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Complex Variables and Laplace Transformations

MAT 215

Lecture Notes

Preface and Acknowledgements

This lecture note compilation has been prepared for aiding the students who are taking the course MAT215 (Complex Variables and Laplace Transform), offered by Brac University. These notes are a compilation of parts taken from two other books (listed below) that have been abridged and altered in a manner best suited for the students who are taking this course. These notes were created under the strict supervision of eminent mathematician, Dr. Syed Hasibul Hasan Chowdhury. The main goal of this compilation is to help keep things organized for the students and ensure continuity of the course content among every section. We are highly indebted to the senior students of the MNS and CSE Department, for helping us out and making this possible. Since this is the first version of this compilation, there may be some errors and typing mistakes. If any mistakes are found please report them to **mishaal.hai@bracu.ac.bd**.

Project Coordinator:

Dr. Syed Hasibul Hasan Chowdhury

Typeset by:

Mishaal Hai & Shaikot Jahan Shuvo *(Editor)*
Rahul Pal *(Template Designer)*
Tahsin Nahian Bin Quddus
Mehedat Orbeen
Nazmoon Falgune Moon
Afia Fahmida Rahman
Shoumik Hossain

Reference Books:

- Complex Variables and Applications, 8th Edition by James W. Brown, Ruel V. Churchill
- Differential Equations with Boundary-Value Problems by Dennis G. Zill, Michael R. Cullen

Contents

Part I

Complex Variables

Chapter 1

Complex Numbers

In this course we shall be looking at the world of complex numbers. We survey the algebraic and geometric structure of the complex number system.

1.1 Sums and Products

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the complex plane. Where the pure real part of the complex number is displayed on the x-axis and the pure imaginary part on the y-axis.

It is convention to denote complex numbers as $z = (x, y)$ where $x = Re\ z$ and $y = Im\ z$. It becomes mathematically easier to work with, however, when we denote complex numbers as $z = x + iy$. Now we can define the addition and multiplication of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as:

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2) \\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\end{aligned} \tag{1.1}$$

1.2 Basic Algebraic Properties

Various properties of addition and multiplication of complex numbers are similar to that of their real number counterpart. Below you will find the more basic of these algebraic properties.

The commutative laws

$$z_1 + z_2 = z_2 + z_1 \tag{1.2}$$
$$z_1 z_2 = z_2 z_1$$

The associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \tag{1.3}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

And the distributive laws

$$z(z_1 + z_2) = z z_1 + z z_2 \tag{1.4}$$

In addition to these, we have the additive, $0 = (0, 0)$, and the multiplicative identity, $1 = (1, 0)$, which are defined as follows:

$$z + 0 = z \tag{1.5}$$
$$z \cdot 1 = z$$

Which automatically leads us to the additive inverse, $-z = (-x, -y)$ such that the equation $z + (-z) = 0$ is satisfied and the multiplicative inverse, z^{-1}, which satisfies the equation $z \cdot z^{-1} = 1$. The most general solution of a multiplicative inverse is as follows:

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \tag{1.6}$$

1.3 Complex Conjugates

The complex conjugate of a complex number $z = x + iy$ is defined as $\bar{z} = x - iy$. We should note here that, $\bar{\bar{z}} = z$ and $|\bar{z}| = |z|$. Therefore we can see that the complex conjugate is simply a reflection on the real axis. A few other properties that can be derived from the concept of complex conjugates are as follows:

For any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

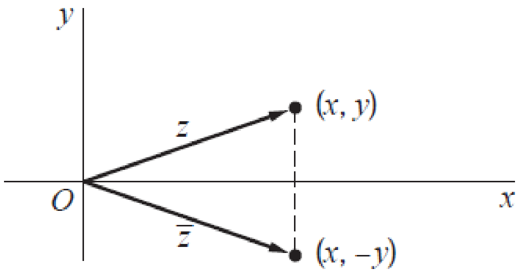


Figure 1.1: *Representation of Complex Conjugation in Argand Diagram*

$$\overline{z_1 + z_2} = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 + \bar{z}_2 \tag{1.7}$$

$$\overline{z_1 - z_2} = (x_1 - iy_1) - (x_2 - iy_2) = \bar{z}_1 - \bar{z}_2 \tag{1.8}$$

$$\overline{z_1 z_2} = z_1\bar{z}_2 \tag{1.9}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \tag{1.10}$$

Another important property to note is for any complex number $z = x + iy$,

$$z \cdot \bar{z} = |z^2| \tag{1.11}$$

1.4 Exponential Form

We can write a complex number $z = x + iy$ in polar form as:

$$z = r(\cos\theta + i\sin\theta) \tag{1.12}$$

Where it is understood that $z \neq 0$ as that would give an undefined θ.

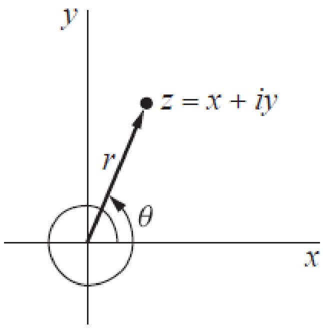


Figure 1.2: *Polar representation of a complex number*

Furthermore, we know according to Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.13}$$

Therefore, using equation (1.12) and (1.13), we can derive the following:

$$z = re^{i\theta} \tag{1.14}$$

1.5 Products and Powers in Exponential Form

We know from trigonometry, $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Therefore, for any two complex numbers, $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, we can write,

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i\theta_1}e^{i\theta_2} = (r_1r_2)\, e^{i(\theta_1+\theta_2)} \tag{1.15}$$

And similarly the following can be written as well,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \tag{1.16}$$

$$z^{-1} = \frac{1}{r}e^{-i\theta}.$$

Intuitively, we can state then state,

$$z^n = r^ne^{in\theta} \tag{1.17}$$

1.6 Roots of Complex Numbers

Two non zero complex numbers, $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal if and only if:

$$\begin{aligned} r_1 &= r_2 \; and \\ \theta_1 &= \theta_2 = 2k\pi \end{aligned} \tag{1.18}$$

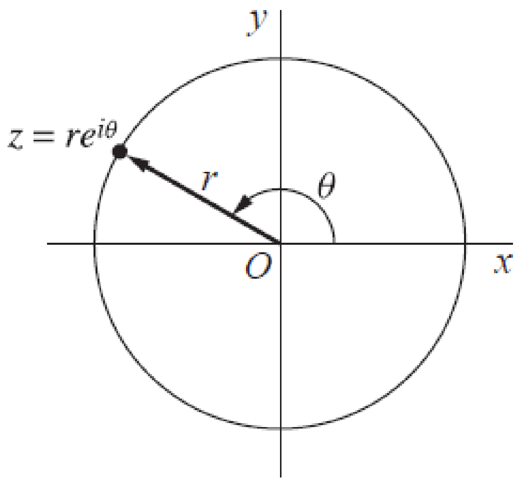


Figure 1.3: *As θ increases by 2π, we arrive at the initial point*

Using this fact and equation (**??**), we can now start the process of finding the n^{th} roots of complex numbers. we can start the process by the fact that an n^{th} root of z_0 is a nonzero number $z = re^{i\theta}$ such that $z_n = z_0$, or:

$$r_n e^{in\theta} = r_0 e^{i\theta_0} \tag{1.19}$$

Which can then be used to derive the following:

$$\begin{aligned} r^n &= r^0 \\ n\theta &= \theta_0 + 2k\pi. \end{aligned} \tag{1.20}$$

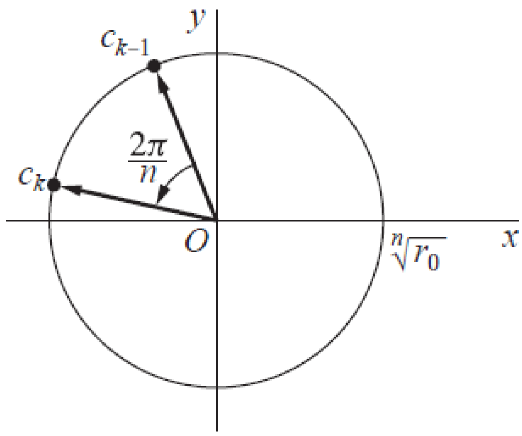
Consequently after simplifying everything and putting them back, we get:

$$z = \sqrt[n]{r_0}\ exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \tag{1.21}$$

are the n^{th} roots of z_0. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians. Evidently all of the distinct roots are obtained only when $k = 0, 1, 2, \ldots, n-1$. We let c_k denote these distinct roots and write:

$$c_k = \sqrt[n]{r_0} exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]. \tag{1.22}$$

As is shown in the figure below:



Chapter 2

Analytic Functions

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

2.1 Functions Of A Complex Variable

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the value of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the domain of definition of f. The following examples should further clarify:

Example 2.1.1.

If f is defined on the set $z \neq 0$ by means of the equation $w = 1/z$, it may be referred to only as the function $w = 1/z$, or simply the function $1/z$.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy) \tag{2.1}$$

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \tag{2.2}$$

If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta}) \tag{2.3}$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r,\theta) + iv(r,\theta) \tag{2.4}$$

Example 2.1.2.

If $f(z) = z^2$, we can say:

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + i2xy \tag{2.5}$$

Hence, the following can be claimed:

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned} \tag{2.6}$$

Hence, re-writing the equations in polar form:

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2\cos 2\theta + ir^2\sin 2\theta \tag{2.7}$$

and we finally get:

$$u(r,\theta) = r^2\cos 2\theta \tag{2.8}$$
$$v(r,\theta) = r^2\sin 2\theta \tag{2.9}$$

Now we can see, if in either of the equations, 2.2 or 2.4, the function v always has value zero, then the value of f is always real. Meaning that f is a *real-valued function* of a complex variable.

2.2 Mappings

We are now going to introduce the concept of mapping. When $w = f(z)$, where z and w are complex, no convenient graphical representation of the function f is available because each of the numbers z and w is located in a plane rather than on a line. However, we can display some information about the function by indicating pairs of corresponding points $z = (x, y)$ and $w = (u, v)$. When a function f is thought of in such a manner, it is often referred to as a *mapping* or a *transformation*.

Terms such as translation, rotation, and reflection are used to convey characteristics of certain mappings. For example, the mapping $w = z+1 = (x+1)+iy$, where $z = x+iy$, can be thought of as a translation of each point z one unit to the right. The following examples should further illustrate what mapping is.

Example 2.2.1.

The mapping $w = z^2$ can be thought of as the transformation

$$u = x^2 - y^2 \tag{2.10}$$
$$v = 2xy \tag{2.11}$$

from the xy plane to the uv plane. This form of the mapping is especially useful in finding the images of certain hyperbolas. It is easy to show, for instance, that each branch of a hyperbola

$$x^2 - y^2 = c_1 \ (c_1 > 0) \tag{2.12}$$

is mapped in a one to one manner onto the vertical line $u = c_1$. We start by noting from the equation (2.2) that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the equation (2.2) tells us that $v = 2y\sqrt{(}y^2+c_1)$ (Just a simple substitution of equation (2.9) after making c_1 the subject and putting it in the equation (2.8)). Thus the image of the right-hand branch can be expressed parametrically as:

$$u = c_1, \ v = 2y\sqrt{(}y^2 + c_1) \text{ where } (-\infty < y < \infty) \tag{2.13}$$

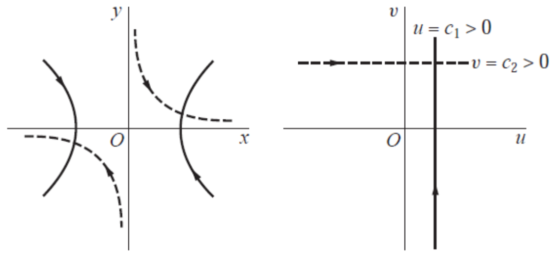


Figure 2.1: *Graphical representation of the mapping $w = z^2$*

On the other hand, each branch of a hyperbola,

$$2xy = c_2 \text{ where } (c_2 > 0) \tag{2.14}$$

is transformed into the line $v = c_2$, as is indicated in figure 2.1. we can verify this by noting the equation (2.8), that $v = c_2$ when (x, y) is a point on either branch. Suppose that (x, y) is on the branch lying in the first quadrant. Then, since $y = \frac{2}{(2x)}$, the equation (2.7) reveals that the branch's image has parametric representation

$$u = x^2 - \frac{c_2^2}{4x^2}, \; v = c_2 \; (0 < x < \infty) \tag{2.15}$$

Since u depends continuously on x, it is clear that as (x, y) travels down the entire upper branch of hyperbola (equation2.10), its image moves to the right along the entire horizontal line $v = c_2$. Inasmuch as the image of the lower branch has parametric representation

$$u = \frac{c_2^2}{4x^2} - y^2, \; v = c_2 \; (\infty < y < 0) \tag{2.16}$$

And since we know that u approaches $-\infty$ as y approaches $-\infty$ and u approaches ∞ as y approaches 0, it follows that the image of a point moving upward along the entire lower branch also travels to the right along the entire line $v = c_2$ (Fig. 2.2)

Example 2.2.2.

We shall now use Example 01 to find the image of a certain *region*.

The domain $x > 0$, $y > 0$, $xy < 1$ consists of all points lying on the upper branches of hyperbolas from the family $2xy = c$, where $0 < c < 2$ (Fig. 2.2). We know from Example 01 that as a point travels downward along the entirety of such a branch, its image under the transformation $w = z^2$ moves to the right along the entire line $v = c$. Since, for all values of c between 0 and 2, these upper branches fill out the domain $x > 0$, $y > 0$, $xy < 1$, that domain is mapped onto the horizontal strip $0 < v < 2$.

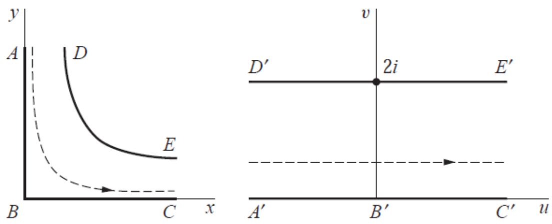


Figure 2.2: *Graphical representation of the mapping* $w = z^2$

In view of equations (2.7) and (2.8), the image of a point $(0, y)$ in the z plane is $(-y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, then, the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is mapped onto the closed strip $0 \leq v \leq 2$, as indicated in Fig. 2.1

Example 2.2.3.

Our last example here illustrates the usefulness of polar co-ordinates in analyzing mappings.

As studied previously, the mapping $w = z^2$ in polar co-ordinates becomes, $w = r^2 e^{i2\theta}$ when $z = re^{i\theta}$. We can then say, that the image $w = \rho e^{i\theta}$ of any non-zero point z is found by the square of the modulus

$r = |z|$ and doubling the value θ of $\arg z$ that it used(refer to chapter 1):

$$\rho = r^2 \; and \; \phi = 2\theta \tag{2.17}$$

We can now clearly see that due to the nature of the mapping, which essentially squares the value of the original number, the points $z = r_0 e^{i\theta}$ o a circle $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $r = r_0^2$. As a point on a circle moves counterclockwise from the positive real (x) axis to the positive imaginary (y) axis, its image at the same time moves counterclockwise from the positive real (x) axis to the negative real (x) axis as is demonstrated in the figure below. So for all values of r_0, the first quadrant and the upper half plane are filled due to the transformation to the z and w planes respectively. The point $z = 0$, however is mapped to the point $w = 0$.

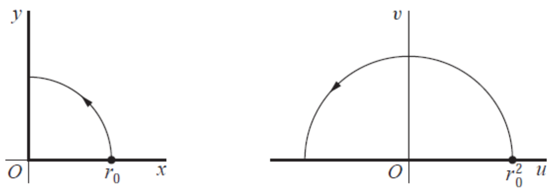


Figure 2.3: *The rotation of a point and its image due to the mapping $w = z^2$*

2.3 Mappings by the Exponential Function

We now set up the basics required for the next chapter, starting with the exponential function:

$$e^z = e^x e^{iy} \; (z = x + iy) \tag{2.18}$$

The two factors e^x and e^{iy} are both well defined (check chapter 1) right now. We shall now use the function e^z to provide further examples of reasonably simple mappings.

Example 2.3.1.

Take the transformation

$$w = e^z = e^{x+iy} \tag{2.19}$$

Therefore, using polar co-ordinates we can make the following deduction:

$$\rho = e^x, \ \phi = y \tag{2.20}$$

Now, consider the image of the point $z = (c_1, y)$ on any vertical line $x = c_1$. In the polar co-ordinates, as is obvious from above, ϕ would remain unchanged while ρ would become e^{c_1} in the w plane. Meaning that this would give an image that moves counterclockwise around a circle and z moves up. And due to the periodic nature of exponential, an infinite number of points would represent any given image on the circle.

Now consider a horizontal line $y = c_2$, in this case ρ would remain unchanged and ϕ would transform as $\phi = c_2$. That is, it would simply give a straight line, or a *ray* that makes an angle with the positive x-axis equal to the value of c_2.

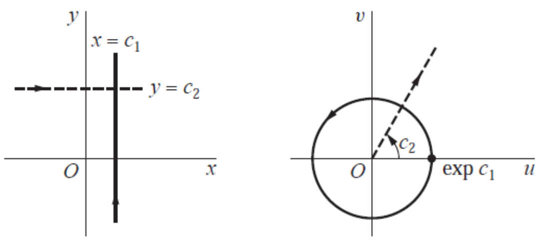


Figure 2.4: *Graphical representation of the mapping $w = e^z$ $w = z^2$*

Example 2.3.2.

We shall now use the images of *horizontal* lines to find the image of a horizontal strip.

When $w = e^z$, the image of the infinite strip $0 \leq y \leq \pi$ is the upper half $v \geq 0$ of the w plane as shown in the figure below. This is seen by recalling from the previous example how a horizontal line is transformed into a ray from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$.

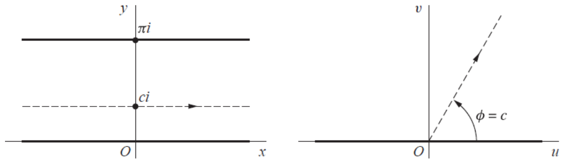


Figure 2.5: *Graphical representation of the effects of the mapping $w = e^z$ on* horizontal *lines.*

2.4 Limits

Let a function f be defined at all points z in some deleted neighborhood of z_0. The statement that the limit of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0 \tag{2.21}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form. The previous statement means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta \tag{2.22}$$

Geometrically, this definition says that for each ϵ neighborhood $|w - w_0| \leq \epsilon$ of w_0, there is a deleted δ neighborhood $0 \leq |z - z_0| \leq \delta$ of z_0 such that every point z in it has an image w lying in the ϵ neighborhood (figure 2.6). Note that even though all points in the deleted neighborhood $0 \leq |z - z_0| \leq \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| \leq \epsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\frac{\delta}{2}$.

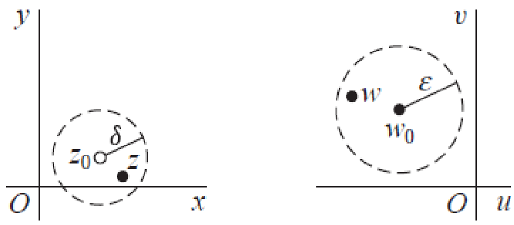


Figure 2.6:

It is easy to show that *when a limit of a function $f(z)$ exists at a point z_0, it is unique.* To do this, we suppose that,

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1 \tag{2.23}$$

Then, for each positive number ϵ, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0 \tag{2.24}$$

and

$$|f(z) - w_1| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1 \tag{2.25}$$

So if $0 < |z - z_0| < \delta$, where δ is any positive number that is smaller than δ_0 and δ_1, we find that

$$|w_1 - w_0| = |[f(z) - w_0] - [f(z) - w_1]| \leq |f(z) - w_0| + |f(z) - w_1| < \epsilon + \epsilon = 2\epsilon.$$

But $|w_1 - w_0|$ is a non-negative constant, and ϵ can be chosen arbitrarily small. Hence

$$w_1 - w_0 = 0, \quad \text{or} \quad w_1 = w_0 \tag{2.26}$$

2.23 requires that f be defined at all points in some deleted neighborhood of z_0. Such a deleted neighborhood, of course, always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that the first of inequalities in 2.22 need to be satisfied by only those points z that lie in both the region and the deleted neighborhood.

Example 2.4.1.

Let us show that if $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2} \tag{2.27}$$

the point in 2.21 being on the boundary of the domain of definition of f. Observe that when z is in the disk $|z| < 1$,

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2} \tag{2.28}$$

Hence, for any such z and each positive number ϵ (see Fig. 2.7),

$$\left| f(z) - \frac{i}{2} \right| < \epsilon \quad \text{whenever} 0 < |z-1| < 2\epsilon \tag{2.29}$$

Thus 2.23 is satisfied by points in the region $|z| < 1$ when δ is equal to 2ϵ or any smaller positive number.

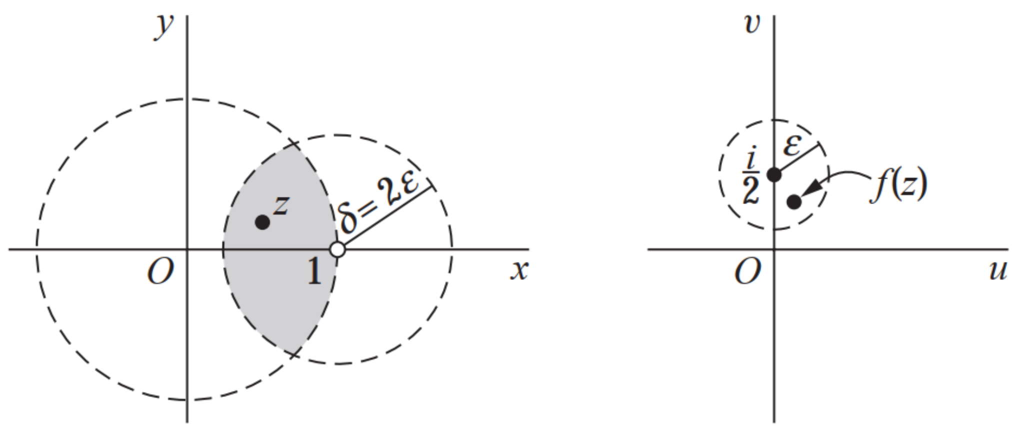


Figure 2.7:

Example 2.4.2.

If

$$f(z) = \frac{z}{\bar{z}} \tag{2.30}$$

the limit

$$\lim_{z \to 0} f(z) \tag{2.31}$$

does not exist. For, if it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner. But when $z = (x, 0)$ is a nonzero point on the real axis (Fig. 2.8),

$$f(z) = \frac{x + i0}{x - i0} = 1; \tag{2.32}$$

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

$$f(z) = \frac{0 + iy}{0 - iy} = 1. \tag{2.33}$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit in 2.31 does not exist.

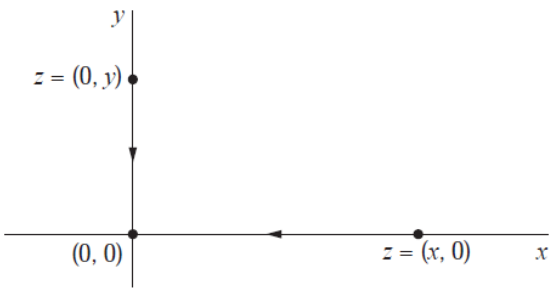


Figure 2.8:

While 2.23 provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit. Theorems on limits, presented in the next section, will enable us to actually find many limits.

2.5 Theorems On Limits

We will present some theorems on limits that will be crucial for our study of complex variables.

Theorem 2.5.1.

$$f(z) = u(x, y) + iv(x, y) \tag{2.34}$$

Where,

$$z = x + iy \tag{2.35}$$

And

$$z_0 = x_0 + iy_0 \tag{2.36}$$
$$w_0 = u_0 + iv_0 \tag{2.37}$$

Then,

$$\lim_{z \to z_0} f(z) = w_0 \tag{2.38}$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \tag{2.39}$$

and

$$\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$$

To prove this theorem, we assume the limits of 2.39 are valid and give us the limit 2.38. From the limits 2.39, we can say that there exists positive numbers δ_1 and δ_2 for each positive number ε, s.t,

When

$$0 \leq \sqrt[2]{(x-x_0)^2+(y-y_0)^2} \leq \delta_1, \ |u-u_0| \leq \frac{\varepsilon}{2} \tag{2.40}$$

and when

$$0 \leq \sqrt[2]{(x-x_0)^2+(y-y_0)^2} \leq \delta_2, \ |v-v_0| \leq \frac{\varepsilon}{2} \tag{2.41}$$

Let δ be a positive value s.t.

$$\delta \leq \delta_1, \delta_2$$

As learnt from prior sections concerning inequalities of complex variables,

$$|(u+iv)-(u_0+iv_0)| = |(u-u_0)+i(v-v_0)| \leq |u-u_0|+|v-v_0| \tag{2.42}$$

and

$$\sqrt[2]{(x-x_0)^2+(y-y_0)^2} = |(x-x_0)+i(y-y_0)| \tag{2.43}$$
$$= |(x+iy)-(x_0+iy_0)| \tag{2.44}$$

Using 2.40 and 2.41

$$|(u-u_0)+i(v-v_0)| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

Replacing with the inequalities denoted in 2.44, we get:

$$|(u_iv)-(u_0+iv_0)| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$
$$\Rightarrow |(u_iv)-(u_0+iv_0)| \leq \frac{\varepsilon}{2}$$

whenever

$$0 \leq |(x+iy)-(x_0+iy_0)| < \varepsilon$$

,
that is when $0 \leq |(x+iy)-(x_0+iy_0)| \leq \delta_1, \delta_2$.
Hence, limit 2.38 is valid.

Now, we shall start by assuming limit 2.38 holds. We also know by now that whenever we have a positive value ε, there exists a positive number δ such that,

$$|(u+iv)-(u_0+iv_0)| \leq \varepsilon \tag{2.45}$$

and

$$0 < |(x+iy)-(x_0+iy_0)| \leq \varepsilon. \tag{2.46}$$

However, prior knowledge of inequalities tells us:

$$|u-u_0| \leq |u-u_0|+|v-v_0| = |(u-u_0)+i(v-v_0)|$$

and,

$$|v-v_0| \leq |u-u_0|+|v-v_0| = |(u-u_0)+i(v-v_0)|$$

and

$$|(x-x_0)+i(y-y_0)| = |(x+iy)-(x_0+iy_0)| = \sqrt[2]{(x-x_0)^2+(y-y_0)^2}$$

Hence from inequalities 2.45 and 2.46 , it can be concluded that $|u-u_0| \leq \varepsilon$ and $|v-v_0| \leq \varepsilon$ when $0 \leq |(x+iy)-(x_0+iy_0)| \leq \varepsilon$.

This proves that the limits 2.39 are valid.

Theorem 2.5.2.

Let

$$\lim_{z \to z_0} f(z) = w_0 \tag{2.47}$$

and,

$$\lim_{z\to z_0} [f(z) + F(z)] = w_0 + W_0 \tag{2.48}$$

and,

$$\lim_{z\to z_0} [f(z)F(z)] = w_0 W_0 \tag{2.49}$$

and if $W_0 \neq 0$,

$$\lim_{z\to z_0} \left[\frac{f(z)}{F(z)}\right] = \frac{w_0}{W_0} \tag{2.50}$$

To verify equation 5.10, we take $f(z) = u(x,y) + iv(x,y), F(z) = U(x,y) + iV(x,y)$

$$z_0 = x_0 + iy_0; w_0 = u_0 + iv_0; W_0 = U_0 + iV_0$$

Then from equation 2.47, we can say that z approaches z_0 when u, v approach u_0, v_0 and U, V approach U_0, V_0 respectively.

If we find the product of $f(z)$ and $F(z)$, $f(z)F(z) = (uU - vV) + i(vU_uV)$. We will see that this product has the limits $u_0U_0 - v_0V_0$ and $u_0V_0 + v_0U_0$ respectively as z approaches z_0. Hence the product itself has the limits $(u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0)$ which is basically equal to w_0W_0. This verifies equation 5.10.

From the previous section,we have seen that

$$\lim_{z\to z_0} c = c, \lim_{z\to z_0} z = z_0 \tag{2.51}$$

where z_0 and c are any complex numbers. Hence, using equation 5.10, we can say that

$$\lim_{z\to z_0} z^n = z_0^n (\text{where n=1,2,3....}) \tag{2.52}$$

So bring into our spotlight equations 2.48 and 5.10, limit of a polynomial can be described as such:

$$P(z) = a_0, a_{1z}, a_{2z}^2 + ... + a_{nz}^n$$

As z approaches z_0, the value of this polynomial shall be:

$$\lim_{z \to z_0} P(z) = P(z_0) \tag{2.53}$$

2.6 Limits Involving the Point at Infinity

Including the *point at infinity*, ie. ∞ on the complex plane forms what is known as the extended complex plane. This shall help us evaluate limits inside the complex plane in which the point z may extend towards ∞. To understand the point at infinity, we must first visualize a sphere, namely a Riemann sphere, with the origin as its centre. There shall be a point P on the surface of the sphere such that the line through a point z on the complex plane and the north pole N passes through the point P. Each point P shall correspond to only one point z on the complex plane. This correspondence is known as the stereographic projection. To make it easier for you to visualize, take a look at 2.9.

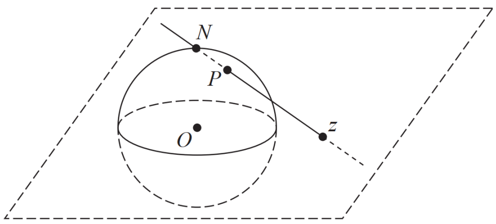


Figure 2.9: *Graphical representation of the* Riemann sphere *and the* stereographic projection

For each small positive number ε, $|z| = \frac{1}{\varepsilon}$ correspond to the circles near N.

Hence $|z| > \frac{1}{\varepsilon}$ is called the ε neighbourhood (i.e. neighbourhood of ∞).

If z_0 or w_0 in the limit:

$$\lim_{z\to\ z_0} f(z) = w_0 \tag{2.54}$$

is replaced with ∞, we shall replace their neighbours (ie. z_0 or w_0 respectively) with neighbours of ∞. We shall see how in the following theorem.

Theorem: If z_0 and w_0 are points in the z and w planes respectively, then

$$\lim_{z\to\ z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to\ z_0} 1/f(z) = 0, \tag{2.55}$$

$$\lim_{z\to\ \infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z\to\ 0} f(1/z) = w_0 \tag{2.56}$$

and

$$\lim_{z\to\ \infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to\ 0} 1/f(1/z) = 0 \tag{2.57}$$

Proof of 2.55: Equation 2.38 can be proven by noting that for each positive number ε, there is a positive number δ s.t.

$$|f(z)| = \frac{1}{\varepsilon} \quad \text{whenever} \quad 0 \leq |z - z_0| \leq \delta \tag{2.58}$$

i.e. the point $w = f(z)$ lies in the deleted neighbourhood $0 \leq |z - z_0| \leq \delta$ of z_0. Since 2.57 can be written as $|1/f(z) - 0| \leq \varepsilon$ whenever $0 \leq |z - z_0| \leq \varepsilon$.

2.6.1 Proof of 2.56

The first of limits of 2.56 suggests that for each positive number ε, a positive number δ exists s.t.

$$|f(z) - w_0| \leq \varepsilon \quad \text{whenever} \quad |z| \geq \frac{1}{\delta} \tag{2.59}$$

Replacing z by $\frac{1}{a}$ in equation 2.59, we arrived at the second part of 2.56:

$$|f(1/z) - w_0| \leq \varepsilon \quad \text{whenever } 0 \leq |z - 0| \leq \delta \tag{2.60}$$

Proof of it comes from for each positive number ε, there exists a positive number δ such that.

$$|f(z)| \leq \frac{1}{\varepsilon} \quad \text{whenever} \quad |z| \geq \frac{1}{\delta} \tag{2.61}$$

The replacement of z by $\frac{1}{z}$ gives us the second part of 2.57:

$$|1/f(z) - 0| \leq \varepsilon \quad \text{whenever} \quad 0 \leq |z - 0| \leq \delta \tag{2.62}$$

and this gives us the second part of limits 2.57

Example 2.6.1.

(i) $\lim_{z\to -1} \frac{iz+3}{z+1} = \infty \Rightarrow \lim_{z\to\ -1} \frac{z+1}{iz+3} = 0$

We achieved the result by finding the reciprocal of the function $\frac{iz+3}{z+1}$ and by applying 2.55

(ii) $\lim_{z\to\ \infty} \frac{2z+i}{z+1} = 2 \Rightarrow \lim_{z\to\ 0} \frac{2\left(\frac{1}{z}\right)+i}{\left(\frac{1}{z}\right)+1} = 2$

Since the point z reached towards infinity, we replaced z by $\frac{1}{z}$ in the limit function.

(iii) $\lim_{z\to\ \infty} \frac{2z^3-1}{z^2+1} = \infty \Rightarrow \lim_{z\to\ \infty} \frac{\frac{1}{z^2}+1}{\frac{1}{z^3}-1} = 0 \Rightarrow \lim_{z\to\ 0} \frac{z+z^3}{2-z^3} =$ 0

This is the example that wraps up the concept of 2.57. Not only did we replace $f(z)$ by $f'(z)$, we also replaced z by $\frac{1}{z}$.

2.7 Derivatives

Let f be a function with a point z_0 whose neighbourhood is $|z - z_0| \leq \varepsilon$. The derivative of f at z_0 is the limit:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)} \tag{2.63}$$

f is differentiable if $f'(z_0)$ exists.

Let $\Delta z = z - z_0$ where $z \neq z_0$. We get,

$$f'(z_0) = \lim_{\Delta z \to 0} \left(\frac{(z_0 + \Delta z) - f(z_0)}{\Delta z} \right) \tag{2.64}$$

The function f shall be defined throughout a small region around z_0, that is, its neighbourhood, and the value $f(z_0 + \Delta z)$ shall be defined for $|\Delta z|$ as shown in 2.10

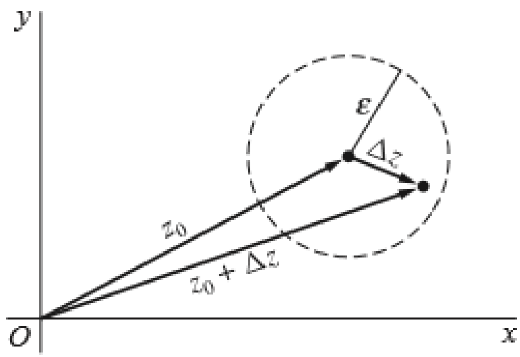


Figure 2.10: *Demonstration of* $|\Delta z|$

Let $\Delta w = f(z+\Delta z) - f(z)$ denote the change in the value of $w = f(z)$ of function f due to a change in Δz at the point at which f is evaluated.

Finding $\frac{dw}{dz}$ for $f'(z)$ gives us:

$$\frac{dw}{dz} = \lim_{\Delta z \to\ 0} (\frac{\Delta w}{\Delta z}) \tag{2.65}$$

Example 2.7.1.

$$f(z) = z^2$$

$$\lim_{\Delta z \to]0} \frac{\Delta w}{\Delta z} = \lim \frac{(z + \Delta z)^2 - z^2}{(\Delta z)} = \lim(2z + \Delta z) = 2z$$

Therefore, $\frac{dw}{dz} = 2z$ or, $f'(z) = 2z$

Example 2.7.2.

$$f(z) = \overline{z}$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{(\overline{z + \Delta z} - \overline{z})}{(\Delta z)} = \frac{(\overline{z})}{(\Delta z)}$$

If the limit does exist, it can be found by letting the point $(\Delta z)\Delta z = (\Delta x, \Delta y)$ approach $(0, 0)$ horizontally through $(\Delta x, 0)$ on the real axis.

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0\Delta z$$

Since $\overline{\Delta z} = \Delta z$,

$$\frac{(\Delta w)}{(\Delta z)} = \frac{(\Delta z)}{(\Delta z)} = 1$$

If the limit at $\frac{(\Delta w)}{(\Delta z)}$ does exist, the value of the point approaching the origin vertically shall be the same as the value obtained when it approaches horizontally, the value of which we have already found out above. That is, in order for the function to be differentiable, the limit, whether approached vertically or horizontally, must bear the same value.

However, when Δz approaches the origin vertically through the points $(0, \Delta y)$ on the imaginary axis,

$$\overline{\Delta z} = \overline{0 + i\Delta y} = -(0 + i\Delta y) = -\Delta z$$

On evaluating the limit function $\frac{(\Delta w)}{(\Delta z)}$, we get:

$$\frac{(\Delta w)}{(\Delta z)} = \frac{(-\Delta z)}{(\Delta z)} = -1$$

It is now evident that the values of $\frac{(\Delta w)}{(\Delta z)}$ obtained are different in each of the cases when the point Δz approaches 0 vertically and horizontally. This proves that the value of $\frac{(\Delta w)}{(\Delta z)}$ does not exist anywhere. 2.11 shows the idea of a point z approaching the origin from the labelled points, which are also used throughout calculations.

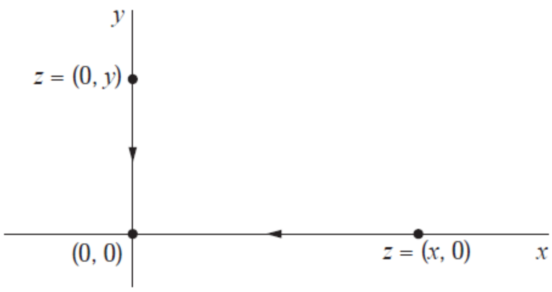


Figure 2.11: *Point z approaching the origin*

Example 2.7.3.

$$f(z) = |z|^2$$

Finding the limit function $\frac{(\Delta w)}{(\Delta z)}$,

$$\begin{aligned}
\frac{(\Delta w)}{(\Delta z)} &= \frac{(|z + \Delta z|^2 - |z|^2)}{(\Delta z)} \\
&= \frac{((z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z})}{(\Delta z)} \\
&= \frac{\overline{z} + \overline{\Delta z} + z\overline{\Delta z}}{(\Delta z)} \qquad (2.66)
\end{aligned}$$

As shown in Example (2.7.2), if we evaluate the value of Δz as it approaches $(0, 0)$ horizontally and vertically respectively, we shall obtain the following values:

$$\frac{(\Delta w)}{(\Delta z)} = \overline{z} + \Delta z + z \quad \text{when} \quad \Delta z = (\Delta x, 0)$$

$$\text{and} \quad \frac{(\Delta w)}{(\Delta z)} = \overline{z} - \Delta z - z \quad \text{when} \quad z = (0, \Delta y)$$

If the limit exists at $\frac{(\Delta w)}{(\Delta z)}$ as Δz approaches $(0, 0)$, then:

$$\overline{z} + z = \overline{z} - z$$

or, $z = 0$.

However, $\frac{(\Delta w)}{(\Delta z)}$ cannot exist at $z = 0$.

To show that $\frac{(\Delta w)}{(\Delta z)}$ exists and that $z = 0$, then the following condition must hold true:

$$\frac{\Delta w}{\Delta z} = \overline{\Delta z} \quad \text{when} \quad z = 0$$

It is now evident that a function $f(z) = u(x, y) + iv(x, y)$ can be differentiable only at a fixed point $z = (x, y)$ but nowhere else (ie. its neighbourhood) as it is not differentiable at points like $(\Delta x, 0)$ or $(0, \Delta y)$.

Since we are dealing with the Riemann sphere, we shall observe its cross-section projected on the real plane as a circle, whereas the area of on the imaginary plane shall cancel out due to its property of the imaginary value.

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0$$

when $f(z) = |z|^2$. Even though from the equations of u and v above it is evident that the function f is continuous at all points and may have continuous *partial derivatives* of varying orders at a point $z = (x, y)$, it may still not be differentiable at that point.

Hence, it can be said that function continuous at a point may not be differentiable at that point. However, *if a function is differentiable at a point, it is most definitely continuous there.* To demonstrate this, we assume that a derivative of f exists at a point z_0

$$\begin{aligned} \lim_{z \to \ z_0} [f(z) - f(z_0)] &= \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \lim_{z \to z_0} (z - z_0) \\ &= f'(z_0) \cdot 0 = 0, \end{aligned}$$

From this, we can conclude that

$$\lim_{z \to z_0} f(z) = f(z_0)$$

This statement proves that the function f is also continuous at the point z_0.

2.8 Cauchy-Riemann Equations

Given that a function f can be expressed as such,

$$f(z) = u(x, y) + iv(x, y) \tag{2.67}$$

and if a derivative of f exists at $z_0 = (x_0, y_0)$, then the first-order partial derivatives of the component functions of f, which are u and v, must satisfy a pair of equations known as the Cauchy-Riemann equations. This section shall explore the idea from scratch.

Let us first take the following:

$$z_0 = x_0 + iy_0 \quad , \quad \Delta z = \Delta x + i\Delta y,$$

and,

$$\begin{aligned}\Delta w &= f(z_0 + \Delta z) - f(z_0)\\ &= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)]\\ &\qquad\qquad + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]\end{aligned}$$

Now, assuming that the following derivative exists:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{2.68}$$

we can say, using prior knowledge from theorem 1 of section 2.5 that:

$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to (0,0)} \left(\mathrm{Re}\frac{\Delta w}{\Delta z}\right) + i \lim_{(\Delta x,\Delta y)\to (0,0)} \left(\mathrm{Im}\frac{\Delta w}{\Delta z}\right) \tag{2.69}$$

In order to simplify 2.69, we shall allow $\Delta z = \Delta x + i\Delta y$ approach $(0,0)$ both the horizontally and vertically sides, for which the point z shall pass through the points $(\Delta x, 0)$ and $(0, \Delta y)$ respectively.

Assuming that it approaches horizontally passing through $\Delta y = 0$, the expression $\frac{\Delta w}{\Delta z}$ turns into:

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Isolating each of the real and imaginary limit components and then finding out their first-order partial derivatives w.r.t. x then gives us:

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to (0,0)} \left(Re\frac{\Delta w}{\Delta z}\right) &= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} \\ &= u_x(x_0, y_0) \end{aligned} \tag{2.70}$$

and

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to (0,0)} \left(Im\frac{\Delta w}{\Delta z}\right) &= \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= v_x(x_0, y_0) \end{aligned} \tag{2.71}$$

On substituting 2.70 and 2.71 into 2.69, we get:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{2.72}$$

On finding the partial first-derivatives of u and v w.r.t y, we get:

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \\ &= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}\end{aligned}$$

Thus the limits of the individual components become:

$$\begin{aligned}\lim_{(\Delta x, \Delta y) \to (0,0)} \left(Re\frac{\Delta w}{\Delta z} \right) &= \lim_{\Delta y \to 0} \frac{v(x_0 + \Delta y, y_0) - v(x_0, y_0)}{\Delta y} \\ &= v_y(x_0, y_0) \qquad (2.73)\end{aligned}$$

and

$$\begin{aligned}\lim_{(\Delta x, \Delta y) \to (0,0)} \left(Im\frac{\Delta w}{\Delta z} \right) &= \lim_{\Delta y \to 0} \frac{u(x_0 + \Delta y, y_0) - u(x_0, y_0)}{\Delta y} \qquad (2.74) \\ &= v_y(x_0, y_0) \qquad (2.75)\end{aligned}$$

Hence, the equations 2.73 and 2.75 abides by 2.69 as such:

$$f'(z_0) = -i[u_y(x_0, y_0) + iv_y(x_0, y_0)]. \qquad (2.76)$$

The equations we have obtained in 2.72 and 2.76 has not only allowed us to achieve the partial derivatives of the component functions u and v, but has also enabled us to figure out the conditions required for $f'(z_0)$ to exist. By individually equating the real parts and imaginary parts of each of these two equations 2.72 and 2.76, we get the *Cauchy-Riemann equations*:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{2.77}$$

We introduce a theorem to summarize the results obtained.

Theorem 2.8.1.

Assuming,

$$f(z) = u(x, y) + iv(x, y) \tag{2.78}$$

and if f is differentiable at $z_0 = x_0+iy_0$, ie. if $f'(z_0)$ exists, then the Cauchy-Riemann equations must be obeyed:

$$u_x = v_y \quad , \quad u_y = -v_x \tag{2.79}$$

where $f'(z_0)$ may also be rewritten as

$$f'(z_0) = u_x + iv_x \tag{2.80}$$

where u_x and v_X are to be evaluated at $z_0 = (x_0, y_0)$.

Example 2.8.1.

In a previous example, we showed that the function $f(z) = z^2 = x^2 - y^2 + i2xy$ is differentiable at all points and that $f'(z) = 2z$. However, to check whether f follows the Cauchy-Riemann equations, we must isolate the component functions as such:

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

Upon further calculations,

$$u_x = 2x = v_y \quad , \quad u_y = -2y = -v_x$$

Hence, the equations are obeyed. Also, following equation 2.80,

$$f'(z) = 2x + i2y = 2(x + iy) = 2z$$

Because the Cauchy-Riemann equations are required to be followed in order for a function f to be differentiable at a point z_0, they can be used to find points at which the function is not differentiable or has no derivatives.

Example 2.8.2.

For $f(z) = |z|^2$, the component functions such that:

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0.$$

For the Cauchy-Riemann equations to be obeyed here, $x = y = 0$ has to be true. Even though $f'(z)$ does not exist at non-zero points, this does not ensure or prove the existence of $f'(0)$.

2.9 Sufficient Conditions for Differentiability

As seen from example 2 of section 2.8, verification of Cauchy-Riemann equations at point $z_0 = (x_0, y_0)$ alone shall not guarantee the existence of $f'(z)$ at that point. Hence, in order to check that, we shall use continuity conditions to come up with the following theorem:

Theorem 2.9.1.

Let f be a function s.t.

$$f(z) = u(x, y) + iv(x, y)$$

f is defined throughout an ε neighbourhood of point $z_0 = x_0 + iy_0$ and assuming:

(a) u_x, u_y, v_x, v_y exist everywhere in the neighbourhood

(b) the derivatives specified above are continuous at $(x_0, y_0$ and satisfies the Cauchy-Riemann equations:

$$u_x = v_y, \quad , \quad u_y = -v_x$$

at (x_0, y_0).

Then $f'(z_0)$ exists and it is:

$$f'(z_0) = u_x + iv_x$$

where the right-hand side shall be solved at (x_0, y_0).

Proof. □

We assume that the hypotheses (a) and (b) are true and take a variable $\Delta z = \Delta x + i\Delta y$, where $0{<}|\Delta z|{<}\varepsilon$ and $\Delta w = f(z_0 + \Delta z) - f(z_0)$. This evaluates to

$$\Delta w = \Delta u + i\Delta v \tag{2.81}$$

if we take $\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$ and $\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$ We assume that $u_x, u_y, v_x and v_y$

are all continuous at the point (x_0, y_0). This means we can write:

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad (2.82)$$

and

$$\Delta v = v_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y \quad (2.83)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 all become close to 0, z_0 approaches $(0, 0)$ in the Δz plane.

Once we substitute equations 2.82 and 2.83 into 2.81, we get:

$$\begin{aligned}\Delta w = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ + i[v_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_3 \Delta x + \varepsilon_4 \Delta y]\end{aligned} \quad (2.84)$$

Because Cauchy-Riemann equations are thought to be obeyed at (x_0, y_0), we can replace u_y and v_y by $-v_x$ and u_x respectively in equation 2.84. Dividing the resultant by $\Delta z = \Delta x + i\Delta y$ then yields:

$$\begin{aligned}\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z} \\ + (\varepsilon_3 + i\varepsilon_4)\frac{\Delta y}{\Delta z}.\end{aligned} \quad (2.85)$$

However, from prior knowledge of inequalities of complex variables, it is known that $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$. Applying this on our limit function quotient, ie. $\frac{\Delta w}{\Delta z}$:

$$\left|\frac{\Delta x}{\Delta z}\right| \leq 1 \quad \text{and} \quad \left|\frac{\Delta y}{\Delta z}\right| \leq 1$$

Using this, we can say that

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{\Delta z}\right| \leq |\varepsilon_1 + i\varepsilon_3| \leq |\varepsilon_1| + |\varepsilon_3|$$

and

$$\left|(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{\Delta z}\right| \leq |\varepsilon_2 + i\varepsilon_4| \leq |\varepsilon_2| + |\varepsilon_4|$$

Since all of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 approach zero as Δz nears zero, the only components remaining in the $\frac{\Delta w}{\Delta z}$ equation in 2.85 are u_x and v_x and this matches with the stated equation for $f'(z_0)$ of the theorem.

2.9.1 Example 1

Let us consider the exponential function

$$f(z) = e^z = e^x e^{iy} \qquad (z = x_i y)$$

If we consider applying Euler's formula on this equation, we get:

$$f(z) = e^x \cos y + ie^x \sin y$$

where y shall be taken in radians. In this case, the real and imaginary components are $e^x \cos y$ and $e^x \sin y$ respectively.

Here, $u_x = v_y$ and $u_y = -v_x$ satisfies for all points and these derivatives, i.e. $f'(z)$, are continuous and exists at all points as well. The condition of the theorem above hence applies for all points across the complex plane for this exponential function.

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y$$

Example 2.9.1.

Considering the function $f(z) = |z|^2$ where

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0$$

We shall observe that the only point this function has a derivative is at $z = 0$ for which both x and y need to be zero because $f'(z) = x + iy$ and $f'(0) = 0 + i0 = 0$. As seen in a previous example, this function is unable to have a derivative at nonzero points because these points do not satisfy the Cauchy-Riemann equations.

2.10 Polar Coordinates

In this section, we shall state the theorem stated in the previous section (ie. section 2.9) using polar coordinates using the coordinate transformation:

$$x = r\cos\theta \quad , \quad y = r\sin\theta \tag{2.86}$$

Whether or not we express the points on the complex plane using

$$z = x + iy \quad \text{or} \quad z = re^{i\theta}(z \neq 0)$$

The only differences are the variables (ie. x and y or r and θ) and the derivative functions. Even though the first-order partial derivatives of both the forms hold the same properties regardless of the variables used, the ones w.r.t r and θ shall have to be found out using the chain rule.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad , \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

We can then evaluate the partial derivatives of u and v w.r.t. r and θ w.r.t the polar coordinates in the following way:

$$u_r = u_x \cos\theta + u_y \sin\theta \quad , \quad u_\theta = -u_x r \sin\theta + u_y r \cos\theta \tag{2.87}$$

and

$$v_r = v_x \cos\theta + v_y \sin\theta \quad , \quad v_\theta = -v_x r \sin\theta + v_y r \cos\theta \tag{2.88}$$

In the polar coordinates plane, the following Cauchy-Riemann equations are still obeyed at z_0:

$$u_x = v_y \quad , \quad u_y = -v_x \tag{2.89}$$

If the partial derivatives of the terms do satisfy the Cauchy-Riemann equations at the point z_0, we can replace the terms

u_x and u_y with terms v_y and $= v_x$ respectively from **??** and get:

$$v_r = -u_y\text{cos}\theta + u_x\text{sin}\theta \quad , \qquad v_\theta = u_y r\,\text{sin}\theta + u_x r\,\text{cos}\theta \tag{2.90}$$

Equating the equations, we get:

$$v_\theta = u_y r\text{sin}\theta + u_x r\text{cos}\theta$$

$$v_\theta = r(u_y\text{sin}\theta + u_x\text{cos}\theta)$$

Since $u_r = u_x \cos\theta + u_y \sin_\theta$, we can place this expression into the previous line and get:

$$v_\theta = r u_r$$

On the other hand,

$$u_\theta = -u_x r\text{sin}\theta + u_y r\text{cos}\theta$$

$$u_\theta = -r(u_y cos\theta - u_x sin\theta)$$

Since $v_r = -u_y\text{cos}\theta + u_x\text{sin}\theta$,

$$u_\theta = -r v_r$$

at the point z_0 as this is the point where the Cauchy-Riemann equations are assumed to be fulfilled. The equations shown here have been built from that basis too. That gives rise to the polar coordinates version of the Cauchy-Riemann equations. They have been proven above and stated below:

$$ru_r = v_\theta \quad , \quad u_\theta = -rv_r \tag{2.91}$$

Theorem 2.10.1.

Let us consider the following equation:

$$f(z) = r(r, \theta) + iv(r, \theta)$$

and let us assume that it is defined throughout an ε neighbourhood of a non-zero point represented by $z_0 = r_0 \exp(i\theta_0)$ and also assume that:
(a) $u_r, u_\theta . v_r, v_\theta$ exist everywhere in the neighbourhood;
(b) the derivatives mentioned in part (a) *are continuous at the point $(r_0,\ \theta_0)$ and satisfy the polar form of the Cauchy-Riemann equations (stated in 2.91)*

In that case, f'(z_0) exists and its value is represented by:

$$f'(z_0) = e^(-i\theta)(u_r + iv_r)$$

where the function has to be solved for the point (r_0, θ_0).

Example 2.10.1.

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^(-i\theta) = \frac{1}{r}(cos\theta - isin\theta) \quad \text{(when z}\neq\text{0)}$$

We know from prior knowledge that

$$u(r,\theta) = \frac{cos\theta}{r} \quad \text{and} \quad v(r,\theta) = -\frac{sin\theta}{r} \quad ,$$

The Cauchy-Riemann equations are satisfied for all points denoted by $z = re^(i\theta)$ in the polar plane. It's been shown below how that is possible:

$$ru_r = -\frac{cos\theta}{r} = v_\theta \quad \text{and} \quad u_\theta = -\frac{sin\theta}{r} = -rv_r$$

Since one of the initial conditions was $z \neq 0$, it can be said that the derivative of f exists. The derivative of the function f here, as per the theorem we have learnt in section 2.7, is as follows:

$$f'(z) = e^{-i\theta}\left(\frac{cos\theta}{r^2} + i\frac{sin\theta}{r^2}\right) = -e^{-i\theta}\frac{e^{-i\theta}}{r^2} = -\frac{1}{(re^{i\theta^2})} = -\frac{1}{z^2}$$

Example 2.10.2.

Show that the following function has a derivative everywhere in its domain of function using the theorem above.

$$f(z) = \sqrt[3]{r}e^{i\theta/3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

where α is a fixed real number. If we isolate the real part, u(r,θ), and imaginary part, v(r,θ), we get:

$$u(r,\theta)\sqrt[3]{r}cos\frac{\theta}{3} \quad \text{and} \quad v(r,\theta) = \sqrt[3]{r}sin\frac{\theta}{3}$$

Let us now use the Cauchy-Riemann equations to verify whether the function does have a derivative at the point (r,θ). If they are satisfied, it does and vis-a-vis.

On finding $ru_r andu_\theta$, we get the following equations:

$$ru_r = \frac{\sqrt[3]{r}}{3}cos\frac{\theta}{3} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt[3]{r}}{3}sin\frac{\theta}{3} = -rv_r$$

This verifies that the Cauchy-Riemann equations are satisfied as the result of ru_r is equal to v_θ whereas that of u_θ is equal to $-rv_r$, which are basically the polar versions of the actual equations.

The theorem also states that $f'(z_0) = e^{-i\theta}(u_r + iv_r)$. We can use this to find the derivative of the function in this example as well and achieve the following:

$$f'(z) = e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}sin\frac{\theta}{3}\right]$$

Example 1 also taught us that the first-order derivative can also be found out using $f'(z) == -\frac{1}{z^2}$. Using this approach gives us:

$$f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}$$

However, as we have learnt from elementary lessons on complex variables and their polynomials, finding a specific point z in the domain of f actually gives us one value of $z^(1/3)$.

Hence, the final form of the derivative can be expressed as such:

$$\frac{d}{dz}z^(1/3) = \frac{1}{3(z^{1/3})^2}$$

2.11 Analytic Functions

A function *f(z)* is said to be *analytical at a point* $z = z_0$ if the function has a derivative at each point in some neighbourhood of z_0, ie. in other words, it has to be analytic at each point in some neighbourhood of $z = z_0$.

If any function f is analytical(differentiable) at some point in every neighbourhood of z_0 but not at the point z_0 itself, we say that z_0 is *singular point or singularity of the function f*.

Let us consider examples to understand when to consider functions as analytic:
(a) The function *f(z)=1/z* is analytic at every point in the finite place except at $z = z_0$. In this case, z_0 is the singular point of the function.
(b) The function $f(z) = |z|^2$ is not analytic since its derivative exists only at a point $z = 0$ and **not throughout any neighbourhood**. Since it is nowhere analytic, it has no singular points.

Entire Function: When a function is analytic at each point in the entire finite plane, we call it an *entire* function. For example, if we consider a polynomial, we shall observe that its derivative exists everywhere. Hence, *every polynomial is an entire function.*

In order for a function f to be analytic in domain D, some necessary, yet insufficient conditions are as follows:
(i) continuity of f throughout D
(ii) satisfaction of the Cauchy-Riemann equations.

The sufficient conditions for a function to be analytic are given in theorems of sections 2.8 and 2.9. However, the points at which two functions have derivatives shall also have derivatives of the sum and product of the two functions at those points as well. As in, *if two functions are analytic in a domain D, their sum and their product are both analytic in D.* To follow up, *the quotient $P(z)/Q(z)$ of two polynomials P and Q is analytic in any domain throughout which $Q(z) \neq 0$, i.e. the denominator Q does not vanish.*

Also, as derived from the chain rule for derivative of a composite function, if a function f(z) is analytic in a domain D, then the image of D under the transformation $g(f(z))$ is analytic in D as well. That is, *a composition of two analytic functions is analytic as well.* The derivative of the composition g[f(z)] is as follows:

$$\frac{d}{dz}g[f(z)] = g'[f(z)]f'(z)$$

Theorem 2.11.1.

If f'(z)=0 everywhere in a domain D, then f(z) must be constant throughout D.

We shall start by assuming $f(z) = u(z, y) + iv(x, y)$ and that f'(z)=0. Considering Cauchy-Riemann equations, we take $u_x + iv_x = 0andv_y - iu_y = 0$. Hence,

$$u_x = u_y = 0 \quad \text{and} \quad v_x = v_y = 0$$

at each point within the domain D.

We shall then show that along any line segment L joining the point P with P', u(x,y) is always a constant and lies completely within the domain D. We shall take s to be the magnitude of the distance between P and other point(s) and shall take **U** as the unit vector along L going towards P'.

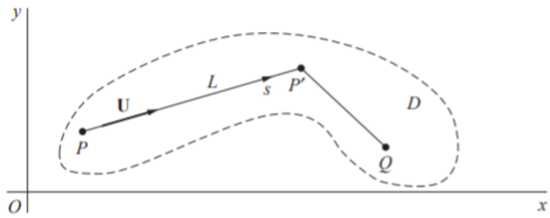


Figure 2.12: *Demonstration of the theorem ideas*

As learnt previously in calculus, we can take the directional derivative du/ds as the following dot product:

$$\frac{du}{ds} = (gradu).\mathbf{U}. \tag{2.92}$$

where grad u is the gradient vector s.t.:

$$gradu = u_x\mathbf{i} + u_y\mathbf{j}. \tag{2.93}$$

Since $u_x = u_y = 0$ in domain D, grad u shall be zero vector at all points on L as well. Hence du/ds is zero along L, meaning u must be a constant on L.

The number of line segments (connecting any two points P and Q in domain D) is always a finite number and hence the values of u at P and Q are always the same. To conclude with, there is always a real constant a such that u(x,y)=a and similarly, v(x,y)=b within domain D. Simply put, we can say that f(z)=a+bi throughout D.

2.12 Harmonic Functions

If throughout a given domain of the xy plane, a real-valued function H of two real variables x and y is said to be *harmonic* if it has continuous partial derivatives of the first and second order and satisfies the following partial differential equation (known as *Laplace's equation*):

$$H_{xx}(x, y) + H_{yy}(x, y) = 0 \tag{2.94}$$

To begin with, harmonic functions are crucial in applied mathematics. It can be used to represent temperatures T(x,y) in the xy plane along with electrostatic potential V(x,y) in the interior of a region of a three-dimensional space that contains no charges.

Example 2.12.1. *Verify that the following function T is harmonic:*

$$T(x, y) = e^{-y} sinx \quad 0 < x < \pi, y > 0$$

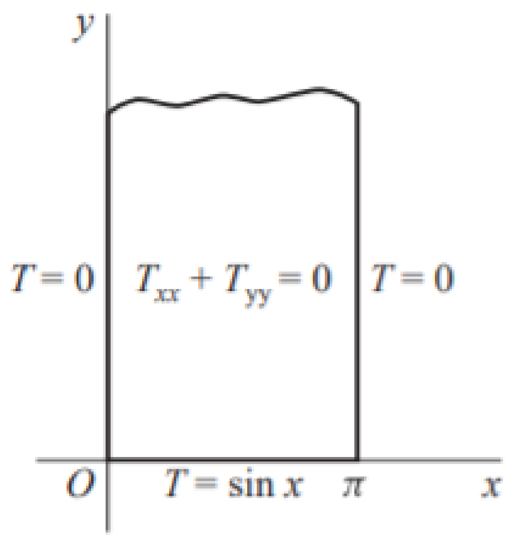


Figure 2.13: *Example 1 Demonstration*

In this case, T_{xx}=$-e^{-y}sinx$ and $T_yy = e^{-y}sinx$. Hence, the conditions are satisfied:

$$T_{x,x}(x,y) + T_{y,y}(x,y) = 0$$

$$T(0,y) = 0 \quad , T(\pi, y) = 0$$

$$T(x,0) = sinx \quad , \quad \lim_{y\to\ \infty} T(x,y) = 0$$

T(x,y) here describes the temperatures of a thin homogenous plate in the xy plane with no heat sources or sinks and is insulated exxcept for the stated conditions along the edges.

Theorem 2.12.1. If a function f(z)=u(x,y)+iv(x,y) is analytic in a domain D, then its component functions u and v are harmonic in D.

To prove this, we need to show that if a function of a complex variable is analytic at a point, then its real and imaginery components have continuous partial derivates of all orders at that point. This shall be further shown in later chapters.

Considering that the function f is analytic in domain D, the first order partial derivatives of its component functions must then satisfy the Cauchy-Riemann equations such that:

$$u_x = v_y \quad , \quad u_y = -v_x \tag{2.95}$$

To verify whether the functions are harmonic, we must find $v_{xx}.v_{yy}, uxxanduyy.$

We can find these values by differentiating the equations in 2.95 w.r.t x and y and are shown respectively:

$$u_{xx} = v_{yx} \quad , \quad u_{yx} = -v_{xx} \tag{2.96}$$

$$u_{xy} = v_{yy} \quad , \quad u_{xx} = -v_{xy} \tag{2.97}$$

As per a theorem in the book "Advanced Calculus" in Taylor-Mann (1983), the continuity of partial derivatives of u and v will always lead to the equations $u_{yx} = u_{xy}$ and $v_{yx} = vxy$. Substituting these in equations 2.96 and 2.97, we get:

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0$$

Hence u and v are harmonic in the domain D.

Example 2.12.2. *Show that the following function is harmonic*

$$f(z) = e^{-y}sinx - ie^{-y}cosx$$

Since this is an *entire function*, it shall be harmonic in every domain of the xy plane.

Example 2.12.3. *Show that the function is analytic whenever z≠0:*

$$\frac{i}{z^2} = \frac{i}{z^2}.\frac{\overline{z}^2}{\overline{z}^2} = \frac{i\overline{z}^2}{(z\overline{z})^2} = \frac{i\overline{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + Y^2)^2}$$

That gives us the real and imaginary parts, u and v respectively:

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad \text{and} \quad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

On finding the derivatives, we can verify that u and v do indeed satisfy the Cauchy-Riemann equations throughout D as well as Laplace's equation and are hence harmonic throughout the domain D. In such a case, we say that v is a *harmonic conjugate* of u.

Theorem 2.12.2. A function f(z)=u(x,y)+iv(x,y) is analytic in a domain D if and only if v is a harmonic conjugate of u

To prove this, we can either:
(a)show that v is a harmonic conjugate of u throughout the plane. This, in turn, verifies that f is analytic in D as per theorems learnt from previous sections.
(b) show that f is analytic since if f is analytic in D, then u and v are harmonic in D as well, hence proving the theorem.

Example 2.12.4. *Consider the equations and check if they're harmonic:*

$$u(x, y) = x^2 - y^2 \quad \textit{and} \quad v(x, y) = 2xy$$

Consider the equations and check if they're harmonic:

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

The equations above are the real and imaginary parts respectively of the function f(z)=z^2. Upon finding the required derivatives, we shall observe that:
(a) v is a harmonic conjugate of u
(b) u is NOT a harmonic conjugate of v

This, in turn, proves that the function is not analytic anywhere.

Example 2.12.5. *Consider the example:*

$$u(x, y) = y^3 - 3x^2 y$$

We shall explore a new method of obtaining the harmonic conjugate of this function here. The function u is harmonic throughout the xy plane as it satisfies Laplace's equation. Since it is harmonic, its harmonic conjugate shall be as such:

$$u_x = v_y \quad , \quad u_y = -v_x \tag{2.98}$$

Using the first equation, we get:

$$v_y(x, y) = -6xy$$

Integrating w.r.t y, keeping x as a constant gives us:

$$v(x, y) = -3xy^2 + \phi(x) \tag{2.99}$$

where $\phi(x)$ is an arbitrary function of x which can be found by equating with the second equation of after differentiating w.r.t x once (keeping y fixed):

$$3y^2 - 3x^2 = 3y^2 - \phi'(x)$$

Hence $\phi'(x) = 3x^2$.

Integrating w.r.t x again,

$\phi(x) = x^3 + C$ where C is a constant.

We are then left with v(x,y), the harmonic conjugate of u:

$$v(x, y) = -3xy^2 + x^3 + C \tag{2.100}$$

The resultant *analytic function* is then:

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C). \tag{2.101}$$

Chapter 3

Elementary Functions

We consider here various elementary functions studied in calculus and thus define corresponding functions of complex variables. We start by defining the complex exponential function and then use it to develop the others.

3.1 The Exponential Function

We define here the exponential function e^z by writing

$$e^z = e^x e^{iy} \qquad (z = x + iy), \tag{3.1}$$

where Euler's formula states

$$e^{iy} = \cos x + i \sin y \tag{3.2}$$

is used and y is to be taken in radians. We see from this definition that e^z reduces to the usual exponential function in calculus when $y = 0$, and according to the definition,

$$e^x e^{iy} = e^{x+iy} \tag{3.3}$$

which is suggested by the additive property of e^x in calculus. The property's extension,

$$e^{z_1} e^{z_2} = e^{z_1+z_2} \tag{3.4}$$

to complex analysis is easy to verify. Observe how this property enables us to write,

$$e^{z_1-z_2} e^{z_2} \;=\; e^{z_1} \quad \text{or} \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}. \tag{3.5}$$

From this, and the fact that $e^0 = 1$, it follows $1/e^z = e^{-z}$. There are a number of other important properties of e^z that are expected. For example,

$$\frac{d}{dz}e^z = e^z \tag{3.6}$$

everywhere in the z plane. Note that the differentiability of e^z for all z tells us that e^z is entire. It is also true that

$$e^z \neq 0$$

for any complex number z. This is evident upon writing definition (3.1) in the form

$$e^z = \rho e^{i\phi} \quad \text{where} \;\; \rho = e^x \;\; \text{and} \; \phi = y \tag{3.7}$$

which tells us that

$$|e^z| = e^x \;\; \text{and} \;\; \arg(e^z) = y + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{3.8}$$

Statement above then follows from the observation that $|e^z|$ is always positive.

Some properties of e^z are, however, not expected. For example, we find that e^z is periodic, with a pure imaginary period of $2\pi i$. For another property of e^z that e^x does not have, we note that while e^x is always positive, e^z can be negative. There are, moreover, values of z such that e^z is any given nonzero complex number.This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

Example 3.1.1.

In order to find numbers $z = x + iy$ such that $e^z = 1 + i$ we write this as

$$e^x e^{iy} = \sqrt{2}\, e^{i\pi/4}$$

Then, in view of the statement regarding the equality of two nonzero complex numbers in exponential form,

$$e^x = \sqrt{2} \;\; \text{and} \;\; y = \frac{\pi}{4} \qquad (n = 0, \pm 1, \pm 2, \dots).$$

because $\ln(e^x) = x$, it follows that

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \text{and} \quad y = (2n + \frac{1}{4})\pi \qquad (n = 0, \pm 1, \pm 2, \dots);$$

and so

$$z = \frac{1}{2}\ln 2 + (2n + \frac{1}{4})\pi i \qquad (n = 0, \pm 1, \pm 2, \dots).$$

3.2 The Logarithmic Function

Our motivation for the definition of the logarithmic function is based on solving the equation,

$$e^w = z \tag{3.9}$$

for w, where z is any nonzero complex number. To do this, we note that when z and w are written $z = re^{i\Theta}$ $(\pi < \Theta \leq \pi)$ and $w = u + iv$, equation above becomes

$$e^u e^{iv} = re^{i\Theta} \tag{3.10}$$

According to the statement about the equality of two complex numbers expressed in exponential form, this tells us that

$$e^u = r \quad \text{and} \quad v = \Theta + 2n\pi \tag{3.11}$$

where n is any integer. Since the equation $e^u = r$ is the same as $u = \ln r$, it follows that equation (3.9) is satisfied if and only if w has one of the values

$$w = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{3.12}$$

So, if we write

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{3.13}$$

equation(3.9) tells us

$$e^{\log z} = z \qquad (z \neq 0). \tag{3.14}$$

which serves to motivate expression (3.13) as the definition of the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$.

3.2.1 Example 1

$$\text{If } z = -1 - \sqrt{3}i, \text{ then } r = 2 \text{ , and } \Theta = -2\pi/3.$$

Example 3.2.1.

$$\textit{If } z = -1 - \sqrt{3}i, \textit{ then } r = 2 \text{ , } \textit{and } \Theta = -2\pi/3.$$

Hence $\log\,(-1-\sqrt{3}i) = \ln\,2 + i\,\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln\,2 + 2\left(n - \frac{1}{3}\right)\,\pi i$

$$n = (0, \pm 1, \pm 2, \dots).$$

It should be emphasized that it is not true that the left-hand side of equation (3.14) with the order of the exponential and logarithmic functions reversed reduces to just z . That is,

$$\log\,(e^z) = z + 2n\pi i \qquad n = (0, \pm 1, \pm 2, \dots). \tag{3.15}$$

The principal value of $\log z$ is the value obtained from equation (3.13) when $n = 0$ there and is denoted by $\text{Log } z$. Thus

$$\text{Log }(z) = \ln\, r + i\,\Theta \tag{3.16}$$

Note that $\text{Log } z$ is well defined and single-valued when $z \neq 0$ and that

$$\log z = \text{Log } z + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{3.17}$$

It reduces to the usual logarithm in calculus when z is a positive real number $z = r$. To see this, one need only write $z = re^{i0}$, in which case equation (3.16) becomes $\text{Log } z = \ln r$. That is, $\text{Log } r = \ln r$. Moreover, From expression (3.10), we find that $\text{Log } 1 = 0$. Furthermore,it is good to reminds us that although we were unable to find logarithms of negative real numbers in calculus, we can now do so.

3.3 Branches and Derivatives of Logarithm

If $z = re^{i\theta}$ is a nonzero complex number, the argument θ has any one of the values $\theta = \Theta + 2n\pi \ \ (n = 0, \pm 1, \pm 2, \dots)$, where $\Theta = \text{Arg } z$. Hence the definition

$$\log z = \ln\, r + i\,(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{3.18}$$

of the multiple-valued logarithmic function can be written

$$\log z = \ln r + i\,\theta \tag{3.19}$$

If we let α denote any real number and restrict the value of θ in expression (3.19) so that $\alpha < \theta < \alpha + 2\pi$, the function

$$\log z = \ln r + i\theta \qquad (r > 0,\ \alpha < \theta < \alpha + 2\pi) \tag{3.20}$$

with components

$$u\,(r, \theta) = \ln r \quad \text{and} \quad v\,(r, \theta) = \theta \tag{3.21}$$

is single-valued and continuous in the stated domain (Fig.3.1).

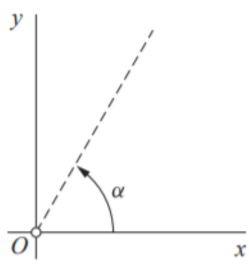


Figure 3.1:

The function (3.19) is not only continuous but also analytic throughout the domain $r > 0,\ \alpha < \theta < \alpha + 2\pi$ since the first-order partial derivatives of u and v are continuous there and satisfy the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r \tag{3.22}$$

of the Cauchy–Riemann equations. Furthermore,

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(\frac{1}{r} + i0) = \frac{1}{re^{i\theta}} \tag{3.23}$$

that is,

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z| > 0,\ \alpha < \arg z < \alpha + 2\pi) \tag{3.24}$$

In particular,

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z| > 0,\ -\pi < \text{Arg}\ z < \pi) \tag{3.25}$$

Principal branch: A branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f . The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f . Observe that for each fixed α, the single-valued function (3.19) is a branch of the multiple-valued function (3.18). The function

$$\log z = \ln r + i\Theta \qquad (r > 0, \, -\pi < \Theta < \pi) \tag{3.26}$$

is called the principal branch. **Branch cut:** A branch cut is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f . Points on the branch cut for F are singular points of F, and any point that is common to all branch cuts of f is called a branch point. The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (3.19) of the logarithmic function. The branch cut for the principal branch (3.26) consists of the origin and the ray $\Theta = \pi$. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Example 3.3.1.

when the principle branch is used we can see,

$$\text{Log}(i^3) = \text{Log}(-i) = \text{ln}1 - i\frac{\pi}{2} = -\frac{\pi}{2}i$$

and

$$3 \text{ Log}\, i = 3 \; (\text{ln}1 + i\frac{\pi}{2}) = \frac{3\pi}{2}i$$

hence

$$\text{Log}(i^3) \neq 3 \text{ Log } i.$$

3.4 Complex Exponents

When $z \neq 0$ and the exponent c is any complex number, the function z^c is defined by means of the equation

$$z^c = e^{c \text{ log } z} \tag{3.27}$$

where log z denotes the multiple-valued logarithmic function. Equation (3.27) provides a consistent definition of z^c in the sense that it is already known to be valid when $c = n \quad (n = 0, \pm1, \pm2, \dots)$ and $c = 1/n \quad (n = \pm1, \pm2, \dots)$. Definition (3.27) is, in fact, suggested by those particular choices of c. If $z = re^i$ and α is any real number, the branch

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi) \tag{3.28}$$

of the logarithmic function is single-valued and analytic in the indicated domain . When that branch is used, it follows that the function $z^c = \exp(c\log z)$ is single-valued and analytic in the same domain. The derivative of such a branch of z^c is found by first using the chain rule to write

$$\frac{d}{dz}z^c = \frac{d}{dz}\exp(c\log z) = \frac{c}{z}\exp(c\log z) \tag{3.29}$$

and then recalling the identity $z = \exp(\log z)$. That yields the result

$$\frac{d}{dz}z^c = c\frac{\exp(c\log z)}{\exp(\log z)} = c\exp[(c-1)\log z] \tag{3.30}$$

or

$$\frac{d}{dz}z^c = c\,z^{(c-1)} \qquad (|z| > 0, \alpha < \arg z < \alpha + 2\pi). \tag{3.31}$$

The principal value of z^c occurs when $\log z$ is replaced by Log z in definition:

$$\text{P.V. } z^c = e^{c \text{ Log } z} \tag{3.32}$$

Equation (3.32) also serves to define the principal branch of the function z^c on the domain $|z| > 0,\ \pi < \text{Arg } z < \pi$.

Example 3.4.1.

The principle value of $(-i)^i$ is

$$\exp[i\,\text{Log}(-i)] = \exp[i\,(\ln 1 - i\frac{\pi}{2})] = \exp\frac{\pi}{2} = \text{P.V. } (-i)^i$$

Example 3.4.2.

The principle branch of $z^{2/3}$ can be written as

$$\exp\left(\frac{2}{3}\text{ Log } z\right) = \exp\left(\frac{2}{3}\ln r + \frac{2}{3}i\Theta\right) = r^{2/3}\exp\left(i\frac{2\Theta}{3}\right)$$

Thus

$$\text{P.V. } z^{2/3} = r^{2/3} \cos\frac{2\Theta}{3} + i\, r^{2/3} \sin\frac{2\Theta}{3}$$

This function is analytic in the domain $r > 0$, $\pi < \Theta < \pi$. According to definition, the exponential function with base c, where c is any nonzero complex constant, is written

$$c^z = e^{z \log c}$$

Note that although e^z is, in general, multiple-valued according to definition, the usual interpretation of e^z occurs when the principal value of the logarithm is taken. This is because the principal value of log e is unity. When a value of log c is specified, c^z is an entire function of z. In fact,

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z \log c} = e^{z \log c} \log c; \tag{3.33}$$

and this shows that

$$\frac{d}{dz}c^z = c^z \log c. \tag{3.34}$$

Chapter 4

Integrals

In this chapter we consider the evaluation of integrals of complex variable functions along appropriate curves in the complex plane. We will see that some of the analysis will resemble to that of functions of real variables. However, for analytic functions, very important new results can be derived,namely Cauchy's Theorem (sometimes called the Cauchy–Goursat Theorem). Complex integration has wide applicability ranging from evaluating difficult definite integrals to engineering applications like power systems, signal processing etc.

4.1 Derivatives and Definite Integrals of Functions $w(t)$

Before we introduce the integrals for a complex variable, we want to take a look at the derivatives of a complex variable. for simplicity we will take the complex variable to be defined by a real parameter t. We denote the complex variable as $w(t)$ as

$$w(t) \;=\; u(t) + iv(t) \tag{4.1}$$

where $u(t)$ and $v(t)$ are both real valued functions of t. Now we want to find it's derivative with respect to t. Since i is just the imaginary unit, we can treat it as a constant for this purpose and write

$$\frac{d}{dt}w(t) \;=\; w'(t) \;= u'(t) + iv'(t). \tag{4.2}$$

Example:

$$\frac{d}{dt}e^{z_0 t} \;=\; z_0\, e^{z_0 t} \tag{4.3}$$

where z_0 is a complex constant. It can be checked easily that various rules learned in calculus for differentiating sums and products will also apply here. However and exception occurs for the mean value theorem. We won't discuss it here. An example can be found at page 119 of Brown and Churchill.

Now we define definite integrals of $w(t)$ over the domain $a \leq t \leq b$ as follows:

$$\int_b^a w(t)\,dt = \int_b^a u(t)\,dt + i\int_b^a v(t)\,dt. \tag{4.4}$$

Example 1:

$$\begin{aligned}
&\int_1^2 \left(\frac{1}{t} - i\right)^2 dt \\
=&\int_1^2 \frac{dt}{t^2} - 2i\int_1^2 \frac{dt}{t} - \int_1^2 dt \\
=&-\frac{1}{2} - i\ln(4)
\end{aligned}$$

Example 2: Assume $Re(z) > 0$

$$\begin{aligned}
&\int_0^\infty e^{-zt}dt \\
=&-\left[\frac{1}{z}e^{-zt}\right]_0^\infty \\
=&\frac{1}{z}
\end{aligned}$$

The reader should think about why the assumption $Re(z) > 0$ is important here. Also, the integral in example 2 should be redone with the decomposition of z into real an imaginary part. The result should coincide with example 2.

In the next section we will take a look at what **contours** are as we will be dealing with contour integrals mainly.

4.2 Contour : Definitions and Examples

Integrals of **complex valued functions** of a **complex variable** are defined on a **contour** rather than on intervals of the real line. Before we discuss about contours we need to know what an **arc** is. **Definition**

of an arc: A set of points $z = (x, y)$ is said to be an arc if

$$x = x(t) \;\; y = y(t) \;\; a \leq t \leq b$$

where $x = x(t)$ and $y = y(t)$ are continuous functions of the real parameter t. through this definition we get a continuous mapping of the interval $a \leq t \leq b$ into the xy plane or the complex plane. The image points $z(t)$ are ordered according to increasing t. The curve is said to be continuous if $x(t)$ and $y(t)$ are continuous functions of t. Similarly, it is said to be differentiable if $x(t)$ and $y(t)$ are differentiable.

A curve C is **simple arc** or a **Jordan arc** corresponds to a curve that does not intersect itself, that is $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, for all $t_1, t_2 \,\epsilon\, [a, b]$,except that $z(b) = z(a)$ is allowed; in the latter case we say that C is a simple closed curve (or Jordan curve). Simply closed curves are positively oriented when it is in the counterclockwise direction and negatively oriented if it is in the clockwise direction. We demonstrate a simple arc with the picture below.

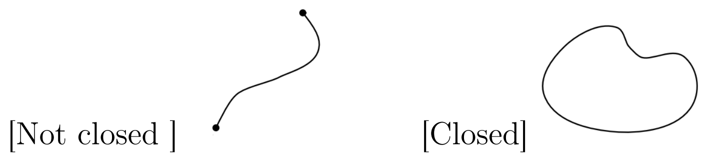


Figure 4.1: Simple Curves

An example of a non simple and not closed curve is the following:

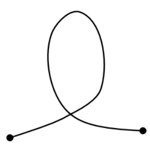


Figure 4.2: Non simple, not closed

The function $f(z)$ is said to be continuous on a curve C if $f(z(t))$ is continuous for $a \leq t \leq b$, and f is said to be piecewise continuous on $[a, b]$ if $[a, b]$ can be broken up into a finite number of subintervals in which $f(z)$ is continuous. A smooth arc C is one for which $z(t)$ is continuous.

Definition of Contour: A contour is an arc consisting of a finite number of connected smooth arcs; that is, a contour is a piecewise smooth arcs joined end to end.

Examples of Contour :

Example 1: The unit circle $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ centered at the origin that is (0,0) is a positively oriented contour.

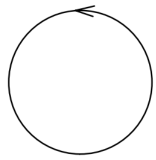


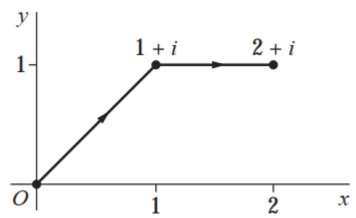
Figure 4.3: Positively oriented contour.

Example 2: The unit circle given by $z' = e^{-i\theta}$ contains the same points as the circle $z = e^{i\theta}$ but they are not the same. z' circle is traversed in the clockwise direction.

Example 3: The polygonal line defined by equations

$$z = \begin{cases} x + ix & \text{when } 0 \leq x \leq 1 \\ x + i & \text{when } 1 \leq x \leq 2 \end{cases} \tag{4.5}$$

represent a simple contour.



In this section we briefly touched upon the definition of contour and saw some examples. From next sections we will be dealing with contour integrals.

4.3 Contour Integrals

The contour integral of a piecewise continuous function on a smooth contour **C** is defined as

$$\int_c f(z)\, dz = f(z(t))z'(t)dt. \tag{4.6}$$

where we have written $dz = \frac{\partial z}{\partial t}dt = z'(t)dt$.

The parametric representation for a given contour C is not necessarily unique. Instead of parameter t defined over the interval $a \leq t \leq b$, let us choose another parameter τ defined over the interval $\alpha \leq \tau \leq \beta$. Let $\Phi(\tau)$ be a map such that

$$t = \Phi(\tau) \tag{4.7}$$

In this case we will assume Φ is a continuous map with a continuous derivative with the property $\Phi'(\tau) > 0$ for each τ. This ensures t increases with τ .

Under this kind of transformation mentioned above, the value of the integral in 4.6 remains invariant. This tells us about an important fact that , one can evaluate the integral with convenient choice of parametrization.

Properties of integration from real valued functions also apply in complex case. That is

$$\int_C [\alpha f(z) + \beta g(z)] dz = \alpha \int_C f(z) dz + \beta \int_C g(z) dz \tag{4.8}$$

where α and *beta* are in general complex constants and $f(z)$ and $g(z)$ are piecewise smooth functions. Now lets take a contour in with one end having a value z_1 and the other end having a value z_2.

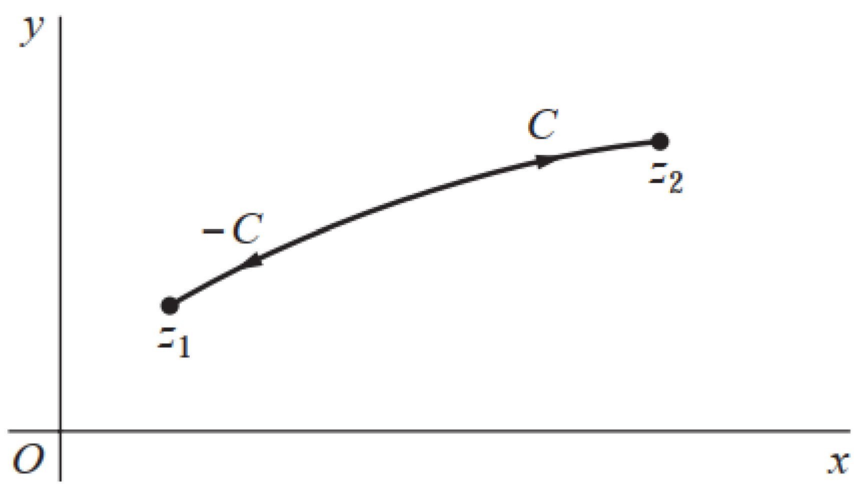
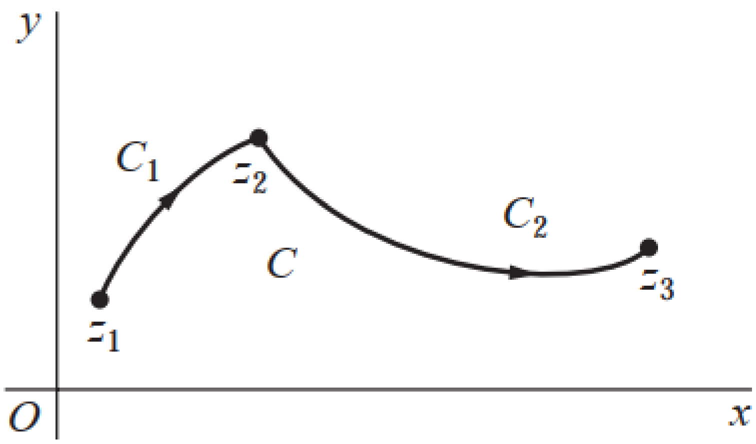


Figure 4.4: A simple contour

Now consider traversing this arc in the direction from z_1 to z_2.This direction is denoted by C. Then if we go from z_2 to z_1 then its denoted by $-C$ and in general the following holds

$$\int_{-C} f(z)dz = -\int_C f(z)dz \tag{4.9}$$

Now consider breaking up the contour as showed in the figure below



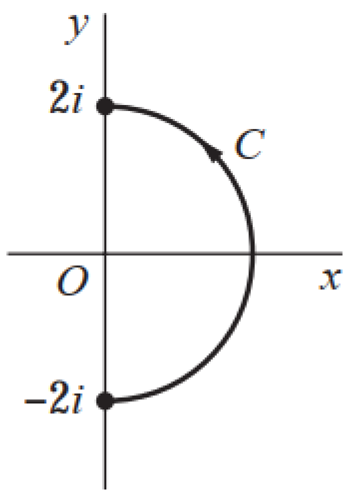
For this case, the integral 4.6 can be broken up into two pieces. That is

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \tag{4.10}$$

Now lets see an example of contour integral.

Example 1: Evaluate the integral $\int_c \bar{z}\, dz$ over the contour given by $z = 2e^{i\theta}$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

First thing to notice here is that the contour is parameterized by θ. The contour looks like the following when drawn



Now we have from 4.6

$$
\begin{aligned}
\int_c f(z)dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{z}z'(\theta)d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{i\theta}}(2e^{i\theta})'d\theta \\
&= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\theta}\, i\, e^{-i\theta}d\theta \\
&= 4i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\
&= 4\pi i
\end{aligned} \tag{4.11}
$$

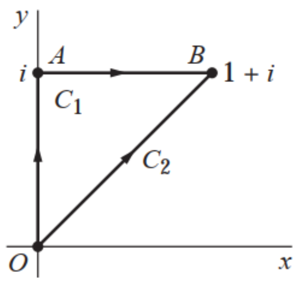
So we finally have $\int_c \bar{z}dz = 4\pi i$.

Now we will take a look at the example where we have $f(z)$in the form

$$f(z) = u(x,y) + i\,v(x,y) \tag{4.12}$$

where $z = x+iy$. We want to do evalute integrals of the form $\int_C f(z)dz$.Notice that $dz = dx + idy$.

Example 2: Take $f(z) = y - x - i\,3x^2$ and C denotes the following contour



We want to evaluate the integral on the polygonal line OAB an dalong the staright line OB.

Evaluating the integral along OAB:

The contour OAB that is C_1can be broken up into two pieces OA and AB. SO we have

$$\int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz \tag{4.13}$$

Along OA we have $x = 0$ and the limit for y starts from 0 to 1. Notice that we are not including i in our limit as this serves a the imaginary unit. So putting $x = 0$ in $f(z)$ and dz we get

$$\int_{OA} f(z)dz = \int_0^1 y\,idy = i\int_0^1 ydy = \frac{i}{2} \tag{4.14}$$

then we do the integral for the line AB. Along AB we have $z = x + i$ as $y = 1$ along this line and x is our variable. So we get

$$\int_{AB} f(z)dz = \int_0^1 (1 - x - i3x^2)\cdot 1dx = \int_0^1 (1-x)dx - 3i\int_0^1 x^2dx = \frac{1}{2} - i \tag{4.15}$$

And finally adding these two results give us

$$\int_{C_1} f(z)dz = \frac{1-i}{2} \tag{4.16}$$

Now if we evaluate the same integral along contour C_2 , where $z = x+iy$ will turn into $z = x + ix$ as along this line which has a slope of 1 , gives us $y = x$. So , $dz = (1+i)\,dx$ and the limit for x runs from 0 to 1. Now ,

$$\int_{C_2} f(z)dz = -3\,i(1+i)\int_0^1 x^2dx = 1 - i. \tag{4.17}$$

Evidently, then, the integrals of $f(z)$ along the two paths C_1 and C_2 have different values even though those paths have the same initial and the same final points. We can subtract 4.16 and 6.5 to get the result for the closed contour integration. The reason for subtracting here is that is that we take 4.16 to be positively oriented and 6.5 to be negatively oriented.

4.4 Antiderivates

In general for a given $f(z)$, 4.6 will depend on the contour. There are certain functions whose integrals from $z1$ to $z2$ have values that are independent of path.Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero. This theorem will provide the notion of **antiderivatives** of complex functions.

Theorem: Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:
(a) $f(z)$ has an antiderivative $F(z)$ throughout D;
(b) The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point $z1$ to any fixed point $z2$ all have the same value, namely

$$\int_{z_1}^{z_2} f(z)dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1) \tag{4.18}$$

Notice that for a closed contour **??** is 0. That is if $f(z)$ is analytic over a region in the complex plane and there is a contour C which encloses that region, we have

$$\oint_C f(z)dz = 0 \tag{4.19}$$

$\oint_C$ denotes a closed contour. That means , the starting point and the ending point are the same. It should be emphasized that the theorem does not claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. Now we will look at some examples.

Example 1 Evaluate $\displaystyle\int_0^{1+i} z^2 dz$.
Since z^2 is a continuous function and it has an antiderivative $\frac{z^3}{3}$ throughout the plane we have

$$\int_0^{1+i} z^2 dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{2}{3}(-1+i) \tag{4.20}$$

This result is independent of the choice of contour running from 0 to $1+i$.
Example 2 Evaluate $\displaystyle\int_C \frac{dz}{z^2}$, where C denotes a contour $z = e^{2i\theta}$ with the origin deleted.

The contour looks like the following

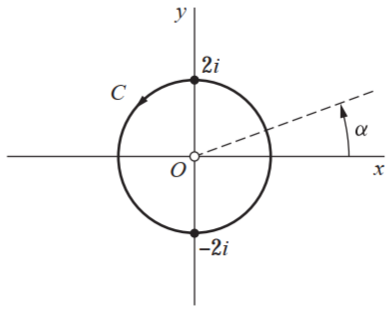


Figure 4.5: Caption

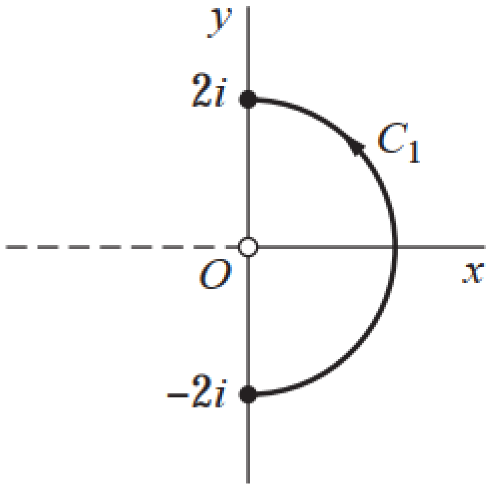
Since the function's antiderivative is continuous everywhere except at the origin , we have

$$\int_C \frac{dz}{z^2} = 0 \quad |z| > 0 \tag{4.21}$$

Note that the integral of the function $f(z) = 1/z$ around the same circle cannot be evaluated in a similar way. For, although the derivative of any branch $F(z)$ of $\log z$ is $1/z$, $F(z)$ is not differentiable, or even defined, along its branch cut. In particular, if a ray = from the origin is used to form the branch cut, $F'(z)$ fails to exist at the point where that ray intersects the circle C in the above figure. So C does not lie in any domain throughout which $F'(z) = 1/z$, and one cannot make direct use of an antiderivative. Example 3 illustrates how a combination of two different antiderivatives can be used to evaluate $f(z) = 1/z$ around C.
Example 3 : In the above figure, the dashed lines denotes the principal branch. If we conside evaluating $\int_{C_1} \frac{dz}{z}$ where C_1 denotes the right half of the circle, then we have $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Now the principal branch of the logarithmic function is $\log z = \ln r + i\Theta$ where $r > 0$ and $-\pi \leq \Theta \leq \pi$. So we have

$$\int_{C_1} \frac{dz}{z} = \int_{-2i}^{2i} \frac{dz}{z} = \log z \Big]_{-2i}^{2i} = \log(2i) - \log(-2i) = \left(\ln 2 + i\frac{\pi}{2}\right) - \left(\ln 2 - i\frac{\pi}{2}\right) \\ = i\pi \tag{4.22}$$

Now we denote the left half of the circle $z = 2e^{i\theta}$ by $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. In this case the limits for z will be run from $2i$ to $-2i$. Pictorially we have



So finally ,

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = \log z \Big]_{2i}^{-2i} = \log(-2i) - \log(2i)$$
$$= \left(\ln 2 + i\frac{3\pi}{2}\right) - \left(\ln 2 + i\frac{\pi}{2}\right) \tag{4.23}$$
$$= i\pi$$

The value of the integral of 1/z around the entire circle $C = C_1 + C_2$ is thus obtained by adding 4.22 and 4.23. Hence we get

$$\int_C \frac{dz}{z} = 2\pi i \tag{4.24}$$

Example 4: Evaluate $\int_C z^{\frac{1}{2}} dz$ where C is any contour from $z = -3$ to $z = 3$ except for the end points. Here the integrand is the branch $f(z) = z^{\frac{1}{2}} = \exp\left(\frac{1}{2}\log z\right) = r^{\frac{1}{2}} e^{i\theta}$ where $r > 0$ and $0 < \theta < 2\pi$. So, we have

$$\int_{C_1} z^{1/2} dz = \int_{-3}^{3} f_1(z) dz = F_1(z)\Big]_{-3}^{3} = 2\sqrt{3}\left(e^{i0} - e^{i3\pi/2}\right)$$
$$= 2\sqrt{3}(1+i) \tag{4.25}$$

In the next section we will talk about an extremely important theorem , namely the Cauchy-Goursat theorem

4.5 Cauchy Goursat Theorem

In previous section, we saw that when a continuous function f has an antiderivative in a domain D, the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero. In this section, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a simple closed contour is zero. The theorem is known as Cauchy Goursat theorem and it is central to the theory of functions of a complex variable.

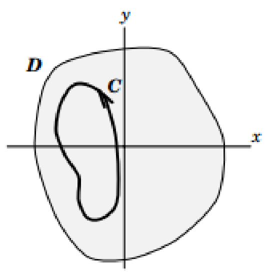
Theorem: If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0 \tag{4.26}$$

We won't prove this theorem here. But we will try to motivate why this is important.

4.6 Simply and Multiply Connected Domain

Simply Connected Domain: We define a simply connected domain D to be one in which every simple closed contour within it encloses only points of D. The points within a circle, square, and polygon are examples of a simply connected domain. We represent it with a picture below



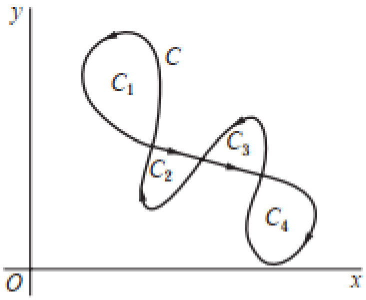
(a) D simply connected

The closed contour in the Cauchy–Goursat theorem need not be simple when the theorem is adapted to simply connected domains. More precisely the contour can cross itself. This is allowed by the following theorem which is an extension of the Cauchy-Goursat theorem.

Theorem: If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)dz = 0. \tag{4.27}$$

for every closed contour. We will sketch the proof here. For that we will consider a contour that intersects itself a finite amount of time.



Now since we assumed that the function is analytic over the contour C,then Cauchy -Goursat theorem ensures that each of the closed loops here are zero.This implies he proof of the theorem stated above.Lets see an example now

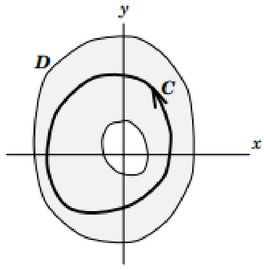
Example1: If C denotes any closed contour lying in the open disk $|z| < 2$ then

$$\int_C \frac{ze^z}{(z^2+9)^5} dz = 0 \tag{4.28}$$

It is easy to see that the singularities od the function $z = \pm 3i$ lies outside the open disk with $|z| < 2$. So the disk is a simply connected domain and using the above theorem we have $\int_C \frac{ze^z}{(z^2+9)^5} dz = 0$.

A corollary of this theorem is the follwoing **Corolllary:**A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D. We begin the proof of this corollary with the observation that a function f is continuous on a domain D when it is analytic there. Consequently, since equation 4.27 holds for the function in the hypothesis of this corollary and for each closed contour C in D, f has an antiderivative throughout D. Note that since the finite plane is simply connected, the corollary tells us that entire functions always possess antiderivatives.

Now we move onto **multiply connected domain**. A domain that is not simply connected is called a multiply connected domain.



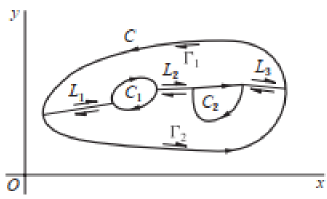
(b) D multiply connected

The following theorem is an adaptation of the Cauchy–Goursat theorem to multiply connected domains.

Theorem: If we have a simply closed contour C described in the counterclockwise direction and we have $C_k\,(k = 0, 1, 2, 3..., n)$ interior to C that are disjoint (meaning they have no overlap with each other/ they have no pints in common /their intersection is 0). If a function $f(z)$ is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k , then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 \tag{4.29}$$

To prove the theorem we will use the following figure.



Notice that, , two simple closed contours Γ_1 and Γ_2 can be formed, each consisting of polygonal paths L_k or L_{-k} and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left. The Cauchy–Goursat theorem can now be applied to f on Γ_1 and Γ_2, and the sum of the values of the integrals over those contours is found to be zero. Since the integrals in opposite directions along each path L_k cancel, only the integrals along C and the $C_k$remain and we arrive at our theorem stated above.

We also get a corollary from it. **Corollary:**Let $C1$ and $C2$ denote positively oriented simple closed contours, where $C1$ is interior to $C2$. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz \tag{4.30}$$

This corollary is known as the principle of deformation of paths. Lets take a look at an example

Example: Take C to be any positively oriented contour surrounding the origin and evaluate $\int_C \frac{dz}{z}$.

Since we can deform paths continuously using the above corollary and make the contour into a circle and from our previous discussion we can immidiately write down the result

$$\int_C \frac{dz}{z} = 2\pi i \tag{4.31}$$

4.7 Cauchy Integral Formula

We will now statet he Cauchy integral formula here. We wont do the proof here. **Theorem(Cauchy Integral Formula:)**Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0} \tag{4.32}$$

this is the Cauchy formula. It can be used to evaluate certain integrals along simple closed contours.

Example: Let C be the positively oriented circle $|z| = 2$. Since the function $f(z) = \frac{z}{9-z^2}$ is analytic on C and since the point $z_0 = -i$ and interior to C. Then our Cacuchy integral formula gives us

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = 2\pi i\left(\frac{i}{10}\right) = \frac{\pi}{5} \tag{4.33}$$

Extension of the Cauchy Formula: The Cauchy integral formula can be extended so as to provide an integral representation for derivatives of f at z_0. We state this without proof and the extension formula is

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}} \tag{4.34}$$

where $n = 1, 2, 3, \dots$.

Example 1: If C is the positively oriented unit circle $|z| = 1$ and $f(z) = e^{2z}$,then evaluate $\int_C \frac{e^{2z}dz}{z^4}$.

We can write the integeral as following

$$\int_C \frac{e^{2z}dz}{z^4} = \int_C \frac{e^{2z}dz}{(z-z_0)^4} = \frac{2\pi i}{6} f'''(0) = \frac{8\pi i}{3} \tag{4.35}$$

Example 2: Evaluate $\int_C \frac{dz}{(z-z_0)^{n+1}}$
Here we can notice that $f(z) = 1$. So using the above formula we can see that $\int_C \frac{dz}{(z-z_0)^{n+1}} = 0$.

4.8 Liouville theorem and the fundamental theorem of algebra

We will state Liouville's theorem down below :

Theorem1: If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane, then $f(z)$ is constant through the plane. To start the proof, we assume that f is as stated and note that since f is entire, can be applied with any choice of z_0 and R. In particular, Cauchy's inequality in that theorem tells us that when $n = 1$

$$|f'(z_0)| \leq \frac{M}{R} \tag{4.36}$$

Moreover, the boundedness condition on f tells us that a nonnegative constant M exists such that $|f(z)| \leq M$ for all z; and, because the constant M_R in inequality is always less than or equal to M, it follows that

$$|f'(z_0)| \leq \frac{M}{R} \tag{4.37}$$

where R can be arbitrarily large. Now the number M in inequality is independent of the value of R that is taken. Hence that inequality holds for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of

z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently f is a constant.

The following theorem, is called *The fundamental theorem of algebra.*

Theorem: For any polynmial

$$P(z) \;=\; a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{4.38}$$

where $a_n \neq 0$. The polynmial of degree $n(n1)$ has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.
Proof: The proof here is by contradiction. Suppose that $P(z)$ is not zero for any value of z. Then the reciprocal

$$f(z) \;=\; \frac{1}{P(z)} \tag{4.39}$$

is clearly entire, and it is also bounded in the complex plane.To show that its is bounded, we first write

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} \tag{4.40}$$

so that

$$P(n) \;=\; (a_n + w)z^n \tag{4.41}$$

Next, we observe that a sufficiently large positive number R can be found such that the modulus of each of the quotients in expression (3) is less than the number |an|/(2n) when |z| > R. The generalized triangle inequality (10), Sec. 4, which applies to n complex numbers, thus shows that

$$|w| < \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R \tag{4.42}$$

Consequently,

$$|a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R \tag{4.43}$$

then we have

$$|P_n(z)| = |a_n + w|\,|z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| > R \tag{4.44}$$

Evidently, then

$$|f(z)| = \frac{1}{|P(z)|} < \frac{2}{|a_n| R^n} \quad \text{whenever} \quad |z| > R \tag{4.45}$$

So f is bounded in the region exterior to the disk |z| $\leq R.Butfiscontinuousinthatclose$

$$P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n) \tag{4.46}$$

And thus we proved the fundamental theorem of Algebra.

Chapter 5

Series

In many scenarios it is, much easier and time saving to evaluate an approximation of a function at a certain point rather than the exact definite value. One such trick of getting the aforementioned approximate values are by the way of Series solutions. We shall look into how they come into play and how to evaluate them in the following sections.

5.1 Taylor and Maclaurin Series

Theorem. *Suppose that a function is analytic throughout a disk* $|z - z_0| < R_0$ *centered at* z_0 *and with radius* R_0 (Figure .5.1). *Then f(z) can take the power series form.*

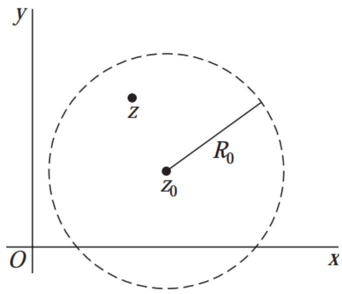


Figure 5.1: Pictorial Representation of a disk around which a complex function is analytical

$f(z)$ then according to the theorem takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{5.1}$$

where

$$a_n = \frac{f^{(n)} z_{(0)}}{n!} \quad (n = 0, 1, 2, \dots) \tag{5.2}$$

where the $f^{(n)}$ is the n^{th} derivative of the function, $f(z)$. Here Eq(5.1) represents the series expansion of $f(z)$ about the point z_0. Of course Eq(5.1)can also be written as

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \quad (|z - z_0| < R_0). \tag{5.3}$$

In Eq(5.3) we have applied with the agreement that $f^{(0)}(z_0) = f(z_0)$ and the fact that $0! = 1$. If a function f is said to be analytic at a point z_0, must have a Taylor series about that point. For, if f is analytic at z_0, it is analytic throughout the neighbourhood of $|z - z_0| < R_0$. It is also an important point to note that any any point z inside the circle with z_0 as the center, with which the Taylor series is constructed will always converge. In the case where our center $z_0 = 0$, for which f is assumed to be analytic throughout the disk $|z| < R_0$, and thus the series in Eq(5.1) becomes the Maclaurin Series.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad (|z| < R_0) \tag{5.4}$$

5.2 Taylor and Maclaurin Series Examples

5.2.1 Example 01

We know from the previous chapter(s) that the function $f(z) = e^z$ is an analytic function for all values of z. Hence we can represent this function as a Maclaurin series function. Using Eq(5.4) we find that

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{5.5}$$

The proof of Eq(5.5) is left to the reader as an exercise (Hint: You can also use Eq(5.3), by setting $z_0 = 0$, and since $f^{(n)}(z) = e^z$ we get $f^{(n)}(0) = 1$ where $n = 0, 1, 2, 3, \dots$. Now, that we have found a series solution for e^z, we can use that to figure out what the series would look like for similar functions. Functions like z^2e^{3z}. Replace the z in Eq(5.5) with $3z$ and multiplying the series with z^2, which yields

$$z^2e^{3z} = z^2 \times \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} \tag{5.6}$$

Then taking $k = n - 2$

$$z^2\mathrm{e}^{3z} = \sum_{k=2}^{\infty} \frac{3^{k-2}}{(k-2)!} z^k \tag{5.7}$$

5.2.2 Example 02

We are going to use the series solution that we got in Eq(5.5) and plugging it into the definition

$$sinz = \frac{e^{iz} - e^{-iz}}{2i} \tag{5.8}$$

we get

$$\sin z = \frac{1}{2i}[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}] = \frac{1}{2i} \sum_{n=o}^{\infty} [1 - (-1)^n] \frac{i^n z^n}{n!} \tag{5.9}$$

But $1 - (-1)^n$, when n is even, so we can replace n by $2k + 1$, giving us

$$\sin z = \frac{1}{2i} \sum_{k=o}^{\infty} [1 - (-1)^{2k+1}] \frac{i^{2k+1} z^{2k+1}}{(2k+1)!} \tag{5.10}$$

We know that $1 - (-1)^{2k+1}$ will result in 2 since for all values of k, and $i^{2k+1} = (i^2)^k \times i$, giving us the expansion for $\sin z$ as

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \quad (|z| < \infty) \tag{5.11}$$

Using the results found in Eq(5.11), you can figure out the Maclaurin series expansion of $cosz$ as well. (Hint: $\frac{d}{dz}\sin z = \cos z$)

5.2.3 Example 03

Another example of a Maclaurin series expansion would be

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + ... = \sum_{n=0}^{\infty} z^n, \quad (|z| < 1). \tag{5.12}$$

Of course Eq(5.12) is the sum of an infinite geometric series. Where the derivative of the function $f(z) = \frac{1}{1-z}$ fails to be analytic at $z = 1$,

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad (n = 0, 1, 2, \dots) \tag{5.13}$$

Furthermore, if we substitute, $-z$ for z in Eq(5.12), noting that $|-z| < 1$, we see that,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad (|z| < 1). \tag{5.14}$$

Also, we can z in Eq(5.12) with $1 - z$, giving us the Taylor expansion for

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad (|z-1| < 1) \tag{5.15}$$

Where in Eq(5.15) we have we stress that $|1 - z|$ is the same as $|z - 1|$

5.2.4 Example 04

For the final example we are given the following function to expand.

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3} . \frac{2+z^2-1}{1+z^2} = \frac{1}{z^3} . \frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right) \tag{5.16}$$

Expanding $\frac{1}{1+z^2}$ in similar fashion, like we did in Example 03 we get

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - ... \quad (|z| < 1 \tag{5.17}$$

Plugging Eq(5.17) back into Eq(5.16) we get the following expansion

$$\begin{aligned} f(z) &= \frac{1}{z^3}\left[2 - (1 - z^2 + z^4 - z^6 + z^8 - ...)\right] \\ &= \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \dots \quad (0 < |z| < 1) \end{aligned} \tag{5.18}$$

5.3 The Laurent Series

We have seen in Section(5.1) how to find out the Taylor and Maclaurin series of complex functions that are analytic at a point z_0. However, to deal with functions that are not, we apply what is called the **Laurent Series**. Moreover, Laurent series's are expanded in both negative and positive power of $z-z_0$. ***Theorem***. *Suppose that a function f is analytic throughout an annular domain $R_1 < |zz_0| < R_2$, centered at z_0 , and let C denote any positively oriented simple closed contour around z_0 and lying in that domain(Figure(5.2)).Then, at each point in the domain, $f(z)$ has a series representation.*

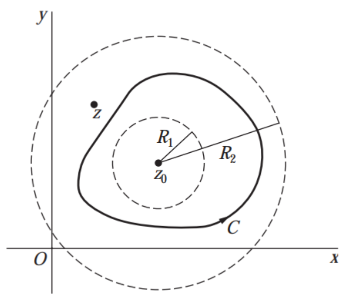


Figure 5.2:

The aforementioned Laurent series takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \quad (R_1 < |z-z_0| < R_2) \tag{5.19}$$

where,

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad (n = 0, 1, 2, \dots) \tag{5.20}$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz, \quad (n = 0, 1, 2, \dots). \tag{5.21}$$

Alternatively by by replacing n by $-n$ in the second term of Eq(5.19), enables us to write

$$\sum_{n=\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}} \tag{5.22}$$

where

$$b_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad (n = -1, -2, \dots). \tag{5.23}$$

Thus, Eq(5.19) changes form to look like the following

$$f(z) = \sum_{-\infty}^{-1} b_{-n}(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad (R_1 < |z-z_0| < R_2), \tag{5.24}$$

if we choose to condense the coefficients a_n and b_{-n} into one coefficient, we have

$$c_n = \begin{cases} b_{-n} & \text{when } n \leqslant -1, \\ a_n & \text{when } n \geqslant 0, \end{cases} \tag{5.25}$$

thus $f(z)$ can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \quad (R_1 < |z-z_0| < R_2), \tag{5.26}$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5.27}$$

Where both Eq(5.19) and Eq(5.28) are valid Laurent series representations of $f(z)$ Observe that the integrand in Eq(5.23) can be written as $f(z)(z-z_0)^{n-1}$, so if f happens to be analytic in the neighbourhood of $|z-z_0| < R_2$, the integrand is too. Thus, the coefficient b_{-n} becomes 0. In a similar fashion the integrand in Eq(5.20) fails to be analytic at $z = z_0$, which is why,

$$\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}, \quad (n = 0, 1, 2, \ldots). \tag{5.28}$$

So we see that Laurent series expansion of $f(z)$ reduces to a Taylor series expansion about z_0. If, however, f fails to be analytic at z0 but is otherwise analytic in the disk $|z - z0| < R_2$, the radius R_1 can be chosen arbitrarily small. Laurent series, is then valid in the punctured disk $0 < |z-z_0| < R_2$. Similarly, if f is analytic at each point in the finite plane exterior to the circle $|z-z_0| = R_1$, the condition of validity is $R_1 < |z-z_0| < \infty$. Note that if f is analytic everywhere in the finite plane except at z_0, the Laurent series is valid at each point of analyticity, or when $0 < |z-z_0| < \infty$.

5.4 Laurent Series Example

5.4.1 Example 01

We know the Maclaurin series expansion of e^z

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (|z| < \infty) \tag{5.29}$$

If however we replace z, with $\frac{1}{z}$ we directly get the Laurent series representation

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots . \quad (0 < |z| < \infty) \tag{5.30}$$

There are no positive powers of z, $(a_n = 0)$ in the above expansion Eq(5.30). We also see that the coefficient of $\frac{1}{z}$ (i.e the negative powers of z) is unity or 1.

5.4.2 Example 02

The function $f(z) = \frac{1}{(z-i)^2}$, is already in the Laurent series form, where $z_0 = i$ as we shall see

$$\frac{1}{(z-i)^2} = \sum_{n=-\infty}^{\infty} c_n (z-i)^n \quad (0 < |z-i| < \infty). \tag{5.31}$$

To evaluate the coefficient c_n, we use the integrand

$$c_n = \frac{1}{2\pi i} \int_C \frac{dz}{(z-i)^{n+3}} \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{5.32}$$

where C is, for instance, any positively oriented circle $|z-i| = R$ about the point $z_0 = i$.

$$\int_c \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0 & \text{when } n \neq -2 \\ 2\pi i & \text{when } n = -2. \end{cases} \tag{5.33}$$

So as we can see only $c_{-2} = 1$, and all other coefficients are zero.

5.4.3 Example 03

Take for example the function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}, \tag{5.34}$$

as we can see above the function has two singular points $z = 1$ and $z = 2$, and is analytic in the following domains

$$|z| < 1, \quad 1 < |z| < 2, \quad \text{and} \quad 2 < |z| < \infty$$

In each of those domains, denoted by D_1, D_2, and D_3, respectively, in Figure(5.3).

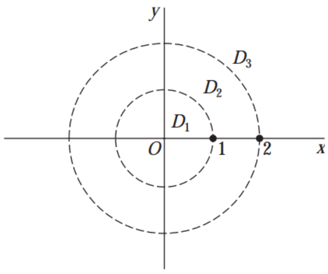


Figure 5.3:

$f(z)$ has series representations in powers of z. They can all be found by making the appropriate replacements for z in the expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1). \tag{5.35}$$

he representation in D_1 is a Maclaurin series. To find it, we observe that

$$|z| < 1 \quad \text{and} \quad |z/2| < 1,$$

and doing some algebraic manipulation we see that

$$f(z) = -\frac{1}{1-z} + \frac{1}{2} \cdot \frac{1}{1-(z/2)} \tag{5.36}$$

This tells us that

$$f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n \quad (|z| < 1) \tag{5.37}$$

Now, we shall shift our gaze towards the D_2 region, where the neighbourhood is $1 < |z| < 2$. When z is a point in D_2, we know that $|1/z| < 1$ and $|z/2| < 1$. Given such point we make appropriate algebraic manipulations such that

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-(1/z)} + \frac{1}{2} \cdot \frac{1}{1-(z/2)} \tag{5.38}$$

For which we employ the same intuition as was employed in Eq(5.37), giving us

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad (1 < |z| < 2) \tag{5.39}$$

Replacing the index of summation n in the first of these series by $n-1$, and then interchange the two series, we arrive at the expansion being as the same form as the one on Laurent's theorem

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n} \quad (1 < |z| < 2) \tag{5.40}$$

Giving us the Laurent series expansion for f in D_2. Finally, we shall evaluate the representation of the function in the unbounded domain D_3, where $2 < |z| < \infty$, is also a Laurent series. Since $|2/z| < 1$ when z is in D_3, it is also true that $|1/z| < 1$. So we write the expression as

$$f(z) = \frac{1}{2} \cdot \frac{1}{1-(1/z)} - \frac{1}{z} \cdot \frac{1}{1-(2/z)} \tag{5.41}$$

we find that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \quad (2 < |z| < \infty). \tag{5.42}$$

Replacing n by $n1$ in this last series then gives the standard form

$$f(z) = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \quad (2 < |z| < \infty). \tag{5.43}$$

Thus giving us the laurent series expansion for the unbounded region, also we see that the a_n coefficient is zero and hence why, we do not see the positive power series as a part of this solution.

Chapter 6

Residues and Poles

In this chapter we will look at various kinds of singularities that arises in complex analysis. We will get to know resuides and poles as they will play a vital role in evaluating different kinds of integras of interest.

6.1 Isolated Singular Points

A point z_0 is called a singular point of a function f if ,f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. A singular point z_0 is said to be isolated if, in addition, there is a deleted neighborhood $0 < |z - z_0| < \epsilon$ of z_0 throughout which f is analytic. That means, if we draw a neighbourhood(a circle) around z_0 with a radius $z - z_0$ which is less that ϵ, then the function is analytic everywhere along and inside that circle except for the point z_0 . Isolated meaning there is no other singular point around z_0.

Example 6.1.1.

The function

$$\frac{z+1}{z^3(z^2+1)} \tag{6.1}$$

has the three isolated singular points $z = 0$ and $z = \pm i$. **EXAMPLE 2.** The origin is a singular point of the principal branch

$$\text{Log } z = \ln r + i\phi \quad (r > 0, -\pi < \phi < \pi) \tag{6.2}$$

of the logarithmic function. It is not an isolated singular point since every deleted neighborhood of it contains points on the negative real axis (see Fig. 82) and the branch is not even defined there. Similar remarks can be made regarding any branch

$$\log z = \ln r + i\theta \quad (r > \theta, < \theta < +2\pi) \tag{6.3}$$

of the logarithmic function.

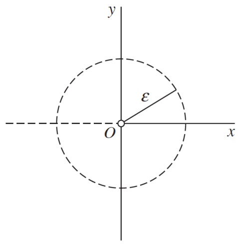


Figure 6.1: Fig82

6.2 Residues

When z_0 is an isolated singular point of a function f , there is a positive number R_2 such that f is analytic at each point z for which $0 < |z - z_0| < R_2$. Consequently, $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_n}{(z - z_0)^n} + ...$$
$$(0 < |zz_0| < R_2)$$

where the coefficients a_n and b_n have certain integral representations. In particular,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, ...) \tag{6.4}$$

where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |zz_0| < R_2$ (Fig. 84). When $n = 1$, this expression for b_n becomes

$$\int_C f(z)dz = 2\pi i b_1. \tag{6.5}$$

The complex number b_1, which is the coefficient of $\frac{1}{(zz_0)}$ in expansion (1), is called the *residue* of f at the isolated singular point z_0, and we shall often write

$$b_1 = \underset{z=z_0}{\mathrm{Res}} f(z). \tag{6.6}$$

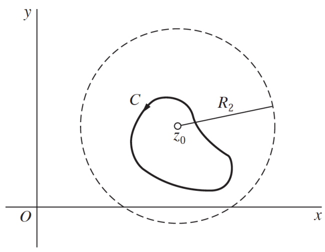


Figure 6.2: Fig 84

So we can write

$$\int_C f(z)dz = 2\pi i \underset{z=z_0}{\mathrm{Res}} f(z). \tag{6.7}$$

Sometimes we simply use B to denote the residue when the function f and the point z_0 are clearly indicated. Equation (3) provides a powerful method for evaluating certain integrals around simple closed contours.

Example 6.2.1.

Consider the integral

$$\int_C z^2 \sin(\frac{1}{z})dz \tag{6.8}$$

where C is the positively oriented unit circle $|z| = 1$. Since the integrand is analytic everywhere in the finite plane except at $z = 0$, it has a Laurent series representation that is valid when $0 < |z| < \infty$. Thus, according to equation (3), the value of integral (4) is $2\pi i$ times the residue of its integrand at $z = 0$.

To determine that residue, we recall the Maclaurin series representation

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ... \quad (|z| < \infty) \tag{6.9}$$

and use it to write

$$z^2 \sin(\frac{1}{z}) = z - \frac{1}{3!}.\frac{1}{z} + \frac{1}{5!}.\frac{1}{z^3} - \frac{1}{7!}.\frac{1}{z^5} + ... \quad (0 < |z| < \infty) \tag{6.10}$$

The coefficient of 1/z here is the desired residue. Consequently,

$$\int_C z^2 \sin(\frac{1}{z})dz = 2\pi i(-\frac{1}{3!}) = -\frac{\pi i}{3} \tag{6.11}$$

EXAMPLE 2. Let us show that

$$\int_C \exp(\frac{1}{z^2})dz = 0 \tag{6.12}$$

when C is the same oriented circle $|z| = 1$ as in Example 1. Since $1/z_2$ is analytic everywhere except at the origin, the same is true of the integrand. The isolated singular point $z = 0$ is interior to C. Now we expand the exponential and get

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \quad (|z| < \infty), \tag{6.13}$$

one can write the Laurent series expansion

$$\exp(\frac{1}{z^2}) = 1 + \frac{z}{1!}\frac{1}{z^2} + \frac{z^2}{2!}\frac{1}{z^4} + \frac{z^3}{3!}\frac{1}{z^6} + ... \quad (0 < |z| < \infty). \tag{6.14}$$

The residue of the integrand at its isolated singular point $z = 0$ is, therefore, *zero* ($b_1 = 0$), and the value of integral (5) is established. **Important Remark:**We are reminded in this example that although the analyticity of a function within and on a simple closed contour C is a sufficient condition for the value of the integral around C to be zero, it is not a *necessary* condition.

6.3 Cauchy's Residue Theorem

We will now look at an important theorem. The Cacuchy's residue theorem. We will state the theorem first and then look at it's proof.

Theorem: *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k ($k = 1, 2, . . . , n$) inside C then*

$$(1) \qquad \int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{6.15}$$

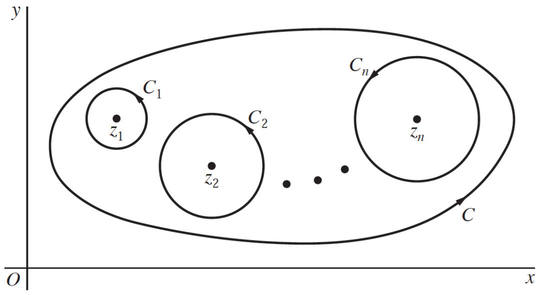


Figure 6.3: Fig 87

Proof: To prove the theorem, let the points $z_k(k = 1, 2, ..., n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circles C_k, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain consisting of the points inside C and exterior to each C_k. Hence, according to the adaptation of the Cauchy–Goursat theorem to such domains

$$\int_C f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 \tag{6.16}$$

and using 6.7 we can see that

$$\int_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \quad (k = 1, 2, ..., n), \tag{6.17}$$

and the proof is complete.

6.4 The Three Types of Isolated Singular Points

We saw in earlier sections that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \ldots \tag{6.18}$$

in a punctured disk $0 < |z - z0| < R_2$. The portion

$$(2) \quad \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \ldots \qquad (6.19)$$

of the series, involving negative powers of $z - z_0$, is called the principal part of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following sections. If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer $m(m1)$ such that

$$b_m \neq 0 \,\text{and}\, b_{m+1} = b_{m+2} = \cdots = 0. \qquad (6.20)$$

That is, expansion (1) takes the form

$$(3) \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$
$$(0 < |z - z_0| < R_2),$$

where $b_m \neq 0$. In this case, the isolated singular point z_0 is called a *pole of order* m^*. A pole of order $m = 1$ is usually referred to as a *simple pole*. **EXAMPLE 1.** Observe that the function

$$\frac{z^2 - 2z + 3}{z-2} = \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2} = 2 + (z-2) + \frac{3}{z-2}$$
$$(0 < |z - 2| < \infty)$$

has a simple pole ($m = 1$) at $_0 = 2$. Its residue b_1 there is 3. When representation (1) is written in the form (see Sec. 60)

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad (0 < |z - z0| < R_2), \qquad (6.21)$$

the residue of f at z_0 is, of course, the coefficient $c1$. **EXAMPLE 2.** From the representation

$$f(z) = \frac{1}{z^2(1+z)} = \frac{1}{z^2} . \frac{1}{1-(-z)} = \frac{1}{z^2}(1 - z + z^2 - z^3 + z^4 - \ldots)$$
$$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \ldots \quad (0 < |z| < 1),$$

one can see that f has a pole of order $m = 2$ at the origin and that

$$\operatorname*{Res}_{z=z_0} f(z) = -1 \tag{6.22}$$

EXAMPLE 4. The point z0 = 0 is a removable singular point of the function

$$f(z) = \frac{1-\cos z}{z^2} \tag{6.23}$$

because

$$f(z) = \frac{1}{z^2}[1-(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots)] = \frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\dots$$
$$(0 < |z| < \infty).$$

When the value $f(0) = 1/2$ is assigned, f becomes entire. If an infinite number of the coefficients b_n in the principal part (2) are nonzero, z_0 is said to be an essential singular point of f. **EXAMPLE 5.** We recall from Example 1 in Sec. 62 that

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}.\frac{1}{z^n} = 1 + \frac{1}{1!}.\frac{1}{z} + \frac{1}{2!}.\frac{1}{z^2} + \dots \quad (0 < |z| < \infty). \tag{6.24}$$

From this we see that $e^{1/z}$ has an essential singular point at $z_0 = 0$, where the residue b_1 is unity.

In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just described. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

6.5 Residues at Poles

In this section, we will develop an alternative characterization of poles and a way of finding residues at poles that is often more convenient. We will start with the following theorem

Theorem: An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{\mathrm{z}=\mathrm{z}_0} f(z)=\phi\left(z_0\right) \quad \text{if } m=1$$

and

$$\operatorname*{Res}_{\mathrm{z}=\mathrm{z}_0} f(z)=\frac{\phi^{(m-1)}\left(z_0\right)}{(m-1)!} \quad \text{if } m \geq 2$$

Observe that expression 6.5 need not have been written separately since, with the convention that $\phi^{(0)}\left(z_0\right)=\phi\left(z_0\right)$ and $0!=1$, expression 6.5 reduces to it when

$$m=1$$

The following ex amples serve to illustrate the use of the theorem stated above. **EXAMPLE 1**. The function

$$f(z)=\frac{z+1}{z^2+9}$$

has an isolated singular point at $z=3i$ and can be written

$$f(z)=\frac{\phi(z)}{z-3i} \quad \text{where} \quad \phi(z)=\frac{z+1}{z+3i}$$

since $\phi(z)$ is analytic at $z=3i$ and $\phi(3i)\neq 0$, that point is a simple pole of the function f; and the residue there is

$$B_1=\phi(3i)=\frac{3i+1}{6i}\cdot\frac{-i}{-i}=\frac{3-i}{6}$$

The point $z=3i$ is also a simple pole of f, with residue

$$B_2=\frac{3+i}{6}. \tag{6.25}$$

EXAMPLE 2. If

$$f(z)=\frac{z^3+2z}{(z-i)^3} \tag{6.26}$$

then

$$f(z)=\frac{\phi(z)}{(z-i)^3} \quad \text{where} \quad \phi(z)=z^3+2z$$

The function $\phi(z)$ is entire, and $\phi(i)=i\neq 0$. Hence f has a pole of order 3 at $z=i$, with residue

$$B=\frac{\phi''(i)}{2!}=\frac{6i}{2!}=3i$$

The theorem can, of course, be used when branches of multiple-valued functions are involved. **EXAMPLE 3.** Suppose that

$$f(z) = \frac{(\log)^3}{z^2+1}, \tag{6.27}$$

where the branch

$$\log z = \ln r + i\theta \quad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function is to be used. To find the residue of f at the singularity $z = i$, we write

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^3}{z+i}$$

The function $\phi(z)$ is clearly analytic at $z = i$; and, since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(\ln 1 + i\pi/2)^3}{2i} = -\frac{\pi^3}{16} \neq 0$$

f has a simple pole there. The residue is

$$B = \phi(i) = -\frac{\pi^3}{16}$$

While the theorem stated above can be extremely useful, the identification of an isolated singular point as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

6.6 Zeros of Analytic Functions

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions. Suppose that a function f is analytic at a point z0. We know from Sec. 52 that all of the derivatives $f(n)(z)(n = 1, 2, ...)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that $f(m)(z_0) \neq 0$ and each derivative of lower order vanishes at z_0 , then f is said to have a zero of order m at z_0. Our first theorem here provides a useful alternative characterization of zeros of order m.

Theorem 1. *Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0 , such that*

$$(1) \quad f(z) = (z - z_0)^m g(z). \tag{6.28}$$

EXAMPLE 1. The polynomial $f(z) = z^3 8 = (z2)(z^2 + 2z + 4)$ has a zero of order $m = 1$ at $z_0 = 2$ since

$$f(z) = (z - 2)g(z), \tag{6.29}$$

where $g(z) = z^2 + 2z + 4$, and because f and g are entire and $g(2) = 12 \neq 0$. Note how the fact that $z_0 = 2$ is a zero of order $m = 1$ of f also follows from the observations that f is entire and that

$$f(2) = 0 \text{ and } f^{'}(2) = 12 \neq 0. \tag{6.30}$$

EXAMPLE 2. The entire function $f(z) = z(e^z 1)$ has a zero of order $m = 2$ at the point $z_0 = 0$ since

$$f(0) = f^{'}(0) = 0 \text{ and } f^{''}(0) = 2 \neq 0. \tag{6.31}$$

In this case, expression becomes

$$f(z) = (z0)^2 g(z), \tag{6.32}$$

where g is the entire function defined by means of the equations

$$g(z) = \begin{cases} (e^z - 1)/z & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases} \tag{6.33}$$

6.7 Zeros and Poles

We will now turn to relating zeros and poles. The following theorem shows how zeros of order m can create poles of order m. ***Theorem 1.*** *Suppose that*
(a) two functions p and q are analytic at a point z_0 ;
(b) $p(z_0) \neq 0$ and q has a zero of order m at z_0 .
Then the quotient $p(z)/q(z)$ has a pole of order m at z_0 .

EXAMPLE 1. The two functions

$$p(z) = 1 \text{ and } q(z) = z(e^z - 1) \tag{6.34}$$

are entire. We notice that $q(z)$ has a pole of order 2 at point $z_0 = 0$.So from theorem 1 we can say the quotient

$$\frac{p(z)}{q(z)} = \frac{1}{z(e^z - 1)} \tag{6.35}$$

has pole of order 2 at that point.

Theorem 1 leads us to another method for identifying *simple* poles and finding the corresponding residues. **Theorem2** Let the functions p and q be analytic at a point z_0. If

$$p(z_0) \neq 0 \; q(z_0) = 0 \text{ and } q'(z_0) \neq 0 \tag{6.36}$$

then z_0 is a simple pole of quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

EXAMPLE 2. Consider the function

$$f(z) = \cot z = \frac{\cos z}{\sin z} \tag{6.37}$$

which is a quotient of the entire functions $p(z) = cosz$ and $q(z) = \sin z$. Its singularities occur at the zeros of q, or at the points

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \dots). \tag{6.38}$$

Since

$$p(n\pi) = (1)^n \neq 0, q(n\pi) = 0, \text{ and } q'(n\pi) = (1)^n \neq 0, \tag{6.39}$$

each singular point $z = n\pi$ of f is a simple pole, with residue

$$B_n = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1. \tag{6.40}$$

EXAMPLE 4. Since the point

$$z_0 = \sqrt{2} e^{\frac{i\pi}{4}} = 1 + i \tag{6.41}$$

is a zero of the polynomial $z^4 + 4$, it is also an isolated singularity of the function

$$f(z) = \frac{z}{z^4 + 4} \tag{6.42}$$

Writing $p(z) = z$ and $q(z) = z^4 + 4$, we find that

$$p(z_0) = z_0 \neq 0, q(z_0) = 0, \text{and} q'(z0) = 4{z_0}^3 \neq 0 \tag{6.43}$$

and hence that z_0 is a simple pole of f . The residue there is, moreover,

$$B_0 = \frac{p\,(z_0)}{q'\,(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8} \tag{6.44}$$

Although this residue can also be found by the method mentioned in the section involving residues at poles, the computation is somewhat more involved.

Chapter 7

Applications of Residues

We turn now to some important applications of the theory of residues, which was developed in Chap. 6. The applications include evaluation of certain types of definite and improper integrals occurring in real analysis and applied mathematics.

7.1 Evaluation of Improper Integrals

In calculus, the improper integral of a continuous function $f(x)$ over the semi infinite interval $0 \leq x < \infty$ is defined by means of the equation

$$\int_0^\infty f(x)\ dx = \lim_{R\to\infty} \int_0^R f(x)\ dx \tag{7.1}$$

When the limit on the right exists, the improper integral is said to *converge* to that limit. If $f(x)$ is continuous for all x, its improper integral over the infinite interval $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^\infty f(x)\ dx = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\ dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\ dx \tag{7.2}$$

and when both of the limits here exist, we say that integral (7.2) converges to their sum. Another value that is assigned to integral (7.2) is often useful. Namely, the *Cauchy principal value* (P.V.) of integral (7.2) is the number

$$\text{P.V.} \int_{-\infty}^\infty f(x)\ dx = \lim_{R\to\infty} \int_{-R}^R f(x)\ dx, \tag{7.3}$$

provided this single limit exists.
If integral (7.2) converges, its *Cauchy principal value* (7.3) exists; and

that value is the number to which integral (7.2) converges. This is because

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx &= \lim_{R\to\infty}\left[\int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx\right]\\ &= \lim_{R\to\infty}\int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty}\int_{0}^{R} f(x)\,dx\end{aligned}$$

and these last two limits are the same as the limits on the right in equation (7.2).

It is not, however, always true that integral (7.2) converges when its Cauchy principal value exists, as the following example shows.

Example 7.1.1.

Observe that

$$\text{P.V.}\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{-R}^{R} = \lim_{R\to\infty} 0 = 0 \tag{7.4}$$

on the other hand,

$$\begin{aligned}\int_{-\infty}^{\infty} x\,dx &= \lim_{R_1\to 0}\int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} x\,dx \qquad (7.5)\\ &= \lim_{R_1\to\infty}\left[\frac{x^2}{2}\right]_{-R_1}^{0} + \lim_{R_2\to\infty}\left[\frac{x^2}{2}\right]_{0}^{R_2}\\ &= -\lim_{R_1\to\infty}\frac{R_1^2}{2} + \lim_{R_2\to\infty}\frac{R_2^2}{2}\end{aligned}$$

and since these last two limits do not exist, we find that the improper integral fails to exist.

But suppose that $f(x)$ $(-\infty < x < \infty)$ is an even function, one where $f(-x) = f(x)$ for all x, and assume that the *Cauchy principal value* (7.3) exists. The symmetry of the graph of $y = f(x)$ with respect to the y axis tells us that

$$\int_{-R_1}^{0} f(x)\,dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)\,dx$$

and

$$\int_0^{R_2} f(x)\ dx = \frac{1}{2}\int_{-R_2}^{R_2} f(x)\ dx.$$

thus

$$\int_{-R_1}^{0} f(x)\, dx + \int_0^{R_2} f(x)\ dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)\ dx + \frac{1}{2}\int_{-R_2}^{R_2} f(x)\ dx.$$

If we let R_1 and R_2 tend to ∞ on each side here, the fact that the limits on the right exist means that the limits on the left do too. In fact,

$$\int_{-\infty}^{\infty} f(x)\ dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx \tag{7.6}$$

Moreover, since

$$\int_0^{R} f(x)\, dx = \frac{1}{2}\int_{-R}^{R} f(x)\, dx$$

it is also true that,

$$\int_0^{\infty} f(x)\, dx = \frac{1}{2}\left[\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx\right] \tag{7.7}$$

We now describe a method involving sums of residues, to be illustrated in the next section, that is often used to evaluate improper integrals of rational functions $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. We agree that $q(z)$ has no real zeros but has at least one zero above the real axis.

The method begins with the identification of all the distinct zeros of the polynomial $q(z)$ that lie above the real axis.
They are, of course, finite in number and may be labeled $z_1, z_2, ..., z_n$, where n is less than or equal to the degree of $q(z)$. We then integrate the quotient

$$f(z) = \frac{p(z)}{q(z)} \tag{7.8}$$

around the positively oriented boundary of the semicircular region shown in Fig. 7.1

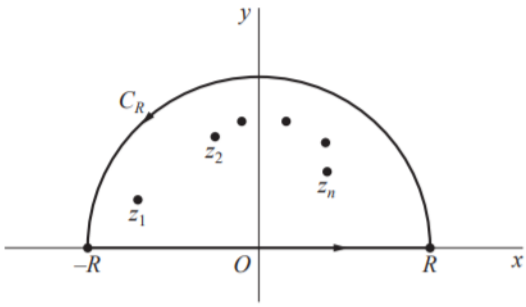


Figure 7.1:

That simple closed contour consists of the segment of the real axis from $z = -R$ to $z = R$ and the top half of the circle $|z| = R$, described counterclockwise and denoted by C_R. It is understood that the positive number R is large enough so that the points $z_1, z_2, ..., z_n$ all lie inside the closed path.

The parametric representation $z = x$ $(-R \leq x \leq R)$ of the segment of the real axis just mentioned and Cauchy's residue theorem in Sec. 6.3 can be used to write

$$\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z),$$

or

$$\int_{-R}^{R} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z) - \int_{C_R} f(z)\, dz \tag{7.9}$$

if

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0,$$

then it follows

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z); \tag{7.10}$$

and if $f(x)$ is *even*, equations (7.6) and (7.7) tell us that

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res} f(z) \tag{7.11}$$

and

$$\int_0^\infty f(x)\,dx = \pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{7.12}$$

7.2 Cauchy Principal Value of an Improper Integral and Related Concepts

We turn now to an illustration of the method in Sec. (7.1) above for evaluating improper integrals.

Example 7.2.1.

In order to evaluate the integral

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx$$

we start with the observation that the function

$$f(z) = \frac{z^2}{x^6+1}$$

has isolated singularities at the zeros of z^6+1, which are the sixth roots of 1, and is analytic everywhere else. The method in Sec. 1.6 for finding roots of complex numbers reveals that the sixth roots of -1 are

$$c_k = \exp\left[i\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)\right] \quad (k = 0, 1, 2, ..., 5)$$

and it is clear that none of them lies on the real axis. The first three roots,

$$c_0 = e^{i\pi/6}, \ \ c_1 = i, \ \text{ and } \ c_2 = e^{5i\pi/6}$$

lie in the upper half plane (Fig. 7.2) and the other three lie in the lower one. When $R > 1$, the points $c_k\,(k = 0, 1, 2)$ lie in the interior of the semicircular region bounded by the segment $z = x\ (-R \leq x \leq R)$ of the real axis and the upper half C_R of the circle $|z| = R$ from $z = R$

to $z = -R$. Integrating $f(z)$ counterclockwise around the boundary of this semicircular region, we see that

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\,(B_0 + B_1 + B_2) \tag{7.13}$$

where B_k is the residue of $f(z)$ at c_k $(k = 0, 1, 2)$.

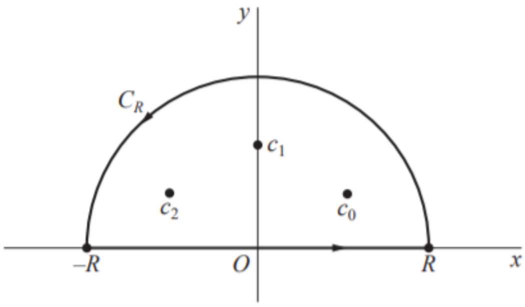


Figure 7.2:

With the aid of Theorem 2 in Sec. 6.8, we find that the points c_k are simple poles of f and that

$$B_k = \operatorname*{Res}_{z=c_k} \frac{z^2}{z^6+1} = \frac{c_k^2}{6c_k^5} = \frac{1}{6c_k^3} \qquad (k = 0, 1, 2)$$

Thus

$$2\pi i(B_0 + B_1 + B_2) = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right)$$

and equation (7.13) can be put in form

$$\int_{-R}^{R} f(x)\,dx = \frac{\pi}{3} - \int_{C_R} f(z)\,dz, \tag{7.14}$$

which is valid for all values of R greater than 1.

Next, we show that the value of the integral on the right in equation (7.14) tends to 0 as R tends to ∞. To do this, we observe that when $|z| = R$,

$$|z^2| = |z|^2 = R^2$$

and

$$|z^6+1| \geq ||z|^6-1| = R^6-1$$

So, if z is any point on C_R,

$$|f(z)| = \frac{|z^2|}{|z^6+1|} \leq M_R \quad \text{where} \quad M_R = \frac{R^2}{R^6-1}$$

and this means that

$$\left| \int_{C_R} f(z)\, dz \right| \leq M_R \pi R, \tag{7.15}$$

πR being the length of the semicircle C_R. Since the number

$$M_R \pi R = \frac{\pi R^3}{R^6-1}$$

is a quotient of polynomials in R and since the degree of the numerator is less than the degree of the denominator, that quotient must tend to zero as R tends to ∞. More precisely, if we divide both numerator and denominator by R^6 and write

$$M_R \pi R = \frac{\frac{\pi}{R^3}}{1-\frac{1}{R^6}}$$

it is evident that $M_R\pi R$ tends to zero. Consequently, in view of the above inequality (7.15)

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

It now follows from the equation (7.14) that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\, dx = \frac{\pi}{3}$$

or

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\, dx = \frac{\pi}{3}$$

Since the integrand here is even,

$$\int_{0}^{\infty} \frac{x^2}{x^6+1}\, dx = \frac{\pi}{6} \tag{7.16}$$

7.3 Jordan's Lemma

Theorem: Suppose that

(a) a function $f(z)$ is analytic at all points in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$;

(b) C_R denotes a semicircle $z = R\,e^{i\theta}$ $(0 \leq \theta \leq \pi)$, where $R > R_0$ (Fig. below);

(c) for all points z on C_R, there is a positive constant M_R such that

$$|f(z)| \leq M_R \quad \text{and} \quad \lim_{R\to\infty} M_R = 0$$

Then for every positive constant a,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{iaz}\, dz = 0$$

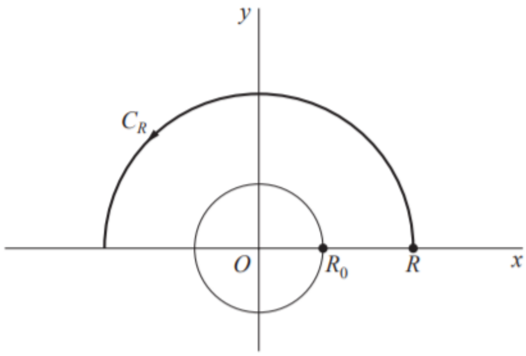


Figure 7.3:

The proof is based on Jordan's inequality:

$$\int_0^{\pi} e^{-R\sin\theta}\, d\theta < \frac{\pi}{R} \quad (R > 0) \tag{7.17}$$

To verify it, we first note from the graphs (Fig. 7.4) of the functions

$$y = \sin\theta \quad \text{and} \quad y = \frac{2\theta}{\pi}$$

that

$$\sin\theta \geq \frac{2\theta}{\pi} \quad \text{when} \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

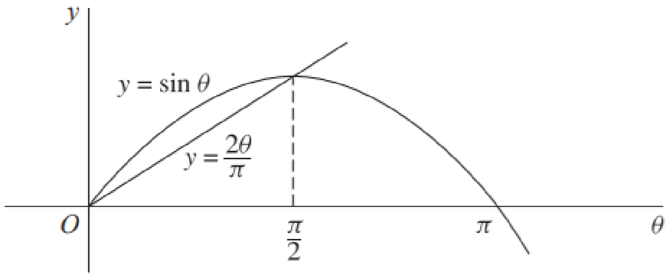


Figure 7.4:

Consequently, if $R > 0$,

$$e^{-R\sin\theta} \leq e^{-2R\theta/\pi} \quad \text{when} \quad 0 \leq \theta \leq \frac{\pi}{2};$$

and so,

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{2R}(1-e^{-R}) \qquad (R>0).$$

Hence

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta = \frac{\pi}{2R} \quad (R>0) \tag{7.18}$$

But this is just another form of the inequality (7.17), since the graph of $y = \sin\theta$ is symmetric with respect to the vertical line $\theta = \frac{\pi}{2}$ on the interval $0 \leq \theta \leq \pi$.
Turning now to the proof of the theorem, we accept statements (a)–(c) there and write

$$\int_{C_R} f(z)e^{iaz}\,dz = \int_0^{\pi} f(Re^{i\theta})\exp(iaRe^{i\theta})Rie^{i\theta}\,d\theta.$$

Since

$$|f(Re^{i\theta})| \leq M_R \quad \text{and} \quad |\exp(iaRe^{i\theta})| \leq e^{-aR\sin\theta}$$

and in view of Jordan's inequality, it follows that

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| \leq M_R R\int_0^{\pi} e^{-aR\sin\theta}\,d\theta < \frac{M_R\pi}{a}$$

The final limit in the theorem is now evident since $M_R \to 0$ as $R \to \infty$.

Example 7.3.1.

Let us find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x\,\sin x\,dx}{x^2+2x+2}.$$

As usual, the existence of the value in question will be established by our actually finding it.
We write

$$f(z) = \frac{z}{z^2+2z+2} = \frac{z}{(z-z_1)(z-\bar{z}_1)},$$

where $z_1 = 1+i$. The point z_1, which lies above the x axis, is a simple pole of the function $f(z)e^{iz}$, with residue

$$B_1 = \frac{z_1 e^{iz_1}}{z_1-\bar{z}_1} \tag{7.19}$$

Hence, when $R > \sqrt{2}$ and C_R denotes the upper half of the positively oriented circle $|z| = R$,

$$\int_{-R}^{R} \frac{xe^{ix}}{x^2+2x+2}\,dx = 2\pi i B_1 - \int_{C_R} f(z)e^{iz}\,dz;$$

and this means that

$$\int_{-R}^{R} \frac{x\sin x\,\,dx}{x^2+2x+2} = \mathrm{Im}(2\pi i B_1) - \mathrm{Im}\int_{C_R} f(z)e^{iz}\,dz \tag{7.20}$$

Now

$$\left|\mathrm{Im}\int_{C_R} f(z)e^{iz}\,dz\right| \leq \left|\int_{C_R} f(z)e^{iz}\,dz\right| \tag{7.21}$$

and we note that when z is a point on C_R,

$$|f(z)| \leq M_R \quad \text{where} \quad M_R = \frac{R}{(R-\sqrt{2})^2}$$

and that $|e^{iz}| = e^y \leq 1$ for such a point. By proceeding as we did in the examples in previous sections, we cannot conclude that the right-hand side of inequality (7.21), and hence its left-hand side, tends to zero as R tends to infinity. For the quantity

$$M_R \pi R = \frac{\pi R^2}{(R-\sqrt{2})^2} = \frac{\pi}{\left(1 - \frac{\sqrt{2}}{R}\right)^2}$$

does not tend to zero. The above theorem does, however, provide the desired limit, namely

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\, dz = 0,$$

since

$$M_R = \frac{\frac{1}{R}}{\left(1 - \frac{\sqrt{2}}{R}\right)^2} \to 0 \; as \; R \to \infty.$$

So it does, indeed, follow from inequality (7.21) above that the left-hand side there tends to zero as R tends to infinity. Consequently, equation (7.20), together with expression (7.19) for the residue $B1$, we get,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x \; dx}{x^2 + 2x + 2} = \text{Im}(2\pi i B_1) = \frac{\pi}{e}(\sin 1 + \cos 1). \tag{7.22}$$

a

Chapter 8

Conformal Mapping

In this chapter, we introduce and develop the concept of a conformal mapping, with emphasis on connections between such mappings and harmonic functions. Applications to physical problems will follow in the following chapters.

8.1 Preservation of Angles

Let C be a smooth arc, represented by the equation

$$z = z(t) \quad (a \leq t \leq b), \tag{8.1}$$

and let $f(z)$ be a function defined at all points z on C. The equation

$$w = f[z(t)] \quad (a \leq t \leq b), \tag{8.2}$$

is a parametric representation of the image Γ of C under the transformation $w = f(z)$.

Suppose that C passes through a point $z_0 = z(t_0) \quad (a < t_0 < b)$ at which f is analytic and that $f'(z_0) \neq 0$. We know from prior knowledge, if $w(t) = f[z(t)]$, then

$$w'(t_0) = f'[z(t_0)]z'(t_0); \tag{8.3}$$

and this means that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(2)(t_0). \tag{8.4}$$

Statement (2) is useful in relating the directions of C and Γ at the points z_0 and $w_0 = f(z_0)$, respectively.

To be specific, let θ_0 denote a value of arg $z'(t_0)$ and let ϕ_0 be a value of arg $w'(t_0)$. According to the discussion of unit tangent vectors **T** near the end of Sec. 4.2, the number θ_0 is the angle of inclination of a directed line tangent to C at z_0 and ϕ_0 is the angle of inclination of a directed line tangent to Γ at the point $w_0 = f(z_0)$. (See Fig. 8.1) In view of statement (2), there is a value ψ_0 of arg $f'[z(t_0)]$ such that

$$\phi_0 = \psi_0 + \theta_0. \tag{8.5}$$

Thus $\phi_0 - \theta_0 = \psi_0$, and we find that the angles ϕ_0 and θ_0 differ by the angle of rotation

$$\psi_0 = \arg f'(4)(z_0). \tag{8.6}$$

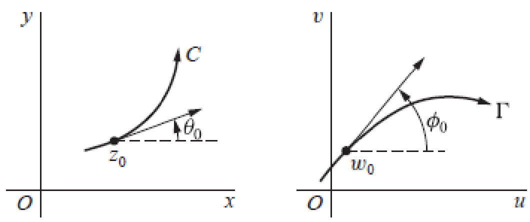


Figure 8.1: *Graphical representation of the mapping* $w = z^2$

Now let C_1 and C_2 be two smooth arcs passing through z_0, and let θ_1 and θ_2 be angles of inclination of directed lines tangent to C_1 and C_2, respectively, at z_0. We know from the preceding paragraph that the quantities

$$\phi_1 = \psi_0 + \theta_1 \ \text{ and } \ \phi_2 = \psi_0 + \theta_2 \tag{8.7}$$

are angles of inclination of directed lines tangent to the image curves Γ_1 and Γ_2, respectively, at the point $w_0 = f(z_0)$. Thus $\phi_2 - \phi_1 = \theta_2 - \theta_1$; that is, the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 is the same in magnitude and sense as the angle $\theta_2 - \theta_1$ from C_1 to C_2. Those angles are denoted by α in Fig. 8.2.

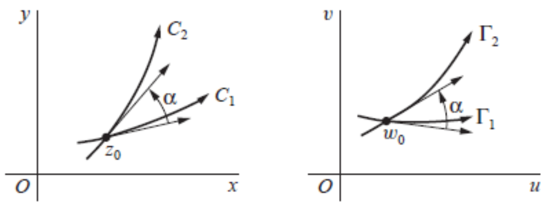


Figure 8.2: *Graphical representation of the mapping $w = z^2$*

Because of this angle-preserving property, a transformation $w = f(z)$ is said to be *conformal* at a point z_0 if f is analytic there and $f'(z0) \neq 0$. Such a transformation is actually conformal at each point in some neighborhood of z_0. For it must be analytic in a neighborhood of z_0; and since its derivative f' is continuous in that neighborhood, A transformation $w = f(z)$, defined on a domain D, is referred to as a conformal transformation, or conformal mapping, when it is conformal at each point in$D$. That is, the mapping is conformal in$D$ if f is analytic in D and its derivative f has no zeros there. Each of the elementary functions studied in Chap. 3 can be used to define a transformation that is conformal in some domain.

8.1.1 Preservation of angles Examples: Example:01

The mapping $w = e^z$ is conformal throughout the entire z plane since $(e^z)' = e^z \neq 0$ for each z. Consider any two lines $x = c_1$ and y $= c_2$ in the z plane, the first directed upward and the second directed to the right. We know from prior knowledge, their images under the mapping $w = e^z$ are a positively oriented circle centered at the origin and a ray from the origin, respectively. As illustrated in Fig. 2.4 (Sec. 2.3), the angle between the lines at their point of intersection is a right angle in the negative direction, and the same is true of the angle between the circle and the ray at the corresponding point in the w plane. The conformality of the mapping $w = e^z$ is also illustrated in Figs. 7 and 8 of Appendix 2.

Example 8.1.1.

Consider two smooth arcs which are level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the real and imaginary components, respectively, of a function

$$f(z) = u(x, y) + iv(x, y) \tag{8.8}$$

and suppose that they intersect at a point z_0 where f is analytic and $f'(z0) \neq 0$. The transformation w=f(z) is conformal at z_0 and maps these arcs into the lines $u = c_1$ and $v = c2$, which are orthogonal at the point $w_0 = f(z_0)$. According to our theory, then, the arcs must be orthogonal at z_0. This has already been verified and illustrated in Exercises 7 through 11 of Sec. 26. A mapping that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called an *isogonal mapping*

Example 8.1.2.

The transformation w=, which is a reflection in the real axis, is isogonal but not conformal. If it is followed by a conformal transformation, the resulting transformation w=f() is also isogonal but not conformal. Suppose that f is not a constant function and is analytic at a point z_0 . If, in addition, $f'(z_0) = 0$, then z0 is called a critical point of the transformation w=f(z).

Example 8.1.3.

The point $z_0 = 0$ is a critical point of the transformation $w = 1 + z_2$, which is a composition of the mappings $Z = z_2$ and $w = 1 + Z$. A ray $\theta = \alpha$ from the point $z_0 = 0$ is evidently mapped onto the ray from the point $w_0 = 1$ whose angle of inclination is 2α, and the angle between any two rays drawn from $z_0 = 0$ is doubled by the transformation. More generally, it can be shown that if z_0 is a critical point of a transformation w=f(z), there is an integer $m(m \geq 2)$ such that the angle between any two smooth arcs passing through z_0 is multiplied by m under that transformation. The integer m is the smallest positive integer such that $f^{(m)}(z_0) \neq 0$. Verification of these facts is left to the exercises.

8.2 Scale Factors

Another property of a transformation w=f(z) that is conformal at a point z_0 is obtained by considering the modulus of f'(z0). From the definition of derivative and a property of limits involving moduli that was derived in Exercise 7, Sec. 18, we know that

$$|f'(z_0)| = \left| \lim_{zz0} \frac{f(z)f(z_0)}{zz_0} \right| = \lim_{zz0} \frac{|f(z)f(z_0)|}{|zz_{0|}} \tag{8.9}$$

Now $|zz_0|$ is the length of a line segment joining z_0 and z, and $|f(z)f(z_0)|$ is the length of the line segment joining the points $f(z_0)$ and f(z) in the w plane. Evidently, then, if z is near the point z_0 , the ratio

|f(z)f(z0)|$_{\overline{|zz0|}}$

of the two lengths is approximately the number $|f'(z_0)|$. Note that $|f(z_0)|$ represents an expansion if it is greater than unity and a contraction if it is less than unity. Although the angle of rotation *arg* f'(z) (Sec. 101) and the scale factor |f'(z)| vary, in general, from point to point, it follows from the continuity of f' (see Sec. 52) that their values are approximately *arg* f'(z0) and |f'(z0)| at points z near z_0. Hence the image of a small region in a neighborhood of z_0 conforms to the original region in the sense that it has approximately the same shape. A large region may, however, be transformed into a region that bears no resemblance to the original one.

8.2.1 Scale Factors Examples: Example:01

When $f(z) = z_2$, the transformation

$$w = f(z) = x^2y^2 + i2xy \tag{8.10}$$

is conformal at the point z = 1 + i, where the half lines

$$y = x(x \geq 0) \text{ and } x = 1(y \geq 0) \tag{8.11}$$

intersect. We denote those half lines by C_1 and C_2 (Fig. 136), with positive sense upward. Observe that the angle from C1 to C2 is /4 at their point of intersection. Since the image of a point z = (x, y) is a point in the w plane whose rectangular coordinates are

$$u = x^2 y^2 \text{ and } v = 2xy \tag{8.12}$$

the half line C_1 is transformed into the curve Γ_1 with parametric representation

$$u = 0\, v = 2x^2 (0 \leq x < \infty) \tag{8.13}$$

Thus Γ_1 is the upper half $v \geq 0$ of the v axis. The half line C_2 is transformed into the curve Γ_2 represented by the equations

$$u = 1y^2\, v = 2y(0 \leq y < \infty) \tag{8.14}$$

Hence Γ_2 is the upper half of the parabola $v^2 = 4(u1)$. Note that in each case, the positive sense of the image curve is upward.

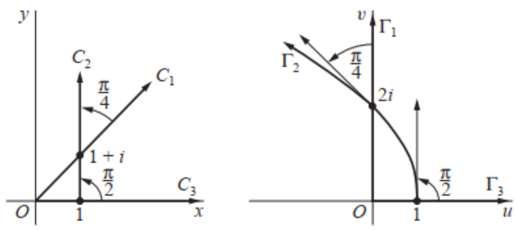


Figure 8.3: $w = z^2$

If u and v are the variables in representation (8.14) for the image curve Γ_2 , then

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v} \tag{8.15}$$

In particular, $dv/du = 1$ when $v = 2$. Consequently, the angle from the image curve Γ_1 to the image curve Γ_2 at the point $w = f(1+i) = 2i$ is $/4$, as required by the conformality of the mapping at $z = 1 + i$. The angle of rotation $/4$ at the point $z = 1 + i$ is, of course, a value of

$$\arg[f'(1+i)] = \arg[2(1+i)] = \frac{\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{8.16}$$

The scale factor at that point is the number

$$|f'(1+i)| = |2(1+i)| = 2\sqrt{2} \tag{8.17}$$

To illustrate how the angle of rotation and the scale factor can change from point to point, we note that they are 0 and 2, respectively, at the point $z = 1$ since $f(1) = 2$. See Fig 8.3, where the curves C_2 and Γ_2 are the ones just discussed and where the nonnegative x axis C_3 is transformed into the nonnegative u axis Γ_3.

8.3 Local Inverses

A transformation $w = f(z)$ that is conformal at a point z_0 has a local inverse there. That is, if $w_0 = f(z_0)$, then there exists a unique transformation $z = g(w)$, which is defined and analytic in a neighborhood N of w_0, such that $g(w_0) = z_0$ and $f[g(w)] = w$ for all points w in N. The derivative of $g(w)$ is, moreover,

$$g'(w) = \frac{1}{f'(z)} \tag{8.18}$$

We note from expression (8.18) that the transformation $z = g(w)$ is itself conformal at w_0.

Assuming that $w = f(z)$ is, in fact, conformal at z_0, let us verify the existence of such an inverse, which is a direct consequence of results in advanced calculus. The conformality of the transformation $w = f(z)$ at z_0 implies that there is some neighborhood of z_0 throughout which f is analytic. Hence if we write

$$z = x + iy, \quad z_0 = x_0 + iy_0, \quad \text{and} \quad f(z) = u(x,y) + iv(x,y) \tag{8.19}$$

we know that there is a neighborhood of the point (x_0, y_0) throughout which the functions $u(x,y)$ and $v(x,y)$, along with their partial derivatives of all orders, are continuous.

Now the pair of equations

$$u = u(x,y), \quad v = v(x,y) \tag{8.20}$$

represents a transformation from the neighborhood just mentioned into the uv plane. Moreover, the determinant

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y \tag{8.21}$$

which is known as the Jacobian of the transformation, is nonzero at the point (x_0, y_0). For, in view of the Cauchy–Riemann equations $u_x = v_y$ and $u_y = v_x$, one can write J as

$$J = (u_x)^2 + (v_x)^2 = |f'(z)|^2 \tag{8.22}$$

and $f(z_0) \neq 0$ since the transformation $w = f(z)$ is conformal at z_0. The above continuity conditions on the functions $u(x, y)$ and $v(x, y)$ and their derivatives,

together with this condition on the Jacobian, are sufficient to ensure the existence of a local inverse of transformation (8.20) at (x_0, y_0). That is, if

$$u_0 = u\left(x_0, y_0\right) \quad \text{and} \quad v_0 = v\left(x_0, y_0\right) \tag{8.23}$$

then there is a unique continuous transformation

$$x = x(u, v), \quad y = y(u, v) \tag{8.24}$$

defined on a neighborhood N of the point (u_0, v_0) and mapping that point onto (x_0, y_0), such that equations (8.20) hold when equations (8.24) hold. Also, in addition to being continuous, the functions (8.24) have continuous first-order partial derivatives satisfying the equations

$$x_u = \frac{1}{J} v_y, \quad x_v = -\frac{1}{J} u_y, \quad y_u = -\frac{1}{J} v_x, \quad y_v = \frac{1}{J} u_x \tag{8.25}$$

throughout N.

If we write $w = u + iv$ and $w_0 = u_0 + iv_0$, as well as

$$g(w) = x(u, v) + iy(u, v) \tag{8.26}$$

the transformation $= g(w)$ is evidently the local inverse of the original transformation $w = f(z)$ at z_0. Transformations (8.20) and (8.24) can be written

$$u + iv = u(x, y) + iv(x, y) \quad \text{and} \quad x + iy = x(u, v) + iy(u, v) \tag{8.27}$$

and these last two equations are the same as

$$w = f(z) \quad \text{and} \quad z = g(w) \tag{8.28}$$

where g has the desired properties. Equations (8.25) can be used to show that g is analytic in N. Details are left to the exercises, where expression (8.18) for $g'(w)$ is also derived.

Example 8.3.1.

We know, that if $f(z) = e^z$, the transformation $w = f(z)$ is conformal everywhere in the z plane and, in particular, at the point $z_0 = 2i$. The image of this choice of z_0 is the point $w_0 = 1$. When points in the w plane are expressed in the form $w = exp(i)$, the local inverse at z_0 can be obtained by writing $g(w) = \log w$, where log w denotes the branch

$$\log w = \ln \rho + i\phi \quad (\rho > 0, \pi < \theta < 3\pi) \tag{8.29}$$

of the logarithmic function, restricted to any neighborhood of w_0 that does not contain the origin. Observe that

$$g(1) = \ln 1 + i2\pi = 2\pi i \tag{8.30}$$

and that when w is in the neighborhood,

$$f[g(w)] = \exp(\log w) = w \tag{8.31}$$

Also

$$g'(w) = \frac{d}{dw} \log w = \frac{1}{w} = \frac{1}{\exp z} \tag{8.32}$$

in accordance with equation (8.18).

Note that if the point $z_0 = 0$ is chosen, one can use the principal branch

$$\log w = \ln \rho + i\phi \quad (\rho > 0, -\pi < \phi < \pi) \tag{8.33}$$

of the logarithmic function to define g. In this case, $g(1) = 0$.

Part II

Laplace Transformation

Chapter 9

Laplace Transformation

9.1 Definition of the Laplace Transform

Integral Transform: If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. Similarly, a definite integral such as $\int_a^b K(s,t)f(t)\,dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an integral transform, where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_a^b K(s,t)f(t)\,dt$ is defined as a limit:

$$\int_0^\infty K(s,t)f(t)\,dt = \lim_{b\to\infty}\int_0^{by} K(s,t)f(t)\,dt \tag{9.1}$$

If the limit in Eq(9.1) exists, then we say that the integral exists or is convergent; if the limit does not exist, the integral does not exist and is divergent. The limit in Eq(9.1) will, in general, exist for only certain values of the variables.

Definition (Laplace transform): The function $K(s,t)$ in Eq(9.1) is called the kernel of the transform. The choice $K(s,t) = e^{s^t}$ as the kernel gives us an especially important integral transform.

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt \tag{9.2}$$

is said to be the Laplace transform of f, provided that the integral converges.

When the defining integral Eq(9.2) converges, the result is a function of s. In general discussion we shall use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace transform.

Example 9.1.1.

Evaluate $\mathcal{L}\{1\}$

Using Eq(9.2)

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^\infty e^{-st}\,dt \\ &= \lim_{b\to\infty}\frac{-e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}\end{aligned}$$

provided that $s > 0$. In other words, when $s > 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$. The use of the limit sign becomes somewhat tedious, so we shall adopt the notation $|_0^\infty$ as a shorthand for writing $\lim_{b\to\infty}()|_0^b$. For example,

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}f(1)\,dt = \lim_{b\to\infty}\int_0^\infty e^{-st}\,dt \\ &= \frac{-e^{-st}}{s}\bigg|_0^b = \frac{1}{s}, \quad s > 0\end{aligned}$$

At the upper limit, it is understood that we mean $e^{-st} \to 0$ as $t \to \infty$.

Example 9.1.2.

Evaluate $\mathcal{L}\{sin2t\}$
From Eq(9.2) and using integration by parts,

$$\begin{aligned}\mathcal{L}\{\sin2t\} &= \int_0^\infty e^{-st}\sin2t\,dt = \frac{-e^{-st}\sin2t}{s}\bigg|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos2t\,dt \\ &= \frac{2}{s}\int_0^\infty e^{-st}\cos2t\,dt, \quad s > 0 \\ &= \frac{2}{s}\frac{-e^{-st}\cos2t}{s}\bigg|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin2t\,dt \\ &= \frac{2}{s^2} - \frac{4}{s^2}\,\mathcal{L}\{\sin2t\}.\end{aligned}$$

At this point we have an equation with $\mathcal{L}\{sin2t\}$ on both sides of the equality. Solving for that quantity yields the result

$$\mathcal{L}\{sin2t\} = \frac{2}{s^2+4}, \quad s > 0 \tag{9.3}$$

Example 9.1.3.

Evaluate $\mathcal{L}\{\cosh(kt)\}$

$$\mathcal{L}\{\cosh(kt)\} = \int_0^b e^{-st}\cosh(kt)dt.$$

Utilizing the identity $\cosh(kt) = \frac{e^{kt}+e^{-kt}}{2}$,

$$\begin{aligned}\mathcal{L}\{\cosh(kt)\} &= \int_0^b \frac{(e^{kt}+e^{=kt})e^{-st}}{2}dt \\ &= \int_0^b \frac{e^{-t(s-k)}+e^{-t(s+k)}}{2} \\ &= \frac{1}{2}\int_0^b e^{-t(s-k)}dt + \frac{1}{2}\int_0^b e^{-t(s+k)}dt,\end{aligned} \tag{9.4}$$

Evaluating the first integrand in Eq(9.4)

$$\begin{aligned}\text{Taking} \quad u &= (s-k)t \\ du &= (s-k)dt,\end{aligned}$$

by making appropriate substitutions

$$\begin{aligned}\frac{1}{2}\int_0^b e^{-t(s-k)}dt &= \frac{1}{2}\int_0^b \frac{1}{(s-k)}e^{-u}du \\ &= \frac{1}{2(s-k)}.\end{aligned}$$

Making similar substitutions we can solve the second integrand and thus arrive to the following solution

$$\mathcal{L}\{\cosh(kt)\} = \frac{1}{2(s-k)} + \frac{1}{2(s+k)} \tag{9.5}$$
$$= \frac{s}{s^2-k^2} \quad s > k$$

Properties of Laplace transform ($\mathcal{L}$ is a linear transform): For a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\,dt = \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s) \tag{9.6}$$

Because of the property given in Eq(9.6), is said to be a linear transform.

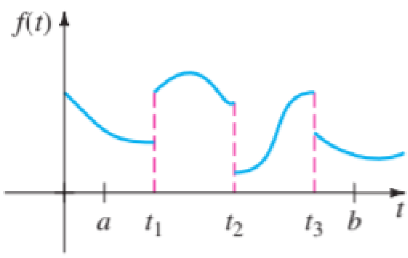
We state the generalization of some of the preceding examples by means of the following table. From this point on we shall also refrain from stating any restrictions on s; it is understood that s is sufficiently restricted to guarantee the convergence of the appropriate Laplace transform.

Transforms of some basic functions:

$a)\ \mathcal{L}\{1\} = \frac{1}{s}$	
$b)\ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$	$c)\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
$d)\ \mathcal{L}\{sinkt\} = \frac{k}{s^2+k^2}$	$e)\ \mathcal{L}\{coskt\} = \frac{s}{s^2+k^2}$
$f)\ \mathcal{L}\{sinhkt\} = \frac{k}{s^2-k^2}$	$g)\ \mathcal{L}\{coshkt\} = \frac{s}{s^2-k^2}$

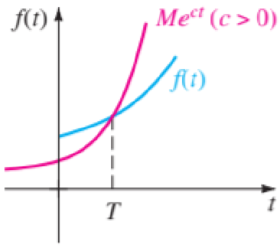
Table 9.1: Caption

Sufficient conditions for existence of $\mathcal{L}\{f(t)\}$: The integral that defines the Laplace transform does not have to converge. For example, neither $\mathcal{L}\{\frac{1}{t}\}$ nor $\mathcal{L}\{e^{t^2}\}$ exists. Sufficient conditions guaranteeing the existence of $\mathcal{L}\{f(t)\}$ are that f be piecewise continuous on $[0, \infty)$ and that f be of exponential order for $t > T$. Recall that a function f is piecewise continuous on $[0, \infty)$ if, in any interval $0 \leq a \leq t \leq b$, there are at most a finite number of points t_k, $k = 1, 2, \ldots, n$ (t_{k-1}, t_k) at which f has finite discontinuities and is continuous on each open interval $(t_{k-1},\ t_k)$. See Figure below.



The concept of exponential order is defined in the following manner.
Exponential Order: A function f is said to be of exponential order c if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$
If f is an increasing function, then the condition $|f(t)| \leq Me^{c^t}$, simply states that the graph of f on the interval (T, ∞)does not grow faster

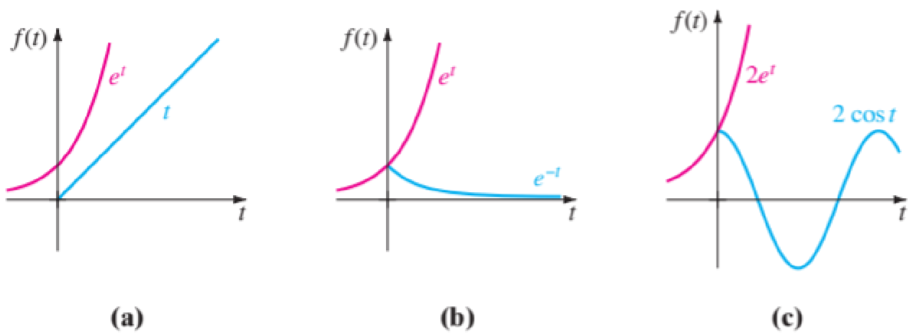
than the graph of the exponential function Me^{c^t}, where c is a positive constant. See Figure below (where f is a exponential order c)



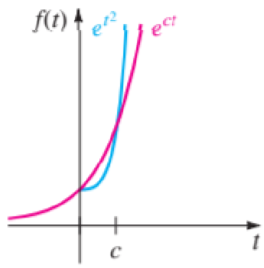
The functions $f(t) = t,\ f(t) = e^{-t},\ and\ f(t) = 2cost$ are all of exponential order $c = 1$ for $t > 0$, since we have, respectively,

$$|t| \le e^t, \quad |e^{-t}| \le e^t, \quad and \quad |2cost| \le 2e^t$$

A comparison of the graphs on the interval $(0, \infty)$ is given in Figure below.



These function are of exponential order $c = 1$
A function such as is not of exponential order, since, as shown in Figure below, its graph grows faster than any positive linear power of e for $t > c > 0$.



A positive integral power of t is always of exponential order, since, for $c > 0$,

$$|t^n| \leq Me^{ct} \quad \text{or} \quad \left|\frac{t^n}{e^{ct}}\right| \quad for\ t > T$$

is equivalent to showing that $\lim_{t\to\infty} \frac{t^n}{e^{ct}}$ is finite for $n = 1, 2, 3, \dots$ The result follows by n applications of L'Hôpital's Rule.

Theorem (Sufficient condition for existence): If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

Proof: By the additive interval property of definite integrals we can write

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt = I_1 + I_2$$

The integral I_1 exists because it can be written as a sum of integrals over intervals on which $e^{st}f(t)$ is continuous. Now since f is of exponential order, there exist constants c, $M > 0$, $T > 0$ so that $|f(t)| \leq Me^{ct}$ for $t > T$. We can then write

$$|I_2| \leq \int_T^\infty |e^{-st} f(t)|\, dt \leq M \int_T^\infty e^{-st} e^{ct}\, dt = M \int_T^\infty e^{-(s-c)t}\, dt = M\frac{e^{-(s-c)T}}{s-c}$$

for $s > c$. Since $\int_T^\infty Me^{-(s-c)t}\, dt$ converges, the integral $\int_T^\infty |e^{-st} f(t)|\, dt$ converges by the comparison test for improper integrals. This, in turn, implies that I_2 exists for $s > c$. The existence of I_1 and I_2 implies that $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ exists for $s > c$.

9.2 Laplace Transform of Derivatives and a Piecewise Continuous Function

Transform a derivative: To do this we need to evaluate quantities such as $\mathcal{L}\{dy/dt\}$ and $\mathcal{L}\{d^2y/dt^2\}$. For example, if f' is continuous for $t \geq 0$, the integration by parts guves

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$
$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$or \quad \mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{9.7}$$

Here we assumed that $e^{-st}f(t) \to 0\ as\ t \to \infty$. Similarly,

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt = e^{-st}f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f'(t)\,dt$$
$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$
$$= s[sF(s) - f(0)] - f'(0)$$

$$or \ \ \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - sf'(0). \tag{9.8}$$

In like manner it can be shown that

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \tag{9.9}$$

The recursive nature of the Laplace transform of the derivatives of a function f should be apparent from the results in (8.11), (8.12), and (8.13). The next theorem gives the Laplace

transform of the nth derivative of f. The proof is omitted **Theorem (Transform of a Derivative:)** If $f, f', \ldots, f^(n-1)$ are continuous on $[0, \infty)$ and are of exponential order and if $f^n(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0),$$

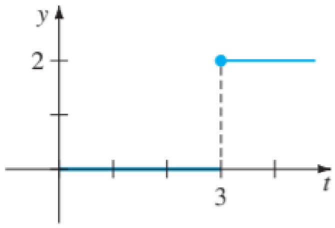
where $F(s) = \mathcal{L}\{f(t)\}$.

Laplace transform of a piecewise continuous function:

Example: Transform a piecewise continuous function.// Evaluate $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$$

Solution: The function f, shown in Figure below, is piecewise continuous and of exponential order for $t > 0$.



Since f is defined in two pieces, $\mathcal{L}\{f(t)\}$ is expressed as the sum of two integrals:

$$\begin{aligned} \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^3 e{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt \\ = 0 + \left. \frac{2e^{-st}}{-s} \right|_3^\infty \\ = \frac{2e^{-3s}}{s}, \quad s > 0, \end{aligned}$$

9.3 Inverse Transform and Solving IVP using Laplace Transforms

The inverse transform problem:

If F(s) represents the Laplace transform of a function f(t), that is, $\mathcal{L}\{f(t)\} - F(s)$, we then say f(t) is the inverse Laplace transform of F(s) and write $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Some inverse transforms

$a)\,1 = \mathcal{L}^{-1}\{\frac{1}{s}\}$	
$b)\,t^n = \mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\},\quad n = 1, 2, 3, \dots$	$c)\,e^{at} = \mathcal{L}^{-1}\{\frac{1}{s-a}\}$
$d)\,sinkt = \mathcal{L}^{-1}\{\frac{k}{s^2+k^2}\}$	$e)\,coskt = \mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\}$
$f)\,sinhkt = \mathcal{L}^{-1}\{\frac{k}{s^2-k^2}\}$	$g)\,coshkt = \mathcal{L}^{-1}\{\frac{s}{s^2-k^2}\}$

9.4 Inverse transform example

Example 9.4.1.

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$$

Solution: To match the form given in the table, we identify $n + 1 = 5\ or\ n = 4$ and then multiply and divide by 4!:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4$$

Properties of inverse transform: $\mathcal{L}^{-1}$ is a linear transform; that is for constants α and β

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}, \tag{9.10}$$

where F and G are the transforms of some functions f and g.

9.5 Termwise division and linearity

Example 9.5.1.

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$$

Solution: We first rewrite the given function of s as two expressions by means of termwise division and then use above linearity of inverse transform,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}\\ &= -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\\ &= -2cos2t+3sin2t\end{aligned}$$

9.6 Partial fractions: Distinct linear factors

Example 9.6.1.

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

Solution: There exist unique real constants A, B, and C so that

$$\begin{aligned}&\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\\ &=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}\\ &=\frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}\end{aligned}$$

Since the denominators are identical, the numerators are identical:

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

By comparing coefficients of powers of s on both sides of the equality, we know that it is equivalent to a system of three equations in the three unknowns A, B, and C. However, there is a shortcut for determining these unknowns. If we set $s = 1$, $s = 2$, $and\, s = -4$ in the equation, we obtain, respectively,

$$16 = A(-1)(5), \quad 25 = B(1)(6), \quad and \quad 1 = C(-5)(-6)$$

and so

$$A = -\frac{16}{5}, \quad B = \frac{25}{6}, \quad and \quad C = \frac{1}{30}$$

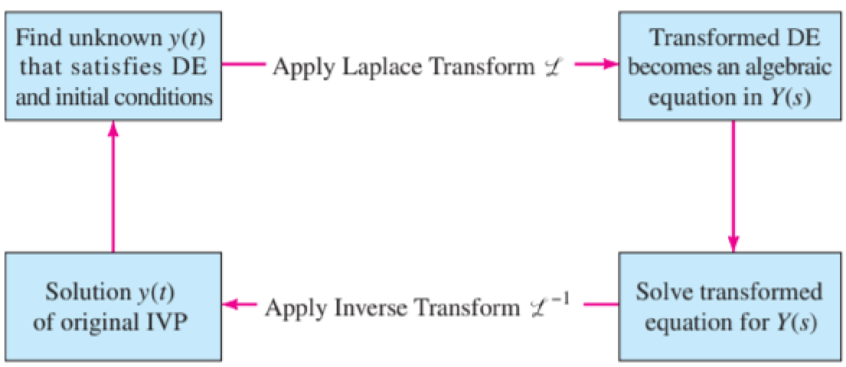
Hence the partial fraction decomposition is

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

and thus, from the linearity of $\mathcal{L}^{-1}$ and part (c) of the table

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$
$$= -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$
$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Solving IVP using Laplace Transforms The process can be shortly summerized by means of following figure:



9.7 Solving a First-Order IVP:

Example 9.7.1.

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13sin2t, \quad y(0) = 6$$

Solution: We first take the transform of each member of the differential equation:

$$\mathcal{L}\{\frac{dy}{dt}\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{sin2t\}$$

From the theorem,

$$\mathcal{L}\{\frac{dy}{dt}\} = sY(s) - y(0) = sY(s) - 6$$

and from the table

$$\mathcal{L}\{sin2t\} = 2/(s^2 + 4)$$

so, above equation becomes,

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4} \quad or \quad (s + 3)Y(s) = 6 + \frac{26}{s^2 + 4}$$

Solving the last equation for Y(s), we get

$$Y(s) = \frac{6}{s + 3} + \frac{26}{(s + 3)(s^2 + 4)} = \frac{6s^2 + 50}{(s + 3)(s^2 + 4)}$$

Since the quadratic polynomial s^2+4 does not factor using real numbers, its assumed numerator in the partial fraction decomposition is a linear polynomial in s:

$$\frac{6s^2 + 50}{(s + 3)(s^2 + 4)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + 4}$$

Putting the right-hand side of the equality over a common denominator and equating numerators gives $6s^2 + 50 = A(s^2 + 4) + (Bs + C)(s + 3)$. Setting $s = -3$ then immediately yields $A = 8$. Since the denominator has no more real zeros, we equate the coefficients of s^2 and $s : 6 = A + B \, and \, 0 = 3B + C$. Using the value of A in the first equation gives

$B = -2$, and then using this last value in the second equation gives $C = 6$. Thus

We are not quite finished because the last rational expression still has to be written as two fractions. This was done by termwise division,

$$y(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

It follows from parts (c), (d), and (e) of the table that the solution of the initial value problem is

$$y(t) = 8e^{-3t} - 2cos2t + 3sin2t$$

9.8 Solving a second order IVP

Example 9.8.1.

Solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$$

Solution:N Proceeding as in Example above, we transform the DE. We take the sum of the transforms of each term, use the theorem, use the given initial conditions, use (c) of table, and then solve for Y(s):

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$\begin{aligned} Y(s) &= \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} \\ &= \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} \end{aligned}$$

The details of the partial fraction decomposition of Y(s) have already been carried out in Example above. So, the solution becomes,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2}t + \frac{1}{30}e^{-4t}$$