

Practice Sheet 2

1. Evaluate the following limits:

$$(i) \lim_{z \rightarrow 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2} \quad (ii) \lim_{z \rightarrow 1+i} \left\{ \frac{z - 1 - i}{z^2 - 2z + 2} \right\}^2 \quad (iii) \lim_{z \rightarrow i} \frac{z^2 + 1}{z^6 + 1}.$$

2. If $f(z) = \frac{2z-1}{3z+2}$, prove that $f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{7}{(3z_0+2)^2}$ provided $z_0 \neq -\frac{2}{3}$.

3. Let $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3+4i$. Is $f(z)$ continuous at $z = 2i$?

4. Find all points of discontinuity for the function $f(z) = \frac{2z-3}{z^2+2z+2}$.

5. Using the definitions, find the derivative of each function at the indicated points

$$(i) f(z) = \frac{2z-i}{z+2i} \quad \text{at } z = -i$$

$$(ii) f(z) = 3z^{-2} \quad \text{at } z = 1+i.$$

6. Evaluate the following limits using L' Hôpital's rule

$$(i) \lim_{z \rightarrow 2i} \frac{z^2+4}{2z^2+(3-4i)z-6i} \quad (ii) \lim_{z \rightarrow 0} \frac{z - \sin z}{z^3}.$$

7. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u + iv$ as an analytic function of z

i) $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ ii) $u(x, y) = xe^x \cos y - ye^x \sin y$

iii) $u(x, y) = e^{-x}(x \sin y - y \cos y)$