25)

a) $L_n(x) = \frac{1}{n!} e^{x} \frac{d^n}{dx^n}(x^n e^{-x})$

$L_2(x) = \frac{1}{2!} e^{x} \frac{d^2}{dx^2}(x^2 e^{-x})$

$\frac{d}{dx}(x^2 e^{-x}) = -x^2 e^{-x} + 2x e^{-x}$

$= e^{-x}[-x^2 + 2x]$

$\frac{d^2}{dx^2}(x^2 e^{-x}) = [-2x+2]e^{-x} - e^{-x}[-x^2+2x]$

$= e^{-x}[x^2 - 4x + 2]$

$L_2(x) = \frac{1}{2} \cancel{e^{x} e^{-x}} [x^2 - 4x + 2]$

$L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$

b) $\frac{1}{2}(x^2 - 4x + 2) = 0$

$x^2 - 4x + 2 = 0$

$x_0 = \frac{4 - \sqrt{16 - 4(1)(2)}}{2} = \frac{4 - \sqrt{8}}{2} = 2 - \sqrt{2}$

$x_1 = \frac{4 + \sqrt{16 - 4(1)(2)}}{2} = \frac{4 + \sqrt{8}}{2} = 2 + \sqrt{2}$

C)

$$w_0 = \int_0^{\infty} e^{-x}\left(\frac{x-(2+\sqrt{2})}{(2-\sqrt{2})-(2+\sqrt{2})}\right)dx = \lim_{t\to\infty}\int_0^t e^{-x}\left(\frac{x-2-\sqrt{2}}{-2\sqrt{2}}\right)dx$$

$$\frac{-2-\sqrt{2}}{-2\sqrt{2}}$$

$$\frac{\cancel{2}}{\cancel{2}\sqrt{2}} + \frac{\sqrt{\cancel{2}}}{2\sqrt{\cancel{2}}}$$

$$\frac{\sqrt{2}}{2} + \frac{1}{2} = \boxed{\frac{\sqrt{2}+1}{2}}$$

$$w_0 = \lim_{t\to\infty}\int_0^t e^{-x}\left(\frac{-x}{2\sqrt{2}} + \frac{\sqrt{2}+1}{2}\right)dx$$

$$w_0 = \lim_{t\to\infty}\frac{-1}{2\sqrt{2}}\int_0^t e^{-x}x\,dx + \lim_{t\to\infty}\frac{\sqrt{2}+1}{2}\int_0^t e^{-x}dx$$

$$w_0 = -\frac{\sqrt{2}}{4}\lim_{t\to\infty}\int_0^t e^{-x}x\,dx + \frac{\sqrt{2}+1}{2}\lim_{t\to\infty}\int_0^t e^{-x}dx$$

$u = x$
$du = dx$
$dv = e^{-x}dx$
$v = -e^{-x}dx$

$$w_0 = -\frac{\sqrt{2}}{4}\left[\lim_{t\to\infty} -xe^{-x}\Big|_0^t + \lim_{t\to\infty}\int_0^t e^{-x}dx\right] + \frac{\sqrt{2}+1}{2}\lim_{t\to\infty} -e^{-x}\Big|_0^t$$

$$w_0 = -\frac{\sqrt{2}}{4}\left[\lim_{t\to\infty}\frac{-t}{e^t} + \lim_{t\to\infty} -e^{-x}\Big|_0^t\right] + \frac{\sqrt{2}+1}{2}\lim_{t\to\infty}\left[\overset{0}{\cancel{\frac{-1}{e^t}}} + \frac{1}{e^0}\right]$$

$$w_0 = -\frac{\sqrt{2}}{4}\left[\overset{\text{L'H:}}{\lim_{t\to\infty}}\overset{0}{\cancel{\frac{-1}{e^t}}} + \lim_{t\to\infty}\left(\overset{0}{\cancel{-\frac{1}{e^t}}} + \frac{1}{e^0}\right)\right] + \frac{\sqrt{2}+1}{2}$$

$$w_0 = -\frac{\sqrt{2}}{4} + \frac{\sqrt{2}+1}{2} = \frac{2\sqrt{2}+2}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{2}+2}{4}}$$

$$w_1 = \int_0^\infty e^{-x}\left(\frac{x-(2-\sqrt{2})}{(2+\sqrt{2})-(2-\sqrt{2})}\right)dx = \lim_{t\to\infty}\int_0^t e^{-x}\left(\frac{x-2+\sqrt{2}}{2\sqrt{2}}\right)dx$$

$$\frac{\cancel{-2}}{\cancel{2}\sqrt{2}}+\frac{\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}$$

$$\frac{1}{2}-\frac{\sqrt{2}}{2}$$

$$\boxed{\frac{1-\sqrt{2}}{2}}$$

$$w_1 = \lim_{t\to\infty}\int_0^t e^{-x}\left(\frac{x}{2\sqrt{2}}+\frac{1-\sqrt{2}}{2}\right)dx$$

$$w_1 = \lim_{t\to\infty}\frac{1}{2\sqrt{2}}\int_0^t e^{-x}x\,dx + \lim_{t\to\infty}\frac{1-\sqrt{2}}{2}\int_0^t e^{-x}dx$$

$$w_1 = \frac{\sqrt{2}}{4}\lim_{t\to\infty}\int_0^t e^{-x}x\,dx + \frac{1-\sqrt{2}}{2}\lim_{t\to\infty}\int_0^t e^{-x}dx$$

$U = x$

$du = dx$

$dV = e^{-x}dx$

$V = -e^{-x}$

$$w_1 = \frac{\sqrt{2}}{4}\left[\lim_{t\to\infty} -e^{-x}x\Big|_0^t + \lim_{t\to\infty}\int_0^t e^{-x}dx\right] + \frac{1-\sqrt{2}}{2}\lim_{t\to\infty}\left[-e^{-x}\Big|_0^t\right]$$

$$w_1 = \frac{\sqrt{2}}{4}\left[\lim_{t\to\infty}\frac{-t}{e^t} + \lim_{t\to\infty}\left(-e^{-x}\Big|_0^t\right)\right] + \frac{1-\sqrt{2}}{2}\lim_{t\to\infty}\left[\overset{0}{\cancel{\frac{-1}{e^t}}}+\frac{1}{e^0}\right]$$

$$w_1 = \frac{\sqrt{2}}{4}\left[\overset{L'H:}{\lim_{t\to\infty}}\overset{0}{\cancel{\frac{-1}{e^t}}} + \lim_{t\to\infty}\left(\overset{0}{\cancel{\frac{-1}{e^t}}}+\frac{1}{e^0}\right)\right] + \frac{1-\sqrt{2}}{2}$$

$$w_1 = \frac{\sqrt{2}}{4}+\frac{1-\sqrt{2}}{2} = \frac{\sqrt{2}}{4}+\frac{2-2\sqrt{2}}{4} = \boxed{\frac{2-\sqrt{2}}{4}}$$

d)
$$\int_0^{\infty} e^{-x} x^3 dx = \sum_{i=0}^{1} w_i f(x_i) = 6$$

Empezamos mostrando que $\int_0^{\infty} e^{-x} x^3 dx = 6$:

Partimos de:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \quad y \quad \Gamma(x) = (x-1)!$$

$$\therefore \int_0^{\infty} e^{-t} t^{x-1} dt = (x-1)!$$

Si elegimos $x=4$: $\int_0^{\infty} e^{-t} t^{4-1} dt = (4-1)!$

$$\int_0^{\infty} e^{-t} t^3 dt = 3! \Rightarrow \int_0^{\infty} e^{-t} t^3 dt = 6$$

Si cambiamos la variable de integración a x:

$$\boxed{\int_0^{\infty} e^{-x} x^3 dx = 6}$$

Ahora, utilizando los pesos y raíces hallados, mostramos que $\sum_{i=0}^{1} w_i f(x_i) = 6$:

Datos: $x_0 = 2-\sqrt{2}$; $x_1 = 2+\sqrt{2}$; $w_0 = \frac{\sqrt{2}+2}{4}$; $w_1 = \frac{2-\sqrt{2}}{4}$; $f(x) = x^3$

$$\sum_{i=0}^{1} w_i f(x_i) = w_0 f(x_0) + w_1 f(x_1) = w_0 (x_0)^3 + w_1 (x_1)^3$$

$$\left(\frac{\sqrt{2}+2}{4}\right)(2-\sqrt{2})^3 + \left(\frac{2-\sqrt{2}}{4}\right)(2+\sqrt{2})^3 = \left(\frac{\sqrt{2}+2}{4}\right)(8-12\sqrt{2}+12-2\sqrt{2}) + \left(\frac{2-\sqrt{2}}{4}\right)(8+12\sqrt{2}+12+2\sqrt{2})$$

$$\left(\frac{\sqrt{2}+2}{4}\right)(20-14\sqrt{2}) + \left(\frac{2-\sqrt{2}}{4}\right)(20+14\sqrt{2}) = \left(\frac{\sqrt{2}+2}{\cancel{4}}\right)\cancel{2}(10-7\sqrt{2}) + \left(\frac{2-\sqrt{2}}{\cancel{4}}\right)\cancel{2}(10+7\sqrt{2})$$

$$\left(\frac{\sqrt{2}}{2}+1\right)(10-7\sqrt{2})+\left(1-\frac{\sqrt{2}}{2}\right)(10+7\sqrt{2})$$

$$5\cancel{\sqrt{2}}-7+10-\cancel{7\sqrt{2}}+10+\cancel{7\sqrt{2}}-5\cancel{\sqrt{2}}-7$$

$$20-14=6$$

$$\therefore \sum_{i=0}^{1} w_i f(x_i)=6$$

Como $\int_0^\infty e^{-x}x^3\,dx=6$ y $\sum_{i=0}^{1} w_i f(x_i)=6$

Se cumple que:

$$\int_0^\infty e^{-x}x^3\,dx=\sum_{i=0}^{1} w_i f(x_i)=6$$