La función costo está definida por:

$$\chi^2(a_0,a_1) = \sum_{i=1}^{n}(y_i - a_0 - a_1 x_i)^2$$

Para minimizar la función, debemos calcular su gradiente e igualarlo al cero:

$$\nabla \chi^2(a_0,a_1) = \left(\frac{\partial \chi^2}{\partial a_0}, \frac{\partial \chi^2}{\partial a_1}\right) = (0,0)$$

$$\therefore \quad \frac{\partial \chi^2}{\partial a_0} = 0 \quad y \quad \frac{\partial \chi^2}{\partial 1} = 0$$

$$\frac{\partial \chi^2}{\partial a_0} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i)(-1) = -2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i)$$

$$\frac{\partial \chi^2}{\partial a_1} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i)(-x_i) = -2\sum_{i=1}^{n}(y_i x_i - a_0 x_i - a_1 (x_i)^2)$$

Minimizando:

$$-2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i) = 0 \Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a_0 - \sum_{i=1}^{n} a_1 x_i = 0$$

Despejando a_0:

$$\sum_{i=1}^{n} y_i - a_1\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_0 \Rightarrow \sum_{i=1}^{n} y_i - a_1\sum_{i=1}^{n} x_i = n a_0$$

$$\frac{\sum_{i=1}^{n} y_i - a_1\sum_{i=1}^{n} x_i}{n} = a_0 \Rightarrow \sum_{i=1}^{n}\frac{y_i}{n} - a_1\sum_{i=1}^{n}\frac{x_i}{n} = a_0 \quad (1)$$

$$\Rightarrow \bar{y} - a_1\bar{x} = a_0 \; ; \text{ donde } \bar{y} = \sum_{i=1}^{n}\frac{y_i}{n} \; y \; \bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}$$

que corresponden a los valores medios de los puntos y sus imágenes.

$$-2\sum_{i=1}^{n}\left(y_i x_i - a_0 x_i - a_1 (x_i)^2\right) = 0$$

$$\sum_{i=1}^{n} y_i x_i - a_0 \sum_{i=1}^{n} x_i - a_1 \sum_{i=1}^{n} (x_i)^2 = 0$$

Despejando a_1:

$$\sum_{i=1}^{n} y_i x_i - a_0 \sum_{i=1}^{n} x_i = a_1 \sum_{i=1}^{n} (x_i)^2$$

Remplazando a_0 por la ecuación 1:

$$\sum_{i=1}^{n} y_i x_i - \left[\frac{1}{n}\left(\sum_{i=1}^{n} y_i - a_1 \sum_{i=1}^{n} x_i\right) \sum_{i=1}^{n} x_i\right] = a_1 \sum_{i=1}^{n} (x_i)^2$$

$$\sum y_i x_i - \frac{\sum y_i \sum x_i}{n} + a_1 \frac{\left(\sum x_i\right)^2}{n} = a_1 \sum (x_i)^2$$

$$\sum y_i x_i - \frac{\sum y_i \sum x_i}{n} = a_1 \sum (x_i)^2 - a_1 \frac{\left(\sum x_i\right)^2}{n}$$

$$\sum y_i x_i - \frac{\sum y_i \sum x_i}{n} = a_1 \left(\sum (x_i)^2 - \frac{\left(\sum x_i\right)^2}{n}\right)$$

Despejando a_1:

$$\frac{\sum y_i x_i - \frac{\sum y_i \sum x_i}{n}}{\sum (x_i)^2 - \frac{\left(\sum x_i\right)^2}{n}} = a_1$$

Para una función de costo cuadrática:

$$\chi^2(a_0,a_1,a_2) = \sum_{i=1}^{n}(y_i - a_0 - a_1x_i - a_2x_i^2)^2$$

$$\frac{\partial \chi^2}{\partial a_0} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1x_i - a_2x_i^2)(-1) \Rightarrow -2\sum_{i=1}^{n}(y_i - a_0 - a_1x_i - a_2x_i^2)$$

Minimizando:

$$-2\sum_{i=1}^{n}(y_i - a_0 - a_1x_i - a_2x_i^2) = 0 \Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n}(a_0 + a_1x_i + a_2x_i^2)$$

$$\sum_{i=1}^{n}\left[a_0 + a_1x_i + a_2x_i^2 = y_i\right]$$

$$\frac{\partial \chi^2}{\partial a_1} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1x_i - a_2x_i^2)(-x_i) = -2\sum_{i=1}^{n}(y_ix_i - a_0x_i - a_1x_i^2 - a_2x_i^3)$$

$$-2\sum_{i=1}^{n}(y_ix_i - a_0x_i - a_1x_i^2 - a_2x_i^3) = 0 \Rightarrow \sum_{i=1}^{n} y_ix_i = \sum_{i=1}^{n}(a_0x_i + a_1x_i^2 + a_2x_i^3)$$

$$\sum_{i=1}^{n}\left[a_0x_i + a_1x_i^2 + a_2x_i^3 = y_ix_i\right]$$

$$\frac{\partial \chi^2}{\partial a_2} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1x_i - a_2x_i^2)(-x_i^2) = -2\sum_{i=1}^{n}(y_ix_i^2 - a_0x_i^2 - a_1x_i^3 - a_2x_i^4)$$

$$-2\sum_{i=1}^{n}(y_ix_i^2 - a_0x_i^2 - a_1x_i^3 - a_2x_i^4) = 0 \Rightarrow \sum_{i=1}^{n} y_ix_i^2 = \sum_{i=1}^{n}(a_0x_i^2 + a_1x_i^3 + a_2x_i^4)$$

$$\sum_{i=1}^{n}\left[a_0x_i^2 + a_1x_i^3 + a_2x_i^4 = y_ix_i^2\right]$$

Regularidad: $$\sum_{i=1}^{n}\left[a_0x_i^{m} + a_1x_i^{m+1} + a_2x_i^{m+2} = y_ix_i^{m}\right]$$

donde $m = \{0, 1, 2\}$