Q.01

a

p	q	$\neg p$	$\neg q$	$\underset{s}{\neg p \vee q}$	$\underset{t}{p \vee \neg q}$	$s \wedge t$	$p \leftrightarrow q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	F	F
F	F	T	T	T	T	T	T

$\therefore$ Logically equivalent.

b $\underbrace{\text{An integer is prime}}_{p}$ $\underbrace{\text{only if}}_{\rightarrow}$ it $\underbrace{\text{has two factors}}_{q}$.

Converse: $\boxed{q \rightarrow p}$ An integer has two factors only if it is prime
OR, If an integer has two factors, then it is prime

Inverse: $\boxed{\neg p \rightarrow \neg q}$ An integer is not prime only if it doesn't have two factors.
OR, If an integer is not prime then it doesn't have two factors.

Contrapositive: $\boxed{\neg q \rightarrow \neg p}$ An integer doesn't have two factors only if it isn't prime.
OR, If then

c i) $\forall x \; L(x, \text{Cindy})$

ii) $\forall x \neg L(\text{Ali}, x)$ OR, $\neg\left(\exists x \; L(\text{Ali}, x)\right)$

iii) $\forall x \; \neg L(\text{Babul}, x)$ OR, $\neg(\exists x \; L(\text{Babul}, x))$

iv) $\exists x \; L(x, \text{Didar})$

Induction (Set B) $\boxed{802}$

$\boxed{a}$ $1\times3 + 2\times3^2 + \cdots + n\times3^n = \frac{3}{4}\left[(2n-1)3^n+1\right] \quad \forall n\in\mathbb{N} \quad \cdots (\mathrm{I})$

Base step: for $n=1$, L.H.S of $P(n=1) = 1\times3 = 3$

R.H.S of $P(n=1) = \frac{3}{4}\left[(2\cdot1-1)3^1+1\right]$

$= \frac{3}{4}\cdot4 = 3$

$\Rightarrow P(n=1)$ is true.

Inductive step:

Assumption: $P(n=K)$ is true.

$\Rightarrow 1\times3 + 2\times3^2 + 3\times3^3 + \cdots + K\cdot3^K = \frac{3}{4}\left[(2K-1)3^K+1\right]$ for some $K\in\mathbb{N}$ $\cdots (\mathrm{II})$

To be proved: $P(n=K+1)$ is true.

$\Rightarrow 1\times3 + 2\times3^2 + 3\times3^3 + \cdots + K\cdot3^K + (K+1)3^{K+1} = \frac{3}{4}\left[(2K+1)3^{K+1}+1\right] \quad \cdots (\mathrm{III})$

L.H.S of (III)

$= \underbrace{1\times3 + 2\times3^2 + 3\times3^3 + \cdots + K\cdot3^K} + (K+1)3^{K+1}$

$= \frac{3}{4}\left[(2K-1)3^K+1\right] + (K+1)\cdot3^{K+1}$ [from (II)]

$= \frac{3}{4}\left[(2K-1)3^K+1 + (4K+4)\cdot3^K\right]$

Or, $= \frac{3}{4}\left[3^K(6K+3)+1\right]$ from $(2K-1+4K+4)3^K$ and the rest is the same.

$= \frac{3}{4}\left[2K\cdot3^K - 3^K + 1 + 4K\cdot3^K + 4\cdot3^K\right]$

$= \frac{3}{4}\left[(6K+3)\cdot3^K+1\right]$

$= \frac{3}{4}\left[3(2K+1)\cdot3^K+1\right]$

$= \frac{3}{4}\left[(2K+1)\cdot3^{K+1}+1\right] =$ R.H.S of (III) [Proved]

$\boxed{b}$ $5^{2n+1} - 12\times16^{2n}$ is divisible by 7 $\forall n\in\mathbb{N}_0$

Base step: for $n=0$, $5^{2\times0+1} - 12\times16^{2\times0} = 5-12 = -7 = 7(-1)+0$

Quotient ↑ ↑ Remainder

$\Rightarrow P(n=0)$ is true.

Inductive step: Assumption: $5^{2K+1} - 12\times16^{2K} = 7m$ for some $K\in\mathbb{N}_0$ & $m\in\mathbb{Z}$ —— (II)

To be proved: $5^{2K+3} - 12\times16^{2K+2} = 7a$ for $a\in\mathbb{Z}$ —— (III)

L.H.S of (III) $= 5^{2K+3} - 12\cdot16^{2K+2} = 25\cdot5^{2K+1} - 12\times16^{2K+2} = 25(7m + 12\cdot16^{2K}) - 12\cdot16^{2K}\cdot256$ [From (II)]

$= 25\times7m + 25\cdot12\cdot16^{2K} - 12\cdot16^{2K}\cdot256 = 7\times25m + 12\times16^{2K}(25-256)$

$= 7\times25m + 12\times16^{2K}\times231 = 7(25m + 12\times16^{2K}\times33) = 7a =$ R.H.S of (III)

Recurrence (Set B) [Q03]

[a] P_n = Principal with compound interest after nth year
P_{n-1} = " " " " " $(n-1)$th "

Then,

$$P_n = P_{n-1} + 10\% \text{ of } P_{n-1} + 1000$$

$$\boxed{P_n = 1.1 P_{n-1} + 1000}$$

[b] $a_{n+1} = -3a_n + 18a_{n-1} + 40a_{n-2}$

Step 1: $r^{n+1} = -3r^n + 18r^{n-1} + 40r^{n-2}$ for $r \neq 0 \in \mathbb{R}$

$\rightarrow r^3 + 3r^2 - 18r - 40 = 0$ Solve it in calculator or just factorize

$(r+5)(r+2)(r-4) = 0$

$\rightarrow r_1 = -5,\ r_2 = -2,\ r_3 = 4$

Step 2: General solution, $a_n = \alpha_1(-5)^n + \alpha_2(-2)^n + \alpha_3(4)^n$ .... (I)

Step 3: Using the boundary conditions and (I) for $n = 0, 2, 4$,
We get,

$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = -2$ .... (II)

~~$a_1 = (-5)\alpha_1 - 2\alpha_2 + 4\alpha$~~

$a_2 = 25\alpha_1 + 4\alpha_2 + 16\alpha_3 = -38$ .... (III)

$a_4 = 625\alpha_1 + 16\alpha_2 + 256\alpha_3 = -1010$ .... (IV)

Solving (II), (III) & (IV),

$\alpha_1 = -2$
$\alpha_2 = -1$
$\alpha_3 = 1$

$$\boxed{\text{So, } a_n = -2(-5)^n - (-2)^n + 4^n}$$

[c] $a_9 - 18a_7 = \left[-2(-5)^9 - (-2)^9 + 4^9\right] - 18\left(-2(-5)^7 - (-2)^7 + 4^7\right]$

$= 1059190$

[Ans]

Note: They can apply brute force to get the result.
Please give 2 out of 2 if the answer is correct!

Number Theory (Set B) [Q.04].

Note that, 229 is a prime

$5^{229-1} \equiv 1 \pmod{229}$

$\Rightarrow 5^{228} \equiv 1 \pmod{229}$

So, $5^{254} \equiv 5^{26} \pmod{229}$ —— (*)

Step 1: $(26)_{10} = (?)_2$

$$\begin{array}{r|l} 2 & 26 \\ 2 & 13 - 0 \\ 2 & 6 - 1 \\ 2 & 3 - 0 \\ 2 & 1 - 1 \\ & 0 - 1 \end{array}$$

$(26)_{10} = (11010)_2$

$\therefore 26 = 2^4 + 2^3 + 2$

$\Rightarrow 5^{26} = 5^{2^4} \cdot 5^{2^3} \cdot 5^{2^1}$ —— (**)

Step 2:

$5^{2^1} \equiv 25 \pmod{229}$ —— (I)

$5^{2^2} \equiv 25^2 \equiv 167 \pmod{229}$ ——

$5^{2^3} \equiv 167^2 \equiv 180 \pmod{229}$ —— (II)

$5^{2^4} \equiv (180)^2 \equiv 111 \pmod{229}$ —— (III)

Step 3: $5^{254} \equiv 5^{26} \equiv 5^{2^4} \cdot 5^{2^3} \cdot 5^{2^1} \equiv 111 \times 180 \times 25 \equiv 51 \pmod{229}$

[b]

$$\begin{array}{l} 570)\,1189\,(2 \\ \quad 1140 \\ \quad\; 49)\,570\,(11 \\ \qquad\;\, 49 \\ \qquad\;\, 80 \\ \qquad\;\, 49 \\ \qquad\; 31)\,49\,(1 \\ \qquad\qquad 31 \\ \qquad\quad 18)\,31\,(1 \\ \qquad\qquad\; 18 \\ \qquad\qquad 13)\,18\,(1 \\ \qquad\qquad\quad 13 \\ \qquad\qquad\; 5)\,13\,(2 \\ \qquad\qquad\quad\; 10 \\ \qquad\qquad\quad 3)\,5\,(1 \\ \qquad\qquad\qquad 3 \\ \qquad\qquad\quad 2)\,3\,(1 \\ \qquad\qquad\qquad 2 \\ \qquad\qquad\quad 1)\,2\,(2 \\ \qquad\qquad\qquad 2 \\ \qquad\qquad\qquad 0 \end{array}$$

$1189 = 570 \times 2 + 49$

$570 = 49 \times 11 + 31$

$49 = 31 \times 1 + 18$

$31 = 18 \times 1 + 13$

$18 = 13 \times 1 + 5$

$13 = 5 \times 2 + 3$

$5 = 3 \times 1 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

$\boxed{\gcd(570, 1189) = 1}$

(Ans)

[c] $a \mid b \Rightarrow b = ak,\ k \in \mathbb{Z}$ — (I)

$b \mid c \Rightarrow c = bm,\ m \in \mathbb{Z}$ — (II)

$7b^2 + 24ac$

$= 7(ak)^2 + 24a(bm)$ [from (I) & (II)]

$= 7a^2k^2 + 24a \cdot akm$ [from (I)]

$= 7a^2k^2 + 24a^2km$

$= a^2(7k^2 + 24km) = a^2 \times \text{int}$

$\Rightarrow a^2 \mid 7b^2 + 24ac$

Number theory (305) Set B

a $24x \equiv 7 \pmod{19}$

$\Rightarrow 24x \equiv 7-19 \equiv -12 \pmod{19}$

$\Rightarrow 2x \equiv -1 \pmod{19}$ [Dividing by 12]

$\Rightarrow 2x \equiv 19-1 \equiv 18 \pmod{19}$

$\Rightarrow x \equiv 9 \pmod{19}$ [Dividing by 2]

b ~~gcd(x,y)~~

Let, $x = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$
$y = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \cdots p_k^{\beta_k}$
for $i=1,2,3\cdots,k$; p_i are distinct prime numbers; $\alpha_i, \beta_i \in \mathbb{N}_0$

Now, $\gcd(x,y) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \cdots p_k^{\min(\alpha_k,\beta_k)}$

$\operatorname{lcm}(x,y) = p_1^{\max(\alpha_1,\beta_1)} p_2^{\max(\alpha_2,\beta_2)} \cdots p_k^{\max(\alpha_k,\beta_k)}$

Also, $xy = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \cdot p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_k^{\beta_k} = p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2} \cdots p_k^{\alpha_k+\beta_k}$ — (i)

$\gcd(x,y) \cdot \operatorname{lcm}(x,y) = p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)+\max(\alpha_2,\beta_2)} \cdots p_k^{\min(\alpha_k,\beta_k)+\max(\alpha_k,\beta_k)}$ — (ii)

Now, for any $i \in \{1,2,\cdots k\}$, $\min(\alpha_i,\beta_i)+\max(\alpha_i,\beta_i) = \alpha_i+\beta_i$

Therefore, $p_i^{\min(\alpha_i,\beta_i)+\max(\alpha_i,\beta_i)} = p_i^{\alpha_i+\beta_i}$ —— (*)

So, Applying (*) in (ii), $\gcd(x,y) \cdot \operatorname{lcm}(x,y) = p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdots p_k^{\alpha_k+\beta_k}$ —— (**)

Comparing (i) & (**), we get,

$\gcd(x,y) \cdot \operatorname{Lcm}(x,y) = xy$ [Proved]

c ~~34334~~

$(34334)_7 = (?)_{10}$

$\Rightarrow (34334)_7 = (3\times7^4 + 4\times7^3 + 3\times7^2 + 3\times7^1 + 4)_{10} = (8747)_{10} = 8747$

$(41303)_7 = (4\times7^4 + 1\times7^3 + 3\times7^2 + 0\times7 + 3)_{10} = (10097)_{10} = 10097$

So, $8747 \times 10097 = 88318459$

$\therefore (88318459)_{10} = (2121460135)_7$ [Like binary conversion]

Ans: $(2121460135)_7$