criterion?

Ans:

For the minimum probability of error criterion, the likelihood ratio test is described by

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0}{p_1}.$$

Thus, we may view the minimum probability of error criterion as a special case of the Bayes criterion with the cost values defined as

$$C_{00} = C_{11} = 0$$
$$C_{10} = C_{01}.$$

That is, the cost of a correct decision is zero, and the cost of an error of one kind is the same as the cost of an error of the other kind.