Q. 01

a

p	q	$\neg p$	$\neg q$	$p \wedge q$ (s)	$\neg p \wedge \neg q$ (t)	$s \vee t$	$p \leftrightarrow q$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

$\therefore$ logically equivalent.

b $\underbrace{\text{An integer has two factors}}_{p}$ $\underbrace{\text{only if}}_{\rightarrow}$ $\underbrace{\text{it is prime}}_{q}$.

Converse: $\boxed{q \rightarrow p}$ An integer is prime only if it has two factors.
OR, if ... then

Inverse: $\boxed{\neg p \rightarrow \neg q}$ An integer doesn't have two factors only if it isn't prime.
OR, if ... then

Contrapositive: $\boxed{\neg q \rightarrow \neg p}$ An integer isn't prime only if it doesn't have two factors.
OR, if ... then

c
i) $\forall x \ L(x, Ali)$
ii) $\forall x \ \neg L(x, Babul)$ OR, $\neg \exists x \ L(x, Babul)$
iii) $\exists x \ L(Cindy, x)$
iv) $\forall x \ \neg L(Didar, x)$ OR, $\neg \exists x \ L(Didar, x)$

Answer sheet

Induction (Set A) $\boxed{802}$

$\boxed{a}$ $1\times2+2\times2^2+3\times2^3+\cdots+n.2^n=(n-1)\cdot2^{n+1}+2 \quad \forall n\in\mathbb{N}$ ---- (I)

Base step: for $n=1$, in $P(n=1)$, $L.H.S = 1\times2=2$

$R.H.S = (1-1)2^{1+1}+2 = 2$

$\Rightarrow P(n=1)$ is true.

Inductive step:

Assumption $P(n=k)$ is true.

$\rightarrow 1\times2+2\times2^2+3\times2^3+\cdots+k.2^k=(k-1)2^{k+1}+2$ ---- (II)

To be proved: $P(n=k+1)$ is true.

$\rightarrow 1\times2+2\times2^2+3\times2^3+\cdots+k.2^k+(k+1).2^{k+1}=k.2^{k+2}+2$ ---- (III)

L.H.S of (III) $= \underbrace{1\times2+2\times2^2+\cdots+k.2^k}+(k+1).2^{k+1}$

$=[(k-1)2^{k+1}+2]+(k+1).2^{k+1}$ [From (II)]

$=k.2^{k+1}-2^{k+1}+2+k.2^{k+1}+2^{k+1}$

$=2k.2^{k+1}+2$

$=k.2^{k+2}+2 =$ R.H.S of (III) [Proved]

Or,

$=(k-1+k+1).2^{k+1}+2$

$=2k.2^{k+1}+2$

$=k.2^{k+2}+2=R.H.S$

[Proved]

$\boxed{b}$ $P(n)$: $4^{2n+1}+10\times11^{2n}$ is divisible by $7 \quad \forall n\in\mathbb{N}_0$ ---- (I)

Base step: for $n=0$, L.H.S of $P(n=0) = 4^{2\times0+1}+10\times11^{2\times0}=14$

Now, $14=7\times2+0$

Remainder

Quotient

$\Rightarrow P(n=0)$ is true.

Inductive step:

Assumption $4^{2k+1}+10\times11^{2k}=7m$ for some $k\in\mathbb{N}_0$ ---- (II) & the quotient, $m\in\mathbb{Z}$

To be proved: $4^{2k+3}+10\times11^{2k+2}=7a$ for $a\in\mathbb{Z}$ ---- (III)

L.H.S of (III) $= 4^{2k+3}+10\times11^{2k+2}$

$=4^{2k+1}.16+10\times11^{2k+2}$

$=(7m-10\times11^{2k}).16+10\times11^{2k}\times121$ [From (II), $4^{2k+1}=7m-10\times11^{2k}$]

$=7m\times16-16\times10\times11^{2k}+10\times11^{2k}\times121$

$=7\times16m+10\times11^{2k}(-16+121)=7\times16m+105\times10\times11^{2k}$

$=7(16m+150\times11^{2k})=7a=R.H.S$

Recurrence: Set A $\boxed{Q03}$

$\boxed{a}$ P_n = Principal with compound interest after nth year
P_{n-1} = " " " " " (n-1)th year

$$P_n = P_{n-1} + 5\% \text{ of } P_{n-1} + 500$$

$$\Rightarrow \boxed{P_n = 1.05P_{n-1} + 500} \text{ (Ans)}$$

$\boxed{b}$ $a_{n+1} = 3a_n + 18a_{n-1} - 40a_{n-2}$

Step 1

$\Rightarrow r^{n+1} = 3r^n + 18r^{n-1} - 40r^{n-2}$

$\Rightarrow r^3 - 3r^2 - 18r + 40 = 0$

$\Rightarrow (r+4)(r-2)(r-5) = 0$ [Using Calculator]

$\Rightarrow r_1 = -4,\ r_2 = 2,\ r_3 = 5$

Step 2: General solution, $a_n = \alpha_1(-4)^n + \alpha_2(2)^n + \alpha_3(5)^n \cdots (I)$

Step 3: Using the base values and (I) for $n = 1, 3, 4$, We get

$a_1 = -4\alpha_1 + 2\alpha_2 + 5\alpha_3 = 16 \cdots (II)$

$a_3 = -64\alpha_1 + 8\alpha_2 + 125\alpha_3 = 322 \cdots (III)$

$a_4 = 256\alpha_1 + 16\alpha_2 + 625\alpha_3 = 1010 \cdots (IV)$

Solving (II), (III) & (IV),

$\alpha_1 = -1$

$\alpha_2 = 1$

$\alpha_3 = 2$

So, the solution, $\boxed{a_n = -(-4)^n + 2^n + 2(5)^n}$ (Ans)

$\boxed{c}$ $a_9 - 18a_7 = \left[-(-4)^9 + 2^9 + 2(5)^9\right] - \left[18\left\{-(-4)^7 + 2^7 + 2(5)^7\right\}\right]$

$\boxed{a_9 - 18a_7 = 1059190}$ (Ans)

Note: If they apply brute force and get the correct answer, please give 2 out of 2!

Number Theory | Set A | Q 04

a Note that, 227 is a prime

Using FLT, $5^{226} \equiv 1 \pmod{227}$

So, $5^{250} \equiv 5^{24} \pmod{227}$ —— $(**)$

Now, Step 1: $(24)_{10} = (?)_2$

$$
\begin{array}{r|l}
2 & 24 \\
2 & 12 - 0 \\
2 & 6 - 0 \\
2 & 3 - 0 \\
2 & 1 - 1 \\
 & 0 - 1
\end{array}
$$

$\Rightarrow (24)_{10} = (11000)_2$

So, $24 = 2^4 + 2^3$

$\Rightarrow 5^{24} = 5^{2^4+2^3} = 5^{2^4} \cdot 5^{2^3}$ —— $(*)$

Step 2:

$5^2 \equiv 25 \pmod{227}$

$5^{2^2} \equiv 25^2 \equiv 171 \pmod{227}$

$5^{2^3} \equiv 171^2 \equiv 185 \pmod{227}$ —— (I)

$5^{2^4} \equiv 185^2 \equiv 175 \pmod{227}$ —— (II)

Using (I) & (II) in $(*)$ & $(**)$,

$5^{250} \equiv 5^{24} \equiv 185 \times 175 \equiv 141 \pmod{227}$

Problem (c)

$a|b \Rightarrow b = ak,\ k \in \mathbb{Z}$ --- (I)

$b|c \Rightarrow c = bm,\ m \in \mathbb{Z}$ --- (II)

So, $4b^2 + 19bc$

$= b(4b + 19c)$

$= b[4ak + 19bm]$ [From (I) & (II)]

$= b[4ak + 19akm]$ [From (I)]

$= ab[4k + 19km]$

$ab \mid 4b^2 + 19bc$ [Proved]

b

491) 1267 (2
982
285) 491 (1
285
206) 285 (1
206
79) 206 (2
158
48) 79 (1
48
31) 48 (1
31
17) 31 (1
17
14) 17 (1
14
3) 14 (4
12
2) 3 (1
2
1) 2 (2
2
0

$1267 = 491 \times 2 + 285$

$491 = 285 \times 1 + 206$

$285 = 206 \times 1 + 79$

$206 = 79 \times 2 + 48$

$79 = 48 \times 1 + 31$

$48 = 31 \times 1 + 17$

$31 = 17 \times 1 + 14$

$17 = 14 \times 1 + 3$

$14 = 3 \times 4 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

So, $\gcd(1267, 491) = 1$

(Ans)

c $a|b \Rightarrow b = ak,\ k \in \mathbb{Z}$ — (I)

$b|c \Rightarrow c = bm,\ m \in \mathbb{Z}$ — (II)

So, $4b^2 + 19bc$

$= 4(ak)^2 + 19(ak)bm$ [From (I), & (II)]

$= 4a^2k^2 + 19ak(ak)m$ [From (I)]

$= 4a^2k^2 + 19a^2k^2m$

$= a$

Number Theory | Set A | Q05

[a] $33x \equiv 4 \pmod{19}$

$\Rightarrow 38x - 5x \equiv 4 - 19 \pmod{19}$ $[38x = 2 \times 19x$ & $-19 \equiv 0 \pmod{19}]$

$\Rightarrow -5x \equiv -15 \pmod{19}$

$\Rightarrow \boxed{x \equiv 3 \pmod{19}}$

(Ans)

[b] Like the previous proof.

[c] $(34334)_5 = (3\times5^4 + 4\times5^3 + 3\times5^2 + 3\times5^1 + 4)_{10}$

$= (1875 + 500 + 75 + 19)_{10}$

$= (2469)_{10} = 2469$

$(41303)_5 = (4\times5^4 + 1\times5^3 + 3\times5^2 + 0\times5^1 + 3)_{10}$

$= (2500 + 125 + 75 + 3)_{10}$

$= (2703)_{10} = 2703$

Now, $(34334)_5 \times (41303)_5 = (2469 \times 2703)_{10} = (6673707)_{10}$

So, $(6673707)_{10} = (3202024312)_5$

(Ans) $(3202024312)_5$

For problem (b)
Alternatively

$a = \prod_{i=1}^{K} p_i^{\alpha_i} \quad b = \prod_{i=1}^{K} p_i^{\beta_i}$

then $ab = \prod_{i=1}^{K} p_i^{\alpha_i} \prod_{i=1}^{K} p_i^{\beta_i} = \prod_{i=1}^{K} p_i^{\alpha_i+\beta_i}$

$\gcd(a,b) = \prod_{i=1}^{K} p_i^{\min(\alpha_i,\beta_i)}$ —— (I) $\quad \operatorname{lcm}(a,b) = \prod_{i=1}^{K} p_i^{\max(\alpha_i,\beta_i)}$ —— (II)

So, (I) × (II) $\Rightarrow \gcd \times \operatorname{lcm} = \prod_{i=1}^{K} p_i^{\min(\alpha_i,\beta_i)+\max(\alpha_i,\beta_i)}$ —— (III)

Now, for any i, $\min(\alpha_i,\beta_i) + \max(\alpha_i,\beta_i) = \alpha_i + \beta_i$ —— (IV)

Using (IV) in (I), $\gcd \times \operatorname{lcm} = \prod_{i=1}^{K} p_i^{\alpha_i+\beta_i}$ = product

[Proved]